Math 115 Professor K. A. Ribet
Final Exam May 18, 1998

☞ The numbers 257 and 661 are prime.

1 *(6 points).* Find a positive integer n such that $n/3$ is a perfect cube, $n/4$ is a perfect fourth power, and $n/5$ is a perfect fifth power.

2 *(5 points).* Prove that there are no whole number solutions to the equation $x^2 - 15y^2 = 31$.

3 *(5 points).* Find the number of solutions to the congruence $x^2 \equiv 9 \bmod 2^3 \cdot 11^2$.

4 *(7 points).* Which positive integers m have the property that there a primitive root mod m? (Summarize what we know about this question, and why we know it. Your answer should be clear enough that one could use it to decide immediately if there is a primitive root modulo $(257)^2$, $4 \cdot 661$, $257 \cdot 661$, ...)

5 *(6 points).* Fermat showed that $2^{37} - 1$ is composite by finding a prime factor p of $2^{37} - 1$ which lies between 200 and 300. Using your knowledge of number theory, deduce the value of p.

6 *(7 points).* The continued fraction expansion of $\sqrt{5}$ is $\langle 2, 4, 4, \ldots \rangle$. If

$$\langle 2, \underbrace{4, 4, \ldots, 4}_{99\ 4'\mathrm{s}} \rangle = h/k$$

(in lowest terms), calculate $h^2 - 5k^2$.

7 *(5 points).* Prove that there are an infinite number of primes congruent to 3 mod 4.

8 *(6 points).* Suppose that $p = a^2 + b^2$, where p is an odd prime number and a is odd. Show that $\left(\dfrac{a}{p}\right) = +1$. (Use the Jacobi symbol.)

9 *(8 points).* Let a and b be positive integers. Show that

$$\phi(ab)\phi(\gcd(a,b)) = \phi(a)\phi(b)\gcd(a,b), \qquad \phi = \text{Euler } \phi\text{-function}.$$

(Example: If $a = 12$ and $b = 8$, the equation reads $32 \cdot 2 = 4 \cdot 4 \cdot 4$.)

10 *(5 points).* Find all solutions in integers y and z to the equation $6^2 + y^2 = z^2$.