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Spring 1995, Math 110, Section 1
First Midterm Exam

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10:10-11:00

1. Suppose V is a *3-dimensional* vector space over a field F, and x_1, x_2, x_3, x_4 are distinct elements of V. Suppose that $\{x_1, x_2, x_4, x_3\}$ spans V, and that these elements satisfy

$$x_1 + x_2 + x_3 + x_4 = 0.$$

(a) (10 points) Show that $\{x_1, x_2, x_3\}$ is a basis of V.

(b) (10 points) Similar reasoning shows that $\{x_1, x_3, x_4\}$ is a basis of V. (You need not write out the argument). Denoting these two bases

$$\alpha = \{x_1, x_2, x_3\}, \qquad \beta = \{x_1, x_3, x_4\},$$

find the matrix that changes coordinates from the basis α to the basis β. Give this matrix at right. (If your answer is correct, you will get full credit. If it is incorrect, you may get partial credit for correct work shown.)

2. (25 points) Prove the following result (a Theorem from the text):
Let S be a linearly independent subset of a vector space V, and let x be an element of V that is not in S. Then $S \cup \{x\}$ is linearly dependent if and only if $x \in \mathrm{span}(S)$.

3. (15 points) Let F be a field, and $a, b, c, d, e, f, p, q, r, s, t, u \in F$. Consider the matrix

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} p & 0 & 0 \\ q & r & 0 \\ s & t & u \end{pmatrix}.$$

Find $\det(A)$. (Advice: you can do this the hard way, by multiplying out and calculating, or more easily, using facts about determinants.)

4. Suppose V and W are vector spaces, and $\mathsf{U}, \mathsf{T}: \mathsf{V} \to \mathsf{W}$ are linear transformations.

(a) (10 points) What is meant by the null space $\mathsf{N}(\mathsf{T})$ of T?

(b) (10 points) Prove $\mathsf{N}(\mathsf{U}+\mathsf{T}) \supseteq \mathsf{N}(\mathsf{U}) \cap \mathsf{N}(\mathsf{T})$. (Recall: $\supseteq$ means "contains" and $\cap$ means "intersection".)

5. Suppose V is a finite dimensional vector space with ordered basis $\alpha = \{x_1, \ldots, x_m\}$, and W a finite dimensional vector space with ordered basis $\beta = \{y_1, \ldots, y_n\}$. Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation.

(a) (5 points) Complete the following definition: The matrix $[\mathsf{T}]_\alpha^\beta$ of T with respect to the ordered bases α and β means the $n \times m$ matrix (a_{ij}) whose entries a_{ij} satisfy the equations . . .

(b) (5 points) Complete the following statement: the basis β^* of W^* dual to the basis β of W consists of the linear functionals $\mathsf{g}_1, \ldots, \mathsf{g}_n$ on W (i.e., the linear maps $\mathsf{W} \to F$) such that for all $i \leq n$ and $j \leq m$, $\mathsf{g}_i(y_j) = \ldots$

(c) (10 points) Prove: The entries a_{ij} of the matrix $[\mathsf{T}]_\alpha^\beta$ may be computed by the formula

$$a_{ij} = \mathsf{g}_i(\mathsf{T}(x_j)).$$