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Spring 1996, Math 185, Section 1 Wednesday, 15 May, 1996

Final Examination 5:10-8:00 PM

1. (32 points) Mark statements ***T*** (true) or ***F*** (false). Each correct answer will count 1 point, each incorrect answer -1 point, each unanswered item 0 points.

_____ There exists a linear fractional transformation φ such that $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi(2) = \infty$.

_____ There exists a linear fractional transformation φ such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(2) = 2$, and $\varphi(3) = \infty$.

_____ If f is a holomorphic function defined in an open set containing the unit circle C, then $\int_C f(z)\,dz$ is an integer multiple of $2\pi i$.

_____ There exists a branch of $z^{1/2}$ on the annulus $\{z \mid 1 < |z| < 2\}$.

_____ There exists a branch of $z^{1/2}$ on the annulus $\{z \mid 1 < |z-3| < 2\}$.

_____ The function $f(z) = |z|$ is holomorphic.

_____ The function $f(z) = \bar{z}$ is holomorphic.

_____ If f and g are holomorphic functions on $\mathbf{C}$, then the function $f(g(z))$ is holomorphic.

_____ If f is a complex-valued function on $\mathbf{C}$, and if $\partial f/\partial x$ and $\partial f/\partial y$ are defined and continuous at all points of $\mathbf{C}$, then f is holomorphic.

_____ If f is a harmonic function on an open subset $G \subseteq \mathbf{C}$, then f^2 (i.e., the function taking z to $f(z)^2$) is also harmonic.

_____ The function e^z is univalent (one-to-one) on the unit disk $\{z \mid |z| < 1\}$.

_____ If f and g are continuous functions on a connected open set $G \subseteq \mathbf{C}$, and $e^{f(z)} = e^{g(z)}$ for all $z \in G$, then $f - g$ is constant.

_____ If f is a harmonic function on an open subset $G \subseteq \mathbf{C}$, and z_0 is a point of G, then there is some open subset $H \subseteq G$ containing z_0 such that on H, f has a harmonic conjugate.

_____ If f is a holomorphic nowhere zero function on an open subset $G \subseteq \mathbf{C}$, then any primitive (i.e., antiderivative) g of the function $f'(z)/f(z)$ is a branch of $\log f(z)$.

_____ If f is a holomorphic nowhere zero function on a connected open subset $G \subseteq \mathbf{C}$, and the function $f'(z)/f(z)$ has a primitive, g, then there exists a branch of $\log f(z)$ on G.

_____ If a_n and b_n are sequences of real numbers, then $\limsup_{n\to\infty} (a_n + b_n) \geq \limsup_{n\to\infty} (a_n) + \limsup_{n\to\infty} (b_n)$.

_____ If a_n and b_n are sequences of real numbers, then $\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} (a_n) + \limsup_{n\to\infty} (b_n)$.

_____ Let C be the unit circle, oriented counterclockwise, and n an integer. Then $\int_C z^n\,dz \neq 0$ if and only if $n = -1$.

_____ Let $G \subseteq \mathbf{C}$ be a domain, and f a holomorphic function on G. Then f has a primitive if and only if for every closed curve γ in G, $\int_\gamma f(z)\,dz = 0$.

____ If f is holomorphic on the right half-plane $P = \{z \mid \operatorname{Re} z > 0\}$, and z_0 is a point of P, then the Taylor series of f at z_0 has radius of convergence at least $\operatorname{Re} z_0$.

____ If f and g are holomorphic functions on the unit disk D, and $f(z) = g(z)$ for infinitely many values of z in G, then $f = g$.

____ Any nonconstant holomorphic function from a domain G to the closed unit disk $\{z \mid |z| \leq 1\}$ will in fact take G into the open unit disk $\{z \mid |z| < 1\}$.

____ If f is a holomorphic function on the annulus $A = \{z \mid 1 < |z| < 2\}$, then there exists a holomorphic function g on the disk $\{z \mid |z| < 2\}$, and a holomorphic function h on the set $\{z \mid |z| > 1\}$, such that $f = g + h$ everywhere on A.

____ The function $z^2/\sin z$ has a removable singularity at 0.

____ The function $z^2/\sin z$ has a removable singularity at π.

____ Any bounded holomorphic function on $\{z \mid \operatorname{Re} z > 0\}$ is constant.

____ If Γ is a contour, and z_0 a point of $\mathbb{C} \setminus \Gamma$, then there is a disk D centered at z_0 such that the winding number function $\operatorname{ind}_\Gamma(z)$ is constant for $z \in D$.

____ If γ_1 and γ_2 are closed curves, then the interior of $\gamma_1 + \gamma_2$ is the intersection of the interior of γ_1 and the interior of γ_2.

____ If γ_1 and γ_2 are closed curves, then the interior of $\gamma_1 + \gamma_2$ is the union of the interior of γ_1 and the interior of γ_2.

____ $\mathbb{C}$ is conformally equivalent to $\mathbb{C} \setminus [0, +\infty)$.

____ The unit disk is conformally equivalent to $\mathbb{C} \setminus [0, +\infty)$.

____ The unit disk is conformally equivalent to $\mathbb{C} \setminus [-1, 1]$.

2. (12 points) How many zeroes (counting multiplicities) does the polynomial $z^{1000} + 5z^{10} + 2z^4 - 1$ have in each of the regions (a) $|z| \leq 1$, (b) $1 < |z| \leq 2$, (c) $|z| > 2$? Briefly indicate your reasoning.

3. (20 points) Evaluate $\int_0^{2\pi} (2 + \cos\theta)^{-1} d\theta$. (Suggestion: Let $z = e^{i\theta}$. Be careful about the relation between $d\theta$ and dz.)

4. (16 points) For what complex numbers c does the function $(z - c)^{-1}$ have a Laurent expansion on the annulus $A = \{z \mid 1 < |z| < 2\}$? Determine this expansion explicitly for all such c.

5. (20 points) Let G be a domain, Γ a contour contained, with its interior, in G, and z_0 a point of the interior of Γ. Suppose f is a holomorphic function on $G \setminus \{z_0\}$. Recall that f is said to have a *removable singularity* at z_0 if there is a holomorphic function e on G which is equal to f everywhere on $G \setminus \{z_0\}$. (We have also seen a criterion in terms of the Laurent expansion of f about z_0.)

Show that f has a removable singularity at z_0 if and only if for every holomorphic function g on G, one has $\int_\Gamma f(z)g(z)\,dz = 0$.