REMARKS ON VASY'S OPERATOR WITH ANALYTIC COEFFICIENTS

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Abstract. We consider Vasy's operator [10] with analytic coefficients. That operator arises in the study of scattering on asymptotically hyperbolic manifolds and in general relativity.

1. General assumptions and conjectures

In his investigation of scattering resolvent on asymptotically hyperbolic manifolds, Vasy [10] introduced a degenerate differential operator switching behaviour from elliptic to hyperbolic along a hypersurface. More general operators with similar properties appear also in the study the of Kerr–de Sitter wave equation. We refer to [4, Chapter 5],[10],[13] and [14, §3.1] for motivation and general results.

In this note we present some remarks about Vasy's operator with analytic coefficients: we ask some general questions about propagation of analytic singularities, prove a conditional analytic hypoellipticity in a model case and give a direct prove of a Cauchy–Kovalevskaya theorem for this operator. After this note was written, Zuily [12] showed that by adapting methods of Bolley–Camus [3], conditional analytic hypoellipticity holds for all such operators settling [14, Conjecture 2]. The method of proof in the model case presented in §2 is however very different and might be open to generalizations.

To introduce Vasy's operator, suppose that M is a compact analytic manifold and

$$(-1,1) \ni x \mapsto Q(x,y,D_y) = \sum_{|\alpha|\leq 2} a_\alpha(x,y) D_y^\alpha, \quad D := \tfrac{1}{i}\partial$$

is an analytic family of self-adjoint (with respect to a density on M) elliptic second order differential operators with analytic coefficients.

We are interested in the following family of operators depending on a parameter$\lambda \in \mathbb{C}$:

$$P(\lambda) := xD_x^2 - (\lambda + i)D_x + \gamma(x,y)xD_x + Q(x,y,D_y). \tag{1}$$

This is the operator which plays a role in Vasy's treatment of scattering on asymptotically hyperbolic manifolds and is close to the operator $e^{i\lambda t}\Box_g e^{-i\lambda t}$ where g is a Kerr–de Sitter-like metric.

Here we assume that the coefficients of Q and the function γ are real analytic in a neighbourhood of $(-1,1)\times M$,

$$a_\alpha, \gamma \in C^\omega([-1,1]\times M).$$

We denote by $\mathrm{WF}(u)$ and $\mathrm{WF}_{\mathrm{a}}(u)$, C^∞ and analytic wave front sets of $u \in \mathcal{D}'((-1,1)\times M)$, respectively – see [6, Chapter 8],[8].

We have the following possible analytic regularity statements of increasing strength: for $u \in C^\infty((-1,1)\times M)$ and $\lambda \notin -i\mathbb{N}^*$,

$$P(\lambda)u = f \in C^\omega((-1,1)\times M) \implies u \in C^\omega((-1,1)\times M); \tag{2}$$

$$P(\lambda)u = f, \quad \mathrm{WF}_{\mathrm{a}}(f)\cap N^*(M\times\{0\}) = \emptyset \implies \mathrm{WF}_{\mathrm{a}}(u)\cap N^*(M\times\{0\}) = \emptyset; \tag{3}$$

$$P(\lambda)u = f \in C^\omega(U), \ (0,y_0)\in U \subset (-1,1)\times M \implies u \in C^\omega(V), \tag{4}$$

for some neighbourhood V of $(0,y_0)$. Finally,

$$P(\lambda)u = f, \quad (0,y_0,1,0)\notin \mathrm{WF}_{\mathrm{a}}(f), \ y_0 \in M \implies (0,y_0,1,0)\notin \mathrm{WF}_{\mathrm{a}}(u). \tag{5}$$

In the local versions of the conjecture we can take $M = \mathbb{R}^n$. The implication (2) is a conditional form of analytic hypoellipticity, that is analytic hypoellipticity under a regularity assumption.

If we assume that $u \in H^{s+1}$ for $s + \frac{1}{2} > -\operatorname{Im}\lambda$ (rather than $u \in C^\infty$) then the statements (2)–(5) hold in the C^∞ category, that is for C^ω replaced by C^∞ and WF_{a} replaced by WF – see [13, §4] for (2),(3) and [5] for (4),(5). Consequently we expect (2)–(5) to be valid under that weaker assumption $u \in H^{s+1}$, $s > -\operatorname{Im}\lambda - \frac{1}{2}$ (which can be further microlocalized).

2. Proof of analytic hypoellipticity in the model case

We will prove (2) in the model case of $\gamma = 0$ and $Q(x,y,D_y) = -\Delta_M$ where $-\Delta_M$ is the Laplacian for a real analytic metric on M. The proof is based on separation of variables and we will quote some results about Fuchsian differential operators – see §3 – as well as some standard results about Bessel function asymptotics.

It is enough to show that u is analytic in $(-\varepsilon,\varepsilon)\times M$ for some $\varepsilon > 0$ as propagation of analytic singularities [8] (which is simply ellipticity for $x > 0$) implies that u is analytic in $(-1,1)\times M$.

We first recall the following consequence of [2, Theorem 1]. In §3 we present a self-contained argument in our special case.

Lemma 2.1. *Suppose that $\lambda \notin -i\mathbb{N}^*$. There exists $\varepsilon > 0$ such that for any $f \in C^\omega((-1,1)\times M))$ we can solve $P(\lambda)u = f$ with $u \in C^\omega((-\varepsilon,\varepsilon)\times M)$.*

Hence we can assume that $f \equiv 0$ and it is enough to show that for $P(\lambda)u = 0$ near $x = 0$, $u(x,y)$ is analytic in $(-\varepsilon,\varepsilon)\times M$ for some $\varepsilon > 0$. For that it is in fact sufficient to show that analyticity holds true in $[0,\varepsilon)\times M$ for some $\varepsilon>0$. This follows from the following observation: a solution analytic in $[0,\varepsilon)$ extends to a solution in $\tilde u$ in $(-\varepsilon_1,\varepsilon)$ for some $\varepsilon_1>0$. Hence the difference $v:=\tilde u-u$ is a smooth solution in $(-\varepsilon_1,\varepsilon)\times M$ supported in $(-\varepsilon_1,0]\times M$. But then [13, Lemma 1] shows that $v\equiv 0$ and hence u is analytic in $(-\varepsilon_1,\varepsilon)\times M$.

Let $\{u_k(y)\}_{k=0}^\infty$ be an orthonormal set of eigenfunctions of $-\Delta_M$:

$$(-\Delta_M-\mu_k^2)u_k(y)=0,\quad \langle u_k,u_\ell\rangle_{L^2(M)}=\delta_{k\ell}.$$

We then write

$$u(x,y)=\sum_{k=1}^\infty a_k(x)u_k(y)$$

where the coefficients $a_k(x)$ are smooth and solve

$$(\partial_x x\partial_x-i\lambda\partial_x-\mu_k^2)a_k(x)=0,\quad |x|<\delta. \tag{6}$$

Let H be the unique solution to

$$(\partial_x x\partial_x-i\lambda\partial_x-1)H_\lambda(x)=0,\quad H_\lambda(0)=1,\quad \lambda\notin -i\mathbb{N}^*,$$

where the uniqueness follows from the indicial equation as in the proof of the following

Lemma 2.2. *For $x\in\mathbb{C}$ with $|\arg x|<\pi$,*

$$H_\lambda(x)=\frac{1}{\sqrt{2\pi}}\Gamma(1-i\lambda)2^{i\lambda}x^{\frac{i\lambda}{2}-\frac12}e^{2\sqrt{x}}(1+\mathcal{O}_\lambda(1/|x|)),\quad |x|\to\infty. \tag{7}$$

In addition, for $x\in\mathbb{C}$ with $|x|>1$,

$$|H_\lambda(x)|\le e^{C_\lambda\sqrt{|x|}}. \tag{8}$$

Proof. If $x=\frac14 z^2$ then (see [13, (2.2)] where the motivation for this change of variables is also presented)

$$x(\partial_x x\partial_x-i\lambda\partial_x-1)=\tfrac14 z^{i\lambda}(z^2\partial_z^2+z\partial_z+\lambda^2-z^2)z^{-i\lambda}$$

which is the modified Bessel equation with the parameter $\alpha=-i\lambda$. It is solved by

$$I_{-i\lambda}(z)=\frac{1}{\Gamma(1-i\lambda)}\left(\frac{z}{2}\right)^{-i\lambda}(1+z^2F(z^2)),\quad z\to 0+$$

Hence

$$H_\lambda(x)=\Gamma(1-i\lambda)x^{\frac{i\lambda}{2}}I_{-i\lambda}(2\sqrt{x}),$$

and the expansion follows from the standard asymptotic expansion of $I_{-i\lambda}(z)$ – see [1, (9.7.1)]. The upper bound (8) is immediate from the recursion relations in the expansion of $H_\lambda(x)$:

$$H_\lambda(x)=\sum_{k=0}^\infty H_{\lambda,k}x^k,\quad H_{\lambda,0}=1,\quad H_{\lambda,k+1}=\frac{1}{(k+1)(k+1-i\lambda)}H_{\lambda,k}.$$

$\square$

Rescaling (6) by $\tilde{x} = \mu_k^2 x$ we see that

$$a_k(x) = a_k(0) H_\lambda(\mu_k^2 x). \tag{9}$$

Since $u(x, y)$ is smooth in x and y it follows that, uniformly for $|x| \leq \frac{1}{2}$,

$$\begin{aligned} a_k(x) &= \langle u(x, \bullet), u_k \rangle_{L^2(M)} = \langle (I - \Delta_M)^N u, (I - \Delta_M)^{-N} u_k \rangle_{L^2(M)} \\ &= \mathcal{O}_N(\langle \mu_k \rangle^{-2N}), \end{aligned}$$

though we only need the case of $N = 0$. These two facts and (7) give, for large values of k,

$$|a_k(0)| = |a_k(\tfrac{1}{2})||H_\lambda(\tfrac{1}{2}\mu_k^2)|^{-1} \leq C_\lambda e^{-\mu_k/2}.$$

Using this, (8) and the analyticity of H_λ in (9) give

$$|\partial_x^\ell a_k(x)| \leq C^{\ell+1} \ell^\ell e^{-\mu_k/3}, \quad |x| \leq \delta, \tag{10}$$

where δ depends only on λ.

On the other hand,

$$|\partial_y^\alpha u_k(y)| \leq C^{1+|\alpha|} \mu_k^{n/2} (\mu_k + |\alpha|)^{|\alpha|}, \tag{11}$$

see for instance [7] for a self-contained presentation. Putting $N(r) = \max\{k : \mu_k \leq r\} \leq C_0 r^n$, $n = \dim M$, we obtain

$$\begin{aligned} |\partial_x^\ell \partial_y^\alpha u| &\leq C^{|\alpha|+\ell+1} \ell^\ell \sum_{k=0}^\infty (\mu_k + |\alpha|)^{|\alpha|} e^{-\mu_k/4} = C^{|\alpha|+\ell+1} \ell^\ell \int_0^\infty (r + |\alpha|)^{|\alpha|} e^{-r/4} dN(r), \\ &\leq C_1^{|\alpha|+\ell+1} \ell^\ell |\alpha|^{|\alpha|}, \end{aligned}$$

since

$$\begin{aligned} \int_0^\infty (r + |\alpha|)^{|\alpha|} e^{-r/4} dN(r) &\leq C_0 |\alpha| \int_0^\infty (1 + r)^n (r + |\alpha|)^{|\alpha|} e^{-r/4} dr \\ &\leq C_0 |\alpha|^{|\alpha|+1} \int_0^\infty (1 + r)^{n+|\alpha|} e^{-r/4} dr \leq C_2^{|\alpha|} |\alpha|^{|\alpha|}. \end{aligned}$$

This completes the proof of (2).

3. A Cauchy–Kovalevskaya theorem for the Vasy operator

We will use the methods of Baouendi–Goulaouic [2] (with simplifications from [9] and [11]) to show existence of analytic solutions of the Vasy operator (1). One could simply quote [2] for the main result but since the situation here is simpler we revisit the approach of [2].

For $\lambda \in \mathbb{C}$ we consider the following family of operators:

$$P(\lambda) = x D_x^2 - (\lambda + i) D_x + \gamma(x, y) x D_x + Q(x, y, D_y), \quad x \in (-1, 1), \quad y \in U \Subset \mathbb{R}^n,$$

where

$$Q(x,y,D_y)=\sum_{|\alpha|\le 2} a_\alpha(x,y)D_y^\alpha,\quad \gamma, a_\alpha\in C^\omega((-1,1)\times U),$$

and γ and a_α are real analytic in (x,y). There are no further assumption on Q and in particular ellipticity plays no role here.

Proposition 3.1. *Suppose that $\lambda\notin -i\mathbb{N}^*$. Then, for any open set $V\Subset U$ there exists $\varepsilon>0$ such that for $u_0\in C^\omega(U)$ and $f\in C^\omega((-1,1)\times U)$ there exists a* unique *$u\in C^\omega((-\varepsilon,\varepsilon)\times V)$ satisfying*

$$P(\lambda)u=f \quad in\ (-\varepsilon,\varepsilon)\times V\ ,\quad u(0,y)=u_0(y),\quad y\in V. \tag{12}$$

We associate to $P(\lambda)$ an *indicial operator*

$$\mathcal{C}(\lambda,x\partial_x):=-(x\partial_x)^2+i\lambda x\partial_x \tag{13}$$

and note that

$$\mathcal{C}(\lambda,s):=x^{-s}\mathcal{C}(\lambda,x\partial_x)x^s=-s(s-i\lambda). \tag{14}$$

To construct $u(x,y)$ we write it as follows,

$$u(x,y)=u_0(y)+\sum_{k=1}^{K-1}\frac{x^k}{k!}\partial_x^k u(0,y)+x^K u_K(x,y). \tag{15}$$

(We will use the same notation for other functions of x: $x^K u_K$ denotes the remainder in the K-term Taylor expansion.)

Since $xP(\lambda)u=xf$, we have

$$\mathcal{C}(\lambda,x\partial_x)u=xf-\gamma x^2D_xu-xQ(x,y,D_y)u,$$

and this gives the following system of equations

$$\begin{aligned}\frac{1}{k!}\mathcal{C}(\lambda,k)\partial_x^k u(0,y)&=\frac{1}{(k-1)!}\partial_x^{k-1}f(0,y)+i\sum_{m+\ell=k-1}\frac{1}{(m-1)!\ell!}\partial_x^\ell\gamma(0,y)\partial_x^m u(0,y)\\ &\quad-\sum_{m+\ell=k-1}\frac{1}{m!\ell!}\partial_x^\ell Q(0,y,D_y)\partial_x^m u(0,y),\quad 1\le k\le K-1.\end{aligned} \tag{16}$$

If $\mathcal{C}(\lambda,k)\neq 0$ for $k\in\mathbb{N}^*$ then we can find $\partial_x^k u(0,y)\in C^\omega(U)$ satisfying (16). In view of (14) that is precisely the condition that $\lambda\notin -i\mathbb{N}^*$.

The equation for u_K becomes

$$xP(\lambda)x^Ku_K=x^KF_K(x,y),$$

where

$$\begin{aligned} F_K(x,y) :=& f_{K-1}(x,y) - (Q_{K-1}(x,y,D_y) - i\gamma_{K-1}(x,y)x\partial_x) \sum_{k=0}^{K-1} \frac{x^k}{k!} \partial_x^k u(0,y) \\ &+ i \sum_{\substack{m+\ell > K-2 \\ m<K-1, \ell<K-1}} \frac{x^{m+\ell-K+1}}{(m-1)!\ell!} \partial_x^\ell \gamma(0,y) \partial_x^m u(0,y) \\ &- \sum_{\substack{m+\ell > K-2 \\ m<K-1, \ell<K-1}} \frac{x^{m+\ell-K+1}}{m!\ell!} \partial_x^\ell Q(0,y,D_y) \partial_x^m u(0,y) \end{aligned} \tag{17}$$

(The specific form of F_K is of course not important – what matters is that it is constructed using $\partial_x^k u(0,y)$, $k \leq K-1$ and analytic in all variables.)

Since $x^{-K}\mathcal{C}(\lambda, x\partial_x)(x^K v) = \mathcal{C}(\lambda, x\partial_x + K)v$, we obtain

$$P_K(\lambda) u_K = F_K, \tag{18}$$

where F_K is given by (17) and

$$\begin{aligned} P_K(\lambda) =& -(x\partial_x + K)^2 + i\lambda(x\partial_x + K) \\ & - ix\gamma(x,y)(x\partial_x + K) + xQ(x,y,D_y). \end{aligned} \tag{19}$$

The advantage now lies in the fact that the indicial equation has two roots $-K, -K+i\lambda$ and for $K > -\operatorname{Im}\lambda$, the real parts of both roots are negative.

Proposition 3.2. *Suppose that*

$$\mathcal{P} := (x\partial_x - s_1)(x\partial_x - s_2) + xb(x,y)(\partial_x x) + xB(x,y,\partial_y),$$

$B(x,y,\partial_y) = \sum_{|\alpha|\leq 2} b_\alpha(x,y)\partial_y^\alpha$, *where* $a_j \in \mathbb{C}$, $b, b_\alpha \in C^\omega([-1,1]\times U)$ *and*

$$\operatorname{Re} s_j < 0.$$

Then for any open set $V \Subset U$ *there exists* $\varepsilon > 0$ *such that for* $f \in C^\omega((-1,1)\times U)$ *there exists a* unique $u \in C^\omega((-\varepsilon,\varepsilon)\times V)$ *solving*

$$\mathcal{P}u = f.$$

Remarks. 1. This works in much greater generality and in particular we could assume that s_j are analytic functions of y. In our case $s_1 = -K$ and $s_2 = -K + i\lambda$ are constant.

2. One can assume continuity of the coefficients and of f in the x variable and still obtain solutions for which $(x\partial_x)^k u$, $k \leq 2$ are continuous with values in the spaces of analytic functions in y – see [2] and [9].

Proof of Proposition 3.2. By passing to a cover (and using a uniqueness argument) and shifting the origin we can assume that

$$V \Subset D_{\mathbb{C}^n}(0,R) \cap \mathbb{R}^n \Subset U, \quad D_{\mathbb{C}^n}(0,R) := \{z \in \mathbb{C}^n : |z_j| < R\},$$

and that the coefficients of $\mathcal{P}$ extend to holomorphic functions in a neighbourhood of $D_{\mathbb{C}}(0,\delta)\times D_{\mathbb{C}^n}(0,R)$, $\delta>0$. For $\eta>0$ sufficiently small we then put

$$G=\{(x,y)\in\mathbb{C}^n : |x|^{\frac{1}{2}}/\eta+|y|<R\},\quad |y|:=\max|y_j|.$$

We define the Banach space $\mathscr{X}=\mathscr{X}_{\eta,p}$ of functions holomorphic in G with the norm

$$\|u\|_{\eta,p}:=\sup_{(x,y)\in G}(R-|y|-|x|^{\frac{1}{2}}/\eta)^p|u(x,y)|,\quad p>1.$$

We put $\mathcal{P}_2:=(x\partial_x-s_1)(x\partial_x-s_2)$ and define

$$\mathcal{H}g(x,y):=\int_0^1\int_0^1\xi_1^{-1-s_1}\xi_2^{-1-s_2}g(\xi_1\xi_2x,y)d\xi_1d\xi_2,$$

so that $\mathcal{P}_2\mathcal{H}g=g$. We also define

$$\mathcal{H}_mg(x,y):=\int_0^1\cdots\int_0^1 g(\xi_1\cdots\xi_mx,y)d\xi_1\cdots d\xi_m,$$

which solves $(\partial_xx)^m\mathcal{H}_mg=g$.

Following [9] we now look for a solution of the form $u=\mathcal{H}_2g$, that is a solution to

$$\mathcal{P}_2\mathcal{H}_2g=f-xb(\partial_xx)\mathcal{H}_2g-xB(x,y,\partial_y)\mathcal{H}_2g.$$

That means we are looking for a fixed point of $\mathcal{T}$ which is defined as

$$\mathcal{T}g:=(\partial_xx)^2\mathcal{H}\left(f-xb\mathcal{H}_1g-xB(x,y,\partial_y)\mathcal{H}_2g\right),$$

and we want to show that, for η small enough, $\mathcal{T}$ is a contraction on $\mathscr{X}$.

We need to show that $\mathcal{T}:\mathscr{X}\to\mathscr{X}$ and we first note that

$$\|(\partial_xx)^2\mathcal{H}u\|_{\eta,p}\leq C_s\|u\|_{\eta,p}. \tag{20}$$

In fact,

$$\begin{aligned}x\partial_x\mathcal{H}u(x,y)&=\int_0^1\int_0^1\xi_1^{-s_1}\xi_2^{-s_2-1}\partial_{\xi_1}[u(\xi_1\xi_2x,y)]d\xi_1d\xi_2\\&=s_1\mathcal{H}u(x,y)+\int_0^1\xi_2^{-s_2-1}u(\xi_2x,y)d\xi_2,\end{aligned}$$

and hence $x\partial_x\mathcal{H}=\mathcal{O}_s(1)_{\mathscr{X}\to\mathscr{X}}$. Writing $(\partial_xx)^2$ using $\mathcal{P}_2$ and $x\partial_x-s_j$ then shows (20). Similarly,

$$\|\mathcal{H}_mg\|_{\eta,p}\leq\|g\|_{\eta,p}. \tag{21}$$

We now have to estimate $\|x\partial_y^\alpha\mathcal{H}_2g\|_{\eta,p}$ for $|\alpha|=1,2$ and the key component is *Nagumo's Lemma*: for $g\in\mathscr{X}$,

$$|\partial_{y_j}g(x,y)|\leq(p+1)e(R-|y|-|x|^{\frac{1}{2}}/\eta)^{-p-1}\|g\|_{\eta,p},\quad j=1,\cdots,n. \tag{22}$$

This follows from a one dimensional inequality: for f holomorphic in $|z|<r$, $z\in\mathbb{C}$,

$$|f(z)|\leq C(r-|z|)^{-p}\longrightarrow|\partial_zf(z)|\leq C(p+1)(1+p^{-1})^p(r-|z|)^{-p-1},$$

which in turn follows from Cauchy's inequality $|\partial_z f(z)| \leq \rho^{-1} \max_{|z-w|=\rho} |f(w)| \leq C\rho^{-1}(r - \rho - |z|)^{-p}$ and optimization in ρ.

We claim that

$$\|x^{\frac{|\alpha|}{2}} \partial_y^\alpha \mathcal{H}_{|\alpha|} g\|_{\eta,p} \leq C\eta^{|\alpha|} \|g\|_{\eta,p}, \quad |\alpha| \leq 2. \tag{23}$$

We first consider the case of $|\alpha| = 1$. Using (22) we see that

$$\begin{aligned} |x|^{\frac{1}{2}} |\partial_{y_j} \mathcal{H}_1 g(x,y)| &\leq C\|g\|_{\eta,p} \int_0^1 \frac{|x|^{\frac{1}{2}}}{(R - |y| - |\xi x|^{\frac{1}{2}}/\eta)^{p+1}} d\xi \\ &\leq C\|g\|_{\eta,p}\, \eta^2 |x|^{-\frac{1}{2}} \int_0^{|x|/\eta^2} (R - |y| - s^{\frac{1}{2}})^{-p-1} ds \\ &\leq C_p\, \eta\, \|g\|_{\eta,p} (R - |y| - |x|^{\frac{1}{2}}/\eta)^{-p}. \end{aligned} \tag{24}$$

Here we used the inequality

$$\int_0^r (\rho - s^{\frac{1}{2}})^{-p-1} ds \leq C_p r^{\frac{1}{2}} (\rho - r^{\frac{1}{2}})^{-p}, \quad 0 \leq r < \rho, \quad p > 1,$$

which follows from differentiating both sides in r (and the fact that the inequality is valid for $r = 0$). This proves (23) for $|\alpha| = 1$. For $|\alpha| = 2$ we again use (22) and obtain

$$\begin{aligned} |x||\partial_{y_j} \partial_{y_k} \mathcal{H}_2 g(x,y)| &\leq \|g\|_{\eta,p} \int_0^1 \int_0^1 |x| (R - |y| - |\xi_1 \xi_2 x|^{\frac{1}{2}}/\eta)^{-p-2} d\xi_1 d\xi_2 \\ &\leq \|g\|_{\eta,p} \prod_{j=1}^2 \int_0^1 |x|^{\frac{1}{2}} (R - |y| - |\xi_j x|^{\frac{1}{2}}/\eta)^{-p/2-1} d\xi_j. \end{aligned}$$

We now argue as in (24) for each term and obtain (23).

From (23) we obtain

$$\|x \partial_y^\alpha \mathcal{H}_2 g\|_{\eta,p} \leq C_p \eta \|g\|_{\eta,p}, \quad |\alpha| \leq 2. \tag{25}$$

Combining this with (20) and (21) and using the fact that $|x| < R\eta^2$ we see that for η sufficiently small $\mathcal{T} : \mathscr{X} \to \mathscr{X}$ is indeed a contraction and that gives solution to $\mathcal{P}u = f$ holomorphic in G. The uniqueness and globalization are standard. $\square$

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