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Spring 2004, Math 185, Sec. 1

Final Examination

17 May, 2004
8:00 AM - 11:00 AM

1. (20 points, 4 points each.) Complete the following definitions. Unless otherwise stated, you may use without defining them any terms or symbols which our text defines before it defines the concept asked for. You do not have to use exactly the same words as Stewart and Tall, but for full credit your statements should be clear, and mean the same as theirs. Something that was *proved* equivalent to the concept in question is not the same as the definition. (a) A domain D is said to be *simply connected* if ...

(b) If D is a domain and $\gamma_0, \gamma_1\colon [a,b] \to D$ are closed paths, then one says that γ_0 and γ_1 are *homotopic via closed paths* in D if ...

(c) If f is a differentiable function on a domain D, z_0 a point of D, and n a positive integer, then one says that f has a *zero of order* n at z_0 if ...

(d) If D is a domain, z_0 a point of D, and f a differentiable function on $D \setminus \{z_0\}$, then $\mathrm{res}(f, z_0)$ (the residue of f at z_0) is defined to be ...

(e) A *Möbius transformation* means ...

2. (30 points, 5 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, name a particular one. If you give a reason why no example exists, don't worry about giving a detailed proof; the key relevant fact will suffice.
(a) Two differentiable functions f and g on $\mathbb{C}$ such that f and g are not everywhere equal, but such that $f(z) = g(z)$ for infinitely many $z \in \mathbb{C}$.

(b) A differentiable function f on $\mathbb{C}$ with a zero of order 2 at ∞.

(c) A differentiable function f on $\mathbb{C} \setminus \{0\}$ with a zero of order 2 at ∞.

(d) A harmonic conjugate $v(z)$ to the function $u(z) = \log|z|$ on the right half-plane. (Recall: Since $|z|$ is positive and real, we understand $\log|z|$ to mean the real-valued logarithm. Hint: Examine what the derivative of $u + iv$ must be.)

(e) A harmonic conjugate $v(z)$ to the function $u(z) = \log|z|$ on $\mathbb{C} \setminus \{0\}$.

(f) A domain D and two distinct analytic functions f_1 and f_2 on D which are analytic continuations of one another.

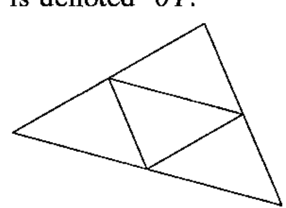
3. (7 points) Compute $\int_0^{2\pi} (5 + 4\sin x)^{-1}\,dx$. Show your work.

4. (a) (5 points) Find the poles of $(z^2+1)^{-2}$, the orders of these poles, and the residues at these poles.

(b) (7 points) Compute $\int_{-\infty}^{\infty} (x^2+1)^{-2}\,dx$, noting briefly the justifications for your computations, including any argument you use saying that a related integral approaches 0. You may assume that the integral asked for converges.

5. (14 points) Below, I give, slightly rearranged, our book's proof of Cauchy's Theorem for a triangle. After certain steps of the proof I have inserted parenthetical questions such as "($\boxed{0}$ Why?)". Answer each of these questions at the bottom of the page, after the corresponding number. You should seldom need as much space as is given for the answers; one key fact or calculation is what is wanted in each case. Note also that if you can't justify some step, you may still assume it in justifying later steps.

We recall that in the statement of that theorem, a "triangle" T is understood to mean the closed region bounded by the line segments $[z_1, z_2]$, $[z_2, z_3]$ and $[z_3, z_1]$ connecting three points z_1, z_2 and z_3 in the plane. The counterclockwise-oriented contour formed by those three line-segments is denoted ∂T.



Theorem. Let f be a differentiable function on a domain D, and T a triangle which lies in D. Then $\int_{\partial T} f = 0$.

Proof. Let $c = |\int_{\partial T} f| \geq 0$. We claim that we can find a sequence of triangles, $T_0 \supset T_1 \supset \ldots \supset T_n \supset \ldots$, such that (i) $T_0 = T$, (ii) for each $n \geq 1$, T_n is one of the four triangles into which T_{n-1} is subdivided by lines connecting the midpoints of its sides, as shown at right, and (iii) for all $n \geq 0$, $|\int_{\partial T_n} f| \geq c/4^n$.

Taking $T_0 = T$, condition (iii) is clear for $n = 0$. ([1] Why?) Assuming $T_0, \ldots, T_{n-1}$ have been chosen, let us call the four triangles into which T_{n-1} is divided as shown above $T^{(1)}, \ldots, T^{(4)}$. Then $\int_{\partial T_{n-1}} f = \Sigma_{r=1}^{4} \int_{\partial T^{(r)}} f$. ([2] Why, briefly?) Hence

$$|\int_{\partial T_{n-1}} f| \leq \Sigma_{r=1}^{4} |\int_{\partial T^{(r)}} f|.$$

([3] What property of the complex numbers do we use to get this inequality from the preceding equality?) Now if all four summands on the right in that display were $< c/4^n$, the left side would be $< 4(c/4^n) = c/4^{n-1}$, contradicting our inductive assumption of (iii), so one or more must be $\geq c/4^n$; we take for T_n the triangle corresponding to one such summand. In this way we successively choose T_n for all $n \geq 0$.

Note that each side of ∂T_n is half the length of the parallel side of ∂T_{n-1}, so by induction the path length $L(\partial T_n)$ equals $L(\partial T_0)/2^n$.

Now by compactness of T, our triangles have some common point z_0. The differentiability of f at z_0 says that as $z \to z_0$, the ratio $(f(z) - f(z_0))/(z - z_0)$ approaches a value $f'(z_0)$, and this says that we can write

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0)\, s(z),$$

where $s(z)$ is a continuous function on $D \setminus \{z_0\}$ which $\to 0$ as $z \to z_0$. ([4] What is the function $s(z)$ that makes this work?)

Let us extend s to all of D by making $s(z_0) = 0$; then s is still continuous, so we can integrate the two sides of the above displayed equation over each of the triangular paths ∂T_n. Now the function given by the first two terms of the right-hand side of that equation, $f(z_0) + (z - z_0) f'(z_0)$, has an antiderivative ([5] namely?), so the integral of those two terms over ∂T_n is zero, hence the integral of the whole equation over this path becomes

$$\int_{\partial T_n} f(z) = \int_{\partial T_n} (z - z_0)\, s(z).$$

By condition (iii), $c/4^n \leq$ the absolute value of the left-hand side of this equation, while on the right-hand side, the factor $z - z_0$ has absolute value $\leq L(\partial T_n)$, so $|(z - z_0)\, s(z)| \leq L(\partial T_n) \max_{z \in \partial T_n} |s(z)|$. So the above equation gives the inequality

$$c/4^n \leq L(\partial T_n)\, L(\partial T_n) \max_{z \in \partial T_n} |s(z)|.$$

([6] What general fact are we using to bound the absolute value of the integral on the right in the preceding display?) Combining the above display with the earlier observation that $L(\partial T_n) = L(\partial T_0)/2^n$, and multiplying both sides by 4^n, we get

$$c \leq L(\partial T_0)^2 \max_{z \in \partial T_n} |s(z)|.$$

Now let ε be any positive real number. As $n \to \infty$, the triangles T_n, which contain z_0, become arbitrarily small; and we know that as $z \to z_0$, $s(z) \to 0$; hence for n sufficiently large, we get $\max_{z \in \partial T_n} |s(z)| < \varepsilon / L(\partial T_0)^2$. The preceding display thus gives $c < \varepsilon$. It follows that $c = 0$ ([7] Why does this follow?), as required.

6. (10 points) Show that if u and v are real-valued functions on a domain D such that $u(z) + i v(z)$ is differentiable, then u satisfies *Laplace's equation*, $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. (Since this is a result from the text, your proof may only make use of results proved in the text *before* this one.)

7. (7 points) Let z_1, z_2 and a be complex numbers. Show that if w_1 is any value of $z_1^{\,a}$, and w_2 is any value of $z_2^{\,a}$, then $w_1 w_2$ is a value of $(z_1 z_2)^a$.

(Note: This is related to, but different from, a problem from your last homework. That problem concerned the complex analog of the law of exponents $z^a z^b = z^{a+b}$; this concerns the analog of $z_1^{\,a} z_2^{\,a} = (z_1 z_2)^a$. Comparing what you are asked to prove here with the result of that homework shows that the answer to "Which quantifications are valid?" is different for these two laws.)