Department of Mathematics, University of California, Berkeley

Mathematics 140
Metric Differential Geometry

Alan Weinstein, Fall, 2001

Final Exam, December 14, 2001

Instructions. Do all six problems below. You may refer to your textbook. Unless specified otherwise, you may use any theorem, proposition, etc. from the book, as well as the result of any calculation in the book (including solutions to exercises), as long as you give a precise page reference.

1. Let $f_a(s) = a + \sin s$, where a is a real number.
A. For which values of a is there a unit speed plane curve whose signed curvature is f_a?
B. For which values of a is there a simple, closed, unit speed plane curve of period 2π whose signed curvature is f_a?

2. Let γ be a unit speed space curve with nowhere vanishing curvature κ. The torsion, unit tangent, principal normal, and binormal are denoted τ, $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ as usual. Given positive real numbers r and ω, define a new curve $\mathbf{c}$ by

$$\mathbf{c}(t) = \gamma(t) + r(\cos\omega t\,\mathbf{n} + \sin\omega t\,\mathbf{b}).$$

A. Suppose that κ and τ are bounded functions: i.e. there is a number C such that $|\kappa(t)| < C$ and $|\tau(t)| < C$ for all t. Prove that $\mathbf{c}$ is a regular curve as long as ω is sufficiently large [how large?] OR r is sufficiently small [how small?].
B. Describe the curve $\mathbf{c}$. You might want to mention its resemblance to a household object and/or draw a sketch. What happens to $\mathbf{c}$ as r approaches zero? What happens as ω approaches infinity?
C. Write as an integral the length of the curve $\mathbf{c}$ on the interval $[a, b]$. There should be no vectors in the integrand, but you do not need to evaluate the integral.

3. Let $f : S_1 \to S_2$ be an isometry of surfaces, let P be a point of S_1, and let γ be a regular curve in S_1. Tell whether each of the following statements is true or false. You should justify a "false" answer by giving a counterexample. You should justify a "true" answer by proving the statement or by citing an appropriate result or results from the book. (Don't forget to include page references.)
A. If the normal curvature of γ in S_1 is zero, then the normal curvature of $f \circ \gamma$ is zero.
B. Same as part A, but for geodesic curvature.
C. Same as part A, but for torsion.
D. If S_1 is a minimal surface, so is S_2.
E. There is a rigid motion which takes S_1 to S_2.
F. If P is an umbilic of S_1, then $f(P)$ is an umbilic of S_2.
G. If S_1 is a compact, connected surface of constant positive gaussian curvature, then S_2 can be obtained from S_1 by a rigid motion.

CONTINUED ON REVERSE SIDE

4. The *geodesic distance* between two points on a compact surface may be defined as the length of the shortest geodesic segment connecting them. The *geodesic circle* of radius r around a point P consists of all points whose geodesic distance from P is equal to r; the *geodesic disc* of radius r consists of all points whose geodesic distance from P is less than or equal to r. Let S be a sphere of radius R, and let C be a geodesic circle of radius r on S which is positively oriented for a surface patch which contains the corresponding geodesic disc. Assume that $0 < r < \pi R$.

A. Find the geodesic curvature of C at each point of C.

B. Find the total length of C.

C. Find the area of the geodesic ball enclosed by C.

D. Verify that the Gauss-Bonnet Theorem (page 248) holds for the curve C and the geodesic disc which it surrounds.

5.

A. Show that, if a surface patch has constant gaussian curvature and constant mean curvature, then either all of its points are umbilics or none of its points are umbilics.

B. Find an example of a surface with constant gaussian curvature and constant mean curvature such that none of its points are umbilics.

C. Show that, if a (connected) surface patch has constant *positive* gaussian curvature and constant mean curvature, then all of its points are umbilics, and conclude that the surface is a part of a sphere. [Hint: you may use some of the ideas used in proving Theorem 10.4 (page 244), but be careful because the surface patch in this problem is not necessarily compact.]

6. The first fundamental form of a surface is given by the expression

$$ds^2 = du^2 + (1 + u^2)dv^2.$$

Can the surface lie on one side of its tangent plane at the point corresponding to $(u, v) = (0, 0)$?