MATH 113: INTRODUCTION TO ABSTRACT ALGEBRA (Section 4)

Final Examination

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- Please be sure to justify your answers, unless instructed otherwise.
- Do all five problems. Within each problem, you may use any preceding parts to answer a question, but not conversely.
- Organization matters. If your answers are scattered around, please let me know.
- Please label your answers.
- Please write legibly; if I cannot read your answers, then I cannot give you points.

Good luck!

1. (a) (*2 points*) Is the alternating group A_5 simple? Just answer yes or no.

   (b) (*4 points*) Let $\sigma = (12345) \in A_5$. Let H be the subgroup generated by σ. Express the non-identity elements of H as products of disjoint cycles.

   (c) (*4 points*) Let H be as above. Is H normal in A_5? If so, then prove the statement. If not, then give an example of $\tau \in A_5$ such that $\tau\sigma\tau^{-1} \notin H$.

   (d) (*4 points*) Consider the set of cosets A_5/H. If H were normal in A_5, then the operation on A_5/H given by

$$(\tau_1 H)(\tau_2 H) = (\tau_1\tau_2)H$$

   would be well-defined. Either prove that the operation is well-defined or give an example where the operation is ill-defined. (**Hint**: Use part (c).)

2. We have previously classified all finite groups of order ≤ 7. In fact, we have also seen all groups of order 8; this problem will guide you to recall, in the Platonic sense, what they are.

   (a) (*5 points*) State the Fundamental Theorem for Finitely Generated Abelian Groups, and prove that $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ iff m and n are coprime positive integers.

   (b) (*4 points*) Using part (a), determine all *non-isomorphic* abelian groups of order 8.

   (c) (*4 points*) Let G be a non-abelian group of order 8, and Z its center. To which group is Z isomorphic? (**Note**: You may state and use any theorem you wish; in particular, you may want to recall a theorem from the second midterm.)

   (d) (*4 points*) Show that the quotient group G/Z is isomorphic to the Klein-4 group.

   (e) (*6 points*) Let $G/Z = \{eZ, aZ, bZ, (ab)Z\}$ and $Z = \{e, u\}$, where e is the identity in G. Show that, up to isomorphism, the only possible multiplication rules for a, b, ab in G are $ab = uba$ and either

$$a^2 = u,\ b^2 = u, \text{ and } (ab)^2 = u$$

   or

$$a^2 = e,\ b^2 = e, \text{ and } (ab)^2 = u.$$

   (**Remark**: Now, you should be able to recall that these two groups correspond to the group of quarternions and the dihedral group D_4.)

3. (a) (*4 points*) Let $\phi_a : \mathbb{Z}_{10}[x] \to \mathbb{Z}_{10}$ be an evaluation homomorphism at $a \in \mathbb{Z}_{10}$. Evaluate $\phi_7(x^{2002} + x^{12} + 2)$.

   (b) (*4 points*) Is $113x^5 + 4x^4 + 2002x^3 + 12x^2 + 12x + 10$ irreducible or reducible over $\mathbb{Q}$?

   (c) (*4 points*) Let D be an integral domain and $I \leq D$ an ideal. Is D/I always an integral domain? If so, then prove the statement; if not, then give a counter-example.

   (d) (*4 points*) Are there any primitive 14th roots of unity in the Galois field $\mathbb{Z}_{113}$? If so, why and how many?

4. (a) (*5 points*) Let R be a commutative ring with $1 \neq 0$. Prove that R is an integral domain if and only if the zero ideal $I = \{0\}$ is a prime ideal in R.

   (b) (*5 points*) Let R be a commutative ring and I an ideal in R. Prove that the radical $\sqrt{I}$ of I defined by

$$\sqrt{I} = \{a \in R \,|\, a^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}$$

   is an ideal in R.

   (c) (*5 points*) Let P be a prime ideal in a commutative ring R. Prove that the radical ideal $\sqrt{P}$ is equal to P.

5. Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{R}$.

   (a) (*4 points*) Find the irreducible polynomial $irr(\alpha, \mathbb{Q})$ for α over $\mathbb{Q}$.

   (b) (*4 points*) What is the degree of the extension field $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$?

   (c) (*4 points*) Find a basis of $\mathbb{Q}(\alpha)$ as a vector space over $\mathbb{Q}$.

   (d) (*4 points*) Express α^6 in the basis you have found in (c).

   (e) (*4 points*) Find the irreducible polynomial $irr(2^{1/3}, \mathbb{Q})$ for $2^{1/3}$ over $\mathbb{Q}$, and thus prove that $2^{1/3} \notin \mathbb{Q}(\alpha)$.

   (f) (*4 points*) Prove that the polynomial $irr(2^{1/3}, \mathbb{Q})$ in part (d) is also irreducible over $\mathbb{Q}(\alpha)$.

   (g) (*4 points*) Find the degree of the extension field $\mathbb{Q}(\alpha, 2^{1/3})$ over $\mathbb{Q}$.

   (h) (*4 points*) Find a basis of $\mathbb{Q}(\alpha, 2^{1/3})$ as a vector space over $\mathbb{Q}$.