Math 128a Final Exam

K. Hare

December 12, 2003

NAME (printed) : ______________________________
(Last Name) (First Name)

Signature : ______________________________

Student Number : ______________________________

(1) Do NOT open this test booklet until told to do so

(2) Do ALL your work in this test booklet

(3) Show ALL your work

(4) Check that there are 15 problems and 17 pages (including this one)

(5) NO CALCULATORS

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	TOTAL

1 a: (4 pts) Evaluate

$$\int_{-1}^{1}\int_{-2}^{2} x^2 + y^2 + xy \; dx \; dy$$

Composite Trapezoidal rule using $n = m = 2$.

b: (4 pts) Describe how to evaluate $\int_1^\infty \frac{e^{-x}}{x^2}\,dx$.
DO NOT EVALUATE.

2 a: (3 pts) Define the local truncation error $\tau_{i+1}(h)$ for an Initial Valued Problem approximation method.

b: (5 pts) Prove that a higher order Taylor approximation of order n has a location truncation error of $\mathcal{O}(h^n)$.

3 a: (4 pts) Given that

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h)$$

Use Richardson's extrapolation to find a 3 point formula with error $\mathcal{O}(h^2)$.

b: (4 pts) Given the values $f(h) = a$ and $f(-2h) = b$:

i: Use a Lagrange polynomial to estimate $f(0)$.

ii: Use a Lagrange polynomial to estimate $f'(0)$.

4: (4 pts) Consider an Initial Valued Problem $y' = f(t, y)$. Find a and b so that the two methods

$$w_{i+1} = w_i + hf(t_i, w_i) + \frac{h^2}{2} f_t(t_i, w_i) + \frac{h^2}{2} f_w(t_i, w_i) f(t_i, w_i)$$

and

$$z_{i+1} = z_i + hf(t_i + a, z_i + b)$$

are equivalent. (i.e. have the same local truncation error.)

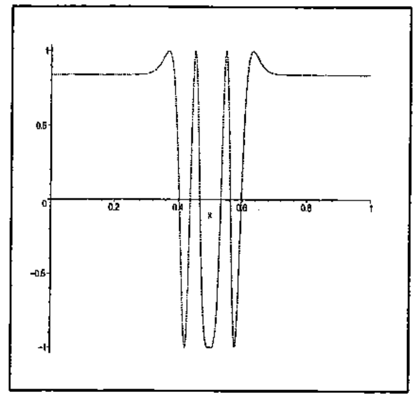
5 a: (4 pts) Use Composite Simpson's rule on

$$\int_{-2}^{2} x^2 - x \, dx$$

with $n = 4$. (i.e. your nodes are $-2, -1, 0, 1$ and 2.) What is the expected error of this method?

b: (4 pts) Consider $\int_0^1 f(x)\, dx$. Let the Trapezoidal method on this integral give 7. Let the Composite Trapezoidal method on this integral with $n = 3$ give 8. Use a variation of Romberg integration to determine a better approximation.

6 a: (4 pts) Consider the function described by the graph given here. Where would you expect the largest number of evaluations to occur, if you were using Adaptive Quadrature method.



b: (4 pts) Given $f(0) = 1$, $f(1) = 1$ and $f'(0) = 1$, estimate $\int_0^1 f(x)\,dx$ to the highest degree of precision.

7 a: (4 pts) Let p_n converge quadratically to 0. (i.e. it converges with order two.) Define $q_n = p_{2n}$. What is the order of convergence of q_n? (Justify your answer.)

b: (4 pts) Consider

$$f(x) = x^4$$

What is the rate of convergence of Newton's method near a root of $f(x)$. [Hint, the Theorem you are thinking of using doesn't apply in this case.]

8: (4 pts) Prove that if $\left|\frac{\partial f}{\partial y}(t,y)\right| \leq L$ for all (t,y) then $f(t,y)$ satisfies the Lipschitz condition with constant L.

9: (4 pts) Consider the Initial Valued Problem

$$y' = y + t \qquad y(0) = 1 \qquad 0 \leq t \leq 3$$

Use the Second Order Taylor method, with step size $h = 1$ to approximate $y(2)$.

10: (4 pts) Consider the Adams-Moulton implicit one-step method

$$w_0 = y(0), \quad \tilde{w}_{i+1} = \tilde{w}_i + \frac{h}{2}[f(t_{i+1}, w_{i+1}) + f(t_i, \tilde{w}_i)].$$

Recall that Euler's method is an explicit method. Use the predictor-corrector method, with these two methods, and with $h = 1$, to estimate $y(2)$ for the initial valued problem

$$y' = 2yt, \qquad y(0) = 1, \qquad 0 \leq t \leq 2$$

11 a: (4 pts) Rewrite

$$y'' + y' + y = t \qquad y(0) = 1 \qquad y'(0) = 2$$

into a system of differential equations. Then use Euler's method with step size 1 to estimate $y(2)$.

b: (4 pts) Using $y(0), y'(0), y(1)$ and $y'(1)$ as calculated above, estimate $y(2)$ using a Hermite polynomial. [If you did not do question **11 a**, use $y(1) = a$ and $y'(1) = b$]

12 a: (4 pts) Assume that A has an LU factorization. Prove that A^T has an LU factorization.

b: (6 pts) Given ideal conditions, what are the rates of convergences for the following methods. State the ideal conditions.

i: Newton's method

ii: Bisection method

iii: Gauss-Seidel method

13 a: (3 pts) Let A and B be $n \times n$ matrices. Prove, or find a counter example. If $AB = 0$ then $A = 0$ or $B = 0$.

b: (3 pts) Let A and B be $n \times n$ matrices. Prove, or find a counter example. If $AB = 0$ then $\det(A) = 0$ or $\det(B) = 0$.

14: (4 pts) Consider the points $(0,0)$, $(2,2)$ and $(2,0)$. Find any parametric curve that goes through these points in order. (It does not need to be a Bezier curve, but at the point $(2,2)$ the curve should have a continuous derivative).

15: (8 pts) For this question we are considering a method of Error control. Consider calculating w_{i+1} and $\tilde{w}_{i+1}$ for an Initial Valued Problem using two different method, based on w_i. Consider calculating $q = \left|\frac{\epsilon h}{\tilde{w}_{i+1} - w_{i+1}}\right|^{1/n}$. Lastly let $h_{\max}$ and $h_{\min}$ be upper bounds and lower bounds for the value of h.

Describe what you would do if

i: $q < 1$ and $hq > h_{\min}$.

ii: $q < 1$ and $hq < h_{\min}$.

iii: $q > 1$ and $hq > h_{\max}$.

iv: $q > 1$ and $hq < h_{\max}$.