Von Neumann Algebras.

Vaughan F.R. Jones [1]

October 1, 2009

[1]Supported in part by NSF Grant DMS93–22675, the Marsden fund UOA520, and the Swiss National Science Foundation.

Chapter 1

Introduction.

The purpose of these notes is to provide a rapid introduction to von Neumann algebras which gets to the examples and active topics with a minimum of technical baggage. In this sense it is opposite in spirit from the treatises of Dixmier [], Takesaki[], Pedersen[], Kadison-Ringrose[], Stratila-Zsido[]. The philosophy is to lavish attention on a few key results and examples, and we prefer to make simplifying assumptions rather than go for the most general case. Thus we do not hesitate to give several proofs of a single result, or repeat an argument with different hypotheses. The notes are built around semester-long courses given at UC Berkeley though they contain more material than could be taught in a single semester.

The notes are informal and the exercises are an integral part of the exposition. These exercises are vital and mostly intended to be easy.

Chapter 2

Background and Prerequisites

2.1 Hilbert Space

A Hilbert Space is a complex vector space $\mathcal{H}$ with inner product $\langle , \rangle : \mathcal{H}\text{x}\mathcal{H} \to \mathbb{C}$ which is linear in the first variable, satisfies $\overline{\langle \xi, \eta \rangle} = \langle \eta, \xi \rangle$, is positive definite, i.e. $\langle \xi, \xi \rangle > 0$ for $\xi \neq 0$, and is complete for the norm defined by test $||\xi|| = \sqrt{\langle \xi, \xi \rangle}$.

Exercise 2.1.1. *Prove the parallelogram identity :*

$$||\xi - \eta||^2 + ||\xi + \eta||^2 = 2(||\xi||^2 + ||\eta||^2)$$

and the Cauchy-Schwartz inequality:

$$|\langle \xi, \eta \rangle| \leq ||\xi||\,||\eta||.$$

Theorem 2.1.2. *If C is a closed convex subset of $\mathcal{H}$ and ξ is any vector in $\mathcal{H}$, there is a unique $\eta \in C$ which minimizes the distance from ξ to C, i.e. $||\xi - \eta'|| \leq ||\xi - \eta|| \quad \forall \eta' \in C$.*

Proof. This is basically a result in plane geometry.

Uniqueness is clear—if two vectors η and η' in C minimized the distance to ξ, then ξ, η and η' lie in a (real) plane so any vector on the line segment between η and η' would be strictly closer to ξ.

To prove existence, let d be the distance from C to ξ and choose a sequence $\eta_n \in C$ with $||\eta_n - \xi|| < d + 1/2^n$. For each n, the vectors ξ, η_n and η_{n+1} define a plane. Geometrically it is clear that, if η_n and η_{n+1} were not close, some point on the line segment between them would be closer than d to ξ. Formally, use the parallelogram identity:

$$||\xi - \frac{\eta_n + \eta_{n+1}}{2}||^2 = ||\frac{\xi - \eta_n}{2} + \frac{\xi - \eta_{n+1}}{2}||^2$$

$$= 2(||\frac{\xi - \eta_n}{2}||^2 + ||\frac{\xi - \eta_{n+1}}{2}||^2 - 1/8||\eta_n - \eta_{n+1}||^2)$$
$$\leq (d + 1/2^n)^2 - 1/4||\eta_n - \eta_{n+1}||^2$$

Thus there is a constant K such that $||\eta_n - \eta_{n+1}||^2 < K/2^n$ or $||\xi - \frac{\eta_n + \eta_{n+1}}{2}||^2$ would be less than d^2.

Thus (η_n) is Cauchy, its limit is in C and has distance d from ξ.

□

Exercise 2.1.3. *If $\phi \in \mathcal{H}^*$ (the Banach-space dual of $\mathcal{H}$ consisting of all continuous linear functionals from $\mathcal{H}$ to $\mathbb{C}$), kerϕ is a closed convex subset of $\mathcal{H}$. Show how to choose a vector ξ_ϕ orthogonal to* $\ker \phi$ *with $\phi(\eta) = \langle \xi_\phi, \eta \rangle$ and so that $\phi \mapsto \xi_\phi$ is a conjugate-linear isomorphism from $\mathcal{H}^*$ onto $\mathcal{H}$.*

We will be especially concerned with *separable* Hilbert Spaces where there is an *orthonormal basis*, i.e. a sequence $\{\xi_1, \xi_2, \xi_3, ...\}$ of unit vectors with $\langle \xi_i, \xi_j \rangle = 0$ for $i \neq j$ and such that 0 is the only element of $\mathcal{H}$ orthogonal to all the ξ_i.

Exercise 2.1.4. *Show that an orthonormal basis always exists (e.g. Gram-Schmidt) and that if $\{\xi_i\}$ is an orthonormal basis for $\mathcal{H}$ then the linear span of the $\{\xi_i\}$ is dense in $\mathcal{H}$.*

A linear map (operator) $a : \mathcal{H} \to \mathcal{K}$ is said to be bounded if there is a number K with $||a\xi|| \leq K||\xi|| \quad \forall \xi \in \mathcal{H}$. The infimum of all such K is called the *n*orm of a, written $||a||$. The set of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$ is written $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and if $\mathcal{H} = \mathcal{K}$ we use $\mathcal{B}(\mathcal{H})$. Boundedness of an operator is equivalent to continuity.

To every bounded operator a between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, by exercise 2.1.3 there is another, a^*, between $\mathcal{K}$ and $\mathcal{H}$, called the *a*djoint of a which is defined by the formula $\langle a\xi, \eta \rangle = \langle \xi, a^*\eta \rangle$.

Exercise 2.1.5. *Prove that*

$$||a|| = \sup_{||\xi|| \leq 1, ||\eta|| \leq 1} |\langle a\xi, \eta \rangle|$$
$$= ||a^*|| = ||a^*a||^{1/2}.$$

Some definitions:

The identity map on $\mathcal{H}$ is a bounded operator denoted 1.

An operator $a \in \mathcal{B}(\mathcal{H})$ is called *self-adjoint* if $a = a^*$.

An operator $p \in \mathcal{B}(\mathcal{H})$ is called a *projection* if $p = p^2 = p^*$.

An operator $a \in \mathcal{B}(\mathcal{H})$ is called *positive* if $\langle a\xi, \xi \rangle \geq 0 \quad \forall \xi \in \mathcal{B}(\mathcal{H})$. We say

$a \geq b$ if $a - b$ is positive.

An operator $u \in \mathcal{B}(\mathcal{H})$ is called an *isometry* if $u^*u = 1$.

An operator $v \in \mathcal{B}(\mathcal{H})$ is called a *unitary* if $uu^* = u^*u = 1$.

An operator $u \in \mathcal{B}(\mathcal{H})$ is called a *partial isometry* if u^*u is a projection.

The last three definitions extend to bounded linear operators between different Hilbert spaces.

If $S \subseteq \mathcal{B}(\mathcal{H})$ then the *commutant* S' of S is $\{x \in \mathcal{B}(\mathcal{H}) | xa = ax \quad \forall a \in S\}$. Also $S'' = (S')'$.

Exercise 2.1.6. *Show that every $a \in \mathcal{B}(\mathcal{H})$ is a linear combination of two self-adjoint operators.*

Exercise 2.1.7. *A positive operator is self-adjoint.*

Exercise 2.1.8. *Find an isometry from one Hilbert space to itself that is not unitary.* (The unilateral shift on $\mathcal{H} = \ell^2(\mathbb{N})$ is a fine example. There is an obvious orthonormal basis of $\mathcal{H}$ indexed by the natural numbers and the shift just sends the $nth.$ basis element to the $(n+1)th.$)

Exercise 2.1.9. *If $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ show that the map $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ which assigns to any point in $\mathcal{H}$ the nearest point in $\mathcal{K}$ is linear and a projection.*

Exercise 2.1.10. *Show that the correspondence $\mathcal{K} \to P_{\mathcal{K}}$ of the previous exercise is a bijection between closed subspaces of $\mathcal{H}$ and projections in $\mathcal{B}(\mathcal{H})$.*

If S is a subset of $\mathcal{H}$, $S^\perp$ is by definition $\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle = 0 \quad \forall \eta \in S\}$. Note that $S^\perp$ is always a closed subspace.

Exercise 2.1.11. *If $\mathcal{K}$ is a closed subspace then $\mathcal{K}^{\perp\perp} = \mathcal{K}$ and $P_{\mathcal{K}^\perp} = 1 - P_{\mathcal{K}}$.*

Exercise 2.1.12. *If u is a partial isometry then so is u^*. The subspace $u^*\mathcal{H}$ is then closed and called the* initial domain *of u, the subspace $u\mathcal{H}$ is also closed and called the* final domain *of u. Show that a partial isometry is the composition of the projection onto its initial domain and a unitary between the initial and final domains.*

The commutator $[a, b]$ of two elements of $\mathcal{B}(\mathcal{H})$ is the operator $ab - ba$.

Exercise 2.1.13. *If $\mathcal{K}$ is a closed subspace and $a = a^*$ then*

$$a\mathcal{K} \subseteq \mathcal{K} \quad iff \quad [a, P_{\mathcal{K}}] = 0.$$

In general $(a\mathcal{K} \subseteq \mathcal{K} \quad and \quad a^*\mathcal{K} \subseteq \mathcal{K}) \iff [a, P_{\mathcal{K}}] = 0.$

2.2 The Spectral Theorem

The *spectrum* $\sigma(a)$ of $a \in \mathcal{B}(\mathcal{H})$ is $\{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}$.

Exercise 2.2.1. (Look up proofs if necessary.) *Show that $\sigma(a)$ is a non-empty closed bounded subset of $\mathbb{C}$ and that if $a = a^*$, $\sigma(a) \subseteq [-||a||, ||a||]$ with either $||a||$ or $-||a||$ in $\sigma(a)$.*

The spectral theorem takes a bit of getting used to and knowing how to prove it does not necessarily help much. If one cannot "see" the spectral decomposition of an operator it may be extremely difficult to obtain—except in a small finite number of dimensions where it is just diagonalisation. But fortunately there is nothing like a course in operator algebras, either C^* or von Neumann, to help master the use of this theorem which is the heart of linear algebra on Hilbert space. The book by Reed and Simon, "Methods of mathematical physics" vol. 1, Functional Analysis, contains a treatment of the spectral theorem which is perfect background for this course. We will make no attempt to prove it here—just give a vague statement which will establish terminology.

The spectral theorem asserts the existence of a *projection valued measure* from the Borel subsets of $\sigma(a)$ (when $a = a^*$ or more generally when a is *normal* i.e. $[a, a^*] = 0$) to projections in $\mathcal{B}(\mathcal{H})$, written symbolically $\lambda \to E(\lambda)$, such that

$$a = \int \lambda dE(\lambda).$$

This integral may be interpreted as a limit of sums of operators (necessitating a topology on $\mathcal{B}(\mathcal{H})$), as a limit of sums of vectors: $a\xi = \int \lambda dE(\lambda)\xi$ or simply in terms of measurable functions $\langle \xi, a\eta \rangle = \int \lambda d\langle \xi, E(\lambda)\eta \rangle$. The projections $E(B)$ are called the *spectral projections* of a and their images are called the *spectral subspaces* of a.

Given any bounded Borel complex-valued function f on $\sigma(a)$ one may form $f(a)$ by $f(a) = \int f(\lambda)dE(\lambda)$.

Exercise 2.2.2. *If μ is a sigma-finite measure on X and $f \in L^\infty(X, \mu)$, the operator $M_f : L^2(X, \mu) \to L^2(X, \mu), (M_f g)(x) = f(x)g(x)$, is a bounded (normal) operator with $||M_f|| = \text{ess-sup}_{x \in X}(|f(x)|)$. If f is real valued then M_f is self adjoint. Find $\sigma(f)$ and the projection-valued measure $E(\lambda)$.*

Exercise 2.2.3. *If $\dim(\mathcal{H}) < \infty$ find the spectrum and projection-valued measure for a (which is a Hermitian matrix).*

The example of exercise 2.2.2 is generic in the sense that there is a version of the spectral theorem which asserts the following. If $\xi \in \mathcal{H}$ is any vector and $a = a^* \in \mathcal{B}(\mathcal{H})$, let $\mathcal{K}$ be the closed linear span of the $\{a^n\xi : n = 0, 1, 2, 3, ...\}$, then a defines a self-adjoint operator on $\mathcal{K}$ and there is a finite measure μ on the spectrum $\sigma(a)$ such that $(\mathcal{K}, a)$ is isomorphic in the obvious sense to $(L^2(\sigma(a), \mu),$ multiplication by $x)$. Continuing such an argument by restricting to $\mathcal{K}^\perp$ one obtains a full spectral theorem.

Exercise 2.2.4. *Show that a self-adjoint operator a is the difference $a_+ - a_-$ of two positive commuting operators called the positive and negative parts of a, obtained as functions of a as above.*

2.3 Polar decomposition

Exercise 2.3.1. *Show that every positive operator a has a unique positive square root $a^{1/2}$.*

Given an arbitrary $a \in \mathcal{B}(\mathcal{H})$ we define $|a| = (a^*a)^{1/2}$.

Exercise 2.3.2. *Show that there is a partial isometry u such that $a = u|a|$, and that u is unique subject to the condition that its initial domain is $ker(a)^\perp$. The final domain of this u is $\overline{Im(a)} = ker(a^*)^\perp$.*

2.4 Tensor product of Hilbert Spaces.

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces one may form their algebraic tensor product $\mathcal{H} \otimes_{alg} \mathcal{K}$ (in the category of complex vector spaces). On this vector space one defines the sesquilinear form $\langle , \rangle$ by:

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \xi, \xi' \rangle \langle \eta, \eta' \rangle$$

and observes that this form is positive definite. The Hilbert space tensor product $\mathcal{H} \otimes \mathcal{K}$ is then the completion of $\mathcal{H} \otimes_{alg} \mathcal{K}$. It is easy to see that if $a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{B}(\mathcal{K})$, there is a bounded operator $a \otimes b$ on $\mathcal{H} \otimes \mathcal{K}$ defined by $a \otimes b(\xi \otimes \eta) = a\xi \otimes b\eta$.

Exercise 2.4.1. *Let $L^2(X, \mathcal{H}, \mu)$ be the Hilbert space of measurable square integrable functions (up to null sets) $f : X \to \mathcal{H}$, with $\mathcal{H}$ a separable Hilbert space. For each $\xi \in \mathcal{H}$ and $f \in L^2(X, \mu)$ let $f_\xi \in L^2(X, \mathcal{H}, \mu)$ be defined by $f_\xi(x) = f(x)\xi$. Show that the map $\xi \otimes f \mapsto f_\xi$ defines a unitary from $\mathcal{H} \otimes L^2(X, \mu)$ onto $L^2(X, \mathcal{H}, \mu)$.*

Chapter 3

The definition of a von Neumann algebra.

3.1 Topologies on $\mathcal{B}(\mathcal{H})$

1. The *norm* or *uniform* topology is given by the norm $||a||$ defined in the previous chapter.

2. The topology on $\mathcal{B}(\mathcal{H})$ of pointwise convergence on $\mathcal{H}$ is called the *strong operator topology.* A basis of neighbourhoods of $a \in \mathcal{B}(\mathcal{H})$ is formed by the

$$N(a, \xi_1, \xi_2, ..., \xi_n, \epsilon) = \{b : ||(b-a)\xi_i|| < \epsilon \quad \forall i = 1, \cdots, n\}$$

3. The *weak operator topology* is formed by the basic neighbourhoods

$$N(a, \xi_1, \xi_2, ..., \xi_n, \eta_1, \eta_2, .., \eta_n, \epsilon) = \{b : |\langle (b-a)\xi_i, \eta_i \rangle| < \epsilon \quad \forall i = 1, \cdots, n\}$$

Note that this weak topology is the topology of pointwise convergence on $\mathcal{H}$ in the "weak topology" on $\mathcal{H}$ defined in the obvious way by the inner product.

The unit ball of $\mathcal{H}$ is compact in the weak topology and the unit ball of $\mathcal{B}(\mathcal{H})$ is compact in the weak operator topology. These assertions follow easily from Tychonoff's theorem.

Exercise 3.1.1. *Show that we have the following ordering of the topologies (strict in infinite dimensions).*

(weak operator topology) < (strong operator topology) < (norm topology)

Note that a weaker topology has less open sets so that if a set is closed in the weak topology it is necessarily closed in the strong and norm topologies.

3.2 The bicommutant theorem.

We will now prove the von Neumann "density" or "bicommutant" theorem which is the first result in the subject. We prove it first in the finite dimensional case where the proof is transparent then make the slight adjustments for the general case.

Theorem 3.2.1. *Let M be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing* 1*, with* $\dim(\mathcal{H}) = n < \infty$*. Then $M = M''$.*

Proof. It is tautological that $M \subseteq M''$.

So we must show that if $y \in M''$ then $y \in M$. To this end we will "amplify" the action of M on $\mathcal{H}$ to an action on $\mathcal{H} \otimes \mathcal{H}$ defined by $x(\xi \otimes \eta) = x\xi \otimes \eta$. If we choose an orthonormal basis $\{v_i\}$ of $\mathcal{H}$ then $\mathcal{H} \otimes \mathcal{H} = \oplus_{i=1}^{n} \mathcal{H}$ and in terms of matrices we are considering the $n \,\mathrm{x}\, n$ matrices over $\mathcal{B}(\mathcal{H})$ and embedding M in it as matrices constant down the diagonal. Clearly enough the commutant of M on $\mathcal{H} \otimes \mathcal{H}$ is the algebra of all $n \,\mathrm{x}\, n$ matrices with entries in M' and the second commutant consists of matrices having a fixed element of M'' down the diagonal.

Let v be the vector $\oplus_{i=1}^{n} v_i \in \oplus_{i=1}^{n} \mathcal{H}$ and let $V = Mv \subseteq \mathcal{H} \otimes \mathcal{H}$. Then $MV \subseteq V$ and since $M = M^*$, $P_V \in M'$ (on $\mathcal{H} \otimes \mathcal{H}$) by exercise 2.1.13. So if $y \in M''$ (on $\mathcal{H} \otimes \mathcal{H}$), then y commutes with P_V and $yMv \subseteq Mv$. In particular $y(1v) = xv$ for some $x \in M$ so that $yv_i = xv_i$ for all i, and $y = x \in M$. □

Theorem 3.2.2. *(von Neumann) Let M be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing* 1*. Then $M'' = \overline{M}$ (closure in the strong operator topology).*

Proof. Commutants are always closed so $\overline{M} \subseteq M''$.

So let $a \in M''$ and $N(a, \xi_1, \xi_2, ..., \xi_n, \epsilon)$ be a strong neighbourhood of a. We want to find an $x \in M$ in this neighbourhood. So let $v \in \oplus_{i=1}^{n} \mathcal{H}$ be $\oplus_{i=1}^{n} \xi_i$ and let $\mathcal{B}(\mathcal{H})$ act diagonally on $\oplus_{i=1}^{n} \mathcal{H}$ as in the previous theorem. Then the same observations as in the previous proof concerning matrix forms of commutants are true. Also M commutes with $P_{\overline{Mv}}$ which thus commutes with a (on $\oplus_{i=1}^{n} \mathcal{H}$). And since $1 \in M$, $av = \oplus a\xi_i$ is in the closure of Mv so there is an $x \in M$ with $||x\xi_i - a\xi_i|| < \epsilon$ for all i. □

Corollary 3.2.3. *If $M = M^*$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ with $1 \in M$, then the following are equivalent:*

1. *$M = M''$*
2. *M is strongly closed.*
3. *M is weakly closed.*

Definition 3.2.4. *A subalgebra of $\mathcal{B}(\mathcal{H})$ satisfying the conditions of corollary 3.2.3 is called a von Neumann algebra.*

(A self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the *norm* topology is called a C^*-algebra.)

3.3 Examples.

Example 3.3.1. *Any finite dimensional *-subalgebra of $\mathcal{B}(\mathcal{H})$ containing* 1.

Example 3.3.2. $\mathcal{B}(\mathcal{H})$ *itself.*

Exercise 3.3.3. *Let (X, μ) be a finite measure space and consider $A = L^\infty(X, \mu)$ as a *-subalgebra of $\mathcal{B}(L^2(X, \mu))$ (as multiplication operators as in exercise 2.2.2). Show that $A = A'$, i.e. A is maximal abelian and hence a von Neumann algebra. (Hint: if $x \in A'$ let $f = x(1)$. Show that $f \in L^\infty$ and that $x = M_f$.)*

Example 3.3.4. If $S \subseteq \mathcal{B}(\mathcal{H})$, we call $(S \cup S^*)''$ the von Neumann algebra generated by S. It is, by theorem 3.2.2 the weak or strong closure of the *-algebra generated by 1 and S. Most constructions of von Neumann algebras begin by considering some family of operators with desirable properties and then taking the von Neumann algebra they generate. But is is quite hard, in general, to get much control over the operators added when taking the weak closure, and all the desirable properties of the generating algebra may be lost. (For instance any positive self-adjoint operator a with $||a|| \leq 1$ is a weak limit of projections.) However, if the desirable properties can be expressed in terms of matrix coefficients then these properties will be preserved under weak limits since the matrix coefficients of a are just special elements of the form $\langle \xi, a\eta \rangle$. We shall now treat an example of this kind of property which is at the heart of the subject and will provide us with a huge supply of interesting von Neumann algebras quite different from the first 3 examples.

Let Γ be a discrete group and let $\ell^2(\Gamma)$ be the Hilbert space of all functions $f : \Gamma \to \mathbb{C}$ with $\sum_{\gamma \in \Gamma} |f(\gamma)|^2 < \infty$ and inner product $\langle f, g \rangle = \sum_{\gamma \in \Gamma} \overline{f(\gamma)} g(\gamma)$. An orthonormal basis of $\ell^2(\Gamma)$ is given by the $\{\varepsilon_\gamma\}$ where $\varepsilon_\gamma(\gamma') = \delta_{\gamma,\gamma'}$ so that $f = \sum_{\gamma \in \Gamma} f(\gamma)\varepsilon_\gamma$ in the ℓ^2 sense. For each $\gamma \in \Gamma$ define the unitary operator u_γ by $(u_\gamma f)(\gamma') = f(\gamma^{-1}\gamma')$. Note that $u_\gamma u_\rho = u_{\gamma\rho}$ and that $u_\gamma(\varepsilon_\rho) = \varepsilon_{\gamma\rho}$. Thus $\gamma \mapsto u_\gamma$ is a unitary group representation called the *left regular representation.* The u_γ are linearly independent so the algebra they generate is isomorphic

to the group algebra $\mathbb{C}\Gamma$. The von Neumann algebra generated by the u_γ goes under various names, $U(\Gamma)$, $\lambda(\Gamma)$ and $L(\Gamma)$ but we will call it $vN(\Gamma)$ as it is the "group von Neumann algebra" of Γ.

To see that one can control weak limits of linear combinations of the u_γ, consider first the case $\Gamma = \mathbb{Z}/n\mathbb{Z}$. With basis $u_0, u_1, u_2, \cdots, u_{n-1}$, the element u_1 is represented by the matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & . & . \\ 0 & 0 & 1 & 0 & 0 & . \\ 0 & . & 0 & 1 & 0 & 0 \\ 0 & . & . & 0 & 1 & 0 \\ 0 & 0 & . & . & 0 & 1 \\ 1 & 0 & 0 & . & . & 0 \end{bmatrix}$$

which is a matrix constant along the "diagonals". Clearly all powers of u_1 and all linear combinations of them have this property also so that an arbitrary element of the algebra they generate will have the matrix form (when $n = 6$):

$$\begin{bmatrix} a & b & c & d & e & f \\ f & a & b & c & d & e \\ e & f & a & b & c & d \\ d & e & f & a & b & c \\ c & d & e & f & a & b \\ b & c & d & e & f & a \end{bmatrix}$$

(Such matrices are known as circulant matrices but to the best of our knowledge this term only applies when the group is cyclic.) If $\mathbb{Z}/n\mathbb{Z}$ were replaced by another finite group the same sort of structure would prevail except that the "diagonals" would be more complicated, according to the multiplication table of the group.

Now let Γ be an infinite group. It is still true that the (γ, ρ) matrix entry of a finite linear combination of the u_γ's will depend only on $\gamma^{-1}\rho$. As observed above, *the same must be true of weak limits of these linear combinations, hence of any element of* $vN(\Gamma)$.

We see that the elements of $vN(\Gamma)$ have matrices (w.r.t. the basis ε_γ) which are constant along the "diagonals" : $\{(\gamma, \rho) : \gamma\rho^{-1} \text{ is constant}\}$.

Exercise 3.3.5. *Check whether it should be* $\gamma^{-1}\rho$ *or* $\gamma\rho^{-1}$ *or some other similar expression.....*

Using the number c_γ on the diagonal indexed by γ we can write, formally at least, any element of $vN(\Gamma)$ as a sum $\sum_{\gamma\in\Gamma} c_\gamma u_\gamma$. It is not clear in what

sense this sum converges but certainly $\sum_{\gamma\in\Gamma} c_\gamma u_\gamma$ must define a bounded linear operator. From this we deduce immediately the following:

(i) The function $\gamma \mapsto c_\gamma$ is in ℓ^2. (Apply $\sum_{\gamma\in\Gamma} c_\gamma u_\gamma$ to ε_{id}.)

(ii) $(\sum_{\gamma\in\Gamma} c_\gamma u_\gamma)(\sum_{\gamma\in\Gamma} d_\gamma u_\gamma) = \sum_{\gamma\in\Gamma}(\sum_{\rho\in\Gamma} c_\rho d_{\rho^{-1}\gamma})u_\gamma$

where the sum defining the coefficient of u_γ on the right hand side converges since $\rho \mapsto c_\rho$ and $\rho \mapsto d_{\rho^{-1}\gamma}$ are in ℓ^2.

Exactly what functions $\gamma \mapsto c_\gamma$ define elements of $vN(\Gamma)$ is unclear but an important special case gives some intuition.

Case 1. $\Gamma = \mathbb{Z}$.

It is well known that the map $\sum c_n\varepsilon_n \to \sum c_n e^{in\theta}$ defines a unitary V from $\ell^2(\Gamma)$ to $L^2(\mathbb{T})$. Moreover $Vu_nV^{-1}(e^{ik\theta}) = Vu_n(\varepsilon_k) = V\varepsilon(k+n) = e^{in\theta}e^{ik\theta}$ so that Vu_nV^{-1} is the multiplication operator $M_{e^{in\theta}}$. Standard spectral theory shows that $M_{e^{in\theta}}$ generates $L^\infty(\mathbb{T})$ as a von Neumann algebra, and clearly if $M_f \in L^\infty(\mathbb{T})$, $\quad V^{-1}M_fV = \sum c_n\varepsilon_n$ where $\sum c_n e^{in\theta}$ is the Fourier series of f. We see that, in this case, the functions $\gamma \mapsto c_\gamma$ which define elements of $vN(\mathbb{Z})$ are precisely the Fourier series of L^∞ functions. In case we forget to point it out later on when we are in a better position to prove it, one way to characterise the functions which define elements on $vN(\Gamma)$ is as <u>all</u> functions which define bounded operators on $\ell^2(\Gamma)$. This is not particularly illuminating but can be useful at math parties.

At the other extreme we consider a highly non-commutative group, the free group on n generators, $n \geq 2$.

Case 2. $\Gamma = F_n$.

"Just for fun" let us compute the centre $Z(vN(\Gamma))$ of $vN(F_n)$, i.e. those $\sum c_\gamma u_\gamma$ that commute with all $x \in vN(\Gamma)$. By weak limits of linear combinations, for $\sum c_\gamma u_\gamma$ to be in $Z(vN(\Gamma))$ it is necessary and sufficient that it commute with every u_γ. This clearly is the same as saying $c_{\gamma\rho\gamma^{-1}} = c_\rho \quad \forall\gamma, \rho$, i.e. the function c is constant on conjugacy classes. But in F_n all conjugacy classes except that of the identity are infinite. Now recall that $\gamma \mapsto c_\gamma$ is in ℓ^2. We conclude that $c_\gamma = 0 \quad \forall\gamma \neq 1$, i.e. $Z(vN(\Gamma)) = \mathbb{C}1$.

Note that the only property we used of F_n to reach this conclusion was that every non-trivial conjugacy class is infinite (and in general it is clear that $Z(vN(\Gamma))$ is in the linear span of the u_γ with γ in a finite conjugacy class.) Such groups are called <u>i.c.c.</u> groups and they abound. Other examples include S_∞ (the group of finitely supported permutations of an infinite set), $PSL(n, \mathbb{Z})$ and $\mathbb{Q} \rtimes \mathbb{Q}^*$.

Unsolved problem in von Neumann algebras:

Is $vN(F_n) \cong vN(F_m)$ *for* $n \neq m$ $\quad$ (*for* n *and* $m \geq 2$)?

Note that it is obvious that the group algebras $\mathbb{C}F_n$ and $\mathbb{C}F_m$ are not isomorphic. Just consider algebra homomorphisms to $\mathbb{C}$. But of course these homomorphisms will *not* extend to $vN(\Gamma)$.

Definition 3.3.6. *A von Neumann algebra whose centre is* $\mathbb{C}1$ *is called a* factor.

Exercise 3.3.7. *Show that* $\mathcal{B}(\mathcal{H})$ *is a factor.*

Exercise 3.3.8. *Suppose* $\mathcal{H} = \mathcal{K}_1 \otimes \mathcal{K}_2$ *and let* $M = \mathcal{B}(\mathcal{K}_1) \otimes 1$ *Show that* $M' = 1 \otimes \mathcal{B}(\mathcal{K}_2)$ *so that* M *and* M' *are factors.*

This exercise is supposed to explain the origin of the term "factor" as in this case M and M' come from a tensor product factorisation of $\mathcal{H}$. Thus in general a factor and its commutant are supposed to correspond to a bizarre "factorisation" of the Hilbert space.

The factor we have constructed as $vN(\Gamma)$ is of an entirely different nature from $\mathcal{B}(\mathcal{H})$. To see this consider the function $tr : vN(\Gamma) \to \mathbb{C}$ defined by $tr(a) = \langle a\varepsilon_1, \varepsilon_1 \rangle$, or $tr(\sum c_\gamma u_\gamma) = c_1$. This map is clearly linear, weakly continuous, satisfies $tr(ab) = tr(ba)$ and $tr(x^*x) = \sum_\gamma |c_\gamma|^2 \geq 0$ (when $x = \sum_\gamma c_\gamma u_\gamma$). It is called a *trace* on $vN(\Gamma)$. If $\Gamma = \mathbb{Z}$ it obviously equals $\frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$ under the isomorphism between $vN(\mathbb{Z})$ and $L^\infty(\mathbb{T})$.

Exercise 3.3.9. *(i)Suppose* $\dim \mathcal{H} < \infty$. *If* $tr : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ *is a linear map with* $tr(ab) = tr(ba)$, *show that there is a constant* K *with* $tr(x) = Ktrace(x)$.

(ii) There is no non-zero weakly continuous linear map $tr : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ *satisfying* $tr(ab) = tr(ba)$ *when* $\dim(\mathcal{H}) = \infty$.

(iii) There is no non-zero linear map $tr : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ *satisfying* $tr(ab) = tr(ba)$ *and* $tr(x^*x) \geq 0$ *when* $\dim(\mathcal{H}) = \infty$.

(iv) (harder) There is no non-zero linear map $tr : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ *satisfying* $tr(ab) = tr(ba)$ *when* $\dim(\mathcal{H}) = \infty$.

Thus our factors $vN(\Gamma)$ when Γ is i.c.c. are infinite dimensional but seem to have more in common with $\mathcal{B}(\mathcal{H})$ when $\dim \mathcal{H} < \infty$ than when $\dim \mathcal{H} = \infty$! They certainly do not come from honest tensor product factorisations of $\mathcal{H}$.

Let us make a couple of observations about these factors.

1)They contain no non-zero finite rank operators, for such an operator cannot be constant and non-zero down the diagonal. (Take x^*x if necessary.)

2)They have the property that $tr(a) = 0 \Rightarrow a = 0$ for a positive element a (a positive operator cannot have only zeros down the diagonal).

3)They have the property that $uu^* = 1 \Rightarrow u^*u = 1$ (i.e. they contain no non-unitary isometries).

Proof. If $u^*u = 1$, uu^* is a projection so $1 - uu^*$ is too and $tr(1 - uu^*) = 1 - tr(u^*u) = 0$. □

Exercise 3.3.10. *Show that in $vN(\Gamma)$, $ab = 1 \Rightarrow ba = 1$. Show that if F is any field of characteristic 0, $ab = 1 \Rightarrow ba = 1$ in $F\Gamma$.*

Hints: 1) You may use elementary property 8 of the next chapter.

2) Only finitely many elements of the field are involved in ab and ba in $F\Gamma$.

As far as I know this assertion is still open in non-zero characteristic. The above exercise is a result of Kaplansky.

The next observation is a remarkable property of the set of projections.

4) If $\Gamma = F_n$, $\{tr(p) : p \text{ a projection in } vN(\Gamma)\} = [0, 1]$.

Proof. It is clear that the trace of a projection is between 0 and 1. To see that one may obtain every real number in this interval, consider the subgroup $\langle a \rangle$ generated by a single non-zero element. By the coset decomposition of F_n the representation of $\langle a \rangle$ on $\ell^2(F_n)$ is the direct sum of countably many copies of the regular representation. The bicommutant of u_a is then, by a matrix argument, $vN(\mathbb{Z})$ acting in an "amplified" way as block diagonal matrices with constant blocks so we may identify $vN(\mathbb{Z})$ with a subalgebra of $vN(\Gamma)$. Under this identification the traces on the two group algebras agree. But as we have already observed, any element $f \in L^\infty(0, 2\pi)$ defines an element of $vN(\mathbb{Z})$ whose integral is its trace. The characteristic function of an interval is a projection so by choosing intervals of appropriate lengths we may realise projections of any trace. □

We used the bicommutant to identify $vN(\mathbb{Z})$ with a subalgebra of $vN(\Gamma)$. It is instructive to point out a problem that would have arisen had we tried to use the weak or strong topologies. A vector in $\ell^2(\Gamma)$ is a square summable sequence of vectors in $\ell^2(\mathbb{Z})$ so that a basic strong neighbourhood of a on $\ell^2(\Gamma)$ would correspond to a neighbourhood of the form $\{b : \sum_{i=1}^\infty ||(a - b)\xi_i||^2 < \epsilon\}$ for a sequence (ξ_i) in $\ell^2(\mathbb{Z})$ with $\sum_{i=1}^\infty ||\xi_i||^2 < \infty$. Thus strong convergence on $\ell^2(\mathbb{Z})$ would not imply strong convergence on $\ell^2(\Gamma)$. This leads us naturally to define two more topologies on $\mathcal{B}(\mathcal{H})$.

Definition 3.3.11. *The topology defined by the basic neighbourhoods of a, $\{b : \sum_{i=1}^\infty ||(a - b)\xi_i||^2 < \epsilon\}$ for any ϵ and any sequence (ξ_i) in $\ell^2(\mathcal{H})$ with $\sum_{i=1}^\infty ||\xi_i||^2 < \infty$, is called the* ultrastrong *topology on $\mathcal{B}(\mathcal{H})$.*

The topology defined by the basic neighbourhoods

$$\{b : \sum_{i=1}^{\infty} |\langle (a-b)\xi_i, \eta_i \rangle| < \epsilon\}$$

for any $\epsilon > 0$ *and any sequences* (ξ_i), (η_i) *in* $\ell^2(\mathcal{H})$ *with*

$$\sum_{i=1}^{\infty} ||\xi_i||^2 + ||\eta_i||^2 < \infty$$

is called the ultraweak *topology on* $\mathcal{B}(\mathcal{H})$.

Note that these topologies are precisely the topologies inherited on $\mathcal{B}(\mathcal{H})$ if it is amplified infinitely many times as $\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{K}}$ with $\dim \mathcal{K} = \infty$.

Exercise 3.3.12. *Show that the ultrastrong and strong topologies coincide on a bounded subset of* $\mathcal{B}(\mathcal{H})$ *as do the weak and ultraweak topologies. That they differ will be shown in 5.1.4.*

Exercise 3.3.13. *Repeat the argument of the von Neumann density theorem (3.2.2) with the ultrastrong topology replacing the strong topology.*

Here are some questions that the inquisitive mind might well ask at this stage. All will be answered in succeeding chapters.

Question 1) If there is a weakly continuous trace on a factor, is it unique (up to a scalar multiple)?

Question 2) If there is a trace on a factor M is it true that $\{tr(p) : p \text{ a projection in } M\} = [0, 1]$?

Question 3) Is there a trace on any factor not isomorphic to $\mathcal{B}(\mathcal{H})$?

Question 4) Are all (infinite dimensional) factors with traces isomorphic?

Question 5) If M is a factor with a trace, is M' also one? (Observe that the commutant of a factor is obviously a factor.)

Question 6) Is $vN(\Gamma)'$ the von Neumann algebra generated by the right regular representation?

Question 7) If $\phi : M \to N$ is a *-algebra isomorphism between von Neumann algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is there a unitary $u : \mathcal{H} \to \mathcal{K}$ so that $\phi(a) = uau^*$ for $a \in M$?

3.4 Elementary properties of von Neumann algebras.

Throughout this chapter M will be a von Neumann algebra on a Hilbert space $\mathcal{H}$.

EP1) If $a = a^*$ is an element of M, all the spectral projections and all bounded Borel functions of a are in M. Consequently M is generated by its projections.

According to one's proof of the spectral theorem, the spectral projections $E(\lambda)$ of a are constructed as strong limits of polynomials in a. Or the property that the spectral projections of a are in the bicommutant of a may be an explicit part of the theorem. Borel functions will be in the bicommutant.

EP2) Any element in M is a linear combination of 4 unitaries in M.

Proof. We have seen that any x is a linear combination of 2 self-adjoints, and if a is self-adjoint, $||a|| \leq 1$, let $u = a + i\sqrt{1 - a^2}$. Then u is unitary and $a = \frac{u+u^*}{2}$. □

EP3) M is the commutant of the unitary group of M' so that an alternative definition of a von Neumann algebra is the commutant of a unitary group representation.

This follows from EP2)

Exercise 3.4.1. *Show that multiplication of operators is jointly strongly continuous on bounded subsets but not on all of $\mathcal{B}(\mathcal{H})$.*

Show that $* : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is weakly continuous but not strongly continuous even on bounded sets.

The following result is well known and sometimes called Vigier's theorem.

Theorem 3.4.2. *If $\{a_\alpha\}$ is a net of self-adjoint operators with $a_\alpha \leq a_\beta$ for $\alpha \leq \beta$ and $||a_\alpha|| \leq K$ for some $K \in \mathbb{R}$, then there is a self-adjoint a with $a = lim_\alpha a_\alpha$, convergence being in the strong topology. Furthermore $a = lub(a_\alpha)$ for the poset of self-adjoint operators.*

Proof. A candidate a for the limit can be found by weak compactness of the unit ball. Then $\langle a_\alpha \xi, \xi \rangle$ is increasing with limit $\langle a\xi, \xi \rangle$ for all $\xi \in \mathcal{H}$ and $a \geq a_\alpha \quad \forall \alpha$. So $lim_\alpha \sqrt{a - a_\alpha} = 0$ in the strong topology. Now multiplication is jointly strongly continuous on bounded sets so $s{-}lim\, a_\alpha = a$. □

Note that we have slipped in the notation $s{-}lim$ for a limit in the strong topology (and obviously $w{-}lim$ for the weak topology).

If a and (a_α) are as in 3.4.2 we say the net (a_α) is *monotone convergent* to a.

EP4) M is closed under monotone convergence of self-adjoint operators.

The projections on $\mathcal{B}(\mathcal{H})$ form an ortholattice with the following properties:

$$p \leq q \iff p\mathcal{H} \subseteq q\mathcal{H}$$

$$p \wedge q = \text{orthogonal projection onto } p\mathcal{H} \cap q\mathcal{H}$$

$$p^\perp = 1 - p$$

$$p \vee q = (p^\perp \wedge q^\perp)^\perp = \text{orthogonal projection onto } \overline{p\mathcal{H} + q\mathcal{H}}.$$

Exercise 3.4.3. *Show that* $p \wedge q = s{-}lim_{\,n\to\infty}(pq)^n$.

The lattice of projections in $\mathcal{B}(\mathcal{H})$ is complete (i.e. closed under arbitrary sups and infs) since the intersection of closed subspaces is closed.

EP5) The projections in M generate M as a von Neumann algebra, and they form a complete sublattice of the projection lattice of $\mathcal{B}(\mathcal{H})$.

Proof. If S is a set of projections in M then finite subsets of S are a directed set and $F \to \bigvee_{p\in F} p$ is a net in M satisfying the conditions of 3.4.2. Thus the strong limit of this net exists and is in M. It is obvious that this strong limit is $\bigvee_{p\in S} p$, the sup being in $\mathcal{B}(\mathcal{H})$. □

Easier proof. For each projection $p \in M$, $p\mathcal{H}$ is invariant under each element of M'. Thus the intersection of these subspaces is also. □

EP6) Let A be a *-subalgebra of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{W}$ be $\bigcap_{a\in A} ker(a)$ and $\mathcal{K} = \mathcal{W}^\perp$. Then $\mathcal{K}$ is invariant under A and if we let $B = \{a|_\mathcal{K} : a \in A\}$, then $1_\mathcal{K}$ is in the strong closure of B, which is thus a von Neumann algebra. Moreover on $\mathcal{K}$, B'' is the strong (weak, ultrastrong, ultraweak) closure of B.

Proof. By the above, if p and q are projections $p \vee q = 1 - (1-p) \wedge (1-q)$ is in the strong closure of the algebra generated by p and q. By spectral theory, if $a = a^*$ the range projection $P_{ker(a)^\perp}$ is in the strong closure of the algebra generated by a so we may apply the argument of the proof of EP5) to conclude that $\bigvee_{a\in A} P_{\ker(a)^\perp}$ is in the strong closure of A. But this is $1_\mathcal{K}$.

Finally, on $\mathcal{K}$, let C be the algebra generated by 1 and B. Clearly $C' = B'$ and just as clearly the strong closure of B is the same as the strong closure of C. So B'' is the strong closure of B by the bicommutant theorem. □

Thus if we were to define a von Neumann algebra as a weakly or strongly closed subalgebra of $\mathcal{B}(\mathcal{H})$, it would be unital as an abstract algebra but its identity might not be that of $\mathcal{B}(\mathcal{H})$ so it would not be equal to its bicommutant. However on the orthogonal complement of all the irrelevant vectors it would be a von Neumann algebra in the usual sense.

EP7) If M is a von Neumann algebra and $p \in M$ is a projection, $pMp = (M'p)'$ and $(pMp)' = M'p$ as algebras of operators on $p\mathcal{H}$. Thus pMp and $M'p$ are von Neumann algebras.

Proof. Obviously pMp and $M'p$ commute with each other on $p\mathcal{H}$. Now suppose $x \in (M'p)' \subseteq \mathcal{B}(p\mathcal{H})$ and define $\tilde{x} = xp(= pxp) \in \mathcal{B}(\mathcal{H})$. Then if $y \in M'$, $y\tilde{x} = yxp = ypxp = (xp)(yp) = xpy = \tilde{x}y$, so $\tilde{x} \in M$ and $x = p\tilde{x}p$. Thus $(pM')' = pMp$ which is thus a von Neumann algebra.

If we knew that $M'p$ were a von Neumann algebra on $p\mathcal{H}$ we would be done but a direct attempt to prove it strongly or weakly closed fails as we would have to try to extend the limit of a net in $M'p$ on $p\mathcal{H}$ to be in M'.

So instead we will show directly that $(pMp)' \subseteq M'p$ by a clever extension of its elements to $\mathcal{H}$. By EP2 it suffices to take a unitary $u \in (pMp)'$. Let $\mathcal{K} \subseteq \mathcal{H}$ be the closure of $Mp\mathcal{H}$ and let q be projection onto it. Then $\mathcal{K}$ is clearly invariant under M and M' so $q \in Z(M)$. We first extend u to $\mathcal{K}$ by

$$\tilde{u} \sum x_i \xi_i = \sum x_i u \xi_i$$

for $x_i \in M$ and $\xi_i \in p\mathcal{H}$. We claim that $\tilde{u}$ is an isometry:

$$\begin{aligned} ||\tilde{u} \sum x_i \xi_i||^2 &= \sum_{i,j} \langle x_i u \xi_i, x_j u \xi_j \rangle \\ &= \sum_{i,j} \langle p x_j^* x_i p u \xi_i, u \xi_j \rangle \\ &= \sum_{i,j} \langle u p x_j^* x_i p \xi_i, u \xi_j \rangle \\ &= \ldots = || \sum x_i \xi_i||^2 \end{aligned}$$

This calculation actually shows that $\tilde{u}$ is well defined and extends to an isometry of $\mathcal{K}$. By construction $\tilde{u}$ commutes with M on $Mp\mathcal{H}$,hence on $\mathcal{K}$. So $\tilde{u}q \in M'$ and $u = \tilde{u}qp$. Hence $(pMp)' = M'p$. □

Corollary 3.4.4. *If M is a factor, pMp is a factor on $p\mathcal{H}$, as is pM'. Moreover the map $x \mapsto xp$ from M' to $M'p$ is a weakly continuous *-algebra isomorphism.*

Proof. As in the proof of the previous result, the projection onto the closure of $Mp\mathcal{H}$ is in the centre of M, hence it is 1 . So if $xp = 0$ for $x \in M'$, $xmp\xi = mxp\xi = 0$ for any $m \in M$, $\xi \in \mathcal{H}$. Hence the map $x \mapsto px$ is an injective $*$-algebra map and pM' is a factor. So by the previous result $(pMp)'$ is a factor and so is pMp. Continuity and the is obvious. □

Corollary 3.4.5. *If M is a factor and $a \in M$ and $b \in M'$ then $ab = 0$ implies either $a = 0$ or $b = 0$.*

Proof. Let p be the range projection of b and apply the previous corollary. □

Exercise 3.4.6. *Show that if M is a von Neumann algebra generated by the self-adjoint, multiplicatively closed subset S, then pSp generates pMp (if p is a projection in M or M'). Show further that the result fails if S is not closed under multiplication.*

Exercise 3.4.7. *Show that if M is a factor and V and W are finite dimensional subspaces of M and M' respectively then the map $a \otimes b \mapsto ab$ defines a linear isomorphism between $V \otimes W$ and the space VW spanned by all vw with $v \in V$ and $w \in W$.*

EP8) If $a \in M$ and $a = u|a|$ is the polar decomposition of a then $u \in M$ and $|a| \in M$.

Proof. By the uniqueness of the polar decomposition both $|a|$ and u commute with every unitary in M'. □

EP9) None of the topologies (except $||*||$) is metrizable on $\mathcal{B}(\mathcal{H})$ but they all are on the unit ball (when $\mathcal{H}$ is separable) and $\mathcal{B}(\mathcal{H})$ is separable for all except the norm topology.

Proof. First observe that a weakly convergent sequence of operators is bounded. This follows from the uniform boundedness principle and 2.1.5 which shows how to get the norm from inner products.

Here is the cunning trick. Let $\{\eta_i, i = 1, \cdots \infty\}$ be an orthonormal basis of $\mathcal{H}$ and let e_i be projection onto $\mathbb{C}\eta_i$. Consider the family $\{e_m + me_n : m, n = 1, \cdots \infty\}$. Let V a basic ultrastrong neighbourhood of 0 defined by ϵ and $\{\xi_i : \sum ||\xi_i||^2 < \infty\}$ and let $|-|_V$ be the corresponding seminorm, then writing

$\xi_i = \sum_j \xi_j^i \eta_j$ we have $\sum_{i,j} |\xi_j^i|^2 < \infty$. Now choose m so that $\sum_i |\xi_m^i|^2 < \epsilon^2/4$ and n so that $\sum_i |\xi_n^i|^2 < \epsilon^2/4m^2$. Observing that $||e_n(\xi_i)||^2 = |\xi_i^n|^2$ we have

$$|e_m + me_n|_V \leq |e_m|_V + m|e_n|_V$$

$$= \sqrt{\sum_i ||e_m\xi||^2} + m\sqrt{\sum_i ||e_n\xi||^2}$$

$$\leq \epsilon/2 + \epsilon/2$$

so that $e_n + me_n \in V$.

On the other hand no subsequence of $\{e_m + me_n : m, n = 1, \cdots \infty\}$ can tend even weakly to 0 since it would have to be bounded in norm which would force some fixed m to occur infinitely often in the sequence, preventing weak convergence! So by the freedom in choosing m and n to force $e_m + me_n$ to be in V, there can be no countable basis of zero for any of the topologies (except of course the norm).

If we consider the unit ball, however, we may choose a dense sequence ξ_i of unit vectors and define $d(a, b) = [\sum_i 2^{-i}||(a - b)\xi_i||^2]^{1/2}$ which is a metric on the unit ball defining the strong topology. (Similarly for the weak topology.)

We leave non-separability of $\mathcal{B}(\mathcal{H})$ in the norm topology as an exercise. □

EP10) An Abelian von Neumann algebra on a separable Hilbert space is generated by a single self-adjoint operator.

Proof. Let $\{e_0, e_1, e_2, \cdots\}$ be a sequence of projections that is strongly dense in the set of all projections in the Abelian von Neumann algebra A. Let $a = \sum_{n=0}^{\infty} \frac{1}{3^n} e_n$. The sum converges in the norm topology so $a \in A$. The norm of the self-adjoint operator $a_1 = \sum_{n=1}^{\infty} \frac{1}{3^n} e_n$ is obviously at most $1/2$ so that the spectral projection for the interval $[3/4, 2]$ for a is e_0. Continuing in this way we see that all the $e_n's$ are in $\{a\}''$. □

This relegates the study of Abelian von Neumann algebras to the spectral theorem. One can show that any Abelian von Neumann algebra on a separable Hilbert space is isomorphic to either $\ell^\infty(\{0, 1, \cdots, n\})$ (where $n = \infty$ is allowed) or $L^\infty([0, 1], dx)$ or a direct sum of the two. This is up to abstract algebra isomorphism. To understand the action on a Hilbert space, multiplicity must be taken into account.

Chapter 4

Finite dimensional von Neumann algebras and type I factors.

4.1 Definition of type I factor.

The first crucial result about factors (remember a factor is a von Neumann algebra with trivial centre) will be the following "ergodic" property.

Theorem 4.1.1. *If M is a factor and p and q are non-zero projections in M there is an $x \in M$ with $pxq \neq 0$. Moreover x can be chosen to be unitary.*

Proof. Suppose that for any unitary $u \in M$, $puq = 0$. Then $u^*puq = 0$ and $\left(\bigvee_{u \in M} u^*pu\right) q = 0$. But clearly $\bigvee_{u \in M} u^*pu$ commutes with all unitaries $u \in M$ so is the identity. □

The reason we have called this an "ergodic" property is because of a pervasive analogy with measure-theoretic dynamical systems (and it will become much more than an analogy). A transformation $T : (X, \mu) \to (X, \mu)$ preserving the measure μ is called *ergodic* if $T^{-1}(A) \subseteq A$ implies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ for a measurable $A \subseteq X$. If T is invertible one can then show that there is, for any pair $A \subset X$ and $B \subset X$ of non-null sets, a power T^n of T such that $\mu(T^n(A) \cap B) \neq 0$. Or, as operators on $L^2(X, \mu)$, $AT^NB \neq 0$ where we identify A and B with the multiplication operators by their characteristic functions. The proof is the same—the union of all $T^n(A)$ is clearly invariant, thus must differ from all of X by a set of measure 0.

Corollary 4.1.2. *Let p and q be non-zero projections in a factor M. Then there is a partial isometry u ($\neq 0$) in M with $uu^* \leq p$ and $u^*u \leq q$.*

p and q had been swapped around in the proof, so we interchanged uu^* and u^*u in the statement.

Proof. Let u be the partial isometry of the polar decomposition of pxq for x such that $pxq \neq 0$. □

Definition 4.1.3. *If M is a von Neumann algebra, a non-zero projection $p \in M$ is called* minimal*, or an* atom*, if $(q \leq p) \Rightarrow (q = 0 \quad or \quad q = p)$.*

Exercise 4.1.4. *Show that p is minimal in M iff $pMp = \mathbb{C}p$.*

Definition 4.1.5. *A factor with a minimal projection is called a* type I factor*.*

4.2 Classification of all type I factors

We will classify all type I factors quite easily. We begin with the model, which we have already seen.

Let $\mathcal{B}(\mathcal{H}) \otimes 1$ be the constant diagonal matrices on $\mathcal{H} \otimes \mathcal{K}$. Its commutant $1 \otimes \mathcal{B}(\mathcal{K})$ will be our model. It is the algebra of all matrices defining bounded operators with every matrix entry being a scalar multiple of the identity matrix on $\mathcal{H}$. A matrix with a single 1 on the diagonal and zeros elsewhere is obviously a minimal projection.

Theorem 4.2.1. *If M is a type I factor of a Hilbert space $\mathcal{L}$, there are Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a unitary $u : \mathcal{L} \to \mathcal{H} \otimes \mathcal{K}$ with $uMu^* = \mathcal{B}(\mathcal{H}) \otimes 1$.*

Proof. Let $\{p_1, p_2, ...\}$ be a maximal family of minimal projections in M such that $p_i p_j = 0$ for $i \neq j$. (We assume for convenience that $\mathcal{L}$ is separable.) Our first claim is that $\bigvee_i p_i = 1$ so that $\mathcal{L} = \oplus_i p_i \mathcal{L}$. For if $1 - \bigvee_i p_i$ were nonzero, by corollary 4.1.2 there would be a $u \neq 0$ with $uu^* \leq p_1$ and $u^*u \leq 1 - \bigvee_i p_i$. By minimality uu^* is minimal and hence so is u^*u contradicting maximality of the p_i. Now for each i choose a non-zero partial isometry e_{1i} with $e_{1i}e_{1i}^* \leq p_1$ and $e_{1i}^* e_{1i} \leq p_i$. By minimality $e_{1i}e_{1i}^* = p_1$ and $e_{1i}^* e_{1i} = p_i$. Then M is generated by the e_{1i}'s, for if $a \in M$ we have $a = \sum_{i,j} p_i a p_j$ the sum converging in the strong topology, and $p_i a p_j = e_{1i}^* e_{1i} a e_{1j}^* e_{1j} \in p_1 M p_1 = \mathbb{C}p_1$. Thus there are scalars λ_{ij} so that $a = \sum_{i,j} \lambda_{ij} e_{1i}^* e_{1j}$. (The details of the convergence of the sum are unimportant—we just need that a be in the strong closure of finite sums.)

If n is the cardinality of $\{p_i\}$, let $X = \{1, 2, ..., n\}$ and define the map $u : \ell^2(X, p_1\mathcal{L}) \to \mathcal{L}$ by

$$uf = \sum_i e_{1i}^* f(i).$$

Observe that u is unitary and $u^* e_{1i} u$ is a matrix on $\ell^2(X, p_1\mathcal{L})$ with an identity operator in the $(1, i)$ position and zeros elsewhere. The algebra generated by these matrices is $\mathcal{B}(\ell^2(X)) \otimes 1$ on $\ell^2(X) \otimes p_1\mathcal{L}$ and we are done. □

Remark 4.2.2. *The importance of being spatial.*

We avoided all kinds of problems in the previous theorem by constructing our isomorphism using a unitary between the underlying Hilbert spaces. In general given von Neumann algebras M and N generated by S and T respectively, to construct an isomorphism between M and N it suffices to construct (if possible !!!) a unitary u between their Hilbert spaces so that T is contained in uSu^*. To try to construct an isomorphism directly on S could be arduous at best.

4.3 Tensor product of von Neumann algebras.

If M is a von Neumann algebra on $\mathcal{H}$ and N is a von Neumann algebra on $\mathcal{K}$ we define $M \otimes N$ to be the von Neumann algebra on $\mathcal{H} \otimes \mathcal{K}$ generated by $\{x \otimes y : x \in M, y \in N\}$.

when you use \h, make sure it's enclosed in $ signs

Exercise 4.3.1. *Show that $M \otimes N$ contains the algebraic tensor product $M \otimes_{alg} N$ as a strongly dense *-subalgebra.*

Definition 4.3.2. *Let M be a von Neumann algebra. A* system of matrix units *(s.m.u.) of size n in M is a family $\{e_{ij} : i, j = 1, 2, ..., n\}$ ($n = \infty$ allowed) such that*

(i) $e_{ij}^* = e_{ji}$.

(ii) $e_{ij}e_{kl} = \delta_{j,k}e_{il}$

(iii) $\sum_i e_{ii} = 1$.

Exercise 4.3.3. *Show that if $\{e_{ij}; i, j = 1, ..., n\}$ is an s.m.u. in a von Neumann algebra M, then the e_{ij} generate a type I factor isomorphic to $\mathcal{B}(\ell^2(\{1, 2, ..., n\}))$ and that M is isomorphic (unitarily equivalent to in this instance) to the von Neumann algebra $e_{11}Me_{11} \otimes \mathcal{B}(\ell^2(\{1, 2, ..., n\}))$.*

4.4 Multiplicity and finite dimensional von Neumann algebras.

Theorem 4.2.1 shows that type I factors on Hilbert space are completely classified by two cardinalities (n_1, n_2) according to:

n_1 = rank of a minimal projection in M, and

n_2 = rank of a minimal projection in M'.

We see that the isomorphism problem splits into "abstract isomorphism" (determined by n_2 alone), and "spatial isomorphism", i.e. unitary equivalence.

A type I_n factor is by definition one for which $n = n_2$. It is abstractly isomorphic to $\mathcal{B}(\mathcal{H})$ with $\dim \mathcal{H} = n$. The integer n_1 is often called the *multiplicity* of the type I factor.

We will now determine the structure of all finite dimensional von Neumann algebras quite easily. Note that in the following there is no requirement that $\mathcal{H}$ be finite dimensional.

Theorem 4.4.1. *Let M be a finite dimensional von Neumann algebra on the Hilbert space $\mathcal{H}$. Then M is abstractly isomorphic to $\oplus_{i=1}^{k} M_{n_i}(\mathbb{C})$ for some positive integers $k, n_1, n_2, ..., n_k$. ($M_n(\mathbb{C})$ is the von Neumann algebra of all $n \times n$ matrices on n-dimensional Hilbert space.) Moreover there are Hilbert spaces $\mathcal{K}_i$ and a unitary $u : \oplus_i \ell^2(X_i, \mathcal{K}_i) \to \mathcal{H}$ (with $|X_i| = n_i$) with $u^* M u = \oplus_i \mathcal{B}(\ell^2(X_i)) \otimes 1$.*

Proof. The centre $Z(M)$ is a finite dimensional abelian von Neumann algebra. If p is a minimal projection in $Z(M)$, pMp is a factor on $p\mathcal{H}$. The theorem follows immediately from theorem 4.2.1 and the simple facts that $Z(M) = \oplus_{i=1}^{k} p_i \mathbb{C}$ where the p_i are the minimal projections in $Z(M)$ (two distinct minimal projections p and q in $Z(M)$ satisfy $pq = 0$), and $M = \oplus_i p_i M p_i$. □

The subject of finite dimensional von Neumann algebras is thus rather simple. It becomes slightly more interesting if one considers *subalgebras* $N \subseteq M$. Let us deal first with the factor case of this. Let us point out that the identity of M is the same as that of N.

Theorem 4.4.2. *If M is a type I_n factor, its type I_m factors are all uniquely determined, up to conjugation by unitaries in M, by the integer (or ∞) $k > 0$ such that pMp is a type I_k factor, p being a minimal projection in the subfactor N and $mk = n$.*

Proof. Let N_1 and N_2 be type I_m subfactors with generating s.m.u.'s $\{e_{ij}\}$ and $\{f_{ij}\}$ respectively. If k is the integer (in the statement of the theorem) for N_1 then $1 = \sum_1^m e_{ii}$ and each e_{ii} is the sum of k mutually orthogonal minimal projections of M, hence $n = mk$. The same argument applies to N_2. Build a partial isometry u with $uu^* = e_{11}$ and $u^*u = f_{11}$ by adding together partial isometries between maximal families of mutually orthogonal projections less than e_{11} and f_{11} respectively. Then it is easy to check that $w = \sum_i e_{j1} u f_{1j}$ is a unitary with $w f_{kl} w^* = e_{kl}$. So $w N_2 w^* = N_1$. □

Now we can do the general (non-factor) case. If $N = \oplus_{i=1}^{n} M_{k_i}(\mathbb{C})$ and $M = \oplus_{j=1}^{m} M_{r_j}(\mathbb{C})$ and $N \subseteq M$ as von Neumann algebras, let p_j be minimal

central projections in M and q_i be those of N. Then for each (i,j), $p_jq_iMq_ip_j$ is a factor and p_jq_iN is a subfactor so we may form the matrix $\Lambda = (\lambda_{ij})$ where λ_{ij} is the integer associated with $p_jq_iN \subseteq p_jq_iMq_ip_j$ by theorem 4.4.2.

Exercise 4.4.3. *Show that the integer λ_{ij} defined above is the following: if e_i is a minimal projection in the factor q_iN, λ_{ij} = trace of the matrix $p_je_i \in M_{r_j}\mathbb{C}$.*

Example 4.4.4. *Let $M = M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$ and N be the subalgebra of matrices of the form:*

$$\begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & z \end{pmatrix} \oplus \begin{pmatrix} X & 0 \\ 0 & z \end{pmatrix}$$

where $z \in \mathbb{C}$ and X is a 2×2 matrix. Then N is isomorphic to $M_2(\mathbb{C})\oplus\mathbb{C}$ and if $p_1 = 1 \oplus 0$, $q_1 = 1 \oplus 0$, etc., we have

$$\Lambda = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The matrix Λ is often represented by a bipartite graph with the number of edges between i and j being λ_{ij}. The vertices of the graph are labelled by the size of the corresponding matrix algebras. Thus in the above example the picture would be:

This diagram is called the *Bratteli diagram* for $N \subseteq M$.

Exercise 4.4.5. *Generalise the above example to show that there is an inclusion $N \subseteq M$ corresponding to any Bratteli diagram with any set of dimensions for the simple components of N.*

4.5 A digression on index.

If $N \subseteq M$ are type I factors we have seen that there is an integer k (possibly ∞) such that M is the algebra of $k \times k$ matrices over N. If $k < \infty$, M is thus a free left N-module of rank k^2. It seems reasonable to call the number k^2 the *index* of N in M and write it $[M : N]$. This is because, if $H < G$ are groups and $\mathbb{C}H \subseteq \mathbb{C}G$ their group algebras, the coset decomposition of G shows that $\mathbb{C}G$ is a free left $\mathbb{C}H$-module of rank $[G : H]$.

Chapter 5

Kaplansky Density Theorem.

5.1 Some simple but telling results on linear functionals.

We begin with a result about linear functionals of independent interest.

Theorem 5.1.1. *Let V be a subspace of $\mathcal{B}(\mathcal{H})$ and let $\phi : V \to \mathbb{C}$ be a linear functional. The following are equivalent:*

(i) There are vectors in $\mathcal{H}$, $\xi_1, \xi_2, ..., \xi_n$ and $\eta_1, \eta_2, ..., \eta_n$ with

$$\phi(x) = \sum_{i=1}^{n} \langle x\xi_i, \eta_i \rangle$$

(ii) ϕ is weakly continuous.

(iii) ϕ is strongly continuous.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious, so suppose ϕ is strongly continuous. One may use the seminorms $\sqrt{\sum_{i=1}^{n} ||a\xi_i||^2}$ as $\{\xi_1, \xi_2, ..., \xi_n\}$ ranges over all finite subsets of $\mathcal{H}$ to define the strong topology. Srong continuity implies that there is an $\epsilon > 0$ and $\{\xi_1, \xi_2, ..., \xi_n\}$ such that $\sqrt{\sum_{i=1}^{n} ||a\xi_i||^2} < \epsilon$ implies $|\phi(a)| \leq 1$. But then if $\sqrt{\sum_{i=1}^{n} ||a\xi_i||^2} = 0$ then multiplying a by large scalars implies $\phi(a) = 0$. Otherwise it is clear that $|\phi(a)| \leq \frac{1}{\epsilon}\sqrt{\sum_{i=1}^{n} ||a\xi_i||^2}$.

Now let $\xi = \xi_1 \oplus ...\xi_n \in \oplus_i \mathcal{H}$ and let $\mathcal{K} = \overline{(V \otimes 1)(\xi)}$. Then define $\tilde{\phi}$ on $V \otimes 1(\nu)$ by $\tilde{\phi}(\oplus_i x\xi_i) = \phi(x)$. Observe that $\tilde{\phi}$ is well-defined and continuous so extends to $\mathcal{K}$ which means there is a vector $\eta = \oplus\eta_i \in \mathcal{K}$ with $\phi(x) = \tilde{\phi}(x \otimes 1)(\eta) = \langle (x \otimes 1)(\xi), \eta \rangle$. □

Exercise 5.1.2. *Replace weak and strong by ultraweak and ultrastrong, and the finite sequences of vectors by ℓ^2-convergent ones in the previous theorem.*

Corollary 5.1.3. *If C is a convex subset of $\mathcal{B}(\mathcal{H})$, its weak and strong closures coincide.*

Proof. Two locally convex vector spaces with the same continuous linear functionals have the same closed convex sets. This is a consequence of the Hahn-Banach theorem to be found in any text on functional analysis. □

Corollary 5.1.4. *If $\dim \mathcal{H} = \infty$ the strong and ultrastrong topologies differ on $\mathcal{B}(\mathcal{H})$.*

Proof. Let (ξ_i) be an orthonormal basis of $\mathcal{H}$ and let $\omega(x) = \sum_i \frac{1}{n^2}\langle x\xi_i, \xi_i\rangle$. Then ω is ultraweakly continuous but not strongly continuous. For if it were weakly continuous it would be of the form $\sum_{i=1}^{n}\langle x\nu_i, \eta_i\rangle$ and $\omega(p) = 0$ where p is the projection onto the orthogonal complement of the vector space spanned by the ν_i. But by positivity $\omega(p) = 0$ forces $p(\xi_i) = 0$ for all i. □

5.2 The theorem

In our discussion of $vN(\Gamma)$ we already met the desirability of having a *norm-bounded* sequence of operators converging to an element in the weak closure of a *-algebra of operators. This is not guaranteed by the von Neumann density theorem. The Kaplansky density theorem fills this gap.

Theorem 5.2.1. *Let A be a *-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the unit ball of A is strongly dense in the unit ball of the weak closure M of A, and the self-adjoint part of the unit ball of A is strongly dense in the self-adjoint part of the unit ball of M.*

Proof. By EP6) we may assume $1 \in M$ and the worried reader may check that we never in fact suppose $1 \in A$. We may further suppose that A is norm-closed, i.e. a C*-algebra. Consider the closure of A_{sa}, the self-adjoint part of A. The * operation is weakly continuous so if x_α is a net converging to the self-adjoint element $x \in M$, $\frac{x_\alpha + x_\alpha^*}{2}$ converges to x so the weak closure of A_{sa} is equal to M_{sa}. Since A_{sa} is convex, the strong closure is also equal to M_{sa} by 5.1.3.

Let us now prove the second assertion of the theorem. Let $x = x^* \in M$, $||x|| < 1$, and $\xi_1, ..., \xi_n, \epsilon > 0$ define a strong neighbourhood of x. We must come up with a $y \in A_{sa}$, $||y|| < 1$, with $||(x-y)\xi_i|| < \epsilon$. The function $t \to \frac{2t}{1+t^2}$ is a homeomorphism of $[-1, 1]$ onto itself. So by the spectral theorem we may choose an $X \in M_{sa}$ with $||X|| \leq 1$, so that $\frac{2X}{1+X^2} = x$. Now by strong density choose $Y \in A_{sa}$ with

$$||Yx\xi_i - Xx\xi_i|| < \epsilon, \text{ and } ||\frac{Y}{1+X^2}\xi_i - \frac{X}{1+X^2}\xi_i|| < \epsilon/4.$$

Put $y = \frac{2Y}{1+Y^2}$ and note that $||y|| \leq 1$.

Now consider the following equalities:

$$y - x = \frac{2Y}{1+Y^2} - \frac{2X}{1+X^2}$$

$$= 2(\frac{1}{1+Y^2}(Y(1+X^2) - (1+Y^2)X)\frac{1}{1+X^2})$$

$$= 2(\frac{1}{1+Y^2}(Y-X)\frac{1}{1+X^2} + \frac{Y}{1+Y^2}(X-Y)\frac{X}{1+X^2})$$

$$= \frac{2}{1+Y^2}(Y-X)\frac{1}{1+X^2} + \frac{1}{2}y(X-Y)x.$$

By the choice of Y, we see that $||(y-x)\xi_i|| < \epsilon$. This proves density for the self-adjoint part of the unit ball.

Now consider a general $x \in M$ with $||x|| \leq 1$. The trick is to form $\left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right) \in M \otimes M_2(\mathbb{C})$. Strong convergence of a net $\left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is equivalent to strong convergence of the matrix entries so $A \otimes M_2(\mathbb{C})$ is strongly dense in $M \otimes M_2(\mathbb{C})$. Moreover if $\left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right)$ strongly then b_α tends strongly to x. And $||b_\alpha|| \leq 1$ follows from $||\left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{smallmatrix}\right)|| \leq 1$ and $\langle b_\alpha \xi, \eta \rangle = \langle \left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 \\ \xi \end{smallmatrix}\right), \left(\begin{smallmatrix} \eta \\ 0 \end{smallmatrix}\right) \rangle$. □

Corollary 5.2.2. *If M is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1 then M is a von Neumann algebra iff the unit ball of M is weakly compact.*

Proof. The unit ball of $\mathcal{B}(\mathcal{H})$ is weakly compact, and M is weakly closed.

Conversely, if the unit ball of M is weakly compact, then it is weakly closed. Let x be in the weak closure of M. We may suppose $||x|| = 1$. By Kaplansky density there is a net x_α weakly converging to x with $||x_\alpha|| \leq 1$. Hence $x \in M$. □

Chapter 6

Comparison of Projections and Type II_1 Factors.

6.1 Order on projections

Definition 6.1.1. *If p and q are projections in a von Neumann algebra M we say that $p \preceq q$ if there is a partial isometry $u \in M$ with $uu^* = p$ and $u^*u \leq q$. We say that p and q are equivalent, $p \approx q$ if there is a partial isometry $u \in M$ with $uu^* = p$ and $u^*u = q$.*

Observe that $\approx$ is an equivalence relation.

Theorem 6.1.2. *The relation $\preceq$ is a partial order on the equivalence classes of projections in a von Neumann algebra.*

Proof. Transitivity follows by composing partial isometries. The issue is to show that $e \preceq f$ and $f \preceq e$ imply $e \approx f$. Compare this situation with sets and their cardinalities.

Let u and v satisfy $uu^* = e, u^*u \leq f$ and $vv^* = f, v^*v \leq e$. Note the picture:

We define the two decreasing sequences of projections $e_0 = e, e_{n+1} = v^*f_nv$ and $f_0 = f, f_{n+1} = u^*e_nu$. The decreasing property follows by induction since $p \to v^*pv$ gives an order preserving map from projections in M less than f to projections in M less than e and similarly interchanging the roles of e and f, v and u. Let $e_\infty = \bigwedge_{i=0}^{\infty} e_i$ and $f_\infty = \bigwedge_{i=0}^{\infty} f_i$. Note that $v^*f_\infty v = e_\infty$ and $f_\infty vv^*f_\infty = f_\infty$ so that $e_\infty \approx f_\infty$. Also $e = (e - e_1) + (e_1 - e_2) + \cdots + e_\infty$ and $f = (f - f_0) + (f_1 - f_2) + \cdots + f_\infty$ are sums of mutually orthogonal projections. But for each even i, $u^*(e_i - e_{i+1})u = f_{i+1} - f_{i+2}$ so $e_i - e_{i+1} \approx f_{i+1} - f_{i+2}$, and

$v^*(f_i - f_{i+1})v = e_{i+1} - e_{i+2}$ so one may add up, in the strong topology, all the relevant partial isometries to obtain an equivalence between e and f. $\square$

Note that if we had been dealing with $vN(\Gamma)$ this argument would have been unnecessary as we could have used the trace:

$$tr(v^*v) \leq tr(e) = tr(uu^*) = tr(u^*u) \leq tr(f) = tr(vv^*) = tr(v^*v)$$

so that $tr(e - v^*v) = 0$ which implies $e = v^*v$. However in general it is certainly possible to get a projection equivalent to a proper subprojection of itself. Just take the unilateral shift on $\mathcal{B}(\ell^2(\mathbb{N}))$ which exhibits an equivalence between 1 and the projection onto the orthogonal complement of the first basis vector. This is analogous to the notion of an infinite set—one which is in bijection with a proper subset of itself.

Definition 6.1.3. *A projection p in a von Neumann algebra M is called* infinite *if $p \approx q$ for some $q < p$, $p \neq q$. Otherwise p is called* finite. *A von Neumann algebra is called* finite *if its identity is finite, and it is called* purely infinite *if it has no finite projections other than* 0. *A factor is called* infinite *if its identity is infinite.*

We will show that purely infinite von Neumann algebras exist though it will not be easy.

Remark 6.1.4. *If $\dim \mathcal{H} = \infty$ then $\mathcal{B}(\mathcal{H})$ is infinite.*

Remark 6.1.5. *A factor with a trace like $vN(\Gamma)$ is finite.*

Remark 6.1.6. *Every projection in a finite von Neumann algebra is finite. Or, more strongly, if $p \leq q$ and q is finite then p is finite.*

For if $p \approx p', \quad p' < p, \quad p \neq p'$ then $p + (q - p) \approx p' + (q - p) \neq q$.

Remark 6.1.7. *If M is any von Neumann algebra,* 1 *is an infinite projection in $M \otimes \mathcal{B}(\mathcal{H})$ if $\dim \mathcal{H} = \infty$.*

Theorem 6.1.8. *If M is a factor and p, q are projections in M, either $p \preceq q$ or $q \preceq p$.*

Proof. Consider the family of partial isometries u with $uu^* \leq p, u^*u \leq q$. This set is partially ordered by $u \leq v$ if $u^*u \leq v^*v$ and $v = u$ on the initial domain $u^*u\mathcal{H}$ of u. This partially ordered set satisfies the requirements for Zorn's lemma so let u be a maximal element in it. If $u^*u = q$ or $uu^* = p$ we are done so suppose $q - u^*u$ and $p - uu^*$ are both non-zero. Then by 4.1.1 there is a $v \neq 0$ with $v^*v \leq q - u^*u$ and $vv^* \leq p - uu^*$. But then $u + v$ is larger than u which was supposed maximal. $\square$

Exercise 6.1.9. *Show that two equivalent projections p and q in a finite factor M are unitarily equivalent, i.e. there is a unitary $u \in M$ with $upu^* = q$.*

We see that the equivalence classes of projections in a factor form a totally ordered set. It is known that, on a separable Hilbert space, the possible isomorphism types for this set are:

1)	$\{0, 1, 2, ..., n\}$ where $n = \infty$ is allowed.	"type I_n"
2)	$[0, 1]$	"type II_1"
3)	$[0, \infty]$	"type II_∞"
4)	$\{0, \infty\}$	"type III"

Strictly speaking this is nonsense as type III is the same as type I_1 and II_1 is the same as II_∞. We mean not only the order type but whether 1 is infinite or not.

Observe that the type II_1 case certainly exists. We saw that $vN(F_2)$ has projections of any trace between 0 and 1. By the previous theorem it is clear that the trace gives an isomorphism between the ordered set of equivalence classes of projections and the unit interval. We will proceed to prove a statement generalising this considerably.

Definition 6.1.10. *A* type II_1 factor *is an infinite dimensional factor M on $\mathcal{H}$ admitting a non-zero linear function $tr : M \to \mathbb{C}$ satisfying*

(i) $tr(ab) = tr(ba)$

(ii) $tr(a^*a) \geq 0$

(iii) tr is ultraweakly continuous.

The trace is said to be normalised if $tr(1) = 1$.

Definition 6.1.11. *In general a linear functional ϕ on a *-algebra A is called* positive *if $\phi(a^*a) \geq 0$ (and $\phi(a^*) = \overline{\phi(a)}$ though this is redundant if A is a C^*-algebra), and* faithful *if $\phi(a^*a) = 0 \Rightarrow a = 0$. A positive ϕ is called a* state *if $1 \in A$ and $\phi(1) = 1$. A linear functional ϕ is called* tracial *(or a trace) if $\phi(ab) = \phi(ba)$.*

It is our job now to show that a II_1 factor has a unique ultraweakly continuous tracial state, which is faithful. First a preliminary result on ideals.

Theorem 6.1.12. *Let $\mathcal{M}$ be an ultraweakly closed left ideal in a von Neumann algebra M. Then there is a unique projection $e \in M$ such that $\mathcal{M} = Me$. If $\mathcal{M}$ is 2-sided, e is in $Z(M)$.*

Proof. $\mathcal{M} \cap \mathcal{M}^*$ is an ultraweakly closed *-subalgebra so it has a largest projection e. Since $e \in \mathcal{M}$, $Me \subseteq \mathcal{M}$. On the other hand if $x \in \mathcal{M}$ let

$x = u|x|$ be its polar decomposition. Since $u^*x = |x|$, $|x| \in \mathcal{M} \cap \mathcal{M}^*$. Hence $|x|e = |x|$ and $x = u|x| = u|x|e \in Me$. So $\mathcal{M} = Me$.

Uniqueness follows easily since $f = xe \Rightarrow f \leq e$.

Moreover if $\mathcal{M}$ is 2-sided, for any unitary $u \in M$, $u\mathcal{M} = \mathcal{M} = u\mathcal{M}u^* = Me = Mueu^*$ so $ueu^* = e$ by uniqueness. Hence $e \in Z(M)$. □

Corollary 6.1.13. *An ultraweakly continuous positive non-zero trace Tr on a II_1 factor is faithful.*

Proof. Let $\mathcal{M} = \{x \in M : Tr(x^*x) = 0\}$. Then since $x^*a^*ax \leq ||a||^2x^*x$, $\mathcal{M}$ is a left ideal and since $Tr(ab) = Tr(ba)$, $\mathcal{M}$ is a 2-sided ideal. Moreover by the Cauchy Schwarz inequality $Tr(x^*x) = 0$ iff $Tr(xy) = 0 \quad \forall y \in M$. Thus $\mathcal{M}$ is ultraweakly closed, being the intersection of the kernels of ultraweakly continuous functionals. Thus $\mathcal{M} = Me$ for some central projection. And e must be zero since M is a factor. □

Corollary 6.1.14. *If M is a type II_1 factor on $\mathcal{H}$ and $p \in M$ is a non-zero projection, pMp is a type II_1 factor on $p\mathcal{H}$.*

Proof. This is clear—a trace on M restricts to a trace on pMp which is non-zero by faithfulness and all the other properties are immediate. Since a minimal projection in pMp would be minimal in M, pMp is infinite dimensional. □

The uniqueness of tr will follow easily once we have gathered some facts about projections in a II_1 factor.

Theorem 6.1.15. *There are non-zero projections in a type II_1 factor of arbitrarily small trace.*

Proof. Let $d = inf\{tr(p) : p \in M, p^2 = p^* = p \neq 0\}$. Suppose $d > 0$. Let p be a projection with $tr(p) - d < d$. Then p is not minimal since we have seen that M is not isomorphic to $\mathcal{B}(\mathcal{H})$. So there is a non-zero projection $q < p$. But then we have $tr(p - q) = tr(p) - tr(q) \leq tr(p) - d < d$. This is a contradiction. So $d = 0$. □

Theorem 6.1.16. *Let M be a type II_1 factor with an ultraweakly continuous positive non-zero trace tr. Then $\{tr(p) : p \in M, p^2 = p^* = p\} = [0, tr(1)]$.*

Proof. For $r \in [0, tr(1)]$ consider $S = \{p : p$ a projection in M and $tr(p) \leq r\}$. Then S is a partially ordered set and if p_α is a chain in S, $p = \bigvee_\alpha p_\alpha \in M$ and p is in the strong closure of the p_α so p is in S. So by Zorn, S has a maximal element, say q. If $tr(q)$ were less than r, then by 6.1.8, $q \prec p$. So choose $q' \cong q, \quad q' < p$. Applying 6.1.14 to $p - q'$ we find a projection strictly between q' and p. □

Corollary 6.1.17. *The map tr gives an isomorphism between the totally ordered set of equivalence classes of projections on a type II_1 factor and the interval $[0, tr(1)]$.*

Proof. By 6.1.16 it suffices to show that the equivalence class of a projection is determined by its trace. This is immediate from 6.1.8. □

Exercise 6.1.18. *Let M be a type II_1 factor. Then for each $n \in \mathbb{N}$ there is a subfactor $N \subseteq M$ with $N \cong M_n(\mathbb{C})$.*

Corollary 6.1.19. *Any two non-zero ultraweakly continuous normalised traces on a type II_1 factor are equal.*

Proof. By the elementary facts it suffices to prove that two such traces Tr and tr agree on projections. We may assume one of them, say tr, is positive. By the previous exercise, 6.1.17, and the uniqueness of the trace on a matrix algebra, tr and Tr are equal on projections for which tr is rational. Given a projection for which $tr(p)$ is irrational build an increasing sequence e_i of subprojections as follows:

Suppose we have already constructed e_i with $tr(e_i) = Tr(e_i)$ and $tr(p) - tr(e_i) < 1/i$. Then $(p - e_i)M(p - e_i)$ is a type II_1 factor so tr and Tr agree on projections in it whose tr is arbitrarily close to $tr(p - e_i)$. So choose in it a projection e_{i+1} between e_i and p, on which tr and Tr agree and with $tr(p) - tr(e_{i+1}) < \frac{1}{i+1}$. Then tr and Tr agree on $\bigvee_i e_i$ which is equal to p by the faithfulness of tr. □

We shall see that a positive trace on a type II_1 factor is norm-continuous and a self-adjoint operator is actually a norm-limit of linear combinations of its spectral projections so in fact an apparently weaker property than ultraweak continuity is all we used in the previous corollary—namely that the trace of the supremum of an increasing net of projections is the supremum of the traces.

Corollary 6.1.20. *Let M be a von Neumann algebra with a positive ultraweakly continuous faithful normalised trace tr. Then M is a type II_1 factor iff $Tr = tr$ for all ultraweakly continuous normalised traces Tr.*

Proof. We just have to show that $Z(M)$ is trivial. But if it were not, choose by faithfulness a projection $p \in Z(M)$ with $0 < tr(p) < 1$. Define $Tr(x) = (\frac{1}{tr(p)})tr(xp)$. Then Tr is an ultraweakly continuous normalized trace different from tr on $1 - p$. □

Exercise 6.1.21. *Let a be a non-zero positive self adjoint operator. Show that there is a bounded piecewise smooth function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $af(a)$ is a non-zero projection.*

Exercise 6.1.22. *A type* II_1 *factor is algebraically simple. (Hint—use the previous exercise to show that a 2-sided ideal contains a projection, then add projections to obtain the identity.)*

6.2 The GNS construction

Thus uniqueness of the trace implies factoriality. This suggests another interesting way to construct a type II_1 factor. If $A = M_2(\mathbb{C})$, A is embedded in $A \otimes A$ as diagonal matrices: $a \hookrightarrow a \otimes 1$. Iterate this procedure to form an increasing sequence A_n of *-algebras with $A_1 = A$ and $A_{n+1} = A_n \otimes A$, and consider the *-algebra $A_\infty = \cup_n A_n$ which could also be called $\otimes_{alg,n=1}^{\infty} A_n$. If we normalise the matrix trace on all matrix algebras so that $tr(1) = 1$ then $tr(a \otimes 1) = tr(a)$ so that tr defines a positive faithful normalised trace on A_∞. Elements of A_∞ can be thought of as linear combinations of tensors of the form $a_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes 1 \otimes 1 \otimes 1 \otimes \cdots$, on which the trace is just the product of the traces of the a_i's. We now turn A_∞ into a von Neumann algebra.

Define an inner product on A_∞ by $\langle x, y \rangle = tr(y^* x)$. Then A_∞ is a pre-Hilbert space and let $\mathcal{H}$ be its completion. Note that $M_n(\mathbb{C})$ is a von Neumann algebra so $tr(y^* x^* xy) \leq ||x||^2 tr(y^* y)$. This means that the operator L_x on A_∞, $L_x(y) = xy$, satisfies $||L_x(\xi)|| \leq ||x|| \cdot ||\xi||$ (where $||x||$ is the operator norm of the matrix x and $||\xi||$ is the Hilbert space norm of ξ) and so extends uniquely to a bounded operator also written L_x on $\mathcal{H}$. One checks that $(L_x)^* = L_{x^*}$ so $x \to L_x$ defines a faithful (=injective) representation of the *-algebra A_∞ on $\mathcal{H}$. Let M be the von Neumann algebra on $\mathcal{H}$ generated by the L_x and identify A_∞ with a subalgebra of M.

The trace on A_∞ is defined by $tr(a) = \langle a\xi, \xi \rangle$ where ξ is the element $1 \in A_\infty$ considered as a vector in $\mathcal{H}$. So tr extends to a trace on M which is ultraweakly continuous, positive and normalised. It is also unique with these properties by the uniqueness of the trace on the ultraweakly dense subalgebra A_∞ of M. If we can show that tr is faithful on M then it follows that M is a type II_1 factor. It is important to note that this does not follow simply from the faithfulness of tr on A. In fact it is true but we need to do something to prove it.

When we showed that L_x was bounded, the same calculation, with $tr(ab) = tr(ba)$, would have shown that R_x, right multiplication by x, is also bounded. Associativity shows that L_x and R_y commute on A_∞, hence on $\mathcal{H}$. Thus M commutes with R_y for each $y \in A_\infty$. Now we can show faithfulness: if

$tr(x^*x) = 0$ for $x \in M$ then for each $a \in A_\infty$ we have

$$||x(a)||^2 = ||xR_a(\xi)||^2 = ||R_a x(\xi)||^2 \leq ||R_a||^2||x\xi||^2 = ||R_a||^2 tr(x^*x) = 0.$$

Since A_∞ is dense, this means $x = 0$. So tr is faithful on M which is thus a type II_1 factor.

Exercise 6.2.1. *Let F_n be the Fibonacci numbers. Show that there is a unique (up to you to figure out in what sense) unital embedding of $M_{F_n}(\mathbb{C}) \oplus M_{F_{n+1}}(\mathbb{C})$ inside $M_{F_{n+1}}(\mathbb{C}) \oplus M_{F_{n+2}}(\mathbb{C})$ for $n \geq 3$. Thus one may form the *-algebra*

$$F_\infty = \cup_{n=1}^\infty M_{F_n}(\mathbb{C}) \oplus M_{F_{n+1}}(\mathbb{C}).$$

Show that there is a unique C^*-norm and unique positive trace on F_∞ so we may repeat the procedure above to obtain another type II_1 factor.

Many points are raised by this example. The easiest to deal with are the properties of the vector ξ which played a prominent role. We used both $\overline{M\xi} = \mathcal{H}$ and $\overline{M'\xi} = \mathcal{H}$.

Definition 6.2.2. *Let M be a von Neumann algebra on $\mathcal{H}$. A vector $\xi \in \mathcal{H}$ is called* cyclic *for M if $\overline{M\xi} = \mathcal{H}$ and* separating *for M if $(x\xi = 0) \Rightarrow (x = 0)$ for all $x \in M$.*

Proposition 6.2.3. *With notation as above, ξ is cyclic for M iff ξ is separating for M'.*

Proof. ($\Rightarrow$) Exercise—in fact done in the discussion of A_∞ above.

($\Leftarrow$) Let p be the projection onto the closure of $M\xi$. Then $p \in M'$. But $(1-p)\xi = 0$ so $p = 1$. □

The construction of M from A_∞ is a special case of what is known as the GNS construction (Gelfand-Naimark-Segal). Given a positive linear functional ϕ satisfying $\phi(a^*) = \overline{\phi(a)}$ on a *-algebra A we let N_ϕ be $\{x \in A : \phi(x^*x) = 0\}$. We also define a sesquilinear form $\langle , \rangle_\phi$ on A by $\langle x, y\rangle_\phi = \phi(y^*x)$. This form is positive semidefinite but this is enough for the Cauchy-Schwartz inequality to hold so that N is the same as $\{x : \langle x, y\rangle_\phi = 0 \quad \forall y \in A\}$ so that N is a subspace and $\langle , \rangle_\phi$ defines a pre-Hilbert space structure on the quotient A/N. Under favourable circumstances, left multiplication by x, L_x defines a bounded linear operator on it. Favourable circumstances are provided by C^*-algebras.

Exercise 6.2.4. *If ϕ is a linear functional on a C^*-algebra satisfying $\phi(a^*a) \geq 0$ show that $\phi(a^*) = \overline{\phi(a)}$. Moreover if A is unital show that ϕ is norm-continuous and in fact $||\phi|| = \phi(1)$.*

Remark 6.2.5. *It is a standard elementary fact in C^*-algebras that one may always adjoin an identity to a C^*-algebra.*

Proposition 6.2.6. *If A is a unital C^*-algebra and $\phi : A \to \mathbb{C}$ is a positive linear functional then*

$$\phi(y^*x^*xy) \leq ||x||^2\phi(y^*y)$$

Proof. Let $\tilde{\phi}(a) = \phi(y^*ay)$. Then $\tilde{\phi}$ is positive so by the exercise $\tilde{\phi}(x^*x) \leq ||x||^2\tilde{\phi}(1)$. □

It follows immediately that, given a positive linear functional ϕ on a unital C^*-algebra, each $x \in A$ determines a bounded linear operator $\pi_\phi(x)$ on the Hilbert space $\mathcal{H}_\phi$ of the GNS construction via left multiplication: $\pi_\phi(x)(y) = xy$. Moreover $||\pi_\phi(x)|| \leq ||x||$ and $\pi_\phi(x^*) = \pi_\phi(x)^*$ since $\langle \pi_\phi(x)y, z\rangle = \phi(z^*xy) = \langle y, \pi_\phi(x^*)z\rangle$. Note that $\phi(x) = \langle \pi_\phi(x)1, 1\rangle$.

To sum up we have the following:

Definition 6.2.7. *If A is a C^*-algebra and ϕ is a positive linear functional on A, the Hilbert space of the GNS construction is written $\mathcal{H}_\phi$ and the representation π_ϕ by left multiplication is called the GNS representation.*

Proposition 6.2.8. *If A is a C^*-algebra on $\mathcal{H}$ and $\xi \in \mathcal{H}$, define $\omega_\xi(a) = \langle a\xi, \xi\rangle$. Then ω_ξ is a positive linear functional and $a \mapsto a\xi$ defines a unitary $u : \mathcal{H}_{\omega_\xi} \to \overline{A\xi}$ such that $u\pi_{\omega_\xi}(a)u^* = a$.*

Proof. Obvious. □

If A is actually a von Neumann algebra, $\pi_\phi(A)$ will not in general be one on $\mathcal{H}_\phi$. This difficulty will be resolved in the next section.

Chapter 7

Normality, complete additivity.

7.1 Normal states.

In quantum mechanics if ξ is a unit vector in a Hilbert space it defines a "state" ϕ. In particular this means that if an observable is given by the self-adjoint operator a then the average value of repeated observations of the system in the state ϕ is $\langle a\xi, \xi\rangle$. For this reason one calls a positive linear functional ϕ a "state" on a unital C*-algebra provided $\phi(1) = 1$.

Definition 7.1.1. *If A is a C*-algebra on $\mathcal{H}$ and ϕ is a state on A we say ϕ is a* vector state *if there is a unit vector $\xi \in \mathcal{H}$ with $\phi = \omega_\xi$, i.e. $\phi(a) = \langle a\xi, \xi\rangle$ for all $a \in A$.*

Not all states are vector states but our goal in this chapter is to show that on von Neumann algebras there is a natural class of states which are automatically vector states provided one amplifies the Hilbert space.

Definition 7.1.2. *(i) If M is a von Neumann algebra a positive linear functional ϕ is called* completely additive *if*

$$\phi(\bigvee_\alpha p_\alpha) = \sum_\alpha \phi(p_\alpha)$$

whenever p_α is a family of mutally orthogonal projections.

(ii) A positive linear map $\Phi : A \to B$ between von Neumann algebras is called normal *if*

$$\Phi(\bigvee_\alpha a_\alpha) = \bigvee_\alpha \Phi(a_\alpha)$$

for any increasing net (a_α) of self-adjoint operators in A.

Our goal in this chapter is to show the following:

Theorem 7.1.3. *If ϕ is state on a von Neumann algebra M on $\mathcal{H}$ the following are equivalent:*
(1) ϕ is normal.
(2) ϕ is completely additive
(3) ϕ is a vector state on $\mathcal{H} \otimes \ell^2(\mathbb{N})$
(4) ϕ is ultraweakly continuous.

The only implication that is not obvious from what we have done is (2) $\implies$ (3). To prove it we will put together some results. The first couple actually establish (4) $\implies$ (3) by 5.1.2.

Lemma 7.1.4. *Let A be a C^*-algebra on $\mathcal{H}$ containing 1. If ψ is a positive linear functional on A and $\xi \in \mathcal{H}$ is a vector with $\psi \leq \omega_\xi$ (i.e. $\omega_\xi - \psi$ is positive), then there is a $s \in A'$ with $\psi = \omega_{s\xi}$.*

Proof. Define a sesquilinear form $(,)$ on $A\xi$ by $(a\xi, b\xi) = \psi(b^*a)$. Cauchy-Schwarz and $\psi \leq \phi_\xi$ give that $|(a\xi, b\xi)| \leq ||a\xi||||b\xi||$ so $(,)$ is well-defined and there is a bounded positive operator t on $\overline{A\xi}$ with $\langle a\xi, tb\xi\rangle = \psi(b^*a)$. But $\langle a\xi, tbc\xi\rangle = \psi(c^*b^*a) = \langle b^*a\xi, tc\xi\rangle = \langle a\xi, btc\xi\rangle$ so that $t \in A'$ on $\overline{A\xi}$. If $p = p_{\overline{A\xi}}$, tp is a positive operator in A' and if $s = \sqrt{t}$, $\psi(a) = \langle a\xi, t\xi\rangle = \langle as\xi, s\xi\rangle = \omega_\xi(a)$. □

.

Corollary 7.1.5. *If ξ and η are vectors such that $\omega(a) = \langle a\xi, \eta\rangle$ is positive (on a C^*-algebra A on $\mathcal{H}$) then there is a vector ν with $\omega = \omega_\nu$.*

Proof. For $a \geq 0$,

$$\begin{aligned}\langle a\xi, \eta\rangle &= 1/4(\langle a(\xi+\eta), \xi+\eta\rangle - \langle a(\xi-\eta), \xi-\eta\rangle) \\ &\leq 1/4\omega_{\xi+\eta}(a).\end{aligned}$$

□

Now we begin to show that complete additivity means that two states cannot disagree too erratically.

Lemma 7.1.6. *Let ϕ_1 and ϕ_2 be completely additive. Suppose $p \in M$ is a projection and $\phi_1(p) < \phi_2(p)$. Then there is a projection $q \leq p$, for which $\phi_1(x) < \phi_2(x) \quad \forall x \geq 0$ with $qxq = x$.*

Proof. Choose a maximal family of mutually orthogonal "bad" projections $e_\alpha \leq p$ for which $\phi_1(e_\alpha) \geq \phi_2(e_\alpha)$. By complete additivity $\bigvee_\alpha e_\alpha$ is bad so let $q = p - \bigvee_\alpha e_\alpha$. By maximality $\phi_1(f) < \phi_2(f)$ for all projections $f \leq q$ and since α is norm continuous, by the spectral theorem $\phi_1(x) < \phi_2(x) \quad \forall x \geq 0$ with $qxq = x$.

□

Next we get vector state behaviour for ϕ on some small projection.

Lemma 7.1.7. *There exists $p > 0$ and $\xi \in \mathcal{H}$ for which*

$$\phi(x) = \langle x\xi, \xi\rangle \quad \forall x \in pMp$$

Proof. Choose $\xi \in \mathcal{H}$ with $\phi(1) = 1 < \langle \xi, \xi\rangle$. Then by the previous lemma there is a $p > 0$ for which $\phi(x) \leq \langle x\xi, \xi\rangle \quad \forall x \in pMp$. By 7.1.4 we are done. □

Now we put together all the little parts and prove that (3) $\implies$ (4) in 7.1.3. So let ϕ be a completely additive state on a von Neumann algebra M acting on $\mathcal{H}$. Let p_α be a maximal family of pairwise orthogonal projections admitting a vector $\xi_\alpha \in p_\alpha\mathcal{H}$ with $\phi(x) = \langle x\xi_\alpha, \xi_\alpha\rangle$ on $p_\alpha M p_\alpha$. Then by the previous lemma $\bigvee_\alpha p_\alpha = 1$. And obviously $||\xi_\alpha||^2 = \phi(p_\alpha)$. Since $\phi(p_\alpha)$ can only be non-zero for countably many α we can assume the set of α's is countable.

By Cauchy-Schwarz, for *any* $x \in M$,

$$|\phi(xp_\alpha)| \leq \phi(p_\alpha x^* x p_\alpha)^{1/2}\phi(p_\alpha)^{1/2} = ||x\xi_\alpha||\phi(p_\alpha)^{1/2}.$$

So the linear functional $x\xi_\alpha \mapsto \phi(xp_\alpha)$ is well-defined and bounded on $M\xi_\alpha$ which means there is a vector η_α, $||\eta_\alpha||^2 = \phi(p_\alpha)$, with

$$\phi(xp_\alpha) = \langle x\xi_\alpha, \eta_\alpha\rangle.$$

Moreover, also by Cauchy-Schwarz, $|\phi(x) - \sum_{\alpha\in F}\phi(xp_\alpha)|$ can be made arbitrarily small by choosing the finite set F sufficiently large since ϕ is completely additive. We conclude that there exist ξ_α, η_α, each of norm $\leq \phi(\alpha)^{1/2}$ with

$$\phi(x) = \sum_\alpha \langle x\xi_\alpha, \eta_\alpha\rangle$$

which is the same as saying that $\phi(x) = \langle (x\otimes 1)\xi, \eta\rangle$ for some $\xi, \eta \in \ell^2(\mathbb{N}, \mathcal{H})$. By corollary 7.1.5 we have proved theorem 7.1.3.

Corollary 7.1.8. *If ϕ is a normal state on the von Neumann algebra M then the GNS representation π_ϕ is ultraweakly continuous onto a von Neumann algebra on $\mathcal{H}_\phi$.*

Proof. We saw in the last theorem that $\phi(x) = \langle x \otimes 1(\nu), \nu\rangle$ on $\mathcal{H} \otimes \ell^2(\mathbb{N})$. The map $x \mapsto x \otimes 1$ is ultraweakly continuous. By 6.2.8 we have that π_ϕ is ultraweakly continuous since the reduction to $\overline{M \otimes 1(\nu)}$ is ultraweakly continuous. So the kernel of π_ϕ is an ultraweakly closed 2-sided ideal, hence of the form Me for some e in the centre of M. It follows that π_ϕ is injective on $M(1-e)$ and since the norm of an operator x is determined by the spectrum of x^*x, the unit ball of the image of M is the image of the unit ball which is weakly compact so by 5.2.2 we are done. □

7.2 Isomorphisms are spatial.

Recall that an isomorphism $\Phi : M \to N$ between von Neumann algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively is called *spatial* if there is a unitary $u : \mathcal{H} \to \mathcal{K}$ such that $\Phi(x) = uxu^*$ for all $x \in M$. Though the title of this section is not literally true, it becomes true on amplification as a result of theorem 7.1.3:

Theorem 7.2.1. *Given an isomorphism $\Phi : M \to N$ between von Neumann algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, there is a Hilbert space $\mathcal{W}$ and a unitary $u : \mathcal{H}\otimes\mathcal{W} \to \mathcal{K}\otimes\mathcal{W}$ with $\Phi(x)\otimes 1 = u(x\otimes 1)u^*$ for all $x \in M$.*

Proof. If $\xi \in \mathcal{H}$ defines the vector state ϕ on M,then since normality (or complete additivity) is defined by algebra, the state $\phi \circ \Phi^{-1}$ is also a vector state on $\mathcal{K} \otimes \ell^2(\mathbb{N})$ given by the vector η. This means that there is a unitary from the closure of $M\xi$ to the closure of $N\eta$ intertwining the actions of x and $\Phi(x) \otimes 1$. One may exhaust $\mathcal{H}$ in this way to obtain an isometry $u : \mathcal{H} \to \oplus_\alpha \mathcal{K} \otimes \ell^2(\mathbb{N})$ intertwining the actions of M. For a big enough $\mathcal{W}$, $\oplus_\alpha \mathcal{K} \otimes \ell^2(\mathbb{N})$ is $\mathcal{K} \otimes \mathcal{W}$ and tensoring again by $\mathcal{W}$ we get an intertwining isometry $u : \mathcal{H} \otimes \mathcal{W} \to \mathcal{K} \otimes \mathcal{W}$. Now consider the action of M on $(\mathcal{H}\otimes\mathcal{W})\oplus(\mathcal{K}\otimes\mathcal{W})$ defined in terms of matrices by $\left(\begin{smallmatrix} x\otimes 1 & 0 \\ 0 & \Phi(x)\otimes 1 \end{smallmatrix}\right)$. To say that u intertwines the actions is precisely the same as saying that $\left(\begin{smallmatrix} 0 & 0 \\ u & 0 \end{smallmatrix}\right)$ is in M'. So $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \preceq \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in M'. Applying this to Φ^{-1} as well we see by theorem 6.1.2 that these two projections are equivalent in M'. But any partial isometry witnessing their equivalence has the form $\left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right)$ with w a *unitary* between $\mathcal{H}$ and $\mathcal{K}$ intertwining the actions. (Note that we never assumed that M was more than a unital *-algebra on $(\mathcal{H} \otimes \mathcal{W}) \oplus (\mathcal{K} \otimes \mathcal{W})$).

□

7.3 Exercises on two projections.

Let p and q be projections onto closed subspaces $\mathcal{H}$ and $\mathcal{K}$ of the Hilbert space $\mathcal{U}$ respectively. Let $M = \{p, q\}''$.

Exercise 7.3.1. *Show that $\mathcal{U} = (\mathcal{H}\cap\mathcal{K})\oplus(\mathcal{H}^\perp\cap\mathcal{K}^\perp)\oplus(\mathcal{H}\cap\mathcal{K}^\perp)\oplus(\mathcal{H}^\perp\cap\mathcal{K})\oplus\mathcal{W}$ for some $\mathcal{W}$ and this decomposition is invariant under p and q.*

Exercise 7.3.2. *Show that, on $\mathcal{W}$, p and q are in "general position", i.e. $p \wedge q = 0$, $p \vee q = 1$, $(1 - p) \wedge q = 0$ and $(1 - p) \vee q = 1$.*

Exercise 7.3.3. *Show that if $a \in \mathcal{B}(\mathcal{H})$, $0 \leq a \leq 1$, $\left(\begin{smallmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{smallmatrix}\right)$ is a projection on $\mathcal{H} \oplus \mathcal{H}$. When is it in general position with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$?*

Exercise 7.3.4. *Let $a = (p-q)^2$ and $A = \{a\}''$. Show that $a \in Z(M)$ and that $\{a_0 + a_1 p + a_2 q + a_3 pq + a_4 qp : a_i \in A\}$ is a *-algebra (which is necessarily weakly dense in M).*

Exercise 7.3.5. *Show that pMp is abelian, generated by pqp.*

>From now on suppose p and q are in general position.

Exercise 7.3.6. *Show that $p \cong q$ in M. (Hint: consider the polar decomposition of pq.)*

Exercise 7.3.7. *Show there is a 2×2 system of matrix units $(e_{ij}) \in M$ with $p = e_{11}$.*

Exercise 7.3.8. *Show that M is spatially isomorphic to $B \otimes M_2(\mathbb{C})$ for some abelian von Neumann algebra B generated by $b, 0 \leq b \leq 1$, with p corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and q corresponding to $\begin{pmatrix} b & \sqrt{b(1-b)} \\ \sqrt{b(1-b)} & 1-b \end{pmatrix}$*

Now drop the hypothesis that p and q are in general position.

Exercise 7.3.9. *Show that $p \vee q - p \cong q - p \wedge q$ in M*

Alternative approach using group representations.

Exercise 7.3.10. *Show that $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$ (infinite dihedral group).*

Exercise 7.3.11. *Classify all unitary representations of $\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$. (Hint—use the spectral theorem for unitaries.)*

Exercise 7.3.12. *Observe that $2p - 1$ and $2q - 1$ are self-adjoint unitaries.*

Exercise 7.3.13. *Obtain the structure of 7.3.8 using the last 3 exercises.*

Chapter 8

The Predual

An ultraweakly continuous linear functional ϕ on a von Neumann algebra M is norm continuous so defines an element of M^*. Our goal in this chapter is to show that the set of all such ϕ is a closed subspace M_* of M^* and that the duality between M_* and M makes M equal to the Banach space dual of M_*. We will first establish this in the special case $M = \mathcal{B}(\mathcal{H})$.

8.1 Trace class and Hilbert Schmidt operators.

The material in this section is standard so we will only prove results as it suits us, otherwise referring any unproved assertions to Reed and Simon.

Lemma 8.1.1. *If $a \in \mathcal{B}(\mathcal{H})$ is positive and (ξ_i) and (η_i) are two orthonormal bases of $\mathcal{H}$, then*

$$\sum_i \langle a\xi_i, \xi_i\rangle = \sum_i \langle a\eta_i, \eta_i\rangle$$

(where ∞ is a possible value for the sum).

Proof. We have

$$\begin{aligned}\sum_i \langle a\xi_i, \xi_i\rangle &= \sum_i ||\sqrt{a}\xi_i||^2 \\ &= \sum_i (\sum_j |\langle \sqrt{a}\xi_i, \eta_j\rangle|^2) \\ &= \sum_j (\sum_i |\langle \sqrt{a}\eta_j, \xi_i\rangle|^2) \\ &= \sum_j ||\sqrt{a}\eta_j||^2\end{aligned}$$

$$= \sum_j \langle a\eta_j, \eta_j \rangle$$

where every number is positive so the order of the sum is immaterial. □

The number $\sum_i \langle a\xi_i, \xi_i \rangle$ of the previous theorem is called the trace of a, written $Trace(a)$.

Definition 8.1.2. *An element $a \in \mathcal{B}(\mathcal{H})$ is said to be of* trace class *if $Trace(|a|)$ is finite.*

If a is trace class and (ξ_i) is an orthonormal basis, the sum

$$\sum_i \langle a\xi_i, \xi_i \rangle$$

converges absolutely and is called the trace, $Trace(a)$, of a.

Theorem 8.1.3. *The trace class operators on $\mathcal{H}$ form a self-adjoint ideal of compact operators, I_1, in $\mathcal{B}(\mathcal{H})$. The function $|a|_1$ defined by $|a|_1 = Trace(|a|)$ defines a norm on I_1 for which it is complete. Moreover $||a|| \leq |a|_1$.*

Proof. The only thing not proved in Reed and Simon is completeness. For this observe that if a_n is a Cauchy sequence in $|-|_1$, it is Cauchy in $||-||$ so what we have to do is show that the norm limit of a $|-|_1$-Cauchy sequence (a_n) is trace class and that the sequence tends to that limit in $|-|_1$. So suppose $\epsilon > 0$ is given. Then for m and n large enough

$$\sum_{i=1}^{\infty} \langle |a_n - a_m|\xi_i, \xi_i \rangle < \epsilon.$$

So for any N,

$$\sum_{i=1}^{N} \langle |a_n - a_m|\xi_i, \xi_i \rangle < \epsilon.$$

Now if b_n tends in norm to b, then $|b_n|$ tends in norm to $|b|$ (obviously $b_n^* b_n \to b^* b$, and approximate the square root function by polynomials on an interval) so for each fixed i,

$$\lim_{n\to\infty} |a_n - a_m|\xi_i = |a - a_m|\xi_i.$$

So $\sum_{i=1}^{N} \langle |a - a_m|\xi_i, \xi_i \rangle < \epsilon$ and letting N tend to ∞ we see that $a \in I_1$ since I_1 is a vector space, and also that $a_n \to a$ in $|-|_1$. □

The trace is independent of the orthonormal basis and if a is trace class and $b \in \mathcal{B}(\mathcal{H})$, $Tr(ab) = Tr(ba)$.

We see that each $h \in I_1$ determines a linear functional ϕ_h on $\mathcal{B}(\mathcal{H})$ by $\phi_h(x) = Trace(xh)$.

Definition 8.1.4. *The trace-class matrix as above is called the density matrix for the state ϕ_h.*

Proposition 8.1.5. *Each ϕ_h is ultraweakly continuous and its norm as an element of $\mathcal{B}(\mathcal{H})^*$ is $|h|_1$.*

Proof. Since h is compact, choose an orthonormal basis (ξ_i) of eigenvectors of $|h|$ with eigenvalues λ_i and let $h = u|h|$ be the polar decomposition. Then

$$\phi_h(x) = \sum_{i=1}^{\infty} \langle xu|h|\xi_i, \xi_i\rangle$$

so ultraweak continuity is apparent, and

$$\phi_h(x) \leq \sum_{i=1}^{\infty} ||x||\,||\,|h|\xi_i||$$

$$= ||x|| \sum_{i=1}^{\infty} \lambda_i$$

$$= ||x||\,|h|_1.$$

Moreover evaluating ϕ_h on u^* gives $||\phi_h|| = |h|_1$. □

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, a bounded operator $x : \mathcal{H} \to \mathcal{K}$ is called *Hilbert-Schmidt* if x^*x is trace class, i.e. $\sum_{i=1}^{\infty} ||x\xi_i||^2 < \infty$ for some (hence any) orthonormal basis (ξ_i) of $\mathcal{H}$. The set of all Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$ is written $\ell^2(\mathcal{H}, \mathcal{K})$ and if x is Hilbert-Schmidt, so is x^*, and x is compact.

Theorem 8.1.6. *If $a \in \mathcal{B}(\mathcal{H})$, $b \in \mathcal{B}(\mathcal{K})$ and $x \in \ell^2(\mathcal{H}, \mathcal{K})$ then $bxa \in \ell^2(\mathcal{H}, \mathcal{K})$. If $x \in \ell^2(\mathcal{H}, \mathcal{K})$ and $y \in \ell^2(\mathcal{K}, \mathcal{H})$ then yx is trace class. With the inner product $\langle x, y\rangle = Trace(y^*x)$, $\ell^2(\mathcal{H}, \mathcal{K})$ is a Hilbert space in which the finite rank operators are dense.*

Proof. See Reed and Simon. □

Exercise 8.1.7. *Prove all the assertions made above about trace-class and Hilbert-Schmidt operators.*

Exercise 8.1.8. *If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces construct a natural map from $\mathcal{K} \otimes \mathcal{H}^*$ to $\ell^2(\mathcal{H}, \mathcal{K})$ and show that it is unitary.*

Let $|x|_2$ be the Hilbert space norm on Hilbert-Schmidt operators.

Lemma 8.1.9. *If $x \in \ell^2(\mathcal{H}, \mathcal{K})$ and $y \in \ell^2(\mathcal{K}, \mathcal{H})$ then $Trace(xy) = Trace(yx)$.*

Proof. First note that the result is true if we suppose that $|x|$ is trace class. For then let $x = u|x|$ be the polar decomposition, choose an orthonormal basis (ξ_i) of the final domain of u and extend it to an orthonormal basis of $\mathcal{K}$. Also extend $(u^*\xi_i)$ to an orthonormal basis of $\mathcal{H}$ by vectors in $\ker(|x|)$. Then

$$\begin{aligned} Trace(xy) &= \sum_i \langle u|x|y\xi_i, \xi_i\rangle \\ &= \sum_i \langle |x|yuu^*\xi_i, u^*\xi_i\rangle \\ &= Trace(|x|(yu)) \\ &= Trace((yu)|x|) \\ &= Trace(yx.) \end{aligned}$$

Now suppose only that x is Hilbert-Schmidt. Let $\epsilon > 0$ be given and choose x' of finite rank with $|x - x'|_2 < \epsilon$. Then

$$|Trace(xy) - Trace(yx)| = |Trace((x - x')y) - Trace(y(x - x'))|$$

which by Cauchy-Schwartz is $\leq 2\epsilon|y|_2$. □

Corollary 8.1.10. *If ω is an ultraweakly continuous linear functional on $\mathcal{B}(\mathcal{H})$, there is a trace class h so that $\omega = \phi_h$.*

Proof. By 5.1.2 there are (ξ_i) and (η_i) in $\ell^2(\mathbb{N}, \mathcal{H})$ so that $\omega(x) = \sum_i \langle x\xi_i, \eta_i\rangle$. Then if we define a and b from $\ell^2(\mathbb{N})$ to $\mathcal{H}$ by $a(f) = \sum_i f(i)\xi_i$ and $b(f) = \sum_i f(i)\eta_i$, a and b are Hilbert Schmidt and $\omega(x) = Trace(b^*xa)$ which is $Trace(xab^*)$ by the previous result. □

Putting everything together so far, we have identified the image of the Banach space I_1 under the map $h \mapsto \phi_h$ with the closed subspace of $\mathcal{B}(\mathcal{H})^*$ consisting of ultraweakly continuous linear functionals. To close the loop we only need to show that the Banach space dual of I_1 is $\mathcal{B}(\mathcal{H})$.

Theorem 8.1.11. *If $\alpha : I_1 \to \mathbb{C}$ is linear and bounded for $|-|_1$, there is an $x \in \mathcal{B}(\mathcal{H})$ so that $\alpha(a) = \phi_a(x)$, and $||\alpha|| = ||x||$.*

Proof. This is rather routine. Two vectors ξ and η define an element x of I_1 by $x(v) = \langle v, \xi \rangle \eta$ so one may define a sesquilinear form on $\mathcal{H}$ by $(\xi, \eta) = \alpha(x)$. Boundedness of x follows from that of α so there is an appropriate $x \in \mathcal{B}(\mathcal{H})$. To show that the norm of x as an element of the dual of I_1 is actually $||x||$, suppose $||x|| = 1$ and choose a unit vector ξ with $||x\xi||$ almost equal to 1. Then $Tr(hx)$ is almost 1 if h is the partial isometry which sends $v \in \mathcal{H}$ to $\langle v, x\xi \rangle \frac{\xi}{||x\xi||}$. □

Exercise 8.1.12. *Fill in the missing details in the previous proof.*

Now we pass to von Neumann algebras though in fact these results work for any ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$.

Theorem 8.1.13. *If V is an ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$ then $V = V^{\perp\perp}$ in the sense that if $\phi(x) = 0$ for every ultraweakly continuous ϕ for which $\phi(V) = 0$ then $x \in V$.*

Proof. This is a simple application of the Hahn-Banach theorem—if $x \notin V$ construct an ultraweakly continuous functional which is zero on V and non-zero on x. □

‘

Exercise 8.1.14. *Exhibit a non-zero trace class operator on $\ell^2(\Gamma)$ which is orthogonal to $vN(\Gamma)$.*

Theorem 8.1.15. *If V is an ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$ then it is canonically the dual Banach space of V_* which is defined as the space of ultraweakly continuous linear functionals on V. Moreover the ultraweak topology on V is the weak-* topology on V as the dual of V_*.*

Proof. If B is a Banach space with dual B^* and V is a weak-* closed subspace of B^* then V is the dual of $B/V^\perp$ (surjectivity of the natural map from V to the dual of $V/B^\perp$ is a result of the previous theorem), so V is a dual space. So we just have to identify the Banach space $B/V^\perp$ with the space of weak-* continuous (as elements of B^{**}) linear functionals on V. This is a simple exercise. Putting $B = I_1$ we are done. □

Exercise 8.1.16. *If V is an ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$, show that V_* is a separable Banach space if $\mathcal{H}$ is a separable Hilbert space.*

8.2 A technical lemma.

Let us prove a lemma which shows what the techniques developed so far can be good for. It will be crucial in our treatment of Tomita-Takesaki theory.

find reference

It is a "Radon-Nikodym" type theorem inspired by one due to Sakai([]).

Lemma 8.2.1. *Let $\lambda \in \mathbb{R}^+$ be given and let ϕ be a faithful ultraweakly continuous state on a von Neumann algebra M. Let $\psi \in M_*$ be such that $|\psi(y^*x)| \leq \sqrt{\phi(x^*x)}\sqrt{\phi(y^*y)}$. Then there is an $a \in M_{1/2}$ (elements of norm $\leq 1/2$) so that*

$$\psi(x) = \lambda\phi(ax) + \lambda^{-1}\phi(xa).$$

Proof. For $a \in M$ let $\theta_a(x) = \phi(\lambda ax + \lambda^{-1}xa)$. Then the map $\alpha : M \to M_*$, $\alpha(a) = \theta_a$, is continuous for the topologies of duality between M and M_*. But we know that this topology on M is the ultraweak topology so that $\alpha(M_1)$ is a compact convex set. By contradiction suppose that ψ is not in $\alpha(M)$.

Then by Hahn-Banach there is an $h \in M$ with $\Re(\psi(h)) > D$ where $D = \sup_{a \in M_{1/2}} \Re(\theta_a(h))$. But if $h = u|h| = |h^*|u$ is the polar decomposition of h, we have

$$\theta_{u^*/2}(h) = 1/2(\lambda\phi(|h|) + \lambda^{-1}\phi(|h^*|))$$

so that

$$2D \geq \lambda\phi(|h|) + \frac{1}{\lambda}\phi(|h^*|) \geq 2\sqrt{\phi(|h|)}\sqrt{\phi(|h^*|)}.$$

But also $D < |\psi(h)| = |\psi(u|h|^{1/2}|h|^{1/2})| \leq \sqrt{\phi(|h|)}\sqrt{\phi(u|h|u^*)}$, a contradiction. □

Chapter 9

Standard form of a II_1 factor and II_∞ factors.

9.1 Standard form.

In this section M will be a von Neumann algebra with an ultraweakly continuous faithful normalized trace tr and $L^2(M, tr)$ will be abbreviated to $L^2(M)$.

In section 6.2 we learned how to construct a von Neumann algebra from a C^*-algebra and a positive linear functional on it. If we apply this construction to $L^\infty(X, \mu)$ (when $\mu(X) < \infty$) with trace given by $\int f d\mu$, the Hilbert space would be $L^2(X, d\mu)$. For this reason, if M is a type II_1 factor we write $L^2(M, tr)$ for the GNS Hilbert space obtained from the trace. In fact one can define L^p spaces for $1 \leq p \leq \infty$ using the L^p norm $||x||_p = tr(|x|^p)^{1/p}$. A noncommutative version of the Holder inequality shows that $|| - ||_p$ is a norm and $L^p(M)$ is the completion. We set $L^\infty(M) = M$ and we shall see that $L^1(M)$ is the predual M_*.

Let us fix on the notation Ω for the vector in $L^2(M)$ which is the identity of M.

Proposition 9.1.1. *If M is a type II_1 factor the $|| - ||$-unit ball of M is a complete metric space for $|| - ||_2$ and the topology defined by $|| - ||_2$ on the unit ball is the same as the strong (and ultrastrong and *-strong) topology.*

Proof. If x_n is Cauchy in $|| - ||_2$ then for each $a \in M$, $x_n a$ is also since $||x_n a||_2 \leq ||a||\,||x_n||_2$. So we can define x on the dense subspace $M\Omega$ of $L^2(M)$ by $x(a\Omega) = lim_{n\to\infty} x_n a\Omega$. Since $||x|| \leq 1$, we have $||x\xi|| \leq ||\xi||$ for $\xi \in M\Omega$ so x extends to a bounded operator on $L^2(M)$ which is obviously in M, and $x\Omega = x = \lim_{n\to\infty} x_n$ in $|| - ||_2$.

The strong topology is obviously no stronger than $||-||_2$ since the single seminorm $a \mapsto ||a\Omega||$ defines the $||-||_2$ topology. Moreover $||xa\Omega|| \leq ||x||_2||a||$ shows that $||-||_2$ controls the strong topology on the unit ball.

Finally note that in the statement of the theorem it does not matter what representation of M is used to define the strong topology on the unit ball as the ultrastrong topology does not change under the manipulations that we used to get the GNS construction from a II_1 factor on an arbitrary Hilbert space. □

The action of M on $L^2(M, tr)$ is called the *standard form* of M. Note that $vN(\Gamma)$ on $\ell^2(\Gamma)$ is already in standard form. (We see that we could have obtained our first example of a II_1 factor by applying the GNS construction to the group algebra $\mathbb{C}\Gamma$ with the trace $tr(\sum_\gamma c_\gamma u_\gamma) = c_{id}$.)

We now want to determine the commutant M' when M is in standard form.

Definition 9.1.2. *Let $J : L^2(M) \to L^2(M)$ be the antilinear unitary involution which is the extension to $L^2(M)$ of the map $x \mapsto x^*$ from M to M.*

Lemma 9.1.3. *For x, a in M, and ξ, η in $L^2(M)$*

(i) $\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle$
(ii) $JxJ(a\Omega) = ax^*\Omega$

Proof. (i) If $\xi = a\Omega$ and $\eta = b\Omega$, $\langle J\xi, J\eta\rangle = tr(ba^*) = \langle \eta, \xi\rangle$.
(ii) $JxJ(a\Omega) = J(xa^*\Omega) = ax^*\Omega$.

□

Corollary 9.1.4. *For M on $L^2(M)$, $JMJ \subseteq M'$.*

Proof. Left and right multiplication commute. □

Lemma 9.1.5. *For M on $L^2(M)$, if $x \in M'$, $Jx\Omega = x^*\Omega$.*

Proof. Take $a \in M$, then

$$\begin{aligned}\langle Jx\Omega, a\Omega\rangle &= \langle Ja\Omega, x\Omega\rangle \\ &= \langle a^*\Omega, x\Omega\rangle \\ &= \langle \Omega, xa\Omega\rangle \\ &= \langle x^*\Omega, a\Omega\rangle.\end{aligned}$$

□

Theorem 9.1.6. *For M on $L^2(M)$, $JMJ = M'$.*

Proof. If we can show that $x \mapsto \langle x\Omega, \Omega\rangle$ is a trace on M' we are done since by the above results $L^2(M)$ can be canonically identified with $L^2(M')$ so that the two J maps coincide. (Note that Ω is cyclic and separating for M hence also for M'.) So we would have $JM'J \subseteq M$.
But for $x, y \in M'$,

$$\begin{aligned}\langle xy\Omega, \Omega\rangle &= \langle y\Omega, x^*\Omega\rangle \\ &= \langle y\Omega, Jx\Omega\rangle \\ &= \langle x\Omega, Jy\Omega\rangle \\ &= \langle x\Omega, y^*\Omega\rangle \\ &= \langle yx\Omega, \Omega\rangle.\end{aligned}$$

□

We see that the commutant of the left regular representation of Γ on $\ell^2(\Gamma)$ is the von Neumann algebra generated by the right regular representation since $Ju_\gamma J\varepsilon_{\gamma'} = \varepsilon_{\gamma'\gamma^{-1}}$. And more generally the commutant of the left action of M on $L^2(M)$ is the *-algebra of right multiplication operators. In particular the commutant of a type II_1 factor M on $L^2(M)$ is also a type II_1 factor. This is *not* the case for M on an arbitrary Hilbert space. For instance we could consider $M \otimes 1$ on $L^2(M) \otimes \mathcal{H}$ for some infinite dimensional $\mathcal{H}$. Then the commutant of $M \otimes 1$ would be $JMJ \otimes \mathcal{B}(\mathcal{H})$—infinite matrices over JMJ.

Definition 9.1.7. *A II_∞ factor is a factor of the form $M \otimes \mathcal{B}(\mathcal{H})$ with M a type II_1 factor and $\dim \mathcal{H} = \infty$.*

Proposition 9.1.8. *Let M be an infinite factor with a projection $p \in M$ so that pMp is a type II_1 factor. Then M is a II_∞ factor.*

Proof. Choose a maximal family $\{p_\alpha\}$ of mutually orthogonal projections in M with $p_\alpha \cong p\ \forall\alpha$. If it were the case that $1 - \sum_\alpha p_\alpha \succeq p$ then we could contradict the maximality of the family $\{p_\alpha\}$. So write $1 = q + \sum_\alpha p_\alpha$ with $q \preceq p$. Since M is infinite the set of indices $\{\alpha\}$ is infinite so we may choose a bijection with itself minus α_0 and write $1 = q + \sum_\alpha p_\alpha \preceq p_{\alpha_0} + \sum_{\alpha\neq\alpha_0} p_\alpha \preceq 1$. We conclude that $\sum_\alpha p_\alpha$ is equivalent to 1 so we may suppose it equal to 1. We may then construct a system of matrix units by using partial isometries implementing the equivalences between the p_α to obtain the result from exercise 4.3.3.

□

Note that we used implicitly in the above proof the fact that the supremum of finitely many finite projections is finite. This will be covered in a series of exercises on projections.

make a reference to section 7.3, if appropriate?!

It could conceivably happen that, given a II_∞ factor M, the type II_1 factor of the form pMp depends on p (obviously only up to equivalence). We now introduce the trace on a II_∞ factor which will make this issue more clear.

If M is a type II_1 factor, define the map tr from $(M \otimes \mathcal{B}(\mathcal{H}))_+$ (the set of positive elements of $M \otimes \mathcal{B}(\mathcal{H})$), to $[0, \infty]$ by

$$tr((x_{ij})) = \sum_{i=1}^{\infty} tr(x_{ii})$$

where we have chosen a basis of the infinite dimensional Hilbert space $\mathcal{H}$ to identify $M \otimes \mathcal{B}(\mathcal{H})$ with certain matrices over M.

Theorem 9.1.9. *Let M be as above.*
(i) $tr(\lambda x) = \lambda tr(x)$ for $\lambda \geq 0$.
(ii) $tr(x + y) = tr(x) + tr(y)$.
(iii) If (a_α) is an increasing net of positive operators with $\bigvee_\alpha a_\alpha = a$ then $tr(\bigvee_\alpha a_\alpha) = \lim_\alpha tr(a_\alpha)$.
(iv) $tr(x^*x) = tr(xx^*)$ $\forall x \in M \otimes \mathcal{B}(\mathcal{H})$.
(v) $tr(uxu^*) = tr(x)$ for any unitary $u \in M \otimes \mathcal{B}(\mathcal{H})$ and any $x \geq 0$ in $M \otimes \mathcal{B}(\mathcal{H})$.
(vi) If p is a projection in $M \otimes \mathcal{B}(\mathcal{H})$ then p is finite iff $tr(p) < \infty$.
(vii) If p and q are projections with p finite then $p \preceq q$ iff $tr(p) \leq tr(q)$.
(viii) $p(M \otimes \mathcal{B}(\mathcal{H}))p$ is a type II_1 factor for any finite projection p.

Proof. The first two assertions are immediate. For (iii), note that the diagonal entries of positive matrices are ordered as the matrices, and all numbers are positive in the sums. (iv) Is obvious using matrix multiplication. (v) follows from (iv) via $uxu^* = (u\sqrt{x})(\sqrt{x}u^*)$. For (vi), if $tr(p) < \infty$ but p is infinite, there is a proper subprojection of p having the same trace as p. The difference would be a projection of trace zero which is clearly impossible. If $tr(p) = \infty$ then if q is a projection of finite trace, $q \preceq p$ and if $q \leq p$ then $tr(p - q) = \infty$ so one may construct an infinite sequence of mutually orthogonal equivalent projections less than p. Using a bijection with a proper subsequence, p dominates an infinite projection so is infinite itself. (vii) follows easily as in the case of a type II_1 factor. For (viii) simply observe that $tr(p) < \infty$ means that $p \preceq q$ for some q whose matrix is zero except for finitely many 1's on the diagonal. And obviously qMq is a type II_1 factor for such a q. □

Corollary 9.1.10. *Let M be a II_∞ factor on a separable Hilbert space and tr be the trace supplied by a decomposition $II_1 \otimes \mathcal{B}(\mathcal{H})$. Then tr defines an isomorphism of the totally ordered set of equivalence classes of projections in M to the interval $[0, \infty]$.*

Proof. Given the previous theorem, we only have to prove that any infinite projection is equivalent to the identity. But if p is infinite choose u with $uu^* = p$ and u^*u strictly less than p. Then $(u^*)^n u^n$ are a strictly decreasing sequence of equivalent projections so we may write p as an orthogonal sum $p = p_\infty + \sum_{i=1}^{\infty} p_i$ with all the p_i equivalent for $i \geq 1$. Now write the identity as a countable orthogonal sum of projections all $\preceq p_1$ (using the decomposition $\mathrm{II}_1 \otimes \mathcal{B}(\mathcal{H})$ if necessary). We see that $1 \leq p$. □

Unlike the II_1 case, or for that matter the $\mathcal{B}(\mathcal{H})$ case, the trace cannot be normalised (by $tr(1) = 1$ in the type II_1 factor case or the trace of a minimal projection being 1 in the $\mathcal{B}(\mathcal{H})$ case). This allows for the possibility of an automorphism α of M with $tr(\alpha(x)) = \lambda tr(x)$ for $x \geq 0$ and $\lambda > 0$, $\lambda \neq 1$.

Exercise 9.1.11. *Show that the trace on a II_∞ factor is unique with properties (i) to (vi), up to a scalar.*

Exercise 9.1.12. *If $\alpha : M \to N$ is a *-homomorphism from a type II_1 factor onto another, then α is an isomorphism, strongly continuous on the unit ball.*

Chapter 10

The Coupling Constant

We want to compare actions of a given II_1 factor on (separable) Hilbert spaces. We will show that they are parameterized by a single number in $[0, \infty]$.

Definition 10.0.13. *If M is a type II_1 factor, by M-*module *we will mean a Hilbert space $\mathcal{H}$ together with an ultraweakly continuous unital *-homomorphism from M to a type II_1 factor acting on $\mathcal{H}$. Thus M acts on $\mathcal{H}$ and we will write that action simply as $x\xi$ for $x \in M$ and ξ in $\mathcal{H}$.*

In fact the ultraweak continuity condition is superfluous. The identity map makes the Hilbert space on which M is defined into an M-module. Given M on $\mathcal{H}$ and another Hilbert space $\mathcal{K}$, $x \mapsto x \otimes id$ makes $\mathcal{H} \otimes \mathcal{K}$ into an M-module. The GNS representation makes $L^2(M)$ into an M-module. (The notion of $M-M$ bimodule is defined similarly as two commuting actions of M on some Hilbert space, $L^2(M)$ being the first example.) There is an obvious notion of direct sum of M-modules. We will compare a given M-module $\mathcal{H}$ with $L^2(M)$ by forming the direct sum of it $\mathcal{H}$ and infinitely many copies of $L^2(M)$.

10.1 Definition of $\dim_M \mathcal{H}$

Theorem 10.1.1. *Let M be a type II_1 factor and $\mathcal{H}$ a separable M-module. Then there is an isometry $u : \mathcal{H} \to L^2(M) \otimes \ell^2(\mathbb{N})$ such that $ux = (x \otimes 1)u$ (i.e. u is M-linear).*

Proof. Form the M-module $\mathcal{K} = \mathcal{H} \otimes L^2(M) \otimes \ell^2(\mathbb{N})$. Let $p = id \oplus 0 \in \mathcal{B}(\mathcal{K})$ be the projection onto $\mathcal{H}$ and $q = 0 \oplus id$ be the projection onto $L^2(M) \otimes \ell^2(\mathbb{N})$. Both p and q are in M' (on $\mathcal{K}$) which is a II_∞ factor since q is clearly infinite in

M' and if e is a rank one projection in $\mathcal{B}(\ell^2(\mathbb{N}))$ then $(0\oplus(1\otimes e))M(0\oplus(1\otimes e))$ is a type II_1 factor, being the commutant of M on $L^2(M)$.

Since q is an infinite projection in M', by 9.1.10 there is a partial isometry in M' with $u^*u = p$ and $uu^* \leq q$. Using the obvious matrix notation for operators on $\mathcal{K}$, let u be represented by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

Then calculating $u^*u = p$ and $uu^* \leq q$ gives $b^*b+d^*d = 0$ and $aa^*+bb^* = 0$ so that

$$u = \left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right)$$

for some isometry $w : \mathcal{H} \to L^2(M) \otimes \ell^2(\mathbb{N})$.
Moreover the fact that u commutes with $\tilde{M}$ is equivalent to $wx = (x \otimes 1)w$ $\forall x \in M$. □

Corollary 10.1.2. *The commutant of a type* II_1 *factor is either a type* II_1 *factor or a type* II_∞ *factor.*

Proof. We leave the proof as an exercise. □

Proposition 10.1.3. *If* $u : \mathcal{H} \to L^2(M) \otimes \ell^2(\mathbb{N})$ *is an* M*-linear isometry then* $uu^* \in M'$ *on* $L^2(M) \otimes \ell^2(\mathbb{N})$ *and* $tr(uu^*)$ *is independent of* u.

Proof. If v were another M-linear isometry then $uu^* = uv^*vu^*$ so by 9.1.9 $tr(uu^*) = tr((vu^*)(uv^*)) = tr(vv^*)$. □

Observe that if M were replaced by $\mathbb{C}$ in the above construction the number $tr(uu^*)$ would be the dimension of $\mathcal{H}$.

Definition 10.1.4. *For a type* II_1 *factor (or the* $n \times n$ *matrices) and an* M*-module* $\mathcal{H}$*, the number* $tr(u^*u)$ *defined by the two previous results is called* $\dim_M \mathcal{H}$*, or the* coupling constant *or the* M-dimension of $\mathcal{H}$.

Simply by reducing by projections in $(M \otimes 1)'$ one obtains Hilbert spaces whose M-dimension is any number in $[0, \infty]$.

Trivial examples
(i) $\dim_M L^2(M) = 1$.
(ii) $\dim_M(L^2(M) \otimes \ell^2(\mathbb{N})) = \infty$

10.2 Elementary properties of $\dim_M \mathcal{H}$

Theorem 10.2.1. *With notation as above,*

(i) $\dim_M(\mathcal{H}) < \infty$ *iff* M' *is a type* II_1 *factor.*
(ii) $\dim_M(\mathcal{H}) = \dim_M(\mathcal{K})$ *iff* M *on* $\mathcal{H}$ *and* M *on* $\mathcal{K}$ *are unitarily equivalent (= spatially isomorphic).*
(iii) If $\mathcal{H}_i$ *are (countably many)* M*-modules,*

$$\dim_M(\oplus_i \mathcal{H}_i) = \sum_i \dim_M \mathcal{H}_i.$$

(iv) $\dim_M(L^2(M)q) = tr(q)$ *for any projection* $q \in M$.
(v) If p *is a projection in* M, $\dim_{pMp}(p\mathcal{H}) = tr_M(p)^{-1} \dim_M(\mathcal{H})$.

For the next two properties we suppose M' *is finite, hence a type* II_1 *factor with trace* $tr_{M'}$.

(vi) If p *is a projection in* M', $\dim_{Mp}(p\mathcal{H}) = tr_{M'}(p) \dim_M \mathcal{H}$.
(vii) $(\dim_M \mathcal{H})(\dim_{M'} \mathcal{H}) = 1$.

Proof. Using an M-linear isometry u we see that M on $\mathcal{H}$ is unitarily equivalent to M on $uu^* L^2(M) \otimes \ell^2(\mathbb{N})$. This makes (i) and (ii) obvious.

To see (iii), choose M-linear isometries u_i from $\mathcal{H}_i$ to $L^2(M) \otimes \ell^2(\mathbb{N})$ and compose them with isometries so that their ranges are all orthogonal. Adding we get an M-linear isometry u with $uu^* = \sum u_i u_i^*$. Taking the trace we are done.

For (iv), choose a unit vector $\xi \in \ell^2(\mathbb{N})$ and define $u(v) = v \otimes \xi$. Then uu^* is $JqJ \otimes e$ where e is a rank one projection.

(v) Let us first prove the relation in the case $\mathcal{H} = L^2(M)q$ where q is a projection in M with $q \leq p$.

Then $pxp\Omega \mapsto p(x\Omega)p$ is a unitary from $L^2(pMp)$ to $pL^2(M)p$ which intertwines the left an right actions of pMp. Hence pMp on $pL^2(M)q$ is unitarily equivalent to pMp on $L^2(pMp)q$. So by (iv), $\dim_{pMp}(p\mathcal{H}) = tr_{pMp}(q) = tr_M(p)^{-1} tr_M(q) = tr_M(p)^{-1} \dim_M \mathcal{H}$.

Now if $\mathcal{H}$ is arbitrary, it is of the form $e(L^2(M) \otimes \ell^2(\mathbb{N}))$ for $e \in (M \otimes 1)'/$ But e is the orthogonal sum of projections all equivalent to ones as in (iv) with $q \leq p$.

(vi) We may suppose $\mathcal{H} = e(L^2(M) \otimes \ell^2(\mathbb{N}))$ so $M' = e(JMJ \otimes \mathcal{B}(\ell^2(\mathbb{N}))e$ and p defines the isometry in the definition of $\dim_M(p\mathcal{H})$. But p is a projection less than e in a $\mathrm{II}\infty$ factor so by uniqueness of the trace, $\dim_M(p\mathcal{H}) = tr_{(M \otimes 1)'}(p) = tr_{(M \otimes 1)'}(p)/tr_{(M \otimes 1)'}(e) \dim_M(\mathcal{H}) = tr_{M'}(p) \dim_M(\mathcal{H})$.

(vii) Observe that, on $L^2(M)$, $\dim_M(\mathcal{H})\dim_{M'}(\mathcal{H}) = 1$ so by (v) and (vi) the result is true for M-modules of the form $L^2(M)p$. Also if one forms $\mathcal{K} = \oplus_{i=1}^k \mathcal{H}$ then $\dim_{M\otimes 1}(\mathcal{K}) = k\dim\mathcal{H}$ and $\dim_{(M\otimes 1)'}\mathcal{K} = k^{-1}\dim_{M'}$ by (v). But any $\mathcal{H}$ can be obtained from $L^2(M)$ as $\oplus_{i=1}^k L^2(M)p$ for suitable k and p. □

Example 10.2.2. If $\Gamma_0 < \Gamma$ are icc groups, $vN(\Gamma_0)$ acts on $\ell^2(\Gamma)$. And if $\gamma \in \Gamma$ the unitary $\rho(\gamma)$ of the right regular representation gives a $vN(\Gamma_0)$-linear unitary between $\ell^2(\Gamma_0)$ and $\ell^2(\Gamma_0\gamma^{-1})$. Hence by the coset decomposition, $\dim_{vN(\Gamma_0)}(\ell^2(\Gamma)) = [\Gamma : \Gamma_0]$.

This inspires the following definition: If $N \subseteq M$ are II_1 factors, the *index* $[M : N]$ of N in M is the real number $\dim_N L^2(M)$.

Exercise 10.2.3. *Show that* $[M : N] = 1$ *implies* $N = M$.

Example 10.2.4. (Due to Atiyah and Schmidt.)

Discrete series representations of locally compact groups.

Reduction by a finite projection in the commutant of a type II_1 factor occurs in the representation theory of locally compact groups. If a discrete series representation is restricted to an icc lattice it generates a type II_1 factor. The coupling constant is given by the ratio of the "formal dimension" and the covolume of the lattice.

We illustrate in the case of $PSL(2,\mathbb{R})$ which is the group of transformations of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$, $z \mapsto \dfrac{az+b}{cz+d}$ defined by invertible real 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. It is well known that there is a fundamental domain D for the action of the subgroup $\Gamma = PSL(2,\mathbb{Z})$ illustrated below:

DO FIGURE

The set D and all its translates under $PSL(2,\mathbb{Z})$ cover $\mathbb{H}$ and are disjoint apart from boundaries which are of Lebesgue measure 0. Thus if μ is an invariant measure equivalent to Lebesgue measure, $L^2(\mathbb{H}, d\mu)$ gives a unitary representation of Γ which is unitarily equivalent to the left regular representation tensored with the identity on $L^2(D, d\mu)$, making $L^2(\mathbb{H}, d\mu)$ into a $vN(\Gamma)$-module whose $vN(\Gamma)$ dimension is infinite.

The measure $\dfrac{dxdy}{y^2}$ is Γ-invariant but we want to vary this procedure slightly. For each $n \in \mathbb{N}$ consider $\dfrac{dxdy}{y^{2-n}}$. This measure is not invariant but we can make the action of $PSL(2,\mathbb{R})$ unitary on $L^2(\mathbb{H}, \dfrac{dxdy}{y^{2-n}})$ by the formula

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) f(z) = \frac{1}{(cz+d)^n} f\left(\frac{az+b}{cz+d}\right)$$

(with perhaps an inverse matrix...—exercise as usual). This changes nothing as far as how the representation looks to $PSL(2,\mathbb{Z})$ so we obtain (unitarily equivalent) $vN(\Gamma)$-modules $\mathcal{H}_n = L^2(\mathbb{H}, \frac{dxdy}{y^{2-n}})$ for each n.

The commutant of $vN(\Gamma)$ on $\mathcal{H}_n$ is a II_∞ factor. But as is well known, holomorphic functions form a closed subspace of L^2 functions which is manifestly invariant under $PSL_2(\mathbb{R})$. The ensuing unitary representation is known to be irreducible and in the *discrete series* of $PSL_2(\mathbb{R})$. It can be shown to be a finite projection in Γ'. Thus we have a concrete example of a $vN(\Gamma)$-module with finite $vN(\Gamma)$-dimension or coupling constant.

In general, if G is a locally compact group with Haar measure dg, the discrete series representations are precisely those irreducible unitary representations π that are direct summands of the left regular representation on $L^2(G, dg)$. So if Γ is a discrete subgroup with a fundamental domain D so that G is covered by the $\gamma(D)$ which are disjoint up to measure zero sets, we may apply the same analysis as above to obtain a $vN(\Gamma)$ module. The obvious question is to calculate its coupling constant. This turns out to be quite simple because of a key property of discrete series representations.

See [ref robert] for the proof that if $\mathcal{H}$ is a Hilbert space affording a discrete series representation π of G, then the functions $g \mapsto \langle \pi_g \xi, \eta \rangle$, the so-called *coefficients* of π are in $L^2(G, dg)$. We may then imitate the usual procedure for finite or compact groups embedding $\mathcal{H}$ in $L^2(G, dg)$. And the usual Schur orthogonality of the coefficients of a representation yields a number d_π such that

$$d_\pi \int_G \langle \pi_g \xi, \eta \rangle \langle \eta', \pi_g \xi' \rangle dg = \langle \xi, \xi' \rangle \langle \eta', \eta \rangle.$$

If G is compact and Haar measure is normalized so that G has measure 1, d_π is the dimension of the vector space $\mathcal{H}$. In general d_π depends on the choice of Haar measure but obviously the product of d_π with the covolume $\int_D dg$ does not. The coefficients give an explicit embedding of $\mathcal{H}$ in $L^2(G, dg)$ and a straightforward calculation of the trace of the projection onto the image of $\mathcal{H}$ in $vN(\Gamma)'$ yields immediately the formula

$$\dim_{vN(\Gamma)}(\mathcal{H}) = d_\pi \, covolume(\Gamma).$$

The detailed calculation from this point of view can be found in [1] pp. 142-148.

Example 10.2.5. Consider the subfactor $N \subseteq N \otimes M_n(\mathbb{C})$ (diagonal embedding). Each matrix unit e_{ij} gives an N-linear isometry of $L^2(N)$ onto a subspace of $L^2(N \otimes M_n(\mathbb{C}))$ which is an orthogonal direct sum of these equivalent subspaces. Hence $[N \otimes M_n(\mathbb{C}) : N] = n^2$.

Proposition 10.2.6. *If M is a type II_1 factor on $\mathcal{H}$ then*

(a) M has a separating vector if $\dim_M(\mathcal{H}) \geq 1$.

(b) M has a cyclic vector if $\dim_M(\mathcal{H}) \leq 1$.

Proof. Both assertions follow immediately by comparing $\mathcal{H}$ to $L^2(M)p$ or a direct sum of copies of it. □

In fact both conditions in the last proposition are iff. For that one needs to control arbitrary vectors in $L^2(M)$. In fact the original definition of the coupling constant by Murray and von Neumann was as follows. Let M on $\mathcal{H}$ be a type II_1 factor whose commutant is a type II_1 factor. Choose any nonzero vector $\xi \in \mathcal{H}$ and let p and q be projections onto the closures of $M\xi$ and $M'\xi$ respectively. Then $p \in M'$ and $q \in M$ and using the normalised traces the coupling constant was defined as the ratio $\dfrac{tr_M(q)}{tr'_M(p)}$, the hard part being to show that this ratio is independent of ξ. Assuming this last statement it is trivial to identify the Murray-von Neumann coupling constant with our $\dim_M(\mathcal{H})$ but at this stage we have nothing to offer in the way of a simplified proof of why this number does not depend on ξ.

Example 10.2.7. *(due to M. Rieffel)* If (X, μ) is a measure space and Γ is a countable group acting by measure preserving transformations on (X, μ) so that Γ acts by unitaries u_γ on $L^2(X, \mu)$ in the obvious way. We say that a measurable subset $D \subseteq X$ is a *fundamental domain* for Γ if $X = \cup_\gamma \gamma(D)$ and $\mu(D\gamma(D)) = 0$ for all $\gamma \in \Gamma$, $\gamma \neq id$. (One may clearly suppose the $\gamma(D)$ are disjoint by removing a set of measure zero.) In this situation the abelian von Neumann algebra $L^\infty(X)^\Gamma$ of Γ-invariant L^∞ functions may be identified with the space $L^\infty(D)$.

Now suppose Γ and Λ are two groups acting on X as above with fundamental domains D and E respectively. We may consider the von Neumann algebra $M_{\Gamma,\Lambda}$ on $L^2(X, \mu)$ defined as $\{\{u_\gamma : \gamma \in \Gamma\} \cup L^\infty(X)^\Lambda\}''$.

Chapter 11

The Crossed Product construction.

Perhaps the most useful way of producing von Neumann algebras from others is the crossed product. We begin by defining a very general notion about which there is not a lot to say, but then examine it carefully in special cases.

11.1 Group actions.

Let M be a von Neumann algebra and G a group. An *action* of G on M is a homomorphism $g \mapsto \alpha_g$ from G to the automorphism group $AutM$ of M (where automorphisms may be assumed ultraweakly continuous if necessary). The algebra of fixed points for the action is denoted M^G and is a von Neumann algebra. A special case of some importance is when the action is a unitary group representation $g \mapsto u_g$ with $u_g M u_g^* = M \ \forall g \in G$. In that case setting $\alpha_g(x) = u_g x u_g^*$ defines an action of G on M (and M'). We say that the action α is *implemented* by the unitary representation u_g. If the u_g are actually in M, we say that the action is *inner* as an inner automorphism of M is by definition one of the form $\mathrm{Ad}\, u(x) = uxu^*$ for u a unitary in M. An automorphism is called *outer* if it is not inner.

Actions are not always implementable though the notion depends on the Hilbert space on which M acts.

Exercise 11.1.1. *If (X, μ) is a measure space and T is a bijection of X which preserves the measure class of μ (i.e. $\mu(A) = 0 \Leftrightarrow \mu(T^{-1}(A)) = 0$.) show how T defines an automorphism α_T of $L^\infty(X, \mu)$. Show further that this automorphism is implemented by a unitary u on $L^2(X, \mu)$.*

A bijection T as above is called *ergodic* if $T(A) = A$ for a measurable subset $A \subseteq X$ implies either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Proposition 11.1.2. *With notation as above T is ergodic iff the only fixed points for α_T are constant functions.*

Proof. ($\Rightarrow$) Let $f \in L^\infty$ and $\alpha_T(f) = f$. After throwing away a null set we may assume that $f(x) = f(T(x))$ for all $x \in X$. Then for every $\epsilon > 0$, by the definition of the essential supremum, $\mu(\{x : ||f|| - |f(x)| < \epsilon\} \neq 0$. But this set is invariant under T so it is equal to X up to a set of measure 0. Letting ϵ tend to 0 we see that $\mu(\{x : |f(x)| \neq ||f||\}) = 0$. So we may assume $f(x) = e^{ig(x)}$ for some measurable g taking values in $[0, 2\pi)$. Repeating the argument for g gives f constant almost everywhere.

($\Leftarrow$) If A is a measurable invariant set then its characteristic function is fixed by α in L^∞ iff A is invariant. □

Exercise 11.1.3. *Let $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be the Pauli spin matrices. Show that Adu_x, Adu_y and Adu_z define an action of the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ on the two by two matrices which is not implementable for $M_2(\mathbb{C})$ on $\mathbb{C}^2$.*

Exercise 11.1.4. *Show that* any *group action is implementable for a type* II_1 *factor in standard form and more generally any automorphism group preserving a faithful normal state is implementable in the GNS representation.*

Exercise 11.1.5. *Show that every automorphism of $\mathcal{B}(\mathcal{H})$ is inner.*

Exercise 11.1.6. *Show that the automorphism of $vN(F_2)$ coming from the group automorphism which exchanges the 2 generators is outer.*

If G is a topological group there are many possible notions of continuity. The most useful is that of pointwise *-strong convergence, i.e. we assume that the map $g \mapsto \alpha(g)(x)$ is *-strong continuous for any $x \in M$. Typically many other notions of continuity will be equivalent to that and even a measurability assumption can be enough to ensure this continuity.

We will always assume pointwise *-strong continuity when referring to an action of a topological group.

Exercise 11.1.7. *Is the action by translation of $\mathbb{R}$ on $L^\infty(\mathbb{R})$ pointwise norm continuous? pointwise strongly continuous? pointwise *-strong continuous?*

Actions of a given group on von Neumann algebras are easy to construct but actions of a group on a given von Neumann algebra may be hard to come by.

Definition 11.1.8. *An action of G on M is said to be* ergodic *if $M^G = \mathbb{C}id$.*

Exercise 11.1.9. *Show that if G acts preserving μ on (X, μ) then the resulting action of G on $L^\infty(X, \mu)$ is ergodic iff the only measurable subsets $A \subseteq X$ which satisfy $\mu(g(A)\Delta A) = 0 \, \forall g \in G$ satisfy either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.*

(Here $A\Delta B$ means $A \setminus B \cup B \setminus A$.)

The following question is an intriguing open problem:

Does $SU(3)$ have any ergodic action on a type II_1 factor?

It is shown in [] that $SU(2)$ has no such action and it is shown in [] that if a compact group acts ergodically on a von Neumann algebra then that von Neumann algebra has a faithful normal trace.

11.2 The crossed product

Suppose α is an action of the locally compact group G with Haar measure dg on the von Neumann algebra M with Hilbert space $\mathcal{H}$. Form the Hilbert space $\mathcal{K} = L^2(G, \mathcal{H}) = L^2(G) \otimes \mathcal{H}$ and let G act on $\mathcal{K}$ by $u_g = \lambda_g \otimes 1$, λ being the left regular representation. Further, let M act on $\mathcal{K}$ by

$$(\tilde{x}f)(g) = \alpha_{g^{-1}}(f(g))$$

.

Exercise 11.2.1. *Show that $x \mapsto \tilde{x}$ is an ultraweakly continuous *-isomorphism of M onto a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$.*

Exercise 11.2.2. *Show that $u_g \tilde{x} u_g^* = \widetilde{\alpha_g(x)}$.*

Note that this gives another way of making a group action implementable, at least when it is locally compact.

Definition 11.2.3. *If M, $\mathcal{H}$, G and α are as above, the* crossed product *$M \rtimes_\alpha G$ is the von Neumann algebra on $\mathcal{K}$ generated by $\{u_g : g \in G\}$ and $\{\tilde{x} : x \in M\}$.*

>From now on we will drop the $\tilde{}$ and identify M with $\tilde{M}$. Note that finite linear combinations $\sum_g x_g u_g$ form a dense *-subalgebra of $M \rtimes_\alpha G$. Moreover the u_g are linearly independent over M in the sense that $\sum_g x_g u_g = 0 \Rightarrow x_g = 0$ for each g in the sum. This dense subalgebra could be called the *algebraic* crossed product.

There is a well-developed theory of $M \rtimes_\alpha G$ when G is compact or abelian, but we shall be mostly interested in the case where G is discrete as then we

may replay the matrix element game that we played for $vN(\Gamma)$ to gain control of weak limits of elements in the algebraic crossed product. (In fact of course $vN(\Gamma)$ is the special case of the crossed product when $M = \mathbb{C}$ and the action is trivial.) Indeed we see immediately as in 3.3.4 that if G is discrete, any element of $M \rtimes_\alpha G$ defines a function $g \mapsto x_g$ so that the sum $\sum_g x_g u_g$ stands for a certain matrix of operators on $\mathcal{K} = \mathcal{H} \otimes \ell^2(G)$. Moreover any matrix of this form which defines a bounded operator on $\mathcal{K}$ is in $M \rtimes_\alpha G$. This is because the sum converges pointwise at least on the dense set of functions of finite support from G to $\mathcal{H}$. In the case where the crossed product is a II_1 factor we know that the commutant consists of right multiplication by elements of $M \rtimes_\alpha G$ so a weakly dense subalgebra of $(M \rtimes_\alpha G)'$ preserves this dense subspace of vectors and on that subspace $\sum_g x_g u_g$ and right multiplication by u_g and $x \in M$ commute. We will return to the general case later on.

Moreover the formulae

$$(\sum x_g u_g)^* = \sum \alpha_g(x_{g^{-1}}) u_g$$

and

$$(\sum x_g u_g)(\sum y_g u_g) = \sum_g \{\sum_h x_h \alpha_h(y_{h^{-1}g})\} u_g$$

are justified by matrix multiplication.

We shall now provide some sufficient conditions for $M \rtimes_\alpha G$ to be a factor—always assuming G is discrete.

Definition 11.2.4. *An action α of G on M is called* outer *if the only g in G for which α_g is inner is the identity.*

Proposition 11.2.5. *If G is a discrete group and α is an outer action of G on the factor M then $M \rtimes_\alpha G$ is a factor.*

Proof. If $x = \sum x_g u_g \in Z(M)$ then equating coefficients in the expression that x commutes with M gives us $yx_g = x_g \alpha_g(y)$ $\forall y \in M$,$g \in G$. By the next lemma this implies $x_g = 0$ for any $g \neq 1$. Thus $x \in M$ and since M is a factor we are done. □

Lemma 11.2.6. *Let $\alpha \in Aut\, M$ for a factor M. Suppose there is an $x \in M$, $x \neq 0$, with*

$$yx = x\alpha(y) \qquad \forall\, y \in M.$$

Then α is inner.

Proof. If x were unitary this would be obvious. So take the adjoint of the relation to obtain $x^*y = \alpha(y)x^*$ $\forall y \in M$. Thus $yxx^* = x\alpha(y)x^* = xx^*y$

and $xx^* \in Z(M)$. Similarly $x^*x \in Z(M)$. But xx^* and x^*x always have the same spectrum so since M is a factor both xx^* and x^*x are equal to the same positive number λ. Dividing by $\sqrt{\lambda}$ converts x into a unitary and we are done. □

These two results prompt the following definition.

Definition 11.2.7. *An automorphism α of a von Neumann algebra M is called free if*

$$yx = x\alpha(y) \qquad \forall\, y \in M \;\; \Rightarrow \;\; x = 0.$$

An action α is called free if α_g is free for every $g \neq id$.

The argument of proposition 11.2.5 shows in fact that if α is a free action on a von Neumann algebra M then $Z(M \rtimes_\alpha G) \subseteq M$, in fact that $M' \cap M \rtimes_\alpha G \subseteq M$.

Theorem 11.2.8. *If α is a free ergodic action of G on a von Neumann algebra M, then $M \rtimes_\alpha G$ is a factor.*

Proof. This follows immediately from the preceding remark. □

To understand the meaning of freeness for automorphisms of the form α_T we need to make a hypothesis on (X, μ) as otherwise one could envisage a T which is non-trivial on X but for which α_T is the identity. So we will suppose from now on that (X, μ) is countably separated. This means there is a sequence B_n of measurable sets with $\mu(B_n) > 0$ for which, if $x \neq y$, there is an n with $x \in B_n$ but $y \notin B_n$. Obviously $\mathbb{R}^n$ is countably separated.

Exercise 11.2.9. *Show that $\alpha_T = id$ means that $Tx = x$ almost everywhere.*

Hint-look at the proof of the next result.

Proposition 11.2.10. *If T is a transformation of (X, μ) then α_T is free iff $\mu(\{x : T(x) = x\}) = 0$.*

Proof. ($\Rightarrow$)If A is any measurable set on which $T = id$ then $\chi_A f = \alpha_T(f)\chi_A$ for all $f \in L^\infty$.

($\Leftarrow$) First throw away any fixed points of T. Then suppose $f_1\alpha_T(f_2) = f_2 f_1 \quad \forall\, f_2 \in L^\infty$. Let A be the support of f_1. Then since T has no fixed points, $A = \cup_n (A \cap B_n \setminus T^{-1}(B_n))$. If f_1 were non-zero in L^∞, we could thus choose an n for which $\mu(A \cap B_n \setminus T^{-1}(B_n)) > 0$. Set $f_2 = \chi_{B_n}$. Then for any $x \in A \cap B_n \setminus T^{-1}(B_n)$ we have $f_1(x)f_2(x) \neq 0$ but $f_1(x)f_2(Tx) = f_1(x)\chi_{B_n}(Tx) = 0$ since $x \notin T^{-1}(B_n)$. Thus $f_1\alpha_T(f_2) \neq f_2 f_1$ in L^∞. So the measure of A must be zero. □

We conclude that if Γ is a countable group acting freely and ergodically on a measure space (X, μ), preserving the class of μ, then the crossed product $L^\infty(X, \mu) \rtimes \Gamma$ is a factor.

Note that if Γ is abelian, ergodic implies free.

Exercise 11.2.11. *Show that freeness of the action actually proves that $L^\infty(X, \mu)$ is maximal abelian in the crossed product.*

The crossed product $M \rtimes \Gamma$ when M is abelian and Γ is discrete is called the *group measure space construction.* Here are several examples.

Example 11.2.12. $X = \mathbb{Z}$, $\Gamma = \mathbb{Z}$ acting by translation, $\mu =$ counting measure.

The action is free and ergodic and $L^\infty(X, \mu) \rtimes \Gamma = \mathcal{B}(\ell^2(\mathbb{Z}))$.

Example 11.2.13. The irrational rotation algebra-von Neumann algebra version.

$(X, \mu) = (\mathbb{T}^1, d\theta)$, $\Gamma = \mathbb{Z}$ generated by the transformation T where $T(z) = e^{i\alpha} z$ and $\alpha/2\pi$ is irrational.

Exercise 11.2.14. *Use Fourier series to show that this T is ergodic.*

Example 11.2.15. Let H be a finite abelian group and $\Gamma = \bigoplus_{n\in\mathbb{N}} H$ be the countable group of sequences (h_n) with h_n eventually the identity. Put $X = G = \prod_{n\in\mathbb{N}} H$ (the set of *all* sequences) with the product topology. Then G is a compact group so has a Haar measure μ. Γ acts on X by left translation. The action is clearly free and ergodic as we shall now argue.

There is a particularly von Neumann algebraic way to view this example without even constructing the space (X, μ) !

Let $A = L^\infty(H) = \mathbb{C}\hat{H}$ be the group algebra of the dual group $\hat{H}$, with its usual trace. As in section 6.2, form the algebraic tensor product $\otimes_{alg, n\in\mathbb{N}} A$ with product trace tr. Then perform the GNS construction with respect to tr to obtain an abelian von Neumann algebra. It may be identified with $L^\infty(G, \mu)$ so the Hilbert space $\mathcal{H}$ of the GNS construction is $L^2(X, \mu)$. But it is clear that an orthornormal basis of $\mathcal{H}$ is given by finite sequences (χ_n) of elements of $\hat{H}$ which define elements $\chi_1 \otimes \chi_2 \otimes \cdots \otimes 1 \otimes 1 \otimes 1 \cdots$ in $\otimes_{alg, n\in\mathbb{N}} A$. The point is that these basis vectors are eigenvectors for the action of Γ on $L^2(X, \mu)$:

$$(h_n)(\chi_1 \otimes \chi_2 \otimes \cdots \otimes 1 \cdots) = (\prod_n \chi_n(h_n))\ \chi_1 \otimes \chi_2 \otimes \cdots \otimes 1 \cdots .$$

Ergodicity follows easily since the only basis element which is fixed by all the (h_n) is the one with all χ_n equal to 1.

Exercise 11.2.16. *Show that if $H = \mathbb{Z}/2\mathbb{Z}$ in this example then the subalgebra of the crossed product generated by $\otimes_{alg,n\in\mathbb{N}}A$ and Γ is the algebraic infinite tensor product of $M_2(\mathbb{C})$.*

Both of the last two examples are special cases of a more general one: X is a compact group with its Haar measure and Γ is a countable dense subgroup acting (freely) by left translation. The Peter Weyl theorem shows that this action is ergodic.

Example 11.2.17. Bernoulli shift.

If Γ is any infinite group and $A = \mathbb{Z}/2\mathbb{Z}$ we may form the tensor product indexed by Γ of a copy of A for each $\gamma \in \Gamma$. The von Neumann algebra thus obtained is once again the L^∞ space of the infinite product measure space, this time with the set indexing the product being Γ. As in the previous example we can obtain a basis of L^2 indexed by functions from Γ to the set $\{0, 1\}$ which are almost always 0. These functions are the same as finite subsets of Γ and the action of Γ on the Hilbert space is by permuting the basis in the obvious way. Ergodicity follows from the fact that the orbit of any non-empty subset is infinite.

One could also chose another trace than the usual one and modify the orthonormal basis of A accordingly. The measures are the obvious ones unless specified.

We give a few more examples of free ergodic actions without supplying proofs of ergodicity.

Example 11.2.18. $SL(2, \mathbb{Z})$ acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ via the linear action on $\mathbb{R}^2$.

Example 11.2.19. $PSL(2, \mathbb{Z})$ acts on $\mathbb{R} \cup \{\infty\}$ by linear fractional transformations.

Example 11.2.20. $SL(2, \mathbb{Z})$ acts on $\mathbb{R}^2$ by linear transformations.

Example 11.2.21. $\mathbb{Q}$ acts on $\mathbb{R}$ by translation.

There are two fairly easy ways to see that this action is ergodic. The first is to reduce it to a dense subgroup of a compact group by observing that an L^∞ function on $\mathbb{R}$ which is invariant under translation by $\mathbb{Z}$ defines an L^∞ function on the quotient $\mathbb{T}$. Then use Fourier series.

The second way is a direct attack which should generalise to show that translation by any countable dense subgroup of a locally compact group is ergodic. If $f \in L^\infty(\mathbb{R})$ is invariant under $\mathbb{Q}$, set things up so that there are sets A and B both of nonzero measure, so that $g(A) \cap g(B) = \emptyset$. Cover A bullshit

and B with intervals of the same width with rational endpoints. Some of these must intersect A and B in non-nul sets. But all these intervals are all translates of each other so g cannot be invariant up to sets of measure zero.

Example 11.2.22. The "$ax + b$" group $\mathbb{Q} \rtimes \mathbb{Q}^*$ acts on $\mathbb{R}$

Example 11.2.23. Same as example 11.2.13 with $H = \mathbb{Z}/2\mathbb{Z}$ but using a normalised trace on $\mathbb{C}H$ which is different from the usual one. Such a trace is specified by its values on the minimal projections of $\mathbb{C}H$ which we could call p and $1-p$ for $0 < p < 1$. The product measure is not absolutely continous with respect to Haar measure, and it is not preserved by group translation so this example is perhaps most easily approached by the von Neumann algebra construction where one can *implement* the action of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ by unitaries. These unitaries come from ones on $L^2(H)$ which exchange two points of unequal weight so they must be correctly scaled.

Exercise 11.2.24. *Work out the details of example 11.2.23*

In the examples we see four different kinds of free ergodic actions:
Type I : Γ acts transitively.11.2.12
Type II_1 : Γ preserves a finite measure. 11.2.13,11.2.15,11.2.17,11.2.18
Type II_∞ : Γ preserves an infinite measure.11.2.20,11.2.21
Type III : Γ preserves no measure equivalent to μ.11.2.19,11.2.22,11.2.23

11.3 The type of the crossed product.

We adopt the notations and conventions of the previous section. The map $E_m : M \rtimes_\alpha \Gamma \to M$ which assigns a_{id} to the element $\sum_{\gamma\in\Gamma}$ is destined to play a big role in the theory. It is called the *conditional expectation* onto M and obviously satisfies the following contitions:

(i) $E_M^2 = E_M$.
(ii) $E_M(x)^* = E_M(x^*)$, $E_M(1) = 1$, $E_M(x^*x) = 0 if f x = 0$.
(iii) $E_M(x^*x) \geq E_M(x^*)E_M(x)$, $||E(x)|| \leq ||x||$.
(iv) $E_M(axb) = aE_M(x)b$ for $a, b \in M$.
(v) E_M is ultraweakly continuous.

So E_M is a projection of norm one in the Banach space sense. The condition (iv) says that E_M is an $M - M$-bimodule map.

Theorem 11.3.1. *If Γ acts non-transitively, freely and ergodically, preserving the finite measure μ then $L^\infty(X,\mu) \rtimes \Gamma$ is a II_1 factor. If Γ preserves the infinite σ-finite measure μ then $L^\infty(X,\mu) \rtimes \Gamma$ is a II_∞ factor unless Γ acts transitively in which case $L^\infty(X,\mu) \rtimes \Gamma$ is type I.*

Proof. (i) It is clearer to prove a more general statement (in the case where Γ preserves μ and $\mu(X) = 1$). So suppose Γ preserves the faithful positive ultraweakly continuous trace tr on the von Neumann algebra A and that its action is free and ergodic. Then we claim $M = A \rtimes \Gamma$ is a type II_1 factor (or a finite dimensional factor). By previous results we need only show that it has an ultraweakly continous positive trace. So define $Tr = tr \circ E_A$ on M. Ultraweak continuity and positivity are obvious so by continuity and linearity it suffices to prove $Tr(au_\gamma bu_\eta) = Tr(bu_\eta au_\gamma)$. For either side of the equation to be non-zero means $\eta = \gamma^{-1}$ amd then the left hand side is $tr(a\alpha_\gamma(b)) = tr(\alpha_\gamma^{-1}(a\alpha_\gamma(b))) = tr(b\alpha^{-1}(a_)$ which is equal to $Tr(bu_\eta au_\gamma)$.

(ii) If μ is infinite and Γ does not act transitively then there are no atoms hence there are subsets Y of X of arbitrary positive measure. Let Y have finite non-zero measure and let ξ be the function $\xi(\gamma) = \delta_{\gamma,id}\ \chi_Y$. Then $\langle au_\gamma\xi, \xi\rangle = \omega_\xi(au_\gamma) = \delta_{id,\gamma}\int_Y a(x)d\mu(x)$. One easily checks that $\omega_\xi((pau_\gamma p)(pbu_\eta p)) = \omega_\xi((pbu_\eta p)(pau_\gamma p))$ so by 3.4.6 ω_ξ defines a positive ultraweakly continuous trace on $p(A \rtimes \Gamma)p$ which is a type II_1 factor. But $A \rtimes \Gamma$ is not itself a type II_1 factor since A contains an infinite family of equivalent mutually orthogonal projections. By 9.1.8 we are done.

(iii) If Γ acts transitively then $(X, \mu) = (\Gamma, \text{counting measure})$ and the characteristic function of a set with one element is a minimal projection in $L^\infty(X, \mu) \rtimes \Gamma$. □

Exercise 11.3.2. *If Γ acts ergodically on (X, μ) preserving the σ-finite measure μ then any other invariant equivalent measure is proportional to μ.*

We now want to show that there are factors that are neither of type I nor type II. Suppose that $M = L^\infty(X, \mu) \rtimes \Gamma$ is a type I or II factor. Then it has a trace $tr : M_+ \to [0, \infty]$. We would like to define an invariant measure on X, absolutely continous with respect to μ, by reversing the procedure of theorem 11.3.1 and defining the measure $\sigma(A)$ to be $tr(\xi_A)$ ($\xi_A \in L^\infty(X, \mu) \subseteq M$). Invariance of the measure σ is no problem. The snag is that $tr(\chi_A)$ could be infinite for every non-null set A. We will show that this is not the case. To this end the concept of lower semicontinuity will be useful.

Definition 11.3.3. *If X is a topological space we say that $f : X \to \mathbb{R}$ is* lower semicontinous *if for every $x \in X$ and $\epsilon > 0$ there is an open set $U \subseteq X$ such that $f(u) > f(x) - \epsilon$ for all $u \in U$.*

Exercise 11.3.4. *Prove that if f is lower semicontinous then*
(a)$f^{-1}((-\infty, K]))$ is closed for every $K \in \mathbb{R}$.
(b)f attains its minimum on any compact subset of X.

Exercise 11.3.5. *If $\mathcal{H}$ is a Hilbert space and $\xi \in \mathcal{H}$, the function $a \mapsto ||a\xi||$ from $\mathcal{B}(\mathcal{H})$ to $\mathbb{R}$ is* weakly *lower semicontinuous.*

Exercise 11.3.6. *If f_α are lower semicontinous then $\vee_\alpha f_\alpha$ is lower semicontinous if it exists.*

Lemma 11.3.7. *Let M be a type I or II factor and $tr : M^+ \to [0, \infty]$ be Trace in type I, as in 9.1.9 in type II_∞ and the trace in type II_1. Then for each $K \geq 0$, $M_{1,K} = \{x : tr(x^*x) \leq K\}$ is weakly compact.*

Proof. Clear in the II_1 case. In a decomposition $M \cong N \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ on $\mathcal{H}$ with N a type II_1 factor or $\mathbb{C}$ we may assume by 10.2.6 that there is a vector $\xi \in e_{11}\mathcal{H}$ with ω_ξ a trace on $e_{11}Me_{11}$. So if $\xi_i = e_{i1}\xi$ we have, up to a scalar, that

$$tr(x) = \sum_{i=1}^{\infty} \langle x\xi_i, \xi_i \rangle.$$

By the previous exercises and weak compactness of the unit ball, we are done. □

Proposition 11.3.8. *With notation as above, for $x \in M_{1,K}$ let $W(x)$ be the weak closure of the convex set of finite sums $\{\sum_i \lambda_i u_i x u_i^* : \sum_i \lambda_i = 1, \lambda_i > 0, u_i$ unitary in $L^\infty(X, \mu)\}$. Then $W(x) \subseteq M_{1,K}$ and if $\phi(y) = tr(y^*y)$ for $y \in W(x)$ then ϕ attains its minimum at a unique point $\mathcal{E}(x)$ of $W(x)$.*

Proof. Note first that $\{z \in M : tr(z^*z) < \infty\}$ is a vector space on which $||z|| = tr(z^*z)$ defines a prehilbert space structure. (Since $(a+b)^*(a+b) \leq 2(a^*a + b^*b)$ as operators, and the parallelogram identity passes to the potentially infinite sum defining tr.) Moreover $W(x)$ is a weakly compact subset of M so by lower semicontinuity ϕ attains its minimum at a point which is unique by two dimensional Euclidean geometry as in 2.1.2. □

Proposition 11.3.9. *Suppose that $M = L^\infty(X, \mu) \rtimes \Gamma$ is a type I or II factor for a free ergodic action of Γ on $L^\infty(X, \mu)$. Let tr be as above and p be a projection in M with $tr(p) < \infty$. Then*

$$\mathcal{E}(p) = E_{L^\infty(X,\mu)}(p)$$

and $0 < tr(\mathcal{E}(p)^2) \leq tr(p)$.

Proof. Let $E = E_{L^\infty(X,\mu)}$. By the uniqueness of $\mathcal{E}(p)$ it commutes with every unitary in L^∞ so it is in L^∞ by 11.2.11. On the other hand $E(y) = E(p)$ for all $y \in W(p)$ by the bimodule linearity of the conditional expectation and its ultraweak continuity. So $E(\mathcal{E}(p)) = E(p) = \mathcal{E}(p)$. But $\phi(\mathcal{E}(p) \leq \phi(p) = tr(p)\infty$. Finally $E(p) = E(p^2)$ which is a positive non-zero self-adjoint operator and hence has non-zero trace. □

Theorem 11.3.10. *Let Γ act freely and ergodically on the countably separated σ-finite measure space (X, μ) so that there is no σ-finite Γ-invariant measure on X absolutely continuous with respect to μ. Then $L^\infty(X, \mu) \rtimes \Gamma$ is a factor not of type I or II.*

Proof. If the crossed product were of type I or II, define the measure ρ on X by $\rho(A) = tr(\chi_A)$. By the previous result $\rho(A)$ would have to be finite and non-zero for some A since the L^∞ function$f = E(p)^2$ must dominate a multiple of χ_A for some A (e.g. let A be those x with $f(x)$ sufficiently close to $||f||$). But then by ergodicity $X = \cup_{\gamma\in\Gamma}\gamma(A)$ (up to null sets) so that ρ is σ-finite. It is automatically absolutely continuous wrt μ. Invariance of ρ under Γ follows from $tr(u_\gamma x u_\gamma^{-1}) = tr(x)$ for $x \geq 0$. □

Definition 11.3.11. *A factor not of type I or II is called a* type III factor.

Example 11.2.22 provides a type III factor since the subgroup $\mathbb{Q}$ acts ergodically so the only possible invariant measure is a multiple of dx by exercise 11.3.2 and this is not invariant under multiplication!

Note that the above technique works in somewhat greater generality than actions of groups on measure spaces.

Exercise 11.3.12. *Adapt the proofs of the results just obtained to show that $M \rtimes_\alpha \mathbb{Z}$ is a type III factor if the action α is generated by a single automorphism of the II_∞ factor scaling the trace by a factor $\lambda \neq 1$.*

Chapter 12

Unbounded Operators and Spectral Theory

There are many naturally arising examples of unbounded operators, some of the most fundamental being multiplication by x and differentiation, the position and momentum operators of quantum mechanics. Our immediate motivation for studying unbounded operators here is to facilitate the study of arbitrary von Neumann algebras acting on GNS Hilbert spaces. Here we establish the necessary preliminaries on unbounded operators. The material closely follows Reed and Simon [2].

12.1 Unbounded Operators

Definition 12.1.1. *An* operator T *on a Hilbert space* $\mathcal{H}$ *consists of a linear subspace* $D(T)$*, the* domain *of* T*, and a linear map from* $D(T)$ *to* $\mathcal{H}$*.*

Example 12.1.2.

(i) M_x*, multiplication by* x *on* $\mathrm{L}^2(\mathbb{R})$*.*

$$D(M_x) = \left\{ f \in \mathrm{L}^2(\mathbb{R}) : \int_{\mathbb{R}} x^2 |f(x)|^2 \mathrm{d}\, x < \infty \right\}.$$

(ii) $T = \frac{\mathrm{d}}{\mathrm{d}\, x}$ *on* $\mathrm{L}^2([0,1])$*.* $D(T) = C^1[0,1]$*.*

In order to do some analysis we want to restrict our attention a little so as not to consider completely arbitrary linear maps.

Definition 12.1.3. *Let* T *be an operator on* $\mathcal{H}$*. The* graph *of* T *is*

$$\Gamma(T) = \{(\xi, T\xi) : \xi \in D(T)\} \subset \mathcal{H} \oplus \mathcal{H}.$$

T *is* closed *if* $\Gamma(T)$ *is closed in* $\mathcal{H} \oplus \mathcal{H}$*.*

Remark 12.1.4. *Note that if T is closed and $D(T) = \mathcal{H}$ then T is bounded by the Closed Graph Theorem.*

Lemma 12.1.5. *A linear subspace $\Gamma \subset \mathcal{H} \oplus \mathcal{H}$ is the graph of an operator iff $(0, \eta) \in \Gamma$ implies $\eta = 0$.*

Proof. Straightforward. □

Many operators are not closed, but can be extended to a closed operator.

Definition 12.1.6. *Let S, T be operators on $\mathcal{H}$. T is an* extension of S, *denoted $S \subset T$, if $\Gamma(S) \subset \Gamma(T)$. Equivalently $D(S) \subset D(T)$ and $T|_{D(S)} = S$.*

Definition 12.1.7. *An operator T is* preclosed *(or* closable*) if it has a closed extension.*

Lemma 12.1.8. *Suppose T is preclosed. Then T has a smallest closed extension $\overline{T}$. $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.*

Proof. Take a closed extension A of T. $\Gamma(A)$ is closed and contains $\Gamma(T)$ so $\overline{\Gamma(T)} \subset \Gamma(A)$. $\overline{\Gamma(T)}$ is the graph of an operator (call it $\overline{T}$) because:

$$(0, \eta) \in \overline{\Gamma(T)} \subset \Gamma(A) \Rightarrow \eta = A(0) = 0.$$

$\overline{T}$ is the smallest closed extension because for all closed extensions A, $\Gamma(\overline{T}) = \overline{\Gamma(T)} \subset \Gamma(A)$. □

Definition 12.1.9. *$\overline{T}$ is called the* closure *of T.*

Remark 12.1.10. *We thus obtain two equivalent definitions of a preclosed operator:*

(i) $(0, \eta) \in \overline{\Gamma(T)} \Rightarrow \eta = 0$.

(ii) *($\xi_n \in D(T)$, $\xi_n \to 0$ and $T\xi_n$ converges)* $\Rightarrow T\xi_n \to 0$.

Exercise 12.1.11.

(i) Define S on $\mathrm{L}^2(\mathbb{R})$ by $D(S) = C_0^\infty(\mathbb{R})$ (infinitely differentiable functions with compact support), $Sf = f'$. Show that S is preclosed.

(ii) Define T from $\mathrm{L}^2(\mathbb{R})$ to $\mathbb{C}$ by $D(T) = \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$, $T(f) = \int_{\mathbb{R}} f$. Show that T is not preclosed.

Definition 12.1.12. *Suppose T is a closed operator. A* core *for T is a linear subspace $D_0 \subset D(T)$ such that $\overline{T|_{D_0}} = T$.*

We can perform basic arithmetic operations with (unbounded) operators as follows: $S+T$ is the operator with domain $D(S+T) = D(S) \cap D(T)$ and $(S+T)\xi = S\xi + T\xi$. ST is the operator with domain $D(ST) = \{\xi \in D(T) : T\xi \in D(S)\}$ and $(ST)\xi = S(T\xi)$. Of particular importance is the adjoint.

Definition 12.1.13. *Let T be a densely defined operator on $\mathcal{H}$. Let*

$$\begin{aligned} D(T^*) &= \{\eta \in \mathcal{H} : \exists \sigma \in \mathcal{H} \text{ such that } \langle T\xi, \eta\rangle = \langle \xi, \sigma\rangle \forall \xi \in D(T)\} \\ &= \{\eta \in \mathcal{H} : \exists C > 0 \text{ such that } |\langle T\xi, \eta\rangle| \leq C||\xi|| \ \forall \xi \in D(T)\}. \end{aligned}$$

For $\xi \in D(T^*)$ note that the element σ is unique (by the density of $D(T)$) and define $T^*\xi = \eta$.

Remark 12.1.14. *Note that if $S \subset T$ then $T^* \subset S^*$.*

Exercise 12.1.15. *Give an example to show that the domain of the adjoint need not be dense. [In fact it can be $\{0\}$].*

Proposition 12.1.16. *Let T be a densely defined operator. Then*

1. *T^* is closed.*
2. *$D(T^*)$ is dense iff T is preclosed. In that case $\overline{T} = T^{**}$.*
3. *If T is preclosed then $(\overline{T})^* = T^*$.*

Proof. Note that $(\eta, \sigma) \in \Gamma(T^*)$ iff $< T\xi, \eta >=< \xi, \sigma >$ for all $\xi \in D(T)$ iff $< (-T\xi, \xi), (\eta, \sigma) >= 0$. Hence

$$\Gamma(T^*) = \{(-T\xi, \xi) : \xi \in D(T)\}^\perp = (u\Gamma(T))^\perp = u\Gamma(T)^\perp,$$

where $u : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is the unitary operator $u(\xi, \eta) = (-\eta, \xi)$. Now:

1. Orthogonal complements are closed, hence $\Gamma(T^*)$ is closed.
2. $\overline{\Gamma(T)} = (\Gamma(T)^\perp)^\perp = u^*\Gamma(T^*)^\perp$, so

$$\begin{aligned} (0, \xi) \in \overline{\Gamma(T)} &\Leftrightarrow (-\xi, 0) \in \Gamma(T^*)^\perp \\ &\Leftrightarrow 0 =< (-\xi, 0), (\eta, T^*\eta) >= - < \xi, \eta > \text{ for all } \eta \in D(T^*) \\ &\Leftrightarrow \xi \in D(T^*)^\perp. \end{aligned}$$

   Hence T is preclosed iff $D(T^*)^\perp = \{0\}$ iff $D(T^*)$ is dense.

   In that case $\Gamma(T^{**}) = u\Gamma(T^*)^\perp = u^2\Gamma(T)^{\perp\perp} = -\overline{\Gamma(T)} = \Gamma(\overline{T})$, so $T^{**} = \overline{T}$.

3. $T^* = \overline{T^*} = T^{***} = (\overline{T})^*$.

Definition 12.1.17. *An operator T is* symmetric *if $T \subset T^*$. Equivalently $< T\xi, \eta >=< \xi, T\eta >$ for all $\xi, \eta \in D(T)$. T is* self-adjoint *if $T = T^*$. A self-adjoint operator T is* positive *if $< T\xi, \xi >\geq 0$ for all $\xi \in D(T)$.*

12.2 Spectral Theory for Unbounded Operators

Definition 12.2.1. *Let T be a closed operator on $\mathcal{H}$. The* resolvent *of T is*

$$\rho(T) = \{\lambda | \lambda 1 - T : D(T) \to \mathcal{H} \text{ is a bijection}\}.$$

The spectrum *of T is $\sigma(T) = \mathbb{C} \backslash \rho(T)$.*

Remark 12.2.2. *Note that if $\lambda 1 - T : D(T) \to \mathcal{H}$ is a bijection then $(\lambda 1 - T)^{-1}$ is bounded by the Closed Graph Theorem.*

Exercise 12.2.3. *The spectrum is highly dependent on the domain. Let $AC[0,1]$ denote the set of absolutely continuous functions on $[0,1]$. Let $T_1 = \frac{\mathrm{d}}{\mathrm{d}x}$, $T_2 = \frac{\mathrm{d}}{\mathrm{d}x}$, with*

$$\begin{aligned} D(T_1) &= \{f \in AC[0,1] : f' \in \mathrm{L}^2([0,1])\} \\ D(T_2) &= \{f \in AC[0,1] : f' \in \mathrm{L}^2([0,1]), f(0) = 0\}. \end{aligned}$$

Show that T_1 and T_2 are closed. Show that $\sigma(T_1) = \mathbb{C}$ while $\sigma(T_2) = \emptyset$.

Proposition 12.2.4. *Let (X, μ) be a finite measure space and F a measureable, real-valued, a.e. finite function on X. Let $D(M_f) = \{g \in \mathrm{L}^2(X,\mu) : fg \in \mathrm{L}^2(X,\mu)\}$ and let $M_f g = fg$. Then M_f is self-adjoint and $\sigma(M_f) =$ ess.range$(f) = \{\lambda \in \mathbb{C} : \mu(\{x : |\lambda - f(x)| < \epsilon\}) > 0 \ \forall \epsilon > 0\}$.*

Exercise 12.2.5. *Prove Prop 12.2.4.*

Theorem 12.2.6 (Spectral Theorem - Multiplier Form)**.** *Let A be a self-adfoint operator on $\mathcal{H}$ with dense domain. Then there exists a finite measure space (X, μ), a real-valued a.e. finite function f on X and a unitary operator $u : \mathcal{H} \to \mathrm{L}^2(X,\mu)$ such that $uAu^* = M_f$*

Proof. See [2]. □

Remark 12.2.7 (Borel Functional Calculus)**.** *Note that the Spectral Theorem allows us to define a Borel functional calculus for self adjoint operators. Given a Borel function h on the spectrum of A, define $h(A) = u^* M_{h \circ f} u$.*

12.3 Polar Decomposition

Theorem 12.3.1. *Let $A : \mathcal{H} \to \mathcal{K}$ be a closed, densely defined operator. Then:*

(i) A^*A *and* AA^* *are positive self-adjoint operators (hence* $(A^*A)^{1/2}$ *and* $(AA^*)^{1/2}$ *exist).*

(ii) There exists a partial isometry with initial space $\overline{\text{Range}}(A^*A)^{1/2}$ *and final space* $\overline{\text{Range}}(A)$ *and*

$$A = v(A^*A)^{1/2}.$$

(iii) If $A = uB$ *for some positive* B *and partial isometry* v *with initial space* $\overline{\text{Range}}(B)$ *then* $u = v$ *and* $B = (A^*A)^{1/2}$.

(iv) In addition $A = (AA^*)^{1/2}v$.

Proof. (i) Since $\Gamma(A)$ is closed, it is a Hilbert space. Let $P : \Gamma(A) \to \mathcal{H}$ be projection onto the first component. Since A is an operator $\text{Ker}(P) = \{0\}$ and hence $\text{Range}(P^*)$ is dense in $\Gamma(A)$ (so $PP^*\mathcal{H}$ is a core for A). Let $\xi \in \mathcal{H}$, $P^*\xi = (\eta, A\eta)$. Then, for all $\sigma \in D(A)$,

$$\begin{aligned} & < \xi, \sigma > = < P^*\xi, (\sigma, A\sigma) > = < \eta, \sigma > + < A\eta, A\sigma > \\ \Rightarrow \quad & < \xi - \eta, \sigma > = < A\eta, A\sigma > \\ \Rightarrow \quad & A\eta \in D(A^*) \text{ and } A^*A\eta = \xi - \eta. \end{aligned}$$

Thus $D(A^*A) \supset PP^*\mathcal{H}$ which is a core for A. In addition $\text{Range}(A^*A + 1) = \mathcal{H}$.

It is easy to see that A^*A is symmetric, so $A^*A + 1 \subset (A^*A + 1)^*$. Let $\xi \in D((A^*A + 1)^*)$. Since $\text{Range}(A^*A + 1) = \mathcal{H}$ there exists $\tilde{\xi} \in D(A^*A + 1)$ with $(A^*A + 1)^*\xi = (A^*A + 1)\tilde{\xi} (= (A^*A + 1)^*\tilde{\xi})$. $\text{Ker}((A^*A + 1)^*) = \{0\}$ because $\text{Range}(A^*A + 1) = \mathcal{H}$, and hence $\xi = \tilde{\xi} \in D(A^*A+1)$. Thus $(A^*A+1)^* = A^*A+1$ and so $(A^*A)^* = A^*A$.

Finally, for $\xi \in D(A^*A)$, $< A^*A\xi, \xi > = < A\xi, A\xi > \geq 0$ so A^*A is positive, i.e. $\sigma(A^*A) \subset [0, \infty)$ (just use the Spectral Theorem).

(ii) As we noted above, $D(A^*A)$ is a core for A. $D(A^*A)$ is also a core for $|A| = (A^*A)^{1/2}$ (use spectral theory). Thus $\overline{AD(A^*A)} = \overline{\text{Range}A}$ and $\overline{|A|D(A^*A)} = \overline{\text{Range}|A|}$. Note that for $\xi \in D(A^*A)$,

$$|||A|\xi||^2 = < A^*A\xi, \xi > = < A\xi, A\xi > = ||A\xi||^2,$$

so that the map $v : |A|\xi \mapsto A\xi$, $\xi \in D(A^*A)$, extends to a partial isometry with initial space $\overline{|A|D(A^*A)} = \overline{\text{Range}|A|}$ and final space $\overline{AD(A^*A)} = \overline{\text{Range}A}$.

For $\xi \in D(|A|)$ take $\xi_n \in D(A^*A)$ with $(\xi_n, |A|\xi_n) \to (\xi, |A|\xi)$. Then $A\xi_n = v|A|\xi_n \to v|A|\xi$ and, as A is closed, $\xi \in D(A)$ and $A\xi = v|A|\xi$.

For $\xi \in D(A)$ take $\xi_n \in D(A^*A)$ with $(\xi_n, A\xi_n) \to (\xi, A\xi)$. Then

$$|A|\xi_n = v^*v|A|\xi_n = v^*A\xi_n \to v^*A\xi.$$

Since $|A|$ is closed, $\xi \in D(|A|)$.

Hence $D(A) = D(|A|)$ and $A = v|A|$.

(iii) If $A = uB$ then $A^* = B^*u^* = Bu^*$. $A^*A = Bu^*uB = B^2$ since u^*u is projection onto $\overline{\text{Range}}(B)$. By uniqueness of the positive square root of a positive operator (Exercise 12.3.3), $(A^*A)^{1/2} = B$. Thus the initial space of u is $\overline{\text{Range}}(|A|)$ and $u|A| = A = v|A|$ so $u = v$.

(iv) $A = v(A^*A)^{1/2}$ so $A^* = (A^*A)^{1/2}v^*$ and hence $AA^* = v(A^*A)^{1/2}(A^*A)^{1/2}v^* = v(A^*A)v^*$ (Exercise 12.3.3). Thus v implements the unitary equivalence of $AA^*|_{\overline{\text{Range}}(A)}$ and $A^*A|_{\overline{\text{Range}}(A^*)}$. Hence $(AA^*)^{1/2} = v(A^*A)^{1/2}v^*$ and then $A = v(A^*A)^{1/2} = (AA^*)^{1/2}v$.

□

Remark 12.3.2. *Note that it was very important in (i) to establish that $D(A^*A)$ contained a core for A and hence was dense. It was not clear a priori that $D(A^*A)$ contained* any *elements other than* 0.

Exercise 12.3.3. *(i) Let T be a positive operator. Show that $T^{1/2}T^{1/2} = T$.*

(ii) Show that a positive operator has a unique positive square-root.

12.4 Unbounded operators affiliated with a von Neumann algebra.

If M is a von Neumann algebra on $\mathcal{H}$, an element $a \in \mathcal{B}(\mathcal{H})$ is in M iff $au = ua$ for every unitary in M'. This inspires the following.

Definition 12.4.1. *If $T : D(T) \to \mathcal{H}$ is a linear operator on the Hilbert space $\mathcal{H}$ and M is a von Neumann algebra on $\mathcal{H}$ we say that T is* affiliated with *M, written $T\eta M$ if, for any unitary $u \in M'$,*

$$uD(T) = D(T) \quad \textit{and}$$

$$uT\xi = Tu\xi \ \forall \xi \in D(T).$$

Lemma 12.4.2. *If T is preclosed with closure $\overline{T}$ then $\overline{T}\eta M$ if $T\eta M$.*

Proof. It is clear that $T\eta M$ iff $u\Gamma(T) = \Gamma(T)$ for all unitaries in M'. But this property passes to the closure of the graph. □

Lemma 12.4.3. *If T is a closed operator affiliated with M then*

1. *The projection onto $\Gamma(T)$ is a 2×2 matrix of operators in M.*
2. *If $T = u|T|$ is the polar decomposition of T then $u \in M$ and $f(|T|) \in M$ for any bounded Borel function of $|T|$.*

Proof. 1. is obvious from the characterisation of affiliation given in the proof of the previous lemma.

2. follows from uniqueness of the polar decomposition and the bicommutant theorem.

□

Chapter 13

Tomita-Takesaki theory.

In chapter 9 we showed that the GNS construction on M using a faithful normal trace produces a perfectly symmetric Hilbert space $\mathcal{H}_{tr}$ with respect to M and its commutant. This is because the map J, which is the extension to $\mathcal{H}_{tr}$ of the * operation on M, is an isometry. So $x \mapsto JxJ$ is the extension to $\mathcal{H}_{tr}$ of right multiplication by x^*. Unfortunately if we use a (normal) non-tracial state ϕ the * operation is no longer an isometry and there is no reason to expect either it or right multiplication by elements of M to have bounded extensions to $\mathcal{H}_\phi$. But as we shall see, the * operation is actually preclosed in the sense of unbounded operators and if $S = J\Delta^{1/2}$ is the polar decomposition of its closure S, we will show that $JMJ = M'$. Quite remarkably, the operator $\Delta^{1/2}$ will satisfy $\Delta^{it} M \Delta^{-it} = M$ so that a state actually gives rise to a dynamics – a one parameter automorphism group of M (and M').

We will prove these results using a method of van Daele for which we have followed some notes of Haagerup ([],[]). But before getting started on this difficult theory it is essential to do some elementary calculations to see how it all works out in the 2×2 matrices.

Exercise 13.0.4. *Let M be $M_2(\mathbb{C})$. Show that any state ϕ on M is of the form $\phi(x) = Trace(hx)$ for some positive h of trace 1. And that ϕ is faithful iff h is invertible. Thus with respect to the right basis,*

$$\phi(x) = Trace(x \begin{pmatrix} \frac{1}{1+\lambda} & 0 \\ 0 & \frac{\lambda}{1+\lambda} \end{pmatrix})$$

for some λ, $0 \leq \lambda \leq 1$.

Exercise 13.0.5. *With notation as in the previous exercise, suppose ϕ is faithful and let S be the * operation on the GNS Hilbert space $\mathcal{H}_\phi$. Calculate*

the polar decomposition $S = J\Delta^{1/2}$ and show that $SMS = JMJ = M'$. Show that $\Delta^z M \Delta^{-z} = M$ for $z \in \mathbb{C}$ so that $\sigma_z^\phi(x) = \Delta^z x \Delta^{-z} = M$ defines a representation of $\mathbb{C}$ as automorphisms of M which are *-automorphisms iff $z \in i\mathbb{R}$.

Exercise 13.0.6. *Generalize the above to the $n \times n$ matrices and in fact any finite dimensional von Neumann algebra.*

13.1 S,F and their graphs.

Throughout this section M will be a von Neumann algebra on $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a cyclic and separating vector for M and hence M'. (The same data as a faithful normal state.) Let S_0 and F_0 be the conjugate linear operators with domains $M\Omega$ and $M'\Omega$ defined by $S_0(x\Omega) = x^*\Omega$ and $F_0(x\Omega) = x^*\Omega$ for $x \in M$ and M' respectively.

Lemma 13.1.1. *In the sense of unbounded operators $F_0 \subseteq S_0^*$ and $S_0 \subseteq F_0^*$ so that S_0 and F_0 have densely defined adjoints and hence are preclosed.*

Proof. To show $S_0^*(a'\Omega)$ is defined if $\langle S_0(a\Omega), a'\Omega\rangle$ extends to a bounded conjugate linear map on all of $\mathcal{H}$. But $\langle S_0(a\Omega), a'\Omega\rangle = \langle (a')^*\Omega, a\Omega\rangle$ which is bounded as a function of $a\Omega$ by Cauchy-Schwartz. Hence $a'\Omega$ is in the domain of S_0^* and $S_0^*(a'\Omega) = (a')^*\Omega = F_0(a'\Omega)$. Interchanging S_0 and F_0 we get the other inclusion. □

Definition 13.1.2. *Let S and F be the closures of S_0 and F_0 respectively. Let $S = J\Delta^{1/2}$ be the polar decomposition of S.*

Observe that $S_0 = S_0^{-1}$ so S is injective and $S^2 = 1$ in the sense of unbounded operators. Thus $\Delta^{1/2}$ has zero kernel, $J^2 = 1$ and $J\Delta^{1/2}J = \Delta^{-1/2}$. The same goes for F and its polar decomposition, but we shall now see that $F = S^*$.

Theorem 13.1.3. *(Takesaki,[].) $S^* = F$, $F^* = S$ and the graph of S is the set of all $(c\Omega, c^*\Omega)$ where c is a closed densely defined operator affiliated with M and $\Omega \in D(c) \cap D(c^*)$.*

Proof. Let $(\xi, F^*\xi)$ be in the graph of F^*. By the definition of F we know that $\langle \xi, (a')^*\Omega\rangle = \langle a'\Omega, F^*\xi\rangle$. Now define operators a and b with domain $M'\Omega$ by $ax'\Omega = x'\xi$ and $bx'\Omega = x'F^*\xi$. Then a and b are closable for if x' and y' are in M' we have

$$\langle a(x'\Omega), y'\Omega\rangle = \langle x'\xi, y'\Omega\rangle = \langle \xi, (x')^*y'\Omega\rangle$$

$$= \langle (y')^* x'\Omega, F^*\xi\rangle = \langle x'\Omega, y'F^*\xi\rangle = \langle x'\Omega, b(y'\Omega)\rangle$$

so that as before $a \subseteq b^*$ and $b \subseteq a^*$.

Let c be the closure of a. Then $c\Omega = a\Omega = \xi$ and $c^* = a^* \supseteq b$ so $c^*\Omega = F^*\xi$. Now by construction the domain of a is invariant under the unitary group of M' and on it a commutes with the unitaries in M'. This means that c is affiliated with M. At this stage we have shown that the graph of F^* consists of all $(c\Omega, c^*\Omega)$ where c is a closed densely defined operator affiliated with M and $\Omega \in D(c) \cap D(c^*)$.

We now want to show that the graph of F^* is contained in the graph of S. This is not hard. Let c be as above and $c = \sqrt{c^*c}$ be its polar decomposition. Then if $f_n(t) = t$ for $0 \leq t \leq n$ and $f_n(t) = 0$ for $t > n$ we have that $f_n(\sqrt{c^*c}) \to \sqrt{c^*c}$ on any vector in the domain of c, and since c is affiliated with M, $f_n(\sqrt{c^*c}) \in M$ so that $uf_n(\sqrt{c^*c})\Omega$ is in the domain of S and tends to ξ. Moreover $f_n(\sqrt{c^*c})u^*\Omega$ tends to $c^*\Omega = F^*\xi$ so $(\xi, F^*\xi)$ is in the graph of S.

Thus $F^* \subseteq S$ and we have already observed that $S \subseteq F^*$. Hence $S = F^*$ and $S^* = F$. □

Corollary 13.1.4. *The polar decomposition of F is $J\Delta^{-1/2}$.*

We now prove a crucial result connecting M and M'.

Lemma 13.1.5. *Let $\lambda \in \mathbb{R}^+$ be given. Then for $a' \in M'$ there is an $a \in M$ with $a\Omega$ in the domain of F and $a'\Omega = (\lambda S + \lambda^{-1}F)a\Omega$.*

Proof. Assuming $||a'|| \leq 1$ we may apply theorem 8.2.1 to the ψ defined by $\psi(x) = \langle x\Omega, a'\Omega\rangle$ and $\phi(x) = \langle x\Omega, \Omega\rangle$ to obtain the existence of an $a \in M$ with

$$\langle x\Omega, a'\Omega\rangle = \lambda\langle ax\Omega, \Omega\rangle + \lambda^{-1}\langle xa\Omega, \Omega\rangle$$

$$= \lambda\langle x\Omega, a^*\Omega\rangle + \lambda^{-1}\langle a\Omega, x^*\Omega\rangle.$$

Provided $a\Omega$ is in the domain of F this equation reads $a'\Omega = (\lambda S + \lambda^{-1}F)a\Omega$.

On the other hand rearranging the equation gives

$$\langle a\Omega, x^*\Omega\rangle = \lambda\langle x\Omega, a'\Omega - \lambda a^*\Omega\rangle$$

so by Cauchy Schwartz $a\Omega$ is in the domain of $F = S^*$. □

Corollary 13.1.6. *For each $\xi \in D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$ there is a sequence $a_n \in M$ with $a_n\Omega \to \xi$, $\Delta^{1/2}a_n\Omega \to \Delta^{1/2}\xi$ and $\Delta^{-1/2}a_n\Omega \to \Delta^{-1/2}\xi$.*

Proof. Set $\eta = (S+F)\xi$ and choose a sequence $a'_n \in M'$ with $a'_n \to \eta$. By the previous lemma there are $a_n \in M$ with $(S+F)a_n\Omega = a'_n\Omega$. But $S + F = J(\Delta^{1/2} + \Delta^{-1/2})$ has bounded inverse (in the usual sense of unbounded operators) so put $\xi_n = (S+F)^{-1}(a'_n\Omega)$. So $a_n\Omega = (S+F)^{-1}a'_n\Omega \to \xi$. Moreover

$$\Delta^{1/2}a_n\Omega = \Delta^{1/2}(\Delta^{1/2} + \Delta^{-1/2})^{-1}Ja'_n\Omega$$

and $\Delta^{1/2}(\Delta^{1/2} + \Delta^{-1/2})^{-1}$ is bounded by spectral theory. So $\Delta^{1/2}a_n\Omega \to \Delta^{1/2}(S+F)^{-1}(S+F)\xi = \Delta^{1/2}\xi$. In the same way $\Delta^{-1/2}a_n\Omega \to \Delta^{-1/2}\xi$. □

We put everything together with a lemma linking M and M' on a dense subspace to which many functions of Δ can be applied.

Lemma 13.1.7. *If ξ and η are in $D(S) \cap D(F)$, a', λ and a as in 13.1.5, then*

$$\lambda\langle SaS\xi, \eta\rangle + \lambda^{-1}\langle FaF\xi, \eta\rangle = \langle a'\xi, \eta\rangle.$$

Proof. By moving one S and F to the other side of the inner products, we see by the previous lemma that we may assume ξ and η are $x\Omega$ and $y\Omega$ respectively, both in $D(F)$, for x and y in M. But on $M\Omega$, SaS acts by right multiplication by a^* so $\langle SaS\xi, \eta\rangle = \langle xa^*\Omega, y\Omega\rangle = \langle Sa\Omega, x^*y\Omega\rangle$. On the other hand, systematically using $F^* = S$ we obtain $\langle FaFx\Omega, y\Omega\rangle = \langle y^*x\Omega, a\Omega\rangle = \langle Sx^*y\Omega, a\Omega\rangle = \langle Fa\Omega, x^*y\Omega\rangle$. Combining these two we see

$$\lambda\langle SaS\xi, \eta\rangle + \lambda^{-1}\langle FaF\xi, \eta\rangle = \langle(\lambda Sa + \lambda^{-1}Fa)\Omega, x^*y\Omega\rangle.$$

But by 13.1.5 this is $\langle a'\Omega, x^*y\Omega\rangle = \langle a'\xi, \eta\rangle$. □

13.2 Proof of the main theorem.

We begin with an easy exercise in contour integration.

Exercise 13.2.1. *Let S be the strip $\{z \in \mathbb{C} : -1/2 \leq \Re(z) \leq 1/2\}$. Suppose f is continuous and bounded on S and analytic on the interior of S. Then*

$$f(0) = \int_{-\infty}^{\infty} \frac{f(1/2+it) + f(-1/2+it)}{2\cosh \pi t} dt$$

Hint: Integrate $\dfrac{f(z)}{\sin \pi z}$ around rectangular contours in S tending to the boundary of S.

Proposition 13.2.2. *With notation as in the previous section*

$$a = \int_{-\infty}^{\infty} \lambda^{2it} \frac{\Delta^{it} J a' J \Delta^{-it}}{2\cosh \pi t} dt$$

Proof. Since $J\Delta^{1/2}J = \Delta^{-1/2}$ we have $J(D(S) \cap D(T)) = D(S) \cap D(T)$ so after a little rearrangement the formula of 13.1.7 reads

$$\langle Ja'J\xi, \eta\rangle = \lambda\langle a\Delta^{-1/2}\xi, \Delta^{1/2}\eta\rangle + \lambda^{-1}\langle a\Delta^{1/2}\xi, \Delta^{-1/2}\eta\rangle.$$

Now let $\mathcal{H}_0$ be the dense subspace of all vectors in $\mathcal{H}$ which is the union of all $\xi_{[a,b]}(\Delta$ for $0 < a < b < \infty$. Certainly $\mathcal{H}_0 \subseteq D(S) \cap D(F)$, $\mathcal{H}_0$ is invariant under J and Δ^z for $z \in \mathbb{C}$, and moreover for $\xi \in \mathcal{H}_0$, $z \mapsto \Delta^z\xi$ is an entire function of z.

For $\xi, \eta \in \mathcal{H}_0$ define the analytic function

$$f(z) = \lambda^{2z}\langle a\Delta^{-z}\xi, \Delta^{\overline{z}}\eta\rangle.$$

Then f is bounded in the strip S of the previous lemma and $f(0) = \langle a\xi, \eta\rangle$. Also $f(1/2 + it) = \langle \Delta^{it}\Delta^{1/2}\xi, \eta\rangle$ so that

$$f(1/2 + it) + f(-1/2 + it) = \lambda^{2it}\langle \Delta^{it} Ja' J\Delta^{-it}\xi, \eta\rangle.$$

So by the previous lemma we are done. □

Theorem 13.2.3. *Let M be a von Neumann algebra on $\mathcal{H}$ and Ω a cyclic and separating vector for M. Suppose S is the closure of $x\Omega \mapsto x^*\Omega$ on $M\Omega$. Let $\Delta = S^*S$, and J be the antiunitary of the polar decomposition $S = J\Delta^{1/2}$. Then*

(i) $JMJ = M'$

(ii) $\Delta^{it}M\Delta^{-it} = M\ \forall t \in \mathbb{R}$

Proof. If $a' \in M'$ we know that

$$\int_{-\infty}^{\infty} \lambda^{2it} \frac{\Delta^{it} Ja' J\Delta^{-it}}{2\cosh \pi t} dt \in M.$$

Conjugating by a unitary $u \in M'$ and writing $\lambda = e^{\frac{i\theta}{2}}$ we see that the Fourier transforms of the strongly continuous rapidly decreasing functions $\dfrac{\Delta^{it} Ja' J\Delta^{-it}}{2\cosh \pi t}$ and $u\dfrac{\Delta^{it} Ja' J\Delta^{-it}}{2\cosh \pi t}u^*$ are equal. Hence $\Delta^{it} Ja' J\Delta^{-it} \in M$ for all real t since it commutes with every unitary $u \in M'$. (Take inner products with vectors if you are not happy with Fourier transforms of operator valued functions.)

Putting $t = 0$ we see $JM'J \subseteq M$ and by symmetry $JMJ \subseteq M'$. Hence $JMJ = M'$ and we are done. □

Definition 13.2.4. *The operator J of the previous result is called the* modular conjugation *and the strongly continuous one-parameter group of automorphisms of M defined by $\sigma_t^\phi(x) = \Delta^{it}x\Delta^{-it}$ is called the* modular automorphism group *defined by ϕ.*

13.3 Examples.

Example 13.3.1. *ITPFI*

The acronym ITPFI stands for "infinite tensor product of finite type I". These von Neumann algebras are formed by taking the *-algebra A_∞ as the union A_∞ of tensor products $A_m = \bigotimes_{k=1}^{m} M_{n_k}(\mathbb{C})$, the inclusion of A_m in A_{m+1} being diagonal. The state ϕ on A_∞ is then the tensor product of states on each M_{n_k}. One may then perform the GNS construction with cyclic and separating vector Ω given by $1 \in A_\infty$, to obtain the von Neumann algebra $M = \bigotimes_{k=1}^{\infty} M_{n_k}(\mathbb{C})$ as the weak closure of A_∞ acting on $\mathcal{H}_\phi$. The case where all the n_k are equal to 2 and all the states are the same is called the "Powers factor" and the product state the "Powers state" as it was R.Powers who first showed that they give a continuum of non-isomorphic type III factors.

A slight snag here is that we do not know that Ω defines a faithful state on M. But if we proceed anyway to construct what have to be J and Δ we will soon see that the state is indeed faithful, i.e. Ω is cyclic for $M'\Omega$.

Recall from exercise 13.0.6 that, for $M_n(\mathbb{C})$, and $\phi_h(x) = trace(xh)$ where h is the diagonal matrix (density matrix) with $h_{ii} = \mu_i, \sum \mu_i = 1, \mu_i > 0$, then $J_n(e_{ij}) = \sqrt{\frac{\mu_j}{\mu_i}}e_{ji}$ and $\Delta_n(e_{ij}) = \frac{\mu_i}{\mu_j}e_{ij}$ (where dependence on h has been suppressed).

To diagonalise the modular operators on $\mathcal{H}_\phi$ completely it is most convincing to choose an orthonormal basis d_i of the diagonal matrices, with $d_1 = 1$. Then a basis for the Hilbert space $\mathcal{H}_\phi$ is formed by tensors $\otimes_{k=1}^{\infty} v_k\Omega$ where $v_k = 1$ for large k, and is otherwise a d_i or an e_{ij} with $i \neq j$.

We can guess that J is, on each basis vector, the tensor product of the J's coming from the matrix algebras. Defining it as such it is clearly an isometry on $A_\infty\Omega$ and thus extends to all of $\mathcal{H}_\phi$. But then, for any $x \in A_\infty$, JxJ is in M' by the finite dimensional calculation! But the linear span of these $JxJ\Omega$ is dense so Ω is cyclic for M' and hence separating for M. We are hence in a position to apply Tomita-Takesaki theory. Each of the basis elements is in $M\Omega$ so $S(\otimes_{k=1}^{\infty} v_k\Omega) = \otimes_{k=1}^{\infty} w_k\Omega$ where w_k is v_k if v_k is diagonal, and e_{ji} if

$v_k = e_{ij}$. So JS is diagonal and hence essentially self-adjoint. We conclude that

$$J(x\Omega) = J_m(x)\Omega \quad \text{and} \quad \Delta(x\Omega) = \Delta_m(x)\Omega \qquad \text{for } x \in A_m,$$

and

$$\sigma_t^{\phi} = \bigotimes_{k=1}^{\infty} \sigma^{\phi_{h_k}} \quad .$$

Example 13.3.2. *Group-measure-space construction.*

Let Γ be a discrete group acting on the finite measure space (X, μ) preserving the class of the finite measure μ. The Hilbert space of the crossed product $L^\infty(X, \mu)$ is $L^2(X, \mu) \otimes \ell^2(\Gamma)$ and as we saw in chapter 11 the vector $1 \otimes \varepsilon_{id}$ is a cylic and separating vector Ω for $M = L^\infty(X, \mu) \rtimes \Gamma$.

Since the class of μ is preserved by the $\gamma \in \Gamma$ the Radon Nikodym theorem guarantees positive L^1 functions h_γ so that $\phi(h_\gamma \alpha_\gamma(y)) = \phi(x)$ where $\phi(y) = \int_X y d\mu$. We know that, if $x \in L^\infty(X, \mu)$ then $S(u_\gamma x) = x^* u_{\gamma^{-1}}$. In general we will not be able to completely diagonalise Δ but the same argument as in the previous example gives that the domain of Δ is

$$\{f : \Gamma \to L^2(X, \mu) : \sum_\gamma \int_X |h_\gamma(x) f(x)|^2 d\mu(x) < \infty\}$$

on which

$$(\Delta f)(\gamma) = h_\gamma f(\gamma),$$

and

$$(Jf)(\gamma) = h_\gamma^{-1/2} f(\gamma).$$

We can now finally answer the question as to which sums $\sum_\gamma x_\gamma u_\gamma$ define elements of $M = L^\infty(X, \mu) \rtimes \Gamma$.

Theorem 13.3.3. *With notation as above, if the function $\gamma \mapsto x_\gamma \in L^\infty(X, \mu)$ is such that $\sum_\gamma x_\gamma u_\gamma$, interpreted as a matrix of operators as in section 11.2, defines a bounded operator, then that operator is in $M = L^\infty(X, \mu) \rtimes \Gamma$.*

Proof. By 13.2.3 it suffices to show that $\sum_\gamma x_\gamma u_\gamma$ commutes with $Jxu_\gamma J$ for all $x \in L^\infty(X, \mu)$ and $\gamma \in \Gamma$. And for this it suffices to check that the commutation holds on functions of the form $f \otimes \varepsilon_\gamma$ for $f \in L^2$. This is just a routine computation. □

Exercise 13.3.4. *Show that example 13.3.1 is in fact a special case of this group-measure-space example in which $L^\infty(X, \mu)$ is provided by the tensor products of the diagonal elements and the group Γ is a restricted infinite Cartesian product of cyclic groups, constructed from the non-diagonal e_{ij}'s. Conclude by the method of 11.2.15 that ITPFI algbras are factors.*

This example brings to light a significant inadequacy of our treatment of Tomita-Takesaki theory. We would like to treat the case where the measure of the space is infinite. Although of course we could choose an equivalent finite measure, this choice may not be natural. To do this we would have to consider the theory of "weights" which are to states as the trace on a II_∞ factor is to the trace on a type II_1 factor. We need the same notion in order to understand the origin of the term "modular" used above as coming from the modular function on a non-unimodular locally compact group. But a serious treatment of weights would take many pages so we simply refer the reader to Takesaki's book [3].

Example 13.3.5. *Hecke algebras à la Bost-Connes.*

If G is a finite group let u_g and v_g be the unitaries of the left and right regular representations respectively. If H is a subgroup, the projection $p_H = \frac{1}{|H|}\sum_{h\in H} v_h$ projects from $\ell^2(G)$ onto functions that are right translation invariant under H, i.e. functions on the quotient space G/H. Thus the so-called "quasi-regular" representation of G on G/H is a direct summand of the left regular representation and we have from EP7 of chapter 3.4 that the commutant of the action of G on $\ell^2(G/H)$ is $p_H\rho(G)p_H$ where $\rho(G)$ is the algebra generated by the right regular representation (of course isomorphic to $\mathbb{C}$). This commutant is spanned by the $p_H v_g p_H$ which, thought of as functions on G, are multiples of the characteristic functions of the double cosets HgH which form the double coset space $H\backslash G/H$. The subalgebra of $\rho(G)$ spanned by these double cosets is the space of $H-H$ bi-invariant functions and we see it is the commutant of G on $\ell^2(G/H)$. It is known as the *Hecke algebra* for the pair (G,H) and has a considerable role to play in the representation theory of finite groups. A famous example is the case where G is the general linear group over a finite field and H is the group of upper triangular matrices. The coset space is then the so-called "flag variety" and the Hecke algebra in this case leads to a lot of beautiful mathemtatics. See Bourbaki [].

Nothing could better indicate how differently things work for infinite discrete groups than how the Hecke algebra works. Far from being direct summands, the quasiregular representations can be totally different from the left regular representations and can even generate type III factors! These Hecke algebras give nice examples where the modular operators can be calculated explicitly.

Definition 13.3.6. *A subgroup H of the discrete group G is called* almost normal *if either of the two equivalent conditions below is satisfied.*

(a) $gHg^{-1}\cap H$ is of finite index in H for all $g\in G$.

(b) Each double coset of H is a finite union of left cosets of H (i.e. the orbits of H on G/H are all finite).

If H is almost normal in G one may construct operators in the commutant of the quasiregular representation of G on $\ell^2(G/H)$ as follows:

Given an element x of G/H let ε_x be the characteristic function of x. These functions form an orthonormal basis of $\ell^2(G/H)$. Moreover each vector ε_x is cyclic for the action of G hence separating for the commutant. If D is a double coset of H define T_D by the matrix

$$(T_D)_{x,y} = \begin{cases} 1 & \text{if } y^{-1}x = D; \\ 0 & \text{otherwise.} \end{cases}$$

check this typesetting

Clearly T_D is bounded since H is almost normal and it obviously commutes with the action of G. From the definition we have

$$T_D^* = T_{D^{-1}}.$$

It is also easy to check that

$$T_D T_E = \sum_F n_{D,E}^F T_F$$

where the structure constants are defined by

$$n_{D,E}^F = \begin{cases} \#(E/H) & \text{if } F \subseteq ED; \\ 0 & \text{otherwise.} \end{cases} x$$

check typesetting here

We will call the von Neumann algebra generated by the T_D's the *Hecke-von Neumann algebra of the pair* $H \subseteq G$ and write it $HvN(G,H)$. The vector state ϕ defined on $HvN(G,H)$ by ε_H is faithful and normal, and $\langle T_D \varepsilon_H, T_{D'} \varepsilon_H \rangle = 0$ unless $D = D'$ so that the T_D's are orthogonal. It is thus easy to calculate the operators for the modular theory on $\mathcal{H}_\phi$ (note that this is <u>not</u> $\ell^2(G/H)$). We guess as usual that $J(T_D\Omega) = (constant)T_{D^{-1}}\Omega$ and by counting cosets in double cosets (or sizes of orbits of H on G/H) we find that the constant has to be $(\#(D/H))^{1/2}(\#(H\backslash D))^{-1/2}$. Thus as before JS is diagonal on the basis $T_D\Omega$ of H_ϕ so essentially self-adjoint and

$$\Delta(T_D\Omega) = \frac{\#(H\backslash D)}{\#(D/H)} T_D\Omega$$

with the obvious domain. Thus

$$\sigma_t^\phi(T_D) = \left(\frac{\#(H\backslash D)}{\#(D/H)}\right)^{it} T_D.$$

Examples of almost normal subgroups are not hard to find. The classical example of Hecke himself is the case where $G = SL(2, \mathbb{Q})$ and $H = SL(2, \mathbb{Z})$. In this case the Hecke algebra is abelian. Bost and Connes in [4] examined the case of the $ax+b$ group over the rationals with the subgroup being integer translations. They showed that $HvN(G, H)$ in this case is a type III factor and made a connection with prime numbers.

13.4 The KMS condition.

In the examples of the previous section the operators of the modular group were easy to calculate explicitly, including the domain of Δ. One can imagine that this is not always so. If we are particularly interested in the modular group σ_t^ϕ it would be useful to be able to guess it and show that the guess is right without bothering about the domain of Δ. The KMS (Kubo-Martin-Schwinger) condition from quantum statistical mechanics allows us to do just that. The modular group came from the non-trace-like property of a state and the KMS condition allows us to correct for that. Let us do a formal calculation assuming that the modular group can be extended to complex numbers (remember that Ω is fixed by S, J and Δ):

$$\begin{aligned} \phi(xy) &= \langle y\Omega, x^*\Omega\rangle \\ &= \langle y\Omega, J\Delta^{-1/2}\Delta x\Delta^{-1}\Omega\rangle \\ &= \langle \Delta x\Delta^{-1}\Omega, Sy\Omega\rangle \\ &= \langle y\Delta x\Delta^{-1}\Omega, \Omega\rangle. \end{aligned}$$

We conclude that

$$\phi(xy) = \phi(y\sigma_i^\phi(x)).$$

Thus the trace is commutative provide we operate by the modular group.

Exercise 13.4.1. *If M is finite dimensional and ϕ is a faithful state, show that $\phi \circ \sigma_t^\phi = \phi$ and that for each x and y in M there is an entire function $F(z)$ with, for $t \in \mathbb{R}$,*

$$\begin{aligned} F(t) &= \phi(\sigma_t^\phi(x)y) \qquad\qquad \textit{and} \\ F(t+i) &= \phi(y\alpha_t(x)). \end{aligned}$$

If M is infinite dimensional we would not expect the function $F(z)$ of the previous exercise to be entire.

Definition 13.4.2. *Let α_t be a strongly continuous one parameter automorphism group of a von Neumann algebra M, and ϕ be a faithful normal state on M. We say that α satisfies the KMS condition for ϕ if $\phi \circ \alpha_t = \phi$ and , for each x and y in M, there is a function F, continuous and bounded on the strip $\{z : 0 \leq \Im m(z) \leq 1\}$, analytic on the interior of the strip and such that for $t \in \mathbb{R}$,*

$$\begin{aligned} F(t) &= \phi(\sigma_t^\phi(x)y) \qquad \text{and} \\ F(t+i) &= \phi(y\alpha_t(x)). \end{aligned}$$

Theorem 13.4.3. *If ϕ is a faithful normal state on a von Neumann algebra M then σ_t^ϕ is the unique one parameter automorphism group satisfying the KMS condition for ϕ.*

This chapter has been heavily technical so we defer the proof, which is by approximation on dense subspaces of the domain of Δ to which the previous calculations can be applied, to an appendix. We content ourselves here with an interesting corollary, identifying a part or M on which ϕ behaves as a trace.

Corollary 13.4.4. *For $a \in M$ the following are equivalent:*

1. *$\phi(ax) = \phi(xa)$ for all $x \in M$.*
2. *$\sigma_t^\phi(a) = a$ for all $t \in \mathbb{R}$.*

Proof. $(1 \Rightarrow 2)$ Observe that for $x \in M$, $\langle x^*\Omega, a\Omega\rangle = \langle \Omega, xa\Omega\rangle = \langle \Omega, ax\Omega\rangle$ (by 1). So $\langle Sx\Omega, a\Omega\rangle = \langle a^*\Omega, x\Omega\rangle$ so that $a\Omega \in D(S^*)$ and $S^*(a\Omega) = \Omega^*$. So $\Delta(a\Omega) = a\Omega$, $\Delta^{it}a\Omega = a\Omega$ and finally $\sigma_t^\phi(a) = a$ for all $t \in \mathbb{R}$.

$(2 \Rightarrow 1)$ $\phi(\sigma_t^\phi(x)a) = \phi(\sigma_t^\phi(xa)) = \phi(xa)$ so that $F(t)$ is constant. Use the Schwarz reflection principle to create a holomorphic function, constant on $\mathbb{R}$, in the union of the strip with its complex conjugate. Thus F is constant on the strip and $\phi(xa) = \phi(ax)$. □

Definition 13.4.5. *The von Neumann subalgebra of M defined by either of the conditions of the previous corollary is called the centraliser of the state ϕ.*

Chapter 14

Connes' theory of type III factors.

14.1 The Connes unitary cocycle Radon-Nikodym theorem.

This result will allow us to extract information from the modular group of a state which is independent of the state.

Theorem 14.1.1. *Let ϕ and ψ be faithful normal states on a von Neumann algebra M. Then there is a strongly continous map $t \to u_t$ from $\mathbb{R}$ to the unitary group of M so that*

$$\sigma_t^\phi = Adu_t \sigma_t^\psi \qquad \forall\, t \in \mathbb{R}.$$

Morevoer u_t satisfies the cocycle condition $u_t \sigma_t^\psi(u_s) = u_{t+s}$.

Proof. We define the faithful normal state Φ on $M \otimes M_2(\mathbb{C})$ by $\Phi((x)_{ij}) = \frac{1}{2}(\phi(x_{11}) + \psi(x_{22}))$. The projection $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is fixed by σ^Φ by 13.4.4. So σ^Φ defines a one parameter automorphism group of $pM \otimes M_2(\mathbb{C})p$ which satisfies the KMS condition for ϕ. Hence $\sigma_t^\Phi(x \otimes e_{11}) = \sigma_t^\phi(x) \otimes e_{11}$. Similarly $\sigma_t^\Phi(x \otimes e_{22}) = \sigma_t^\psi(x) \otimes e_{22}$. Let $V_t = \sigma_t^\Phi(1 \otimes e_{21})$. Then $V_t V_t^* = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $V_t^* V_t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Hence $V_t = \left(\begin{smallmatrix} 0 & 0 \\ v_t & 0 \end{smallmatrix}\right)$ for some unitary $v_t \in M$. Routine computations give the rest. □

Corollary 14.1.2. *If M is a factor and σ_t^ϕ is outer for any ϕ and t then M is of type III.*

Proof. By the previous result it suffices to exhibit a single faithful normal state on a type II factor with inner modular group. In the II_1 case use the trace and in the II_∞ case choose a faithful normal state ϕ on $\mathcal{B}(\mathcal{H})$ and use $tr \otimes \phi$, using the KMS condition (if necessary) to very that the modular group for the tensor product is the tensor product of the modular groups. □

Corollary 14.1.3. *The subgroup of all $t \in \mathbb{R}$ for which σ_t^ϕ is inner is independent of the faithful normal state ϕ.*

Definition 14.1.4. *The subgroup of the previous corollary, which is an invariant of M, is called $T(M)$.*

We shall now calculate $T(M)$ for the Powers factor R_λ where this refers to the ITPFI factor with all $n_k = 2$ and all states having the same density matrix $h = \begin{pmatrix} \frac{1}{1+\lambda} & 0 \\ 0 & \frac{\lambda}{1+\lambda} \end{pmatrix}$.

Theorem 14.1.5.

$$T(R_\lambda) = \frac{2\pi}{\log \lambda}\mathbb{Z}$$

.

Proof. By the formula for the modular group $\sigma^\phi_{\frac{2\pi}{\log\lambda}} = id$ so $\frac{2\pi}{\log\lambda}\mathbb{Z} \subseteq T(R_\lambda)$. For the other direction it suffices to show that an automorphism α of the form

$$\alpha = \otimes_{k=1}^\infty \mathrm{Ad}u$$

is outer whenever the unitary u is not a scalar.

For this first define $u_k = \otimes_1^k u$ and observe that if $\alpha = \mathrm{Ad}v$ then $(u_k \otimes 1)^{-1}v = id$ on the matrix algebra $A_k = \otimes_1^k M_2(\mathbb{C})$. By exercise 4.3.3 this means that $v = u_k \otimes w$. Now it is clear from our basis that we can choose $\otimes_{j=1}^p x_i \otimes 1\Omega$ with non-zero inner procuct with $v\Omega$. But then fixing p and letting k tend to infinity we see that

$$\langle(\otimes_{j=1}^p x_i \otimes 1)\Omega, v\Omega\rangle = \prod_{j=1}^p \langle x_i, u\rangle\langle 1, u\rangle^{k-p}\langle 1, w\rangle.$$

The left hand side does not depend on k and $|\langle 1, w\rangle| \leq 1$ so we must have $|\langle 1, u\rangle| = 1$ which means that u is a scalar multiple of 1 by the Cauchy-Schwarz inequality. □

We conclude that the Powers factors R_λ are type III factors, mutually non-isomorphic for different values of λ.

14.2 Type III$_\lambda$.

The spectrum of the modular operator Δ is easy to calculate for an ITPFI factor. It is simply the closure of the set of all ratios $\frac{\mu_i}{\mu_j}$ as μ varies over all the density matrices defining the product state. Apart from being closed

under the inverse operation this set of non-negative real numbers has no particular structure and can be altered easily by making changes in finitely many density matrices which of course do not change the factor.

Definition 14.2.1. *If M is a von Neumann algebra the invariant $S(M)$ is the intersection over all faithful normal states ϕ of the spectra of their corresponding modular operators Δ_ϕ.*

Theorem 14.2.2. *A factor M is of type III iff $0 \in S(M)$.*

Theorem 14.2.3. *(Connes-van Daele) $S(M) \setminus \{0\}$ is a closed subgroup of the positive real numbers.*

There are only three kinds of closed subgroups of $\mathbb{R}^+$.

Definition 14.2.4. *A factor M is called type III_λ for $0 \leq \lambda \leq 1$ if*

$$\begin{aligned} \lambda = 0 &: S(M) = \{0\} \cup \{1\} \\ 0 < \lambda < 1 &: S(M) = \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\} \\ \lambda = 1 &: S(M) = \{0\} \cup \mathbb{R}^+ \end{aligned}$$

Theorem 14.2.5. *The Powers factor R_λ is of type III_λ.*

In his thesis, Connes showed that every type III_λ factor for $0 < \lambda < 1$ is canonically isomorphic to the crossed product of a type II_∞ factor with an action of $\mathbb{Z}$ whose generator scales the trace by λ. Connes thesis

If A is a locally compact abelian group with an action α on a von Neumann algebra M, there is an action $\hat{\alpha}$ of the Pontryagin dual $\hat{A}$ on the crossed product $M \rtimes_\alpha A$ satisfying

$$\begin{aligned} \hat{\alpha}_a(x) &= x && \text{for } x \in M \\ \hat{\alpha}_{\hat{a}}(u_a) &= \hat{a}(a)u_a && \text{if } u_a \text{ are the unitaries defining the crossed product.} \end{aligned}$$

The existence of the so-called "dual action" $\hat{\alpha}$ is trivial proved since it is implemented by the obvious unitary representation of $\hat{A}$ on $L^2(A)$.

Exercise 14.2.6. *If A is finite consider the projection $p = \sum_a u_a \in M \rtimes A$. Show that $pM \rtimes Ap = M^A p$ and thus show that $(M \rtimes_\alpha A) \rtimes_{\hat{\alpha}} \hat{A}$ is isomorphic to $M \otimes M_{|A|}(\mathbb{C})$.*

Observe that the crossed product of a von Neumann algebra M on $\mathcal{H}$ by the modular group σ^ϕ does not depend, up to isomorphism, on the faithful normal state ϕ. This follows from theorem 14.1.1 by defining the unitary V on $L^2(\mathbb{R}, \mathcal{H})$ by

$$Vf(t) = u_t f(t)$$

where ψ is another faithful normal state with unitary one-cocycle u_t. Conjugating the operators that generate $M \rtimes_{\sigma^\phi} \mathbb{R}$ by V one obtains the generators of $M \rtimes_{\sigma^\psi} \mathbb{R}$.

Theorem 14.2.7. *The crossed product of M by the modular group admits a trace satisfying the properties of 9.1.9*

Definition 14.2.8. *The action of $\hat{\mathbb{R}}$ on $Z(M \rtimes_{\sigma^\phi} \mathbb{R})$ is called the "flow of weights" of M.*

Theorem 14.2.9. *(Takesaki duality) The crossed product*

$$(M \rtimes_{\sigma^\phi} \mathbb{R}) \rtimes_{\widehat{\sigma^\phi}} \hat{\mathbb{R}}$$

is isomorphic to the tensor product $M \otimes \mathcal{B}(\mathcal{H})$ for $\mathcal{H} = L^2(\mathbb{R})$.

Thus if M is a factor the flow of weights is ergodic.

Theorem 14.2.10. *If M is a factor of type III_λ the flow of weights is*

III_1: *The trivial flow on a one point set if M is III_1.*

III_λ: *The transitive flow on the circle with period $\frac{2\pi}{\lambda}$ if M is of type III_λ, $0 < \lambda < 1$.*

III_0: *Ergodic non-transitive if M is of type III_0.*

Moreover any ergodic non-transitive flow arises as the flow of weights for some type III_0 factor.

Chapter 15

Hyperfiniteness

Definition 15.0.11. *A von Neumann algebra M on a separable Hilbert space is called* hyperfinite *if there is an increasing sequence A_n of finite dimensional *-subalgebras of M which generates M as a von Neumann algebra.*

15.1 The hyperfinite type II_1 factor R

The first main result about hyperfiniteness was proved by Murray and von Neumann in []. We will use R to denote the hyperfinite II_1 factor whose uniqueness they proved.

Theorem 15.1.1. *Up to abstract isomorphism there is a unique hyperfinite II_1 factor.*

Sketch of proof. One works with the norm $||x||_2 = tr(x^*x)^{1/2}$ on M. It is not hard to show that a von Neumann subalgebra N of M is strongly dense in M iff it is dense in $||-||_2$. Given a subalgebra A of M and a subset $S \subseteq M$ one says

$$\begin{array}{ccc} S & \subseteq & A \\ & \varepsilon & \end{array}$$

if for each $x \in S$ there is a $y \in A$ with $||x - y||_2 < \varepsilon$.

The hyperfiniteness condition then implies:

*For every finite subset $S \subseteq M$ and every $\varepsilon > 0$ there is a finite dimensional *-subalgebra A of M with*

$$\begin{array}{ccc} S & \subseteq & A. \\ & \varepsilon & \end{array}$$

The next step is to show that the A in the preceeding condition can be chosen to be the $2^n \times 2^n$ matrices for some (possibly very large) n. This part uses what might be described as "II_1 factor technique". One begins with A and approximates all its minimal projections $\{e_i\}$ by ones whose traces are numbers of the form $k/2^n$. The matrix units of A can be changed a little bit in $|| - ||_2$ so that, together with matrix units conecting projections of trace $1/2^n$ less than the e_i, they generate a $2^n \times 2^n$ matrix algebra containing, up to ε, the matix units of A. Perturbation of the matrix units will involve results of the form:

If $u \in M$ satisfies $||(uu^*)^2 - uu^*||_2 < \epsilon$ then there is a partial isometry $v \in M$ with $||v - u||_2 < F(\epsilon)$
(for some nice function f with $f(0) = 0$).

or:

If p and q are projections with $||pq||_2 < \epsilon$ then there is a projection q' with $pq' = 0$ and $||q - q'|| < F(\epsilon)$.

or:

If f_{ij} are "almost $n \times n$ matrix units", i.e.

(a) $||f_{ij} - f_{ji}||_2 < \epsilon$

(b) $||f_{ij} f_{kl} - \delta_{j,k} f_{il}||_2 < \epsilon$

(c) $||1 - \sum_{i=1}^{n} fii||_2 < \epsilon$

then there are $n \times n$ matrix units e_{ij} with $||e_{ij} - f_{ij}|| < F(\epsilon)$ where F depends only on n and $F(0) = 0$.

Such results are proved by a skilful use of the polar decomposition and spectral theorem.

Thus one shows that in a hyperfinite type II_1 factor one has:

Property $*$: *For every finite subset $S \subseteq M$ and every $\varepsilon > 0$ there is a $2^n \times 2^n$ matrix subalgebra of M with*

$$S \underset{\varepsilon}{\subseteq} A.$$

One may now proceed to prove the theorem by choosing a $|| - ||_2$-dense sequence x_k in M and inductively constructing an increasing sequence of $2^{n_k} \times 2^{n_k}$ matrix algebras A_k with the property that

$$\text{For each } k = 1, 2, 3, ..., \qquad \{x_1, x_2, ..., x_k\} \underset{1/k}{\subseteq} A_k \ .$$

The union of the A_k's is clearly dense in $||-||_2$. This is enough to prove the theorem since the A_k's can be used to give an isomorphism of M with the type II_1 factor $\otimes^\infty M_2(\mathbb{C})$ constructed in section 6.2.

To construct A_{k+1} from A_k one needs to arrange for the new algebra to contain the one already constructed. So to the elements $x_1, x_2, ..., x_{k+1}$, add matrix units e_{ij} for A_{k+1}. Now use property * to obtain a B almost containing the x_i and the matrix units, with ϵ small enough to control sums over the matrix units e_{ij}. In B we know there are approximate matrix units close to the e_{ij} so by a technical lemma, exact matrix units f_{ij} close to the e_{ij}. Now choose a unitary close to the identity which conjugates the f_{ij} to the e_{ij} and use this unitary to conjugate B to a superalgebra of A_k. This superalgebra is A_{k+1} and it contains the x_i up to epsilon since u is close to the identity.

This completes the sketch of the proof. The technique involved is considered standard in von Neumann algebras and the details can be found in . dixmier

Corollary 15.1.2. *If S_∞ is the group of finitely supported permutations of a countably infinite set then $vN(S_\infty) \cong \otimes^\infty M_2(\mathbb{C})$.*

Proof. The subgroups of S_∞ permuting an increasing sequence of finite sets show that $vN(S_\infty)$ is hyperfinite. □

It is surprising at first sight that the type II_1 factor $L^\infty(X,\mu) \rtimes \mathbb{Z}$ obtained from an ergodic measure-preserving transformation T is hyperfinite. This can be shown by Rokhlin's tower theorem which asserts that, for each $n \in \mathbb{N}$ and each $\epsilon > 0$ there is a measurable subset $A \subseteq X$ with

(1) $T^i(A) \cap T^j(A) = \emptyset$ for $1 \leq i < j \leq n$, and

(2) $\mu(X \setminus \cup_{i=0}^n T^i(A)) < \epsilon$.

The unitary u_1 of the crossed product and the characteristic function of A can be combined, with a little perturbation to get the identity, to get a $n \times n$ matrix algebra. Careful application of the tower theorem will allow one to get any element of $L^\infty(X,\mu)$, and u_1, in this matrix algebra up to some ϵ. This was first proved by Henry Dye in who went on to prove that in fact all Dye groups of polynomial growth give hyperfinite II_1 factors in this way.

The ultimate result in this direction is the celebrated "Injective factors" theorem of Connes who showed that hyperfiniteness for a von Neumann algebra M on $\mathcal{H}$ is equivalent to "injectivity" which means there is a projection in the Banach space sense of norm one from $\mathcal{B}(\mathcal{H})$ onto M. This theorem, whose proof is a great, great tour de force, has a raft of corollaries, many of

which were open questions. Let us just mention the fact that it follows easily that any subfactor of R which is infinite dimensional is in fact isomorphic to R. It also implies that $vN(\Gamma)$, as well as $L^\infty(X,\mu) \rtimes \Gamma$ is hyperfinite as soon as Γ is amenable.

15.2 The type III case.

Connes actions

Connes Injective factors

The complete classification of injective(=hyperfinite) factors is a triumph of 20th. century mathematics. Connes showed in that there is only one trace-scaling automorphism of $R \otimes \mathcal{B}(\mathcal{H})$ for each scaling factor $\lambda \neq 1$ up to conjugacy. Together with this shows that for each λ with $0 < \lambda < 1$ there is a unique injective factor of type III_λ.

krieger

injective

Using results of Krieger in , his thesis and , Connes showed that hyperfinite type III_0 factors are classified by their flow of weights (up to conjugacy of flows, not orbit equivalence). This means that there is a rather large number of III_0 factors but their classification is in the realm of ergodic theory rather than von Neumann algebras.

uffeIIIone

The remaining case of injective type III_1 factors was solved by Haagerup in . There is just one such factor and a hyperfinite factor is "generically" of type III_1.

Chapter 16

Central Sequences.

16.1 Generalities.

Definition 16.1.1. *If M is a type II_1 factor, a central sequence in M is a norm bounded sequence (x_n) with $\lim_{n\to\infty} ||[x_n, a]||_2 = 0$. A central sequence is said to be trivial if $\lim_{n\to\infty} ||x_n - tr(x_n)id||_2 = 0$. M is said to have property Γ if there is a central*

The collection of all central sequences is clearly a C*-subalgebra of $\ell^\infty(\mathbb{N}, M)$. If ω is a free ultrafilter on $\mathbb{N}$, the subalgebra $\mathcal{I}_\omega$ of $\ell^\infty(\mathbb{N}, M)$ consisting of sequences with $\lim_{n\to\omega} ||x_n||_2 = 0$ is a 2-sided ideal of $\ell^\infty(\mathbb{N}, M)$. Note also that M is embedded in $\ell^\infty(\mathbb{N}, M)$ as constant sequences.

Definition 16.1.2. *With notation as above, the ultraproduct of M along ω is the quotient of $\ell^\infty(\mathbb{N}, M)$ by $\mathcal{I}_\omega$. It is written M^ω. The algebra of (ω-)central sequences is the centraliser $M_\omega = M' \cap M^\omega$ of M in $\ell^\infty(\mathbb{N}, M)$.*

By compactness, the trace on M defines a trace on M^ω by

$$tr((x_n)) = \lim_{n\to\omega} tr(x_n)$$

and by definition it is faithful on M^ω.

Exercise 16.1.3. *Show that the unit ball (in the C* norm) of M^ω is complete in $|| - ||_2$ so that M^ω and M_ω are von Neumann algebras.*

16.2 Central sequences in R

All models for R exhibit central sequences in abundance. The most obvious situation is that of $\otimes^\infty M_2(\mathbb{C})$. Fixing $x \in M_2(\mathbb{C})$ we can define the sequence

$x_n = 1 \otimes 1 \otimes 1 ... x \otimes 1 \otimes 1 ...$ with the x in the nth slot in the tensor product. For large enough n, x_n will commute with any element in the algebraic tensor product so by the obvious (in the II_1 case!) inequality $|||[x_n, a]||| \leq 2||x_n||\,||a||_2$ we see that (x_n) is central and clearly non-trivial if x is not a scalar. Just as clearly the central sequence algebra is non-commutative as we only need to choose x and y that do not commute and construct the sequences (x_n) and (y_n) as above. In fact it is not hard to show that R_ω is a factor.

Theorem 16.2.1. *The central sequence algebra R_ω is a type II_1 factor.*

finish proof!

Proof. If (x_n) represents an element $X \in R_\omega$, □

16.3 Type II_1 factors without property Γ.

Theorem 16.3.1. *Let Γ be an icc group possessing a subset Δ not containing the identity and three elements α, β and γ such that*

(a)$\Gamma = \{1\} \cup \Delta \cup \alpha\Delta\alpha^{-1}$

(b)$\Delta, \beta\Delta\beta^{-1}$ *and* $\gamma\Delta\gamma^{-1}$ *are mutually disjoint.*

then for $x \in vN(\Gamma)$,

$$||x - tr(x)id||_2 \leq 14 \max\{|||[x, u_\alpha]|||_2, |||[x, u_\beta]|||_2, |||[x, u_\gamma]|||_2\}.$$

Proof. Write x as $\sum_{\nu\in\Gamma} x_\nu u_\nu$. We will frequently use the formula

$$|||[x, u_\rho]|||_2^2 = ||u_{\rho^{-1}} x u_\rho - x||^2 = \sum_{\nu\in\Gamma} |x_\nu - x_{\rho\nu\rho^{-1}}|^2.$$

By replacing x by $x - tr(x)id$ it suffices to prove the result if $tr(x) = 0$ and we may suppose $||x||_2 = 1$ so that for such an x we must show $1 \leq 14 \max\{|||[x, u_\alpha]|||_2, |||[x, u_\beta]|||_2, |||[x, u_\gamma]|||_2\}$.

We first make a simple estimate. If Λ is any subset of Γ and $\rho \in \Gamma$ then

$$\begin{aligned} |\sum_{\nu\in\Lambda} |x_\nu|^2 - \sum_{\nu\in\Lambda} |x_{\rho\nu\rho^{-1}}|^2| &= \sum_{\nu\in\Lambda} (|x_\nu| + |x_{\rho\nu\rho^{-1}}|)||x_\nu| - |x_{\rho\nu\rho^{-1}}|| \\ &\leq \sum_{\nu\in\Lambda} (|x_\nu| + |x_{\rho\nu\rho^{-1}}|)(|x_\nu - x_{\rho\nu\rho^{-1}}|) \\ &\leq 2||x||_2 (\sum_{\nu\in\Lambda} |x_\nu - x_{\rho\nu\rho^{-1}}|^2)^{1/2} \end{aligned}$$

so that if $\rho \in \{\alpha, \beta, \gamma\}$ we have

$$|\sum_{\nu\in\Lambda} |x_\nu|^2 - \sum_{\nu\in\Lambda} |x_{\rho\nu\rho^{-1}}|^2| \leq 2\epsilon$$

where $\epsilon = \max\{|||[x, u_\alpha]|||_2, |||[x, u_\beta]|||_2, |||[x, u_\gamma]|||_2\}$.

Let us now first overestimate $||x||^2 = 1$:

$$\begin{aligned} 1 &\leq \sum_{\nu\in\Delta} |x_\nu|^2 + \sum_{\nu\in\Delta} |x_{\alpha\nu\alpha^{-1}}|^2 \\ &\leq 2\sum_{\nu\in\Delta} |x_\nu|^2 + 2\epsilon. \end{aligned}$$

Now underestimate it:

$$\begin{aligned} 1 &\geq \sum_{\nu\in\Delta} |x_\nu|^2 + \sum_{\nu\in\Delta} |x_{\beta\nu\beta^{-1}}|^2 + \sum_{\nu\in\Delta} |x_{\gamma\nu\gamma^{-1}}|^2 \\ &\geq 3\sum_{\nu\in\Delta} |x_\nu|^2 - 4\epsilon. \end{aligned}$$

Let $y = \sum_{\nu\in\Delta} |x_\nu|^2$ and eliminate y from the inequalities $1 \leq 2y + 2\epsilon$ and $1 \geq 3y - 4\epsilon$ to obtain

$$\epsilon \geq 1/14$$

as desired. □

It is easy to come up with groups having subsets as in the previous theorem. For instance if $G = F_2$, free on generators g and h, let Δ be the set of all reduced words ending in a non-zero power of g. Let $\alpha = g$, $\beta = h$ and $\gamma = h^{-1}$. The same works for more than two generators. We conclude:

Theorem 16.3.2. *The type* II_1 *factor* $vN(F_n)$ *for* $n \geq$ *does not have property* Γ.

Chapter 17

Bimodules and property T

Chapter 18

Fermions and Bosons:CAR and CCR

According to physics lore, the states of a quantum system are given by (the one-dimensional subspaces of) a Hilbert space $\mathcal{H}$ and if two systems have state spaces $\mathcal{H}$ and $\mathcal{K}$, the joint system has state space $\mathcal{H} \otimes \mathcal{K}$. Fermions are particles such that "the wave function of several fermions is antisymmetric" which means that it is the antisymmetric tensor product $\Lambda^n\mathcal{H}$ which describes n identical fermions. Bosons are particles whose wave functions are symmetric so it is the symmetric tensor power $S^n\mathcal{H}$ which describes n identical bosons. In order to treat families with an unlimited number of fermions and Bosons we need the fermionic and bosonic Fock spaces (which are Hilbert space direct sums):

$$\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty}\Lambda^n\mathcal{H}$$

and

$$S(\mathcal{H}) = \oplus_{n=0}^{\infty}S^n\mathcal{H}.$$

The passage from $\mathcal{H}$ to $\mathcal{F}(\mathcal{H})$ or $S(\mathcal{H})$ is known as "second quantisation'.

We will not attempt to explain the physics above but will define properly these two Fock spaces and how they give rise to interesting von Neumann algebras related to physics.

Both these Fock spaces are subspaces of the "full Fock space" or tensor algebra

$$\mathcal{T}(\mathcal{H}) = \oplus_{n=0}^{\infty} \otimes^n \mathcal{H}$$

$\mathcal{T}(\mathcal{H})$ is related to quantum physics also though so far in a less fundamental way through the large N behaviour of random $N \times N$ matrices and Voiculescu's free probability.

18.1 The Fock spaces.

18.1.1 Full Fock space

Definition 18.1.2. *If $\mathcal{H}$ is a Hilbert space the* full Fock space $\mathcal{T}(\mathcal{H})$ *is the Hilbert space direct sum $\oplus_{n=0}^{\infty} \otimes^n \mathcal{H}$. By definition $\otimes^0 \mathcal{H}$ is one dimensional, spanned by the "vacuum" vector Ω.*

For each n and $f \in \mathcal{H}$ the operator $\ell(f) : \otimes^n \mathcal{H} \to \otimes^{n+1} \mathcal{H}$ given by

$$\ell(f)(\xi_1 \otimes \xi_2 \cdots \xi_n) = f \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$$

is clearly bounded by $||f||$ so extends to an operator we will call $\ell(f)$ on all of full Fock space.

Exercise 18.1.3. *(i) Show that $\ell(f)^*(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \langle \xi_1, f \rangle \xi_2 \otimes \xi_3 \cdots \xi_n$ and $\ell(f)^*(\xi) = \langle \xi, f \rangle \Omega$ for $\xi \in \otimes^1 \mathcal{H}$. (ii) Show that*

$$\ell(f)^* \ell(g) = \langle g, f \rangle$$

18.1.4 Fermionic Fock space.

Given a Hilbert space $\mathcal{H}$, the *nth.* exterior or antisymmetric power of $\mathcal{H}$ is the Hilbert space $\Lambda^n \mathcal{H} = p(\otimes^n \mathcal{H})$ where p is the antisymmetrisation projection

$$p(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \xi_{\sigma(1)} \otimes \xi_{\sigma(1)} \otimes \xi_{\sigma(2)} \cdots \otimes \xi_{\sigma(n)}$$

Definition 18.1.5. *The fermionic Fock space of $\mathcal{H}$ is the Hilbert space direct sum*

$$\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty} \Lambda^n \mathcal{H}$$

.

Given $\xi_1, ..\xi_n \in \mathcal{H}$ we set

$$\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n = \sqrt{n!} \quad p(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n)$$

Exercise 18.1.6. *Show that $\langle \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n, \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_n \rangle$ is the determinant of the matrix whose (i, j) entry is $\langle \xi_i, \eta_j \rangle$.*

Obviously if $\sigma \in S_n$,

$$\xi_{\sigma(1)} \wedge \xi_{\sigma(1)} \wedge \xi_{\sigma(2)} \cdots \wedge \xi_{\sigma(n)} = (-1)^\sigma \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n.$$

Exercise 18.1.7. *For $f \in \mathcal{H}$ define $A(f) : \otimes^n \mathcal{H} \to \Lambda^{n+1}\mathcal{H}$ by*

$$A(f)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \frac{1}{\sqrt{n!}} f \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n,$$

show that $A(f)(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) = f \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$.

Exercise 18.1.8. *The previous exercise shows that for each $f \in \mathcal{H}$ there is a bounded linear map from $\Lambda^n\mathcal{H}$ to $\Lambda^{n+1}\mathcal{H}$ defined by:*

$$a(f)(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) = f \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n.$$

Show that

$$a(f)^*(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \xi_1 \wedge \cdots \hat{\xi}_i \cdots \wedge \xi_{n+1}$$

We have sloppily left out the n on our operators $a(f)$. But we can put them all together to form the densely defined operators $a(f)$ and $a(f)^*$ on $\mathcal{F}(\mathcal{H})$ whose domain is the algebraic direct sum of the $\Lambda^n\mathcal{H}$.

Exercise 18.1.9. *Show that these densely defined operators satisfy the CAR relations.*

Exercise 18.1.10. *Show that $||a(f)\xi|| \leq ||\xi||$ for ξ in the domain of $a(f)$ so $a(f)$ an $a(f)^*$ extend to bounded operators on $\mathcal{F}(\mathcal{H})$ which are one another's adjoints and satisfy the CAR relations.*

18.2 CAR algebra, CCR algebra and the (extended) Cuntz algebra.

18.2.1 CAR algebra

Definition 18.2.2. *If $\mathcal{H}$ is a complex Hilbert space the CAR (canonical anticommuation relations) algebra $CAR(\mathcal{H})$ is the unital *-algebra with generators $a(f)$ for each $f \in \mathcal{H}$ subject to the following relations:*
(i) The map $f \mapsto a(f)$ is linear.
(ii) $a(f)a(g) + a(g)a(f) = 0 \quad \forall f, g \in \mathcal{H}$.
(iii) $a(f)a(g)^* + a(g)^*a(f) = \langle f, g \rangle \quad \forall f, g \in \mathcal{H}$.

(the identity is implicit on the right hand side of (iii))

It is clearly going to be useful to define the anticommutator of two operators as

$$\{a, b\} = ab + ba.$$

Exercise 18.2.3. *Show that if* $\dim \mathcal{H} = 1$, $CAR(\mathcal{H})$ *is isomorphic to* $M_2(\mathbb{C})$. *Show further that if* $\dim \mathcal{H} = n < \infty$, $CAR(\mathcal{H}) \cong M_{2^n}(\mathbb{C})$.

This exercise immediately implies that there is a unique C^*-norm on $CAR(\mathcal{H})$ so it is neither here nor there to consider it a C^*-algebra.

Exercise 18.2.4. *Show that* $||a(f)|| = ||f||$.

Exercise 18.2.5. *A unitary* u *on* $\mathcal{H}$ *obviously defines an automorphism* α_u *of* $CAR(\mathcal{H})$ *(sometimes called a Bogoliubov automorphism) by functorially extending* $\alpha_u(a(f)) = a(uf)$. *In particular choosing* $u = -1$ *makes* $CAR(\mathcal{H})$ *into a* $\mathbb{Z}/2\mathbb{Z}$*-graded algebra. Define a notion of graded product* $A \otimes_{\mathbb{Z}/2\mathbb{Z}} B$ *for* $\mathbb{Z}/2\mathbb{Z}$*-graded algebras* A *and* B. *Show that if* V *and* W *are orthogonal Hilbert subspaces of* $\mathcal{H}$ *then* $CAR(V \oplus W) \cong CAR(V) \otimes_{\mathbb{Z}/2\mathbb{Z}} CAR(W)$.

18.2.6 CCR algebra

18.2.7 Cuntz algebra

Definition 18.2.8. *Given the Hilbert space* $\mathcal{H}$, *let the* extended Cuntz algebra $\mathcal{C}(\mathcal{H})$ *be the unital* $*$*-algebra with generators* ℓf *for each* $f \in \mathcal{H}$ *subject to the following relations:*
(i) The map $f \mapsto \ell(f)$ *is linear.*
(ii) $\ell(f)^*\ell(g) = \langle g, f\rangle \quad \forall f, g \in \mathcal{H}$.

Exercise 18.2.9. *If* $\xi_1, \xi_2, ..., \xi_n$ *are orthogonal unit vectors then* $\ell(\xi_i)$ *are isometries with orthogonal ranges, and the projection*

$$\sum_{i=1}^{n} \ell(\xi_i)\ell(\xi_i)^*$$

depends only on the space spanned by $\xi_1, \xi_2, ..., \xi_n$.

18.3 Quasi-free states

18.4 Complex structure

One way to obtain the existence of the quasi-free state ϕ_P for a projection P is to change the complex structure on $\mathcal{H}$.

Definition 18.4.1. *If $\mathcal{K}$ is a* real *Hilbert space, a complex structure on $\mathcal{K}$ is an orthogonal transformation J with $J^2 = -1$.*

Lemma 18.4.2. *A real Hilbert space $\mathcal{K}$ with inner product $(,)$ and a complex structure J can be made into a complex Hilbert space by defining the action of $\mathbb{C}$ as $(x + iy)\xi = x\xi + yJ\xi$ and the inner product*

$$\langle \xi, \eta \rangle = (\xi, \eta) - i(J\xi, \eta)$$

Proof. The vector space structure is routine as is sesquilinearity of $\langle , \rangle$. But $\langle \xi, \xi \rangle = (\xi, \xi) - i(J\xi, xi)$ and $(J\xi, xi) = -(\xi, J\xi) = -(J\xi, xi)$ which is therefore zero. Hence $<,>$ is positive definite and defines the same norm as $(,)$ so completeness is unchanged. □

Definition 18.4.3. *The Clifford algebra of a real Hilbert space kh is the (complex) *-algebra generated by $c(f)$ subject to:*
(i) The map $f \mapsto c(f)$ is real linear.
(ii) $c(f) = c(f)*$ $\quad \forall f \in \mathcal{K}$.
(iii) $\{c(f), c(g)\} = 2(f, g)$ $\quad \forall f, g \in \mathcal{H}$.

(clearly $c(f)^2 = (f, f)$ is equivalent to (iii)).

Proposition 18.4.4. *If kh is a real Hilbert space with complex structure J, defining $a(f) = c(f) + ic(Jf)$ defines a representation of the CAR algebra of the complex Hilbert space in the Clifford algebra of $\mathcal{K}$.*

Proof. □

Proposition 18.4.5. *If $\mathcal{H}$ is a complex Hilbert space, sending $c(f)$ to $a(f) + a(f)^*$ defines an isomorphism of the Clifford algebra of the real Hilbert space defined by the real part of $\langle , \rangle$ with $CAR(\mathcal{H})$*

Given a complex Hilbert space $\mathcal{H}$ and a projection $p \in \mathcal{B}(\mathcal{H})$ we can change the complex structure simply by changing the sign of i on $p\mathcal{B}(\mathcal{H})^\perp$ thus: consider $\mathcal{H}$ as a real Hilbert space with $(\xi, \eta) = Re(\langle \xi, \eta \rangle)$ and define and J by $J = ip - i(1 - p)$ (which is actually $\mathbb{C}$-linear). Call $\mathcal{H}_J$ the Hilbert space with the new complex structure. Then by the previous proposition $a(f)$ defines an isomorphism of $CAR(\mathcal{H}_J)$ with $CAR(\mathcal{H})$ and hence a representation of $CAR(\mathcal{H}_J)$ every time we have one of $CAR(\mathcal{H})$, in particular the Fock representation.

The question thus becomes: what does $\mathcal{H}_J$ look like? If $\mathcal{H}$ is a Hilbert space let $\overline{\mathcal{H}}$ be the dual Hilbert space of $\mathcal{H}$ and $\xi \mapsto \overline{\xi}$ be the canonical antilinear map from $\mathcal{H}$ to its dual.

Proposition 18.4.6. *Let $q = (1 - p)$. Then the map $\xi \mapsto p\xi \oplus \overline{q\xi}$ is a $\mathbb{C}$-linear isomorphism from $\mathcal{H}$ to $\mathcal{H}_J$.*

Chapter 19

Subfactors

Chapter 20

Free probability

Appendix A

Proof of the KMS condition.

Theorem A.0.7. *Let ϕ be a faithful normal state on a von Neumann algebra M. Then the modular group σ_t^ϕ is the unique one parameter automorphism group of M which satisfies the KMS condition for ϕ.*

Proof. Perform the GNS construction with canonical cyclic and separating vector Ω and modular operators $S = J\Delta^{1/2}$. Recall that $f(\Delta)\Omega = \Omega$ for any function of Δ with $f(1) = 1$. In particular $\phi(\sigma_t^\phi(x) = \langle(\Delta^{it}x\Delta^{-it}\Omega, \Omega\rangle$ so σ_t^ϕ preserves ϕ.

Now let us check the rest of the KMS condition. We have

$$\phi(\sigma_t^\phi(x)y) = \langle\Delta^{-it}y\Omega, x^*\Omega\rangle$$

and

$$\begin{aligned}\phi(y\sigma_t^\phi(x)) &= \langle y\sigma_t^\phi(x)\Omega, \Omega\rangle \\ &= \langle J\Delta^{1/2}\sigma_t^\phi(x^*)\Omega, J\Delta^{1/2}y\Omega\rangle \\ &= \langle\Delta^{1/2}y\Omega, \Delta^{1/2}\Delta^{it}x^*\Omega\rangle \\ &= \langle\Delta^{1/2-it}y\Omega, \Delta^{1/2}x^*\Omega\rangle\end{aligned}$$

So let $\xi = y\Omega$, $\eta = x^*\Omega$ and let p_n be the spectral projection for Δ for the interval $[1/n, n]$ so that p_n tends strongly to 1 and $\Delta^{\pm 1}$ are bounded on $p_n\mathcal{H}_\phi$. The functions

$$F_n(z) = \langle\Delta^{-iz}p_n\xi, \eta\rangle$$

are then entire and

$$\begin{aligned}|F_n(t) - \phi(\sigma_t^\phi(x)y)| &= |\langle\Delta^{-it}(1-p_n)\xi, \eta\rangle| \leq ||(1-p_n)\xi||\ ||\eta|| \\ |F_n(t+i) - \phi(y\sigma_t^\phi(x))| &= |\langle\Delta^{1/2-it}(1-p_n)\xi, \Delta^{1/2}\eta\rangle| \leq ||(1-p_n)\Delta^{1/2}\xi||\ ||\Delta^{1/2}\eta||.\end{aligned}$$

Hence the F_n are bounded and continuous on the strip $\{z : 0 < \Im mz < 1\}$ and converge uniformly on its boundary. By the Phragmen-Lindelof theorem we are done.

Now let us prove uniqueness of the modular group with the KMS condition.

Let α_t be another continous one-parameter automorphism group satisfying KMS for ϕ. The fact that α_t preserves ϕ means we can define a strongly continous one-parameter unitary group $t \mapsto u_t$ by $u_t x\Omega = \alpha_t(x)\Omega$. By Stone's theorem it is of the form $t \mapsto D^{it}$ for some non-singular positive self-adjoint operator A. The goal is to prove that $D = \Delta$. As a first step we construct a dense set of analytic vectors in $M\Omega$ by Fourier transform. Let A be the set of all operators of the form

$$\int_{-\infty}^{\infty} \hat{f}(t)\alpha_t(x)dx$$

for all C^∞ functions f of compact support on $\mathbb{R}$. The integral converges strongly so

$$f(\log(D))x\Omega = \int_{-\infty}^{\infty} \hat{f}(t)D^{it}(x\Omega)dx$$

is in $A\Omega$. Thus the spectral projections of D are in the strong closure of A and $A\Omega$ is dense. Moreover $z \mapsto D^z x\Omega$ is analytic for $x \in A$ since $x\Omega$ is in the spectral subspace of A for a bounded interval. Also $A\Omega$ is invariant under D^z by the functional calculus. To compare with ϕ define, for x and y in A, the entire function

$$F_1(z) = \langle D^{-iz}y\Omega, x^*\Omega\rangle.$$

Let F be the function, analytic inside the strip and continuous and bounded on it, guaranteed for x and y by the KMS condition. Then if we define $G(z)$ for $-1 \leq \Im mz \leq 1$ by

$$G(z) = \begin{cases} F(z) - F_1(z) & \text{if } \Im mz \geq 0; \\ \\ \overline{F(\overline{z}) - F_1(\overline{z})} & \text{if } \Im mz \leq 0. \end{cases}$$

check typesetting

Since F and F_1 agree on the real line G is analytic for $-1 < \Im mz < 1$, hence equal to 0, and since both F and F_1 are continous on the strip, $\phi(y\sigma_t(x)) = F(t+i) = F_1(t+i) = \langle D^{1-it}y\Omega, x^*\Omega\rangle$. In particular putting $t = 0$ we get

$$\begin{aligned} \langle Dy\Omega, x^*\Omega\rangle &= \phi(yx) \\ &= \langle x\Omega, y^*\Omega\rangle \\ &= \langle J\Delta^{1/2}x^*\Omega, J\Delta^{1/2}y\Omega\rangle \\ &= \langle \Delta^{1/2}y\Omega, \Delta^{1/2}x^*\Omega\rangle \end{aligned}$$

So $\Delta^{1/2} y\Omega$ is in the domain of $\Delta^{1/2}$ and $\Delta y\Omega = Dy\Omega$.

Thus D and Δ agree on $A\Omega$. But multiplication by the function $e^z + 1$ is a linear isomorphism of C_c^∞ so by functional calculus $(D+1)A\Omega = A\Omega$ which is thus dense. Since $D + 1$ is invertible by spectral theory, any $(\xi, (D + 1)\xi)$ in the graph of $D+1$ can be approximated by $(A_n\Omega, (D+1)A_n\Omega)$. Thus D is essentially self-adjoint on $A\Omega$, and both Δ and D are self-adjoint extensions of the restriction of D to this domain. Thus $D = \Delta$. □

Bibliography

[1] F.M. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter graphs and towers of algebras*, Springer-Verlag, 1989.

[2] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, 1972.

[3] M. Takesaki, *Theory of Operator Algebras II*. EMS series on Operator Algebras, 2002.

[4] J.-B. Bost and A.Connes, *Hecke algbebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series),1, (1995), 411-457.