MATH 126 — FINAL EXAM S06 L. Evans

Problem #1. Recall from the calculus of variations that minimizers of the energy

$$E[v] = \int_D F(x, u, \nabla u)\, dx$$

satisfy the *Euler-Lagrange equation*

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial F(x,u,\nabla u)}{\partial p_i}\right) + \frac{\partial F(x,u,\nabla u)}{\partial u} = 0.$$

Find a function $F = F(x, u, p)$ for which the corresponding Euler-Lagrange equation is the nonlinear Poisson equation

$$\Delta u = \phi(u),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is given.

Problem #2. Write down the explicit formula for the solution $u = u(x, t)$ of the heat equation

$$\begin{cases} u_t = \Delta u & \text{for } x \in \mathbb{R}^n, t > 0 \\ u = \phi & \text{for } x \in \mathbb{R}^n, t = 0. \end{cases}$$

Problem #3. Write down a formula for the solution $u = u(x, y)$ of

$$\begin{cases} \Delta u = 0 & \text{in } B(0, a) \\ u = 5\sin 8\theta + \cos\theta & \text{on } \partial B(0, a). \end{cases}$$

Problem #4. Show that for an arbitrary function f,

$$u(r, t) = \frac{1}{r} f(t - r) \qquad (r > 0)$$

solves the wave equation for $n = 3$ space dimensions, for $c = 1$.

Problem #5. Show that if u solves the KDV equation

$$u_t + u_{xxx} + 6uu_x = 0 \qquad \text{for } x \in \mathbb{R}, t > 0,$$

then the energy

$$\int_{-\infty}^{\infty} \frac{1}{2} u_x^2 - u^3\, dx$$

is constant in time.

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Problem #6. Prove that if the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

has a solution $u \not\equiv 0$, then $\lambda > 0$.

Problem #7. Show that if u solves the wave equation

$$u_{tt} = \Delta u \qquad \text{for } x \in \mathbb{R}^n, t > 0,$$

then

$$\frac{d}{dt}\left(\frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + |\nabla u|^2 \, dx\right) \leq 0$$

for $0 \leq t \leq t_0$. Here $\nabla u = (u_{x_1}, \ldots, u_{x_n})$.

Problem #8. Suppose that

$$\Delta u = 0$$

within a bounded domain $D \subset \mathbb{R}^n$. Show that

$$\max_{D} |\nabla u| = \max_{\partial D} |\nabla u|;$$

that is, the length of ∇u attains its maximum on ∂D.

(Hint: Let $v = |\nabla u|^2$ and compute Δv.)

Problem #9. A weak solution of Burgers' equation

$$u_t + uu_x = 0 \qquad \text{for } x \in \mathbb{R}, t > 0$$

has the form

$$u(x,t) = \begin{cases} \frac{x}{t+1} & \text{for } x < \xi(t) \\ 0 & \text{for } x > \xi(t), \end{cases}$$

where $\xi(t)$ is (curved) shock wave starting at $\xi(0) = 1$.

Find a formula for $\xi(t)$.

(Hint: Use the Rankine-Hugoniot condition to find an ODE that $\xi(t)$ satisfies.)

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Problem #10. Fill in the details for this alternative proof of the *mean value property* for harmonic functions in $n = 3$ dimensions:

(i) Prove the calculus formula

$$\frac{d}{ds}\left(\frac{1}{s^2}\int_{\partial B(x,s)} u\, dS(y)\right) = \frac{1}{s^2}\int_{\partial B(x,s)} \frac{\partial u}{\partial n}\, dS(y),$$

by changing variables by $y = x + sz$, for $z \in \partial B(0,1)$.

(ii) Next, use this formula to show

$$\frac{d}{ds}\left(\frac{1}{s^2}\int_{\partial B(x,s)} u\, dS\right) = 0,$$

if u is harmonic. Deduce

$$u(x) = \frac{1}{4\pi r^2}\int_{\partial B(x,r)} u\, dS$$

if u is harmonic in the ball $B(x,r)$.