Definability in Degree Structures

Theodore A. Slaman and W. Hugh Woodin

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Relative Definability and Degree Structures

In this chapter, we describe our notation and make some preliminary observations. We do not intend this chapter to be a rigorous treatment of the fundamentals of recursion theory; see instead Enderton (2001) or Shoenfield (2001) for the basic concepts and Soare (1987) for more advanced topics. Rather, we use this chapter as a catalog of facts that we will use as we go along.

1.1 Recursive Sets and Turing functionals

We use ω to denote the set of natural numbers $0, 1, 2, \dots$ and 2^ω to denote the set of subsets of ω. When convenient, we will view an element of 2^ω as a Boolean valued function so that $A(n) = 0$ is equivalent to $n \notin A$ and $A(n) = 1$ is equivalent to $n \in A$.

An n-ary relation R on ω is *recursive* if and only if there is a program P such that, for each $n \in \omega$, if P is started in its initial state with input n, then eventually P returns the value $R(n)$. A subset W of ω is *recursively enumerable* if and only if it is the projection of a recursive relation. That is, for some recursive R, $y \in W \iff (\exists x_1, \dots, x_n)[(y, x_1, \dots, x_n) \in R]$.

Similarly, if A is a set, then B is *recursive in* A, $B \leq_T A$, if and only if there is a program P which can query A such that, for each $n \in \omega$, if P is started in its initial state with input n, then eventually P returns the value $B(n)$. A and B are *Turing equivalent*, $A \equiv_T B$, if and only if each is recursive in the other.

Definition 1.1.1 The *Turing degrees* are the $\equiv_T$-equivalence classes, which are ordered by evaluating $\geq_T$ on representatives. We let $\mathfrak{D}$ denote this partially ordered set.

If A and B are subsets of ω, we let $A \oplus B$ denote their recursive join:

$$A \oplus B = \{2n : n \in A\} \cup \{2n + 1 : n \in B\}.$$

If a and b are the degrees of A and B, then we let $a \vee b$ denote the Turing degree of $A \oplus B$. Note that $a \vee b$ is the least upper bound of the pair $\{a, b\}$ in $\mathfrak{D}$.

Definition 1.1.2 A subset $\mathcal{I}$ of $\mathfrak{D}$ is an *ideal* if and only if $\mathcal{I}$ is closed under $\leq_T$ ($x \in \mathcal{I}$ and $y \leq_T x$ implies $y \in \mathcal{I}$) and closed under $\vee$ ($x \in \mathcal{I}$ and $y \in \mathcal{I}$ implies $x \vee y \in \mathcal{I}$).

Definition 1.1.3 1. A *Turing functional* Φ is a set of sequences (x, y, σ) such that x is a natural number, y is either 0 or 1, and σ is a finite binary sequence. Further, for all x, for all y_1 and y_2, and for all compatible σ_1 and σ_2, if $(x, y_1, \sigma_1) \in \Phi$ and $(x, y_2, \sigma_2) \in \Phi$, then $y_1 = y_2$ and $\sigma_1 = \sigma_2$.

2. Φ is *use-monotone* if the following conditions hold.
   (a) For all (x_1, y_1, σ_1) and (x_2, y_2, σ_2) in Φ, if σ_1 is a proper initial segment of σ_2, then x_1 is less than x_2.
   (b) For all x_1 and x_2, y_2 and σ_2, if $x_2 > x_1$ and $(x_2, y_2, \sigma_2) \in \Phi$, then there are y_1 and σ_1 such that $\sigma_1 \subseteq \sigma_2$ and $(x_1, y_1, \sigma_1) \in \Phi$.
3. We write $\Phi(x, \sigma) = y$ to indicate that there is a τ such that τ is an initial segment of σ, possibly equal to σ, and $(x, y, \tau) \in \Phi$. If $X \subseteq \omega$, we write $\Phi(x, X) = y$ to indicate that there is an ℓ such that $\Phi(x, X \restriction \ell) = y$, and write $\Phi(X)$ for the function evaluated in this way.

Note that in Definition 1.1.3, we did not require that Φ be definable. Consequently, if Φ is a Turing functional and $X \subseteq \omega$, then $\Phi(X)$ is recursive only in the join of Φ and X. The program to compute $\Phi(X)$ from X would be for input n to search in Φ for a triple of the form (n, y, σ) such that $\sigma \subset X$ and output the value y for the first such triple found.

Definition 1.1.4 A *recursive functional* is a Turing functional Φ such that Φ is recursively enumerable.

Note, for subsets A and B of ω, B is recursive in A if and only if there is a recursive functional Φ such that $\Phi(A) = B$.

1.2 Preliminary Observations

Let σ_0 denote the null sequence. If B is a recursive set, then the functional $\Phi = \{(x, B(x), \sigma_0) : x \in \omega\}$ is recursive. Further, for every A, $\Phi(A) = B$. Consequently, the recursive sets form a Turing degree and that degree is below every other degree. We let 0 denote the Turing degree of the recursive sets, the least element of $\mathfrak{D}$.

Theorem 1.2.1 1. *For each x in $\mathfrak{D}$, there are at most countably many y in $\mathfrak{D}$ such that $x \geq_T y$. ($\mathfrak{D}$ is* locally countable.*)*
2. *$\mathfrak{D}$ has cardinality the continuum.*

Proof: The first claim follows from the observation that there are only countably many recursive functionals. The second claim follows similarly from the observation that 2^ω has size the continuum and $\mathfrak{D}$ is is the quotient of 2^ω by an equivalence relation in which each equivalence class is countable.

Definition 1.2.2 1. For A and B contained in ω, define $A \oplus B$ as follows.

$$A \oplus B = \{2n : n \in A\} \cup \{2n + 1 : n \in B\}$$

$A \oplus B$ is called the *recursive join* of A and B.
2. If a and b are the Turing degrees of A and B, respectively, then $a + b$ is the Turing degree of $A \oplus B$.

Theorem 1.2.3 *For Turing degrees a and b, $a + b$ is the least upper bound in $\mathfrak{D}$ of the pair $\{a, b\}$.*

Proof: Let X, A, and B be representatives of x, a, and b, respectively.

Suppose that A and B are recursive relative to X. It follows that $A \oplus B$ is recursive in X, and so $x \geq_T a + b$. Conversely, if $x \geq_T a + b$ then $A \oplus B$ is recursive in X, and so both of A and B are recursive in X. Then, $x \geq_T a$ and $x \geq_T b$, as required. ∎

1.3 The Arithmetic Hierarchy

Let $\mathbb{N}$ be the structure with universe ω, the natural numbers, with binary operations addition, $+$, and multiplication, $\times$, the binary order relation, $>$, and distinguished elements 0 and 1. We define the set of first order terms and formulas in this language as usual, for example as in (Enderton, 2001).

Definition 1.3.1 1. The *bounded formulas* are the subset of the formulas in the first order language of $\mathbb{N}$ obtained by closing the atomic formulas under Boolean operations and bounded quantification, $\forall x < t$ or $\exists x < t$, where t is an arithmetic term which does not mention x. Here, $(\forall x < t)\varphi$ is the formula $\forall x(x < t \to \varphi)$, and $(\exists x < t)\varphi$ is the formula $\exists x(x < t \to \varphi)$.

2. We define the collections of formulas Σ^0_n and Π^0_n by recursion.
   (a) Σ^0_0 and Π^0_0 are both equal to the class of bounded formulas.
   (b) A formula is Σ^0_{n+1} if and only if it is of the form $\exists x_1 \ldots \exists x_n \theta$, where $\theta \in \Pi^0_n$.
   (c) A formula is Π^0_{n+1} if and only if it is of the form $\forall x_1 \ldots \forall x_n \theta$, where $\theta \in \Sigma^0_n$.

Definition 1.3.2 1. For R a relation on ω, we say that R is Σ^0_n or Π^0_n if and only if R is definable in $\mathbb{N}$ by a formula of the corresponding type.

2. R is Δ^0_n if and only if it is both Σ^0_n and Π^0_n.
3. R is *arithmetically definable* if there is an n such that R is Σ^0_n.

The first few levels of the arithmetic hierarchy appear naturally as recursion theoretic classes. For any sets R and W, R is Δ^0_1 if and only if it is recursive, and W is Σ^0_1 if and only if it is recursively enumerable.

1.3.1 Arithmetic definability relative to a real parameter

Suppose that R is a subset of ω, and consider the extension of $\mathbb{N}$ obtained by adding a predicate for R. The first order language appropriate for this structure is the language for $\mathbb{N}$ with an additional unary predicate symbol R. When t is a term, $R(t)$ is an additional atomic formula, and we generate the language for the expanded structure as above. Of course, we are not limited on adding only one predicate, we could add arbitrarily many of them.

Theorem 1.3.3 *Let φ be a bounded sentence. There is a number b, computed uniformly from φ, such that if R_1 and R_2 are subsets of ω which have exactly the same elements less than b, then $(\mathbb{N}, R_1) \models \varphi$ if and only if $(\mathbb{N}, R_2) \models \varphi$.*

Proof: We prove Theorem 1.3.3 by induction on the definition of satisfaction for φ.

If φ is an atomic sentence, then φ is one of an equality $t_1 = t_2$ between terms, an instance of order $t_1 > t_2$ between terms, or an assertion that some term satisfies the predicate $R(t)$. In the first two cases, the satisfaction of φ does not depend on the interpretation of R and we may let b equal 0 to verify the claim. Consider the third case. Since φ is a sentence, the term mentioned in φ is a closed expression in the constants 0 and 1 and the function symbols $+$ and $\times$. The value v of this term is recursively determined, and does not depend on the interpretation of R. Let b equal $v + 1$. If R_1 and R_2 agree on all of the numbers less than b, then $v \in R_1$ if and only if $v \in R_2$. Consequently, $(\mathbb{N}, R_1) \models \varphi$ if and only if $(\mathbb{N}, R_2) \models \varphi$.

If φ is a Boolean combination of simpler formulas φ_1 and φ_2, then we may take the supremum b of the bounds b_1 and b_2 computed for the subformulas of φ and argue that if R_1 and R_2 agree on the numbers less than b, then $(\mathbb{N}, R_1) \models \varphi$ if and only if $(\mathbb{N}, R_2) \models \varphi$.

Finally, suppose that φ is of the form $(\exists x < t)\varphi_0$. We repeat the argument from above. Since φ is a sentence, there cannot be any variables in t. Consequently, t is a closed term, the interpretation m of t is recursively determined from t. Then $(\exists x < t)\varphi_0$ is equivalent to the disjunction of $\varphi_0(0), \varphi_0(1), \ldots, \varphi_0(m-1)$. For each of these disjuncts $\varphi_0(j)$, we can compute a number b_j as required by the theorem. Then let b be the supremum of $b_0, \ldots, b_{m-1}$. If R_1 and R_2 agree on all numbers less than b, then for each j less than m, $(\mathbb{N}, R_1) \models \varphi_0(j)$ if and only if $(\mathbb{N}, R_2) \models \varphi_0(j)$. Consequently, $(\mathbb{N}, R_1) \models \varphi$ if and only if $(\mathbb{N}, R_2) \models \varphi$. ∎

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Generic Sets

2.1 Forcing

Some general comments about forcing.

2.2 Forcing in Set Theory

Definition 2.2.1 Let T be the fragment of *ZFC* which includes only the instances of replacement and comprehension in which the defining formula is Σ_1.

Our choice of T is not critical. We will only need to know that T is strong enough to formulate definitions for and prove the basic facts about the natural numbers, the reals, and the power set of the reals, and that T is consistent. Any sufficiently strong recursively axiomatized fragment of set theory would be acceptable.

2.3 Adding Cohen Reals

Let P_{ω_1} be the partial order to add ω_1-many Cohen (1966) reals to V. A condition in P_{ω_1} is a finite map with domain contained in $\omega_1 \times \omega$ and range contained in $\{0, 1\}$. Conditions are ordered by inclusion, the stronger condition being defined at more points.

Suppose that $\mathcal{G}$ is P_{ω_1}-generic over V. We will identify $\mathcal{G}$ with the function from $\omega_1 \times \omega$ into $\{0, 1\}$ given by its union. We can equivalently view $\mathcal{G}$ as an ω_1-sequence of reals $(G_\alpha : \alpha < \omega_1)$ by setting $G_\alpha(n)$ equal to $\mathcal{G}(\alpha, n)$.

Suppose that $X \in 2^\omega$ is an element of $V[\mathcal{G}]$.

- There is a set $\mathcal{G}_X$ such that $\mathcal{G}_X$ is P_{ω_1} generic over $V[X]$ and $V[X][\mathcal{G}_X] = V[\mathcal{G}]$.
-

2.4 Forcing in Arithmetic

In recursion theory, forcing is typically used to construct exotic subsets of ω. It is also typical that their exotic properties are arithmetically definable.

For example, Kleene and Post (1954) constructed a pair of sets with incomparable Turing degree, Friedberg (1957) inverted the Turing jump on the Turing degrees above $0'$, and Spector (1956) constructed a set with minimal Turing degree. Even though this observation came after the fact, the proofs of these theorems turn on the analysis of the forcing relation for the appropriate partial orders. In the first two cases the partial order is $2^{<\omega}$, and in the third case the partial order is that of recursive perfect trees ordered by inclusion.

Two hallmarks of recursion theoretic forcing are visible in these examples. First, it is possible to build sets which are generic with respect to one set of criteria and fail to be generic with respect to another, usually more complex, criterion. For example, in the proof of the Friedberg Jump Inversion Theorem, one builds a set G which is sufficiently generic so that atomic facts about G' are decided by forcing, but not so generic that the Turing degree of G' is incomparable with the given set above $0'$ upon which one wishes to invert the jump. Second, it is possible to use forcing and not be interested in the generic filter, but rather in a set derived from it. For example, Spector's (1956) minimal degree is the unique path through the set of perfect trees in the generic filter.

2.4.1 The forcing relation for arithmetic formulas

We consider the special case in which P is a partial order on a subset of ω. In this context, we refer to elements of P as conditions and use $>_P$ to denote the ordering between conditions. The contents of this section are well-known. However, since much of the machinery in the sections and chapters which follow depends on the fine analysis of forcing, we will fix a particular presentation and associated system of notation.

Definition 2.4.1 We define the *strong forcing relation* $\Vdash^*$ between elements p of P and arithmetic sentences $\varphi(\boldsymbol{m}, P, \mathcal{G})$. In this context, we syntactically identify a sequence of natural numbers $\boldsymbol{m}$ with the sequence of symbols which refer to it. That is to say that we will write $m_1+m_2 = m_3$ to refer to the sentence asserting that the sum of the term referring to m_1 with the term referring to m_2 is equal to the term referring to m_3. Similarly, we identify the partial order P with symbol to refer to it, and we introduce a unary symbol $\mathcal{G}$, which will refer to a generic filter G.

1. If $\varphi(\boldsymbol{m}, P, \mathcal{G})$ is an atomic sentence which does not mention $\mathcal{G}$, such as $m \in P$, $m_1 >_P m_2$, $m_1 + m_2 = m_3$, and so forth, then $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$ if and only if φ is true in the standard model of arithmetic with an additional predicate symbol referring to P.
2. $p \Vdash^* q \in \mathcal{G}$ if and only $q \geq_P p$.
3. $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$ & $\psi(\boldsymbol{m}, P, \mathcal{G})$ if and only if $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$ and $p \Vdash^* \psi(\boldsymbol{m}, P, \mathcal{G})$.
4. $p \Vdash^* (\exists x)\psi(x, \boldsymbol{m}, P, \mathcal{G})$ if and only for some $k \in \omega$, $p \Vdash^* \psi(k, \boldsymbol{m}, P, \mathcal{G})$.
5. $p \Vdash^* \neg\theta(\boldsymbol{m}, P, \mathcal{G})$ if and only if, for all q, if $p \geq_P q$, then it is not the case that $q \Vdash^* \theta(\boldsymbol{m}, P, \mathcal{G})$.

The definition of the strong forcing relation is by recursion on the complexity of φ. In other words, when we specify whether $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$, we may assume that we have specified the forcing relation between any condition in P and any instance of a subformula of $\varphi(\boldsymbol{m}, P, \mathcal{G})$. That the recursion has this form is especially obvious in the clause defining forcing for a negation.

Theorem 2.4.2 1. *For any $p \in P$ and any sentence $\varphi(\boldsymbol{m}, P, \mathcal{G})$, if $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$, then it is not the case that $p \Vdash^* \neg\varphi(\boldsymbol{m}, P, \mathcal{G})$.*
2. *For any $p \in P$ and any sentence $\varphi(\boldsymbol{m}, P, \mathcal{G})$, if $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$ and $p \geq_P q$, then $q \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$.*

Proof: The first claim follows directly from the fifth clause in the definition of $\Vdash^*$.

The second claim is proven by induction on the definition of $\Vdash^*$. ■

Theorem 2.4.3 *For each sentence $\varphi(\boldsymbol{m}, P, \mathcal{G})$ as above, the set of p in P such that $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$ is arithmetically definable relative to P.*

Proof: In Definition 2.4.1, we specified the relation $\Vdash^*$ for all arithmetic sentences using a recursion of length ω in which each step of the recursion is arithmetic relative to P. Fixing $\varphi(\boldsymbol{m}, P, \mathcal{G})$, there is a finite step in this recursion during which we specified for all p whether $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$. Thus, the set of p such that $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$ is arithmetically definable relative to P. ■

Definition 2.4.4 If p is a condition in P and $\varphi(\boldsymbol{m}, P, \mathcal{G})$ is a sentence, then p *strongly decides* $\varphi(\boldsymbol{m}, P, \mathcal{G})$ if and only if either $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$ or $p \Vdash^* \neg\varphi(\boldsymbol{m}, P, \mathcal{G})$.

Definition 2.4.5 A condition p in P *forces* an arithmetic sentence $\varphi(\boldsymbol{m}, P, \mathcal{G})$, written $p \Vdash \varphi(\boldsymbol{m}, P, \mathcal{G})$, if and only if $p \Vdash^* \neg\neg\varphi(\boldsymbol{m}, P, \mathcal{G})$.

2.4.2 Generic filters

Definition 2.4.6 Suppose that D is a subset of the partially order set P.

1. D is *open* if and only if for all p and q in P, if $p \geq_P q$ and $p \in D$, then $q \in D$.
2. D is *dense* if and only if for all $p \in P$ there is a $d \in D$ such that $p \geq_P d$.

Definition 2.4.7 Suppose that G is a subset of the partially order set P.

1. G is a *filter* on P if and only if the following conditions hold.
   (a) $G \neq \emptyset$.
   (b) For all p and q, if $p \geq_P q$ and $q \in G$, then $p \in G$.
   (c) For all p and q, if $p \in G$ and $q \in G$, then there is an $r \in G$ such that $p \geq_P r$ and $q \geq_P r$.
2. If $\boldsymbol{D}$ is a collection of dense open subsets of P, then G is a $\boldsymbol{D}$-*generic filter* on P if and only if G is a filter on P and for all $D \in \boldsymbol{D}$, $G \cap D \neq \emptyset$.

When the partial order P is understood, we will simply say that G is generic for $\boldsymbol{D}$.

Theorem 2.4.8 *Suppose that P is partial ordering on ω, $p \in P$, and $\boldsymbol{D} = (D_n : n \in \omega)$ is a countable collection of dense open subsets of P. Then there is a $G \subseteq P$ such that $p \in G$ and G is a $\boldsymbol{D}$-generic filter on P.*

Proof: We define a sequence of conditions $(p_n : n \in \omega)$ by recursion on n. Let $p_0 = p$. Let p_{n+1} be the least natural number such that $p_n \geq_P p_{n+1}$ and $p_{n+1} \in D_n$. There is such a p_{n+1} since D_n is a dense subset of P. Now, let G be defined by the following equation.

$$G = \{q : (\exists n)[q \geq_P p_n]\}$$

By definition, G is closed upwards in P. If q_1 and q_2 belong to G, then there are n_1 and n_2 such that $q_1 \geq_P p_{n_1}$ and $q_2 \geq_P p_{n_2}$. For m the supremum of n_1 and n_2, p_m is a common extension of q_1 and q_2 which belongs to G. Consequently, G is a filter on P. It is $\boldsymbol{D}$-generic since for each n, $p_{n+1} \in G \cap D_n$. ∎

Definition 2.4.9 G is an *arithmetically in P* generic filter on P if and only if G is a filter on P and for every D, if D is a dense open subset of P and D is arithmetically definable relative to P, then $G \cap D \neq \emptyset$.

When P is an arithmetic partial order, we will abbreviate by saying that that G is arithmetically P-generic.

Theorem 2.4.10 *Suppose that G is arithmetically P-generic. For each arithmetic sentence $\varphi(\boldsymbol{m}, P, \mathcal{G})$, there is a $p \in G$ such that either $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$ or $p \Vdash^* \neg\varphi(\boldsymbol{m}, P, \mathcal{G})$.*

Proof: By Theorem 2.4.3, for each sentence $\varphi(\boldsymbol{m}, P, \mathcal{G})$, the collection of $p \in P$ such that p strongly decides $\varphi(\boldsymbol{m}, P, \mathcal{G})$ is open and arithmetically definable relative to P. It is also dense, since either there is an extension of p which strongly forces $\varphi(\boldsymbol{m}, P, \mathcal{G})$, or p strongly forces $\neg\varphi(\boldsymbol{m}, P, \mathcal{G})$. By assumption, at least one such p must belong to G. ■

Theorem 2.4.11 *Suppose that $\varphi(\boldsymbol{m}, P, \mathcal{G})$ is a sentence, G is a filter on P, and for each subformula $\varphi_0(\boldsymbol{x}, \boldsymbol{m}, P, \mathcal{G})$ of $\varphi(\boldsymbol{m}, P, \mathcal{G})$ and for each $\boldsymbol{k}$ from ω there is a condition $p \in G$ such that p strongly decides $\varphi_0(\boldsymbol{k}, \boldsymbol{m}, P, \mathcal{G})$. Then, the following conditions are equivalent.*

1. *There is a $p \in G$, $p \Vdash \varphi(\boldsymbol{m}, P, \mathcal{G})$.*
2. $\varphi(\boldsymbol{m}, P, G)$

Proof: We first note that if $p \in G$ and $p \Vdash \varphi(\boldsymbol{m}, P, \mathcal{G})$, q in G and q strongly decides $\varphi(\boldsymbol{m}, P, \mathcal{G})$, then $q \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$. Otherwise, p and q would have a common extension r in G which as an extension of p would strongly force $\neg\neg\varphi(\boldsymbol{m}, P, \mathcal{G})$ and as an extension of q would strongly force $\neg\varphi(\boldsymbol{m}, P, \mathcal{G})$. The existence of such an r would contradict Theorem 2.4.2. Thus, we may replace the first condition above by the one which asserts that there is a $p \in G$, $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$.

We prove the equivalence between the two conditions by induction on the complexity of instances of subformulas of $\varphi(\boldsymbol{m}, P, \mathcal{G})$.

Atomic sentences. Suppose that $\varphi(\boldsymbol{m}, P, \mathcal{G})$ is an atomic sentence. If it does not refer to $\mathcal{G}$, then strongly forcing $\varphi(\boldsymbol{m}, P, \mathcal{G})$ is defined to be identical with $\varphi(\boldsymbol{m}, P, \mathcal{G})$'s being true, and the claim holds on trivial grounds. Now, consider the atomic sentence $m \in \mathcal{G}$.

Suppose that $p \in G$ and $p \Vdash^* (m \in \mathcal{G})$, that is $m \geq_P p$. Since G is a filter, m is an element of G, as required.

For the converse, suppose that $m \in G$. Then, m itself is a condition in G which strongly forces the sentence $m \in \mathcal{G}$.

Conjunction. Suppose that $\varphi(\boldsymbol{m}, P, \mathcal{G})$ is the conjunction of $\psi_1(\boldsymbol{m}, P, \mathcal{G})$ and $\psi_2(\boldsymbol{m}, P, \mathcal{G})$.

Suppose that $p \in G$ and $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$. Then, $p \Vdash^* \psi_1(\boldsymbol{m}, P, \mathcal{G})$ and $p \Vdash^* \psi_2(\boldsymbol{m}, P, \mathcal{G})$. By induction, both $\psi_1(\boldsymbol{m}, P, G)$ and $\psi_2(\boldsymbol{m}, P, G)$ are satisfied. Of course, this implies that their conjunction is satisfied, as required.

Conversely, suppose that the conjunction of $\psi_1(\boldsymbol{m}, P, G)$ and $\psi_2(\boldsymbol{m}, P, G)$ is satisfied. Then each conjunct is satisfied. By induction, there are p_1 and p_2 in G such that for each i, p_i strongly forces $\psi_i(\boldsymbol{m}, P, G)$. Since G is a filter, let p be an element of G which extends both p_1 and p_2. By Theorem 2.4.2, $\Vdash^*$ is a monotone relation. Thus, for each i either 1 or 2, p strongly forces $\psi_i(\boldsymbol{m}, P, G)$. Consequently, p is an element of G such that $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$, as required.

Existential quantification. Suppose that $\varphi(\boldsymbol{m}, P, \mathcal{G})$ is the existential sentence $(\exists x)\psi(x, \boldsymbol{m}, P, \mathcal{G})$.

Suppose that $p \in G$ and $p \Vdash^* \varphi(\boldsymbol{m}, P, \mathcal{G})$. Then there is a k in ω such that $p \Vdash^* \psi(k, \boldsymbol{m}, P, \mathcal{G})$. By induction, G satisfies $\psi(k, \boldsymbol{m}, P, \mathcal{G})$, and therefore G satisfies $(\exists x)\psi(x, \boldsymbol{m}, P, \mathcal{G})$, as required.

Conversely, suppose that G satisfies $(\exists x)\psi(x, \boldsymbol{m}, P, \mathcal{G})$. Fix k such that G satisfies $\psi(k, \boldsymbol{m}, P, \mathcal{G})$. By induction, there is a $p \in G$ such that $p \Vdash^* \psi(k, \boldsymbol{m}, P, \mathcal{G})$. Then, for this p, $p \Vdash^* (\exists x)\psi(x, \boldsymbol{m}, P, \mathcal{G})$, as required.

Negation. Suppose that $\varphi(\boldsymbol{m}, P, \mathcal{G})$ is the negation $\neg\psi(\boldsymbol{m}, P, \mathcal{G})$.

Suppose that $p \in G$ and $p \Vdash^* \neg\psi(\boldsymbol{m}, P, \mathcal{G})$. Then, there is no extension q of p such that $q \Vdash^* \psi(\boldsymbol{m}, P, \mathcal{G})$. Since G is a filter, for every $r \in G$ there is a q in G such that q is a common extension of p and r. Consequently, no element of G can strongly force $\psi(\boldsymbol{m}, P, \mathcal{G})$. By induction, G does not satisfy $\psi(\boldsymbol{m}, P, \mathcal{G})$, and hence G satisfies $\neg\psi(\boldsymbol{m}, P, \mathcal{G})$, as required.

Conversely, suppose that G satisfies $\neg\psi(\boldsymbol{m}, P, \mathcal{G})$. Then G does not satisfy $\psi(\boldsymbol{m}, P, \mathcal{G})$, and so there is no p in G such that $p \Vdash^* \psi(\boldsymbol{m}, P, \mathcal{G})$. By assumption, there is a p in G such that p strongly decides $\psi(\boldsymbol{m}, P, \mathcal{G})$, and this p must strongly force $\neg\psi(\boldsymbol{m}, P, \mathcal{G})$, as required. ∎

Theorem 2.4.12 *For $p \in P$ and $\varphi(\boldsymbol{m}, P, \mathcal{G})$ an arithmetic sentence, the following are equivalent.*

1. $p \Vdash \varphi(\boldsymbol{m}, P, \mathcal{G})$
2. *For every arithmetically P-generic filter G, if $p \in G$ then $\varphi(\boldsymbol{m}, P, G)$ is true.*

Proof: Suppose that $p \Vdash \varphi(\boldsymbol{m}, P, \mathcal{G})$, G is an arithmetically P-generic filter, and $p \in G$. Then $p \Vdash^* \neg\neg\varphi(\boldsymbol{m}, P, \mathcal{G})$, and by Theorem 2.4.11, $\neg\neg\varphi(\boldsymbol{m}, P, G)$ is true. But then $\varphi(\boldsymbol{m}, P, G)$ is also true, as required.

Conversely, if $p \nVdash \varphi(\boldsymbol{m}, P, \mathcal{G})$, then there is a q extending p in P such that $q \Vdash^* \neg\varphi(\boldsymbol{m}, P, \mathcal{G})$. By Theorem 2.4.8, let G be arithmetically P-generic such that q is an element of G. By Theorems 2.4.10 and 2.4.11, the arithmetic statements true of P and G are exactly those which are strongly forced by conditions in G. But then G satisfies $\neg\varphi(\boldsymbol{m}, P, G)$, and is the required generic filter doing so. ∎

2.5 Cohen forcing

In some cases, the forcing relation can be analyzed more efficiently than by directly unraveling Definition 2.4.1.

Let $2^{<\omega}$ denote the set of finite binary sequences ordered by extension. To fit into the above paradigm of forcing with an ordering on a set of natural numbers, we could think of a natural number as representing the sequence of digits in its binary expansion and order the numbers by extension on the sequences that they represent. Thus, 3 represents the sequence (11), 6 represents (110), and $3 >_P 6$. However, in this section, we will suppress the recursive coding of $2^{<\omega}$ and refer directly to the sequences rather than to their numeric codes. By the relevant discussion in (Enderton, 2001) this ordering on ω is defined by a Δ^0_1 formula[1] Consequently, we can omit the symbol for the partial order from the forcing language in the following.

2.5.1 Analyzing the forcing relation for Cohen forcing

Theorem 2.5.1 *If $\varphi(\boldsymbol{m}, \mathcal{G})$ is a bounded sentence, then the set of $p \in 2^{<\omega}$ such that $p \Vdash^* \varphi(\boldsymbol{m}, \mathcal{G})$ is Δ^0_1.*

Proof: Let p be a Cohen condition and let $\varphi(\boldsymbol{m}, \mathcal{G})$ be a bounded sentence. To be precise, let $\varphi(\boldsymbol{m}, \mathcal{G})$ be a sentence inductively constructed from atomic formulas by applications of conjunction, negation, and bounded existential quantification. (We choose these logical operations since they are the ones for which we have explicitly defined the forcing relation.)

[1] .

By Theorem 1.3.3, there is a number b such that b is uniformly computable from $\varphi(\boldsymbol{m}, \mathcal{G})$ and such that for all G, the values of G at numbers less than or equal to b recursively determine whether $(\mathbb{N}, G) \models \varphi(\boldsymbol{m}, \mathcal{G})$. Let b be fixed for the remainder of this proof.

We proceed by induction on the complexity of φ to prove the theorem. If $\varphi(\boldsymbol{m}, \mathcal{G})$ is an atomic sentence then the definition of the strong forcing relation is clearly recursive. Similarly, if $\varphi(\boldsymbol{m}, \mathcal{G})$ is a conjunction of sentences and we can recursively decide the forcing relation for the two conjuncts, then we can recursively decide the forcing relation for $\varphi(\boldsymbol{m}, \mathcal{G})$.

Now, we consider the two nontrivial cases in the induction. First, suppose that $\varphi(\boldsymbol{m}, \mathcal{G})$ is $(\exists x < t)\psi(x, \boldsymbol{m}, \mathcal{G})$; or equivalently suppose that $\varphi(\boldsymbol{m}, \mathcal{G})$ is $(\exists x)(x < t \;\&\; \psi(\boldsymbol{m}, \mathcal{G}))$, where t is a term with no free variables. Then, p strongly forces $\varphi(\boldsymbol{m}, \mathcal{G})$ if and only if there is a k such that p strongly forces $(k < t \;\&\; \psi(k, \boldsymbol{m}, \mathcal{G}))$. In turn, this is equivalent to p's strongly forcing $k < t$ and also $\psi(k, \boldsymbol{m}, \mathcal{G})$. p strongly forces $k < t$ if and only if k is actually less than the value of t. So there are only finitely many possible values for k, and we can use recursion to settle whether p strongly forces $\psi(k, \boldsymbol{m}, \mathcal{G})$ for one of these values.

Finally, suppose that $\varphi(\boldsymbol{m}, \mathcal{G})$ is $\neg\psi(\boldsymbol{m}, \mathcal{G})$. Then, p strongly forces $\varphi(\boldsymbol{m}, \mathcal{G})$ if and only if there is no condition q extending p such that q strongly forces $\psi(\boldsymbol{m}, \mathcal{G})$. Now we have a sequence of equivalences. There is a q extending p which strongly forces $\psi(\boldsymbol{m}, \mathcal{G})$ if and only if there is a q extending p which forces $\psi(\boldsymbol{m}, \mathcal{G})$, if and only if there is a q extending p and an arithmetically generic G containing q which satisfies $\psi(\boldsymbol{m}, \mathcal{G})$. But since $\psi(\boldsymbol{m}, \mathcal{G})$ is bounded whether G satisfies $\psi(\boldsymbol{m}, \mathcal{G})$ depends recursively on just the values of G below b. Consequently, p strongly forces $\varphi(\boldsymbol{m}, \mathcal{G})$ if and only if there is no extension of p of length greater than or equal to b which ensures that $\psi(\boldsymbol{m}, \mathcal{G})$ is true, and this last condition is uniformly recursive in p and $\varphi(\boldsymbol{m}, \mathcal{G})$.

Theorem 2.5.2 *For $n \geq 1$, the strong forcing relation for $2^{<\omega}$ has the following complexity.*

1. *The set*

$$\{(p, \varphi(\boldsymbol{m}, \mathcal{G})) : \varphi(\boldsymbol{m}, \mathcal{G}) \text{ is a } \Sigma^0_n \text{ sentence and } p \Vdash^* \varphi(\boldsymbol{m}, \mathcal{G})\}$$

   is Σ^0_n.

2. *The set*

$$\{(p, \varphi(\boldsymbol{m}, 2^{<\omega}, \mathcal{G})) : \varphi(\boldsymbol{m}, \mathcal{G}) \text{ is a } \Pi^0_n \text{ sentence and } p \Vdash^* \varphi(\boldsymbol{m}, \mathcal{G})\}$$

is Π^0_n. *Note, since we defined the strong forcing relation only for the existential quantifier, when we say that* $p \Vdash^* \forall x\,\psi$, *we mean that* $p \Vdash^* \neg\exists x \neg\psi$.

Proof: We prove Theorem 2.5.2 by induction on n.

First, we consider the case when n is equal to 1. Suppose that $\varphi(\boldsymbol{m}, \mathcal{G})$ is a Σ^0_1 sentence $\exists \boldsymbol{x}\,\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$ in which ψ is a bounded formula. Then $p \Vdash^* \varphi$ if and only if there is a sequence $\boldsymbol{k}$ such that $p \Vdash^* \psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$. By Theorem 2.5.1, strongly forcing a bounded sentence is a Δ^0_1 property, $p \Vdash^* \varphi$ if and only if there is a sequence $\boldsymbol{k}$ with a Σ^0_1 property relative to p and $\varphi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$. Thus, the forcing relation for Σ^0_1 sentences is itself Σ^0_1. Now consider the case when φ is a Π^0_1 sentence $\forall \boldsymbol{x}\,\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$. Then, by definition, $p \Vdash^* \forall \boldsymbol{x}\,\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$ if and only if $p \Vdash^* \neg\exists \boldsymbol{x} \neg\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$, if and only if for all $q <_{2^{<\omega}} p$ and all $\boldsymbol{k}$, $q \nVdash^* \neg\psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$. Whether q strongly forces $\neg\psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$ is a Δ^0_1 property of q and $\neg\psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$, and so the later condition is a Π^0_1 property of p and φ.

Now, assume that Theorem 2.5.2 holds for n.

Suppose that $\varphi(\boldsymbol{m}, \mathcal{G})$ is of the form $(\exists \boldsymbol{x})\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$, where $\boldsymbol{x}$ is a finite sequence of variables and $\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$ is a Π^0_n formula in those variables. Then $p \Vdash^* (\exists \boldsymbol{x})\psi(\boldsymbol{m}, \mathcal{G})$ if and only if there is a sequence of natural numbers $\boldsymbol{k}$ such that $p \Vdash^* \psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$. By induction, $p \Vdash^* \psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$ is a Π^0_n property of p and $\psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$. Consequently, $p \Vdash^* (\exists \boldsymbol{x})\psi(\boldsymbol{m}, \mathcal{G})$ if and only if there is a sequence $\boldsymbol{k}$ with a Π^0_n property relative to p and $\psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$. Thus, whether $p \Vdash^* (\exists \boldsymbol{x})\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$ is a Σ^0_{n+1} property of p and $\varphi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$.

Now, suppose that $\varphi(\boldsymbol{m}, \mathcal{G})$ is of the form $\neg(\exists \boldsymbol{x})\neg\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$, where $\boldsymbol{x}$ is a finite sequence of variables and $\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$ is a Σ^0_n formula in those variables. Then, $p \Vdash^* \neg(\exists \boldsymbol{x})\neg\psi(\boldsymbol{x}, \boldsymbol{m}, \mathcal{G})$ if and only if for every q extending p in P and for every $\boldsymbol{k}$ from ω, it is not the case that $q \Vdash^* \neg\psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$. This condition on q holds if and only if there is an r extending q such that $r \Vdash^* \psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$. By induction, the condition $r \Vdash^* \psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$ is Σ^0_n. Thus, $p \Vdash^* \varphi(\boldsymbol{m}, \mathcal{G})$ if and only if for every q extending p in P and every $\boldsymbol{k}$ from ω, there is an r extending q such that r has a property that is Σ^0_n relative to q and $\psi(\boldsymbol{k}, \boldsymbol{m}, \mathcal{G})$. Thus, $p \Vdash^* \varphi(\boldsymbol{m}, \mathcal{G})$ if and only if a Π^0_{n+1} condition holds of p and $\varphi(\boldsymbol{m}, \mathcal{G})$, as required. ∎

2.5.2 The universal role of Cohen forcing

Theorem 2.5.3 *Suppose that P is a countable partial order which is presented recursively in Z. There is a function $\lambda p.\pi(p, Z)$ which is recursive in Z, maps $2^{<\omega}$ to P, preserves order, and has the property that if D is a dense subset of P, then $\pi^{-1}(D)$, the pointwise inverse image of D, is a dense subset of $2^{<\omega}$.*

Proof: First, consider the case when P is $\omega^{<\omega}$ ordered by extension. This is a recursive partial order, so we may omit mentioning Z for the moment. We define a function p_ω mapping $2^{<\omega}$ to $\omega^{<\omega}$.

Given $p \in 2^{<\omega}$, define $p_\omega(p) \in \omega^{<\omega}$ and an auxiliary sequence $h \in \omega^{<\omega}$ by a finite recursion. We obtain h from the numbers on which p changes value and obtain p_ω as the sequence of lengths of intervals on which p is constant.

Let $h(0) = 0$. If there is a number $\ell > h(n)$ such that $p(h(n)) \neq p(\ell)$, then let $h(n+1)$ be the least such number and let $p_\omega(n) = h(n+1) - h(n)$. Otherwise, end the recursion and let h and p_ω be as has already been determined. In particular, if $h(1)$ is not defined, then $p\omega(p)$ is the empty element of $\omega^{<\omega}$.

Now, define D_0 to be the set of $p \in 2^{<\omega}$ such that for $\ell + 1$ the length of p, $p(\ell) \neq p(\ell - 1)$. That is $p \in D_0$ if and only if last value of p is different from the second-to-last value. For $p \in D_0$, let $n + 1$ be the cardinality of the set of m's such that either $m = 0$ or $m + 1$ is in the domain of p and $p(m) \neq p(m + 1)$. Then, $p_\omega(p)$ is defined on $[0, n]$.

Clearly, D_0 is dense in $2^{<\omega}$. Further, for any $p \in D_0$ and any q^* extending $p_\omega(p)$ in $\omega^{<\omega}$, there is a q extending p such that $p_\omega(q) = q^*$. Such a q is obtained by recursion, using the values of q^* to determine the lengths of the intervals on which q is constant valued. So, if D^* is a dense subset of $\omega^{<\omega}$, then $p_\omega^{-1}(D^*) = \{p : p_\omega(p) \in D^*\}$ is a dense subset of $2^{<\omega}$.

Now, we consider the general case, suppose that P is a partial order and P is recursively presented relative to Z. We may assume that the presentation of P has all of ω as its domain.

We will define a function $p : \omega^{<\omega} \to P$ by taking the limit of the following finite recursion. Given $p^* \in \omega^{<\omega}$ define $\lambda n.p(n, p^*)$ by recursion; the domain of this function will be a finite initial segment of ω and its range will be a decreasing sequence in P.

Let $p(0, p^*)$ be a fixed element of P. Suppose n_0 is given so that $p(n_0, p^*)$ is defined and $p(n_0 + 1, p^*)$ is not yet defined. If there is a number m such that m is greater than n_0 and $p^*(m)$ is defined and less than $p(n_0, p^*)$ in the ordering of P, then let m_0 be the least such number, for each number n in $[n_0 + 1, m_0]$ define $p(n, p^*)$ to be equal to $p^*(m_0)$, and go to the next step in the recursion. If there is no such m, then let $p(p^*) = p(n, p^*)$ and end the recursion without defining $p(m, p^*)$ for any m greater than n_0.

Any $p^* \in \omega^{<\omega}$ determines a condition $p(p^*) \in P$. Further, if $q \in P$ and $p(p^*) >_P q$ in P, then there is a q^* in $\omega^{<\omega}$ such that $p^* >_{\omega^{<\omega}} q^*$ and $p(q^*) = q$. We obtain q^* by appending the number q to the sequence p^*. We may conclude, if D^P is a dense subset of P, then $p^{-1}(D^P) = \{p^* : p(p^*) \in D^P\}$ is a dense subset of $\omega^{<\omega}$.

Thus, if we let $\pi : 2^{<\omega} \to P$ be the composition of p_ω from $2^{<\omega}$ to $\omega^{<\omega}$ with p from $\omega^{<\omega}$ to P, then the theorem is proven. ∎

Corollary 2.5.4 *Suppose that P is a countable partial order which is presented recursively in Z. Let $\lambda p.\pi(p, Z)$ be the Z-recursive function of Theorem 2.5.3 mapping 2^ω to P. If $\boldsymbol{D} = (D_n : n \in \omega)$ is a collection of dense open subsets of P, then $\pi^{-1}(\boldsymbol{D})$, the set of pointwise inverse images of the D_n's, is a collection of dense open subsets of $2^{<\omega}$. Further, if G is a $\pi^{-1}(\boldsymbol{D})$ generic filter on $2^{<\omega}$, then the upward closure of the range of π on G is a $\boldsymbol{D}$ generic filter on P.*

Proof: Let $\boldsymbol{D} = (D_n : n \in \omega)$ be a collection of dense open subsets of P. It is immediate from Theorem 2.5.3 that for each n, $\pi^{-1}(D_n)$ is a dense subset of $2^{<\omega}$.

Let G be a $\pi^{-1}(\boldsymbol{D})$ generic filter on $2^{<\omega}$. Then $\pi(G)$ the pointwise image of G under π has nontrivial intersection with each element of $\boldsymbol{D}$. Let G^* be the upward closure of $\pi(G)$ in P. Since G^* is closed upwards and meets every element of $\boldsymbol{D}$, it is sufficient to show that for each pair p^* and q^* of G^*, there is an $r^* \in G^*$ such that $p^* \geq_P r^*$ and $q^* \geq_P r^*$. Fix p^* and q^* from G^*. Let p and q be elements of G such that $p^* \geq_P \pi(p)$ and $q^* \geq_P \pi(q)$. Since G is a filter, fix r so that $p \geq_{2^{<\omega}} r$ and $q \geq_{2^{<\omega}} r$. Then, since π preserves order, $\pi(r)$ is a common extension of p^* and q^*. Since $\pi(G) \subseteq G^*$, $\pi(r) \in G^*$, and the theorem is proven. ∎

2.5.3 Adding Cohen reals

Definition 2.5.5 Let $P_{\omega,\omega}$ be the partial order consisting of binary valued functions with domain a finite subset of $\omega \times \omega$.

$P_{\omega,\omega}$ is the partial order to add countably many Cohen generic reals with finite support. If G is a sufficiently generic filter on $P_{\omega,\omega}$, then we can derive a sequence of reals from G, by letting $C_i(n) = j$ if and only if there is a $p \in G$ such that $p((i, n)) = j$.

Theorem 2.5.6 *Let B be a subset of ω. Let $\boldsymbol{C}$ be a set of reals derived from a filter on $P_{\omega,\omega}$ (as above) which is generic with regard to all the dense subsets of $P_{\omega,\omega}$ that are arithmetic in B. For any A_0 and A_1 which are recursive in B and any C_i and $C_{i_1}, \ldots, C_{i_n}$ belonging to $\boldsymbol{C}$, $A_0 \oplus C_i$ is recursive in $A_1 \oplus_{j \le n} C_{i_j}$ if and only if A_0 is recursive in A_1 and there is a j such that $1 \le j \le n$ and C_i is equal to C_{i_j}.*

Proof: If A_0 is recursive in A_1 and there is an j between 1 and n such that C_i is equal to C_{i_j}, then it is clear that $A_0 \oplus C_i$ is recursive in $A_1 \oplus_{j \le n} C_{i_j}$.

For the converse, suppose that $A_0 \oplus C_i$ is recursive in $A_1 \oplus_{j \le n} C_{i_j}$.

First, suppose that C_i is not among the $C_{i_1}, \ldots, C_{i_n}$. Let p be a condition in $P_{\omega,\omega}$ and let Θ be a recursive functional. Let m be the least number such that (i, m) is not in the domain of p. Suppose that q a condition extending p which forces a value y for $\Theta(m, A_1 \oplus_{j \le n} C_{i_j})$ and is defined at every number which is queried during the course of the computation of this value. Then the computation refers only to the values of A_1 and of the $C_{i_1}, \ldots, C_{i_n}$. Let q^* be the possibly weaker condition obtained by removing all points from the domain of q which are not in the domain of p and are not of the form (i_j, x), for some $j \le n$. The condition q^* forces the same atomic statements to hold of the $C_{i_1}, \ldots, C_{i_n}$ that were forced by q. Thus, q^* forces $\Theta(m, A_1 \oplus_{j \le n} C_{i_j}) = y$. Now we can extend q^* to $q^* \cup \{((i, m), 1 - y)\}$ and force $\Theta(m, A_1 \oplus_{j \le n} C_{i_j}) = y \ne C_i(m)$. Thus, there is a dense set of conditions q^* such that for some $m \in \omega$ either there is no extension of q^* which forces $\Theta(A_1 \oplus_{j \le n} C_{i_j})$ to be defined at m or q^* forces $\Theta(m, A_1 \oplus_{j \le n} C_{i_j}) \ne C_i(m)$ and fixes a computation in Θ which establishes the inequality. Hence, it is forced by the empty condition that $\Theta(A_1 \oplus_{j \le n} C_{i_j}) \ne C_i$. Since $P_{\omega,\omega}$ is a recursive partially ordered set and the C_{i_j}'s are arithmetically definable from a arithmetically generic filter on $P_{\omega,\omega}$-generic, any statement which is arithmetic relative to B and $\boldsymbol{C}$ and which is forced by the null condition will be true of B and $\boldsymbol{C}$. Thus, assuming that C_i is not among the $C_{i_1}, \ldots, C_{i_n}$ contradicts the hypothesis that C_i is recursive in $A_1 \oplus_{j \le n} C_{i_j}$.

Now, we consider A_0. Suppose that p is a condition in $P_{\omega,\omega}$, that Θ is a recursive functional, and that p forces $\Theta(A_1 \oplus_{j \le n} C_{i_j}) = A_0$. Then, for each x and y, the following conditions are equivalent.

1. There is a q extending p in $P_{\omega,\omega}$ such that q forces $\Theta(x, A_1 \oplus_{j\leq n} C_{i_j}) = y$ by a computation that mentions only values of C_{i_j}'s that are included in the domain of q.
2. $A_0(x) = y$.

A counterexample to the implication from (1) to (2) would be a counterexample to p's forcing $\Theta(A_1 \oplus_{j\leq n} C_{i_j}) = A_0$. Similarly, because p has forced the above equality, if $A_0(x) = y$ then there is a condition q which extends p and specifies enough of $C_{i_1}, \ldots, C_{i_n}$ to fix the computation setting $\Theta(x, A_1 \oplus_{j\leq n} C_{i_j}) = y$. Thus, the values of A_0 can be computed from A_1 just be searching for conditions q as above.

Consequently, if A_0 is not recursive in A_1, then for each Θ, the null condition forces $\Theta(A_1 \oplus_{j\leq n} C_{i_j}) \neq A_0$. As above, these statements which are arithmetic relative to B and $\boldsymbol{C}$, hence true of these sets. This verifies Theorem 2.5.6. ∎

2.5.4 Adding reals with limited Cohen genericity

Definition 2.5.7 A filter G on $2^{<\omega}$ is *n-generic* if and only if for every Σ^0_n-sentence φ, there is a $p \in G$ such that p strongly decides φ. Similarly, G is *n-generic relative to X* if and only if for every $\Sigma^0_n(X)$-sentence $\varphi(X)$, there is a $p \in G$ such that p strongly decides $\varphi(X)$.

Theorem 2.5.8 *There is a recursive functional G such that for all $n \in \omega$ and all X, $G(n, X^{(n)})$ is n-generic relative to X.*

Proof: By Theorem 2.5.2 , the strong forcing relations for Σ^0_n and Π^0_n sentences are Σ^0_n and Π^0_n, respectively. Similarly, the strong forcing relations for $\Sigma^0_n(X)$ and $\Pi^0_n(X)$ sentences are $\Sigma^0_n(X)$ and $\Pi^0_n(X)$, respectively, and therefore recursive in $X^{(n)}$, uniformly in X. Let $(\varphi_m(X) : m \in \omega)$ be a recursive enumeration of the sentences which are $\Sigma^0_n(X)$; for each m, let $D_m(X)$ be the dense open set of conditions p, such that p strongly decides φ_m; and let $\boldsymbol{D}(X) = (D_m(X) : m \in \omega)$. As in the proof of Theorem 2.4.8, we construct a sequence of conditions from $2^{<\omega}$ so that for each m, $p_m \in D_m(X)$ and $p_m \geq_{2^{<\omega}} p_{m+1}$. We let $G(n, X^{(n)})$ be the filter $\{q : \exists m(q \geq_{2^{<\omega}} p_m)\}$. We observed that $\boldsymbol{D}(X)$ is uniformly recursive in $X^{(n)}$. Further, $G(n, X^{(n)})$ is also uniformly recursive in $X^{(n)}$, since for each q, the sentence $q \in \mathcal{G}$ is decided by one of the p_m's in a way which is recursively determined relative to $X^{(n)}$.

Finally, since the argument given above was uniformly recursive in n and $X^{(n)}$, we have actually specified a recursive function G on pairs $(n, X^{(n)})$, as required. ■

In Theorem 2.5.8, we could just as easily build a filter on any other recursive partial ordering of ω and ensure that it is generic for any Δ^0_{n+1} countable sequence of dense subsets of that partial order.

Theorem 2.5.9 *Let B be a subset of ω. Let $\boldsymbol{C}$ be a set of reals derived from a filter G on $P_{\omega,\omega}$ as in Theorem 2.5.6 and such that every $\Sigma^0_1(B)$-sentence about G is decided by a condition in G. For any A_0 and A_1 which are recursive in B and any C_i and $C_{i_1}, \dots, C_{i_n}$ belonging to $\boldsymbol{C}$, $A_0 \oplus C_i$ is recursive in $A_1 \oplus_{j \leq n} C_{i_j}$ if and only if A_0 is recursive in A_1 and there is an j between 1 and n such that C_i is equal to C_{i_j}.*

Proof: In Theorem 2.5.6, we proved the same claim under the assumption that G meets every dense subset of $P_{\omega,\omega}$ which is arithmetic relative to B. However, if one inspects the proof of Theorem 2.5.6, one sees that for every dense set D which appears in the proof there is a $\Sigma^0_1(B)$ formula $\varphi(\mathcal{G}, B)$ such that D is the set of conditions which decide $\varphi(\mathcal{G}, B)$. ■

Definition 2.5.10 1. A subset T of $2^{<\omega}$ is a *tree* if and only if for all $q \in T$ and all q_0, if q_0 is an initial segment of q, then $q_0 \in T$.

2. Such a tree T is *perfect* if and only if for all $p \in T$, there are incompatible q_1 and q_2 in T such that $p \geq_{2^{<\omega}} q_1$ and $p \geq_{2^{<\omega}} q_2$.

Theorem 2.5.11 *Suppose that G is $n+1$-generic. Then there is a perfect binary tree T such that T is recursive in G and for any two distinct infinite paths G_1 and G_2 in T, G_1 is n-generic relative to G_2.*

Proof: Let P be the partial order of finite subtrees of $2^{<\omega}$. For $q \in T$, say that q is a terminal node in T if there is no proper extension of q in T. Order P by end extension: if T_1 and T_2 belong to P, then $T_1 >_P T_2$ if and only if $T_1 \subseteq T_2$ and for all $q_2 \in T_2 \setminus T_1$ there is a terminal node q_1 of T_1 such that q_1 is an initial segment of q_2.

By Corollary 2.5.4, let π be a recursive order-preserving function from $2^{<\omega}$ to P such that for each dense $D \subseteq P$, $\pi^{-1}(D)$ is a dense subset of $2^{<\omega}$.

Suppose that G^* is an $(n+1)$-generic filter on $2^{<\omega}$. Let G be the filter on P generated from $\pi(G^*)$:

$$G = \{T : \exists p \in G^*(T \geq_P \pi(p))\}.$$

Let T_G be the tree determined by the union of G. We will show that T_G has the following properties.

1. T_G is recursive in G^*.
2. T_G is a perfect tree.
3. For every infinite path G_1 in T_G, G_1 is n-generic.
4. For every pair G_1 and G_2 of distinct paths in T_G, G_2 is n-generic relative to G_1.

We compute T_G from G^* as follows. Suppose that $p \in 2^{<\omega}$. Let q be the least element of G^* such that every terminal node in $\pi(q)$ has length greater than the length of p. Since the set of such trees is recursive and open dense in P, the set of q's such that $\pi(q)$ is such a tree is a recursive and open dense subset of $2^{<\omega}$, and there will be such a q in G^*. Then $p \in T_G$ if and only if $p \in \pi(q)$.

Similarly, T_G is a perfect tree since for each n and each p, the set of trees T such that either $p \notin T$ or p has incompatible extensions in T is a recursive dense open subset of P.

The argument for the third claim is a special case of the argument for the fourth. We leave it for the reader to extract it from the one we give below.

Now, we consider the fourth claim. Let $\psi(G_1, G_2)$ be a Σ^0_n statement about G_2 relative to G_1.

Consider the partial order $P_{2,\omega} = 2^{<\omega} \times 2^{<\omega}$, used to add two Cohen generic subsets of ω. A condition in $P_{2,\omega}$ is a pair (p_1, p_2) in which p_1 and p_2 are each elements of $2^{<\omega}$. A filter G on $P_{2,\omega}$ is equivalent to a pair of filters (G_1, G_2), each on $2^{<\omega}$.

For G_2 to be n-generic relative to G_1, for every Σ^0_n-sentence $\psi(G_1, \mathcal{G}_2)$, there must be a condition $p \in 2^{<\omega}$ which is an initial segment of G_2 an which strongly decides $\psi(G_1, \mathcal{G}_2)$. Working back to an equivalent statement about conditions in $P_{2,\omega}$, there must be a pair (p_1, p_2) in G such that

$$p_1 \Vdash_{2^{<\omega}} \text{“}p_2 \text{ strongly decides } \psi(\mathcal{G}_1, \mathcal{G}_2)\text{”}.$$

Let D^2_ψ denote the set of such pairs (p_1, p_2). By Theorem 2.5.2, p_2's strongly deciding $\psi(G_1, \mathcal{G}_2)$ is defined by a disjunction of $\Sigma^0_n(G_1)$ and

$\Pi^0_n(G_1)$ statements. Consequently, by invoking Theorem 2.5.2 again, D^2_ψ is a Δ^0_{n+1} set.

For ψ a Σ^0_n sentence about G_1 and G_2, let D_ψ be the collection of conditions T in P such that for any two distinct terminal nodes p_1 and p_2 of T, $(p_1, p_2) \in D^2_\psi$. Since D^2_ψ is open and dense in $P_{2,\omega}$ and since every element of P has only finitely many pairs of terminal nodes, if $T_0 \in P$, then there is a T, obtained by extending the terminal nodes of T_0 to meet D^2_ψ pairwise, such that $T_0 >_P T$ and $T \in D_\psi$. Consequently, D_ψ is a dense open subset of P. By the previous paragraph, D_ψ is Δ^0_{n+1}-definable. Since G^* is $(n+1)$-generic and $\pi^{-1}(D_\psi)$ is Δ^0_{n+1}, there must be a condition p in G^* which strongly decides whether $G^* \cap \pi^{-1}(D_\psi)$ is empty. Since $\pi^{-1}(D_\psi)$ is dense, no condition can strongly force this intersection to be empty. Consequently, p must strongly force the statement that $G^* \cap D_\psi$ is not empty. But then, since G^* is $n+1$-generic, this statement is true; consequently, $G \cap D_\psi$ is not empty. We may conclude that there is a $T \in D_\psi$ which is a subtree of T_G.

Now, let p_1 and p_2 be initial segments of G_1 and G_2, respectively, such that $(p_1, p_2) \in D^2_\psi$. As G_1 is n-generic and p_1 strongly forces that p_2 strongly decides $\psi(\mathcal{G}_1, \mathcal{G}_2)$, p_2 does strongly decide $\psi(G_1, \mathcal{G}_2)$.

Since ψ was arbitrary, for each Σ^0_n sentence about G_2 relative to G_1 there is an initial segment of G_2 which strongly decides ψ relative to G_1. Thus, G_2 is n-generic relative to G_1, as required. ■

Theorem 2.5.12 *For all $n \in \omega$ and all $G \in 2^\omega$, if G is n-generic relative to Z, then $(Z \oplus G)^{(n)} \equiv_T Z^{(n)} \oplus G$.*

Proof: Let Z be a subset of ω, and suppose that G is n-generic relative to Z. It is sufficient to exhibit an algorithm which is recursive relative to $Z^{(n)}$ and G and which determines, for a given Σ^0_n-sentence ψ, whether ψ is true of $Z \oplus G$. Our proof that there is such an algorithm goes back to Friedberg (1957).

Let $\psi(Z \oplus G)$ be a Σ^0_n-sentence. Since G is n-generic relative to Z, there is an m such that $G \upharpoonright m$, the element of $2^{<\omega}$ determined by the first m values of G, strongly decides ψ. The set of conditions which strongly decide ψ is $\Delta^0_{n+1}(Z)$ and so is recursive in $Z^{(n)}$. Consequently, the function mapping ψ to the least m as above is recursive in $Z^{(n)} \oplus G$. In addition, the function which takes $G \upharpoonright m$ to the Boolean value which it decides for

ψ is recursive in $Z^{(n)}$. Since G is n-generic relative to Z, $\psi(Z \oplus G)$ is true if and only if $G \upharpoonright m$ strongly forces $\psi(Z \oplus G)$.

In short, we can compute whether $\psi(Z \oplus G)$ is true by using $Z^{(n)}$ to recognize the shortest initial segment of G which strongly decides $\psi(Z \oplus G)$, and then noting that $\psi(Z \oplus G)$ is true if and only if this condition forces $\psi(Z \oplus G)$. ■

3

Countable Representations

We have two goals in this chapter. First, we will show that structure of the hereditarily countable sets can be faithfully interpreted in $\mathfrak{D}$. We will then draw some preliminary conclusions concerning the global properties of $\mathfrak{D}$. In particular, we will prove the Nerode and Shore theorem that every automorphism of $\mathfrak{D}$ is fixed on a cone of degrees.

3.1 Coding Theorem

Our first step is to interpret quantifiers over countable relations in the first order language of $\mathfrak{D}$. The following result of Slaman and Woodin (1986) provides the mechanism by which elements of $\mathfrak{D}$ can act as *codes* for countable relations on $\mathfrak{D}$.

Definition 3.1.1 A countable n-place relation $\mathcal{R}$ on $\mathfrak{D}$ is a countable subset of the n-fold Cartesian product of $\mathfrak{D}$ with itself. In other words, R is a countable subset of the set of length n sequences of elements of $\mathfrak{D}$.

Theorem 3.1.2 (The Coding Theorem) *For every n there is a first order formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that for every countable n-place relation $\mathcal{R}$ on $\mathfrak{D}$ there is a sequence of degrees $\boldsymbol{p} = (p_1, \ldots, p_m)$ such that for all sequences of degrees $\boldsymbol{d} = (d_1, \ldots, d_n)$,*

$$\boldsymbol{d} \in \mathcal{R} \iff \mathfrak{D} \models \varphi(\boldsymbol{d}, \boldsymbol{p}).$$

Definition 3.1.3 A set of degrees A is an *antichain* if any two elements of A are incomparable.

We present the proof of the Coding Theorem in a sequence of lemmas. We first show that every countable antichain is uniformly coded in $\mathfrak{D}$. Then we reduce the problem of coding a general countable relation to that of coding a countable antichain.

We make use of the following lemma of Dekker and Myhill.

Lemma 3.1.4 (Dekker and Myhill (1958)) *Suppose that X is a subset of ω. There is a $Y \subseteq \omega$ with the same Turing degree as X, such that Y is recursive in each of its infinite subsets.*

Proof: Given X, let Y be the set of integers which represent sequences $\sigma \in 2^{<\omega}$ such that X extends σ. Y is recursive in X, as X can compute the set of its own initial segments. X is recursive in any infinite subset of Y, as any atomic question about X can be answered by examining any sufficiently long initial segment of X. ■

3.1.1 Coding Antichains

Definition 3.1.5 Let $\boldsymbol{A} = (A_i : i \in \omega)$ be a sequence of subsets of ω such that for every element A of $\boldsymbol{A}$, A is recursive in all of its infinite subsets and such that the Turing degrees of the elements of $\boldsymbol{A}$ form an antichain. We define the partial ordering P as follows.

Conditions. A *condition* p is a triple $(p_1, p_2, F(p))$. Here p_1 and p_2 are finite binary sequences of equal length and $F(p)$ is a finite initial segment of $\boldsymbol{A}$. The set of conditions is denoted by P.

Order. For p and q in P, we say that q is *stronger* than p if q_1 extends p_1, q_2 extends p_2 and $F(q)$ extends $F(p)$. In addition, if k is less than the length of $F(p)$ and A is the kth element of $F(p)$ then the following condition holds. If a is an element of A and (k, a) is less or equal to the common length of q_1 and q_2 but greater than the common length of p_1 and p_2, then q_1 and q_2 have the same value at (k, a).

Definition 3.1.6 Suppose that $p = (p_1, p_2, F(p))$ is a condition in P.

1. Say that m is a *coding location for p* if m is greater than the common length of p_1 and p_2, m is equal to (k, a) for some k less than the length of $F(p)$, and a an element of A_k.
2. Suppose that $F(p)$ has length greater than k. The set of coding locations for p of the form (k, a) is recursive in A_k. We will refer to this set as the set of coding locations in the kth column.

We can read Definition 3.1.5 as saying that in order for q to extend p, q_1 and q_2 must agree at all of the coding locations for p.

Lemma 3.1.7 *Let $\boldsymbol{A} = (A_i : i \in \omega)$ be a sequence of reals whose degrees form a countable antichain in $\mathfrak{D}$ and let B be an upper bound on the elements of $\boldsymbol{A}$. There are reals G_1 and G_2 with the following properties.*

1. *For every A_i in $\boldsymbol{A}$, there is a C such that C is recursive in $G_1 \oplus A_i$ and recursive in $G_2 \oplus A_i$, but C is not recursive in A_i.*
2. *For every Y below B, either*
   (a) *for every Z, if Z is recursive in $G_1 \oplus Y$ and recursive in $G_2 \oplus Y$, then Z is recursive in Y*
   (b) *or there is an A_i in $\boldsymbol{A}$ such that $Y \geq_T A_i$.*

Proof: By Lemma 3.1.4, we may assume that each A_i is recursive in all of its infinite subsets. Let P be the notion of forcing described above for $\boldsymbol{A}$. Let G be arithmetically P-generic and let G_1 and G_2 be the pair of reals obtained by taking the limits of the first two coordinates of the elements of G. We will show that both statements 1 and 2 are forced by the empty condition and therefore true of these G_1 and G_2.

Definition 3.1.8 For $k \in \omega$, let $D(k)$ be the set of conditions p such that $F(p)$ has length greater than k.

Clearly, $D(k)$ is arithmetic and dense in P. Suppose that p^k is an element of $D(k) \bigcap G$. Define $C(k)$ as follows.

$$C(k) = \left\{ m : \begin{array}{c} \text{The } m\text{th coding location for } p^k \text{ in the} \\ k\text{th column is an element of } G_1. \end{array} \right\}$$

Since every extension of p^k is required to make its first two coordinates agree at all coding locations for p^k, $C(k)$ is recursive in both $G_1 \oplus A_k$ and $G_2 \oplus A_k$. It remains to show that $C(k)$ is not recursive in A_k. Let e be an index for a Turing reduction.

Definition 3.1.9 Let $E(e, k)$ be the set of conditions p such that for some n less than the length of p_1, either $\{e\}(n, A_k) \uparrow$, or $\{e\}(n, A_k) \downarrow$ and its value is unequal to the value of p_1 at the nth coding location in p_1's kth column.

If there is an n greater than the length of p_1 such that $\{e\}(n, A_k) \downarrow$, we can define q extending p to disagree with $\{e\}(n, A_k)$ at the nth coding location in the ith column. Thus, $E(e, k)$ is dense in P. In addition, any condition in $E(e, k)$ has fixed an argument at which $\{e\}(A_k)$ and $C(k)$ disagree. Thus, we have verified statement 1 of Lemma 3.1.7.

It remains to show that if Y is a real below B then

$$\Vdash \begin{pmatrix} \exists(A_k \in \boldsymbol{A})[Y \geq_T A_k] \text{ or} \\ \{Z : Y \oplus G_1 \geq_T Z\} \cap \{Z : Y \oplus G_2 \geq_T Z\} = \{Z : Y \geq_T Z\}. \end{pmatrix}$$

Let Y be fixed below B. Suppose that e_1 and e_2 are indices for Turing reductions.

Definition 3.1.10 Define $M(Y, e_1, e_2)$ as follows.

$$M_0(Y, e_1, e_2) = \left\{ p : (\exists n) \begin{pmatrix} \{e_1\}(n, p_1 \oplus Y)\downarrow,\ \{e_2\}(n, p_2 \oplus Y)\downarrow, \\ \text{and } \{e_1\}(n, p_1 \oplus Y) \neq \{e_2\}(n, p_2 \oplus Y) \end{pmatrix} \right\}$$

$$M_1(Y, e_1, e_2) = \{p : (\forall q > p)[q \notin M_0(Y, e_1, e_2)]\}$$

$$M(Y, e_1, e_2) = M_0(Y, e_1, e_2) \cup M_1(Y, e_1, e_2)$$

$M(Y, e_1, e_2)$ is a dense subset of P. Any condition p in $M_0(Y, e_1, e_2)$ forces that (e_1, e_2) is not a pair of indices showing that a single set is recursive in both $G_1 \oplus Y$ and $G_2 \oplus Y$.

Consider a condition p in $M_1(Y, e_1, e_2)$. We show that either there is an extension of p which forces the possible common values of $\{e_1\}(G_1 \oplus Y)$ and $\{e_2\}(G_2 \oplus Y)$ to be non-total or recursive in Y, or else there is an element of $F(p)$ that is recursive in Y.

There are two cases to consider.

Case 1. There is a q extending p such that for every n, there is an m such that the following equation holds.

$$q \Vdash (\forall z)\,(\{e_1\}(n, G_1 \oplus Y) = z \rightarrow z = m)$$

If q is a condition as in Case 1, q forces that if $\{e_1\}(G_1 \oplus Y)$ is total then it is recursive in Y. Below q, Y can compute $\{e_1\}(G_1 \oplus Y)$ at n by finding any extension r_1 of q_1 such that $\{e_1\}(n, r_1 \oplus Y) \downarrow$ with use no greater than the length of r_1. For any such sequence r_1, there is a condition r extending q and having r_1 as its first coordinate. One can obtain such a r by extending q_2 to agree with r_1 on all numbers in the domain of r_1 but not in the domain of q_2. The value of $\{e_1\}(G_1 \oplus Y)$ at n must be the same as $\{e_1\}(n, r_1 \oplus Y)$ since any two extensions of q assigning a value to $\{e_1\}(n, G_1 \oplus Y)$ necessarily assign the same value. Thus, in case 1, $\{e_1\}(G_1 \oplus Y)$ is recursive relative to Y.

Case 2. For any q extending p in P, there are r and r' extending q and an integer n such that r and r' force different values for $\{e_1\}(n, G_1 \oplus Y)$.

Suppose that r and r' force different values for $\{e_1\}(n, G_1 \oplus Y)$. We begin either by finding two conditions forcing different values for $\{e_1\}(n, G_1 \oplus Y)$ such that their first coordinates have exactly one point of disagreement or by finding one condition r extending p such that r forces $\{e_1\}(n, G_1 \oplus Y)$ to be undefined.

We may assume that both r and r' force that the use of $\{e_1\}(n, G_1 \oplus Y)$ is less than the minimum of their two lengths. We start with an x such that r_1 and r_1' disagree at x. Obtain r_1'' by changing r_1 at the one point x to agree with r_1' and then extending to decide the value of $\{e_1\}(n, G_1 \oplus Y)$, if possible. If it is impossible to extend r_1' to make $\{e_1\}(n, G_1 \oplus Y)$ defined, then we can find a condition r extending p with first coordinate r_1'; this condition forces $\{e_1\}(n, G_1 \oplus Y)$ to be undefined and we are done. Otherwise, extend the domain of r_1 and r_1' to agree with r_1'' on all of the new points in their domains. Either r_1'' and r_1 disagree at exactly one point and force incompatible values for $\{e_1\}(n, G_1 \oplus Y)$ or r_1'' and r_1' force incompatible values and have less points of disagreement than exist between r_1 and r_1'. By induction, there is a pair with exactly one point of disagreement forcing incompatible values for $\{e_1\}(n, G_1 \oplus Y)$ or there is a condition forcing $\{e_1\}(n, G_1 \oplus Y)$ to be undefined.

Assume that r_1 and r_1' disagree exactly at the one point (k, a) and force different values for $\{e_1\}(n, G_1 \oplus Y)$. Additionally, we may assume that the computations involved have use no greater than the length of the first coordinate of their associated conditions. (This property is dense.)

For the sake of a contradiction, suppose a does not belong to the kth element of $F(p)$. If q is a stronger condition than p then $q_1(m)$ is not required to agree with $q_2(m)$. In this case, we find a condition extending p and forcing that $\{e_1\}(n, G_1 \oplus Y)$ disagree with $\{e_2\}(n, G_2 \oplus Y)$ as follows. First, find an extension $\hat{r}$ of r deciding the value of $\{e_2\}(n, G_2 \oplus Y)$ and forcing the use of the computation to be less than the length of its second coordinate. Either $\hat{r}$ forces a disagreement between $\{e_1\}(n, G_1 \oplus Y)$ and $\{e_2\}(n, G_2 \oplus Y)$ or the condition resulting from $\hat{r}$ by changing the value of $\hat{r}_1$ at (k, a) forces this disagreement. Either case contradicts the original assumption that p forces the two functions to be equal.

Thus, for any two conditions which extend p, force different values for $\{e_1\}(G_1 \oplus Y)$ at some argument, and have first coordinates which disagree at exactly one point, that point of disagreement is a coding location for p.

Namely, if the first coordinates disagree at (k, a), then a must be an element of the kth element of $F(p)$. By the condition of Case 2, no single condition above p can decide all the values of $\{e_1\}(G_1 \oplus Y)$. By the argument given above, Y can compute an infinite set of points (k, a) such that each is unique point of difference in the first coordinates of a pair of conditions forcing contradictory values for $\{e_1\}(G_1 \oplus Y)$. As $F(p)$ is finite, infinitely many of these points must involve the same k. Consequently, there is an infinite subset of that A_k which is recursive in Y. The elements of $\boldsymbol{A}$ were chosen to be recursive in any of their infinite subsets. Thus, Y can compute some element of $F(p)$.

The two cases exhaust all of the possibilities, and in either case statement 2 follows. ∎

3.1.2 Coding Relations

The next step is to reduce defining an arbitrary countable set of Turing degrees to defining an antichain. The following lemma is a standard application of Cohen forcing. See Theorem 2.5.6.

Lemma 3.1.11 *Suppose that $\mathcal{S}$ is a countable set of Turing degrees and b is an upper bound on the elements of $\mathcal{S}$. Let B be an element of b and suppose that $\boldsymbol{C}$ is a countable set of reals that are mutually Cohen generic with regard to meeting every dense set that is arithmetic in B. If ψ is a bijection between $\mathcal{S}$ and the set degree$(\boldsymbol{C})$ of degrees of elements of $\boldsymbol{C}$, then $\mathcal{S}$ and ψ are definable in parameters in $\mathfrak{D}$.*

Proof: Let $\mathcal{A}$ be the set of degrees of the form $x \vee \psi(x)$ where x is a degree in $\mathcal{S}$. Theorem 2.5.6 implies that both $degree(\boldsymbol{C})$ and $\mathcal{A}$ are antichains. Lemma 3.1.7 states that each of these sets can be defined in $\mathfrak{D}$ uniformly using finitely many parameters.

This implies the definability of $\mathcal{S}$ and ψ by the following equations.

$$x \in \mathcal{S} \iff \Big(x < b \text{ and } (\exists c \in degree(\boldsymbol{C}))(\exists z \in \mathcal{A})\big(x \vee c = z\big)\Big)$$

$$\psi(x) = c \iff \Big(x \in \mathcal{S} \text{ and } c \in degree(\boldsymbol{C}) \text{ and } (x \vee c) \in \mathcal{A}\Big)$$

∎

We can now finish the proof of the Coding Theorem 3.1.2 and show that any countable relation $\mathcal{R}$ on $\mathfrak{D}$ is definable from parameters in $\mathfrak{D}$.

Proof: Suppose that $\mathcal{R}$ is a countable subset of $\mathfrak{D}^n$. For each m smaller than n, let $\mathcal{R}(m)$ be defined by

$$\mathcal{R}(m) = \Big\{a : (\exists(v_1, \ldots, v_n) \in \mathcal{R})\Big(v_m = a\Big)\Big\}.$$

Let b be a uniform upper bound on all of the $\mathcal{R}(m)$; let B be an element of b. Let $\boldsymbol{C}$ be the Turing degrees of a set of reals that are mutually Cohen generic with regard to meeting all of the dense sets in the Cohen partial order arithmetically definable in B, so that C has the same cardinality as the disjoint union of the $\mathcal{R}(m)$. Write $\boldsymbol{C}$ as a disjoint union of sets $\boldsymbol{C}(m)$, each of which has the same cardinality as $\mathcal{R}(m)$.

Fix bijections $\psi_m : \mathcal{R}(m) \to \boldsymbol{C}(m)$.

By the preceding Lemmas 3.1.7 and 3.1.11, each ψ_m, $\mathcal{R}(m)$ and $\boldsymbol{C}(m)$ is definable from parameters in $\mathfrak{D}$. Define $\mathcal{S}$ by

$$\mathcal{S} = \{g_1 \vee g_2 \ldots \vee g_n : (\psi_1^{-1}(g_1), \psi_2^{-1}(g_2), \ldots, \psi_n^{-1}(g_n)) \in \mathcal{R}\}.$$

By Theorem 2.5.6 , each element $g_1 \vee \ldots \vee g_n$ of $\mathcal{S}$ uniquely determines the sequence $(g_1, \ldots, g_n)$ which joins to it. $\mathcal{S}$ is definable from parameters in $\mathfrak{D}$ by Lemma 3.1.11. Now $\mathcal{R}$ can be defined by

$$\mathcal{R} = \left\{ (a_1, \ldots, a_n) : \begin{array}{c} (\forall m \leq n)\Big(a_m \in \mathcal{R}(m)\Big) \text{ and} \\ \psi_1(a_1) \vee \ldots \vee \psi_n(a_n) \in \mathcal{S} \end{array} \right\}$$

Finally, the bounded quantifier "$\forall m \leq n$" can be replaced by an n-fold conjunction to produce a formula in the language of $\mathfrak{D}$. ■

We can use the Coding Theorem to give a short calculation of the Turing degree of the first order theory of $\mathfrak{D}$.

Theorem 3.1.12 (Simpson (1977)) *There is a recursive interpretation of the second order theory of arithmetic in the first order theory of* $\mathfrak{D}$.

Proof: The usual second order characterization of a standard model of arithmetic involves specifying a countable set N, a distinguished element "0", and a unary "successor" function s, such that $\mathbb{N} = (N, 0, s)$ satisfies finitely many first order properties (P^-) together with second order induction. A countable model $\mathbb{N}$ can be represented by finitely many countable relations on $\mathfrak{D}$. In $\mathfrak{D}$, we can define a class of isomorphic copies of the standard model by considering the collection of countable models of P^- (coded as structures on the Turing degrees) for which every countable subset has a least element. The quantifier over countable sets can be expressed in $\mathfrak{D}$ through the codes. Finally, the second order quantifiers over N are also interpreted by first order quantifiers over codes for subsets of N. ∎

Consequently, the first order theory of $\mathfrak{D}$ and the second order theory of arithmetic are recursively isomorphic.

We can also apply the Coding Theorem to obtain some information about the substructures of $\mathfrak{D}$.

Definition 3.1.13 Say that a partial order $\mathfrak{D}^*$ is a *substructure* of $\mathfrak{D}$ if $\mathfrak{D}^*$ consists of a subset of the Turing degrees ordered by Turing reducibility. $\mathfrak{D}^*$ is an *elementary substructure* if $\mathfrak{D}^*$ is a substructure of $\mathfrak{D}$ and for every sequence $\boldsymbol{p}$ from $\mathfrak{D}^*$ and every first order formula φ,

$$\mathfrak{D} \models \varphi(\boldsymbol{p}) \iff \mathfrak{D}^* \models \varphi(\boldsymbol{p}).$$

Definition 3.1.14 A substructure $\mathfrak{D}^*$ is *cofinal* in $\mathfrak{D}$ if for every x in $\mathfrak{D}$ there is a y in $\mathfrak{D}^*$ such that $x \leq_T y$.

Theorem 3.1.15 (Slaman and Woodin (1986)) *Suppose that $\mathfrak{D}^*$ is a cofinal elementary substructure of $\mathfrak{D}$. Then $\mathfrak{D}^* = \mathfrak{D}$.*

Proof: We must show that every element of $\mathfrak{D}$ is an element of D^*.

Let x be a degree. Since $\mathfrak{D}^*$ is cofinal in $\mathfrak{D}$, there is a degree a in $\mathfrak{D}^*$ such that $x \leq_T a$. By the Coding Theorem, let $\boldsymbol{c}$ be a finite sequence of degrees which codes a standard model of arithmetic $\mathbb{N} = (N, 0, s)$; let f be a function from N onto the degrees below a which is coded by $\boldsymbol{d}$. The statement that there is a sequence $\boldsymbol{c}$ which codes a standard model of arithmetic and there is a sequence $\boldsymbol{d}$ which codes a counting of the degrees below a is a first order statement in $\mathfrak{D}$ about a. Since the same statement is true about a in $\mathfrak{D}^*$, we may choose $\boldsymbol{c}$ and $\boldsymbol{d}$ so that they belong to $\mathfrak{D}^*$.

We chose a so that x is recursive in a. Therefore, in $\mathfrak{D}$ there is an integer n such that x is the value of f at the nth element of $\mathbb{N}$. Since every natural number is definable in $\mathbb{N}$, x is definable from a, $\boldsymbol{c}$ and $\boldsymbol{d}$. Since $\mathfrak{D}^*$ is an elementary substructure of $\mathfrak{D}$, x must be an element of $\mathfrak{D}^*$. ∎

3.2 Coding parameter

In Section 3.1, we began with a countable relation $\mathcal{R}$ and produced parameters $\boldsymbol{p}$ such that $\mathcal{R}$ was first order definable in $\mathfrak{D}$ relative to the parameters $\boldsymbol{p}$.

In this section, we will analyze more closely the dependence of the coding parameters on the presentation of $\mathcal{R}$. We will then use the more effective results to infer some preliminary limitations on the possible automorphisms of $\mathfrak{D}$.

Definition 3.2.1 Suppose that $\mathcal{R}$ is a countable relation on $\mathfrak{D}$. A *presentation R* of $\mathcal{R}$ is a relation on a countable subset of 2^ω such that:

1. For all $(x_1, \ldots, x_n) \in \mathcal{R}$, there is a sequence $(X_1, \ldots, X_n) \in R$ such that for all $j \leq n$, x_j is the Turing degree of X_j.
2. For all $(X_1, \ldots, X_n) \in R$, the sequence $(x_1, \ldots, x_n)$, formed by taking the sequence of Turing degrees in $(X_1, \ldots, X_n)$ is in $\mathcal{R}$.

As usual, we may regard a real as a countable sequence of reals and hence as determining a presentation of a countable relation on $\mathfrak{D}$.

Theorem 3.2.2 (Effective Coding Theorem) *Suppose that there is a presentation of the countable relation $\mathcal{R}$ which is recursive in the set R, and let r be the Turing degree of R. There are parameters $\boldsymbol{p}$ which code $\mathcal{R}$ in $\mathfrak{D}$ such that the elements of $\boldsymbol{p}$ are below r'.*

Proof: We divided the proof of the Coding Theorem into two distinct pieces. We reduced the coding of $\mathcal{R}$ to the coding a finite collection of countable antichains and we showed that any countable antichain is definable from finitely many parameters.

In reducing the general coding problem to the one of coding antichains, we introduced a family of Cohen generic reals relative to an upper bound on $\mathcal{R}$. For our present purposes, we use r as the upper bound on $\mathcal{R}$. We needed to know that these Cohen generic reals where independent over the ideal below r. That is we needed to know that if a_0 and a_1 are degrees below r and $c_0, \ldots, c_n$ is a sequence of degrees of mutually Cohen generic reals then $a_0 \vee c_0 \leq_T a_1 \vee c_1 \vee \ldots \vee c_n$ if and only if $a_0 \leq_T a_1$ and there is an i in $[1, n]$ such that $c_0 = c_i$. This independence property is guaranteed for any set of reals which are mutually 1-generic relative to R. In particular, we could have used the countable collection of reals obtained by taking the columns of a real C which is 1-generic relative to R. We can build such a C recursively in R' by making the finite conditions on C decide all $\Sigma^0_1(R)$ statements about C. This C has the additional property that $(R \oplus C)'$ is equal to R'.

Now consider the problem of going from the presentation of an antichain to a sequence of codes for it. In Definition 3.1.5, we defined a notion of forcing P to produce two generic reals G_0 and G_1. We argued that any two sufficiently generic reals would do as parameters to define the given antichain. First, note that in our application P is recursive in $R \oplus C$ since the antichains involved in coding $\mathcal{R}$ are presented recursively in $R \oplus C$. We will construct a filter recursively in $(R \oplus C)'$, hence recursively in R'.

Upon inspection of the proof, $M(Y, e_0, e_2)$ was the only nontrivial dense subset of P which appeared in it. In the present setting, Y is recursive in $R \oplus C$, and $M(Y, e_0, e_2)$ is the collection of conditions which either force a disagreement between convergent values of $\{e_0\}(Y \oplus G_0)$ and $\{e_2\}(Y \oplus G_1)$ or force that there is no such convergent disagreement. More abstractly, $M(Y, e_0, e_2)$ is the set of conditions which decide whether the generic filter $\mathcal{G}$ meets a subset of P which is $\Sigma^0_1(R \oplus C)$. In particular, we need not determine whether the recursive function which is to compute Y from $R \oplus C$ is defined everywhere. Thus, given any condition p in P, it is possible to find an extension q of p in $M(Y, e_0, e_2)$ uniformly recursively in R'.

In the case that q forces a disagreement between convergent values of $\{e_0\}(Y \oplus G_0)$ and $\{e_2\}(Y \oplus G_1)$, no further attention is needed for the pair (e_0, e_2). Otherwise, there were two possibilities. In the first case, there is a condition with decides all the convergent values of e_1. In the second, either there is a condition extending p forcing $\{e_1\}(n, G_1 \oplus Y)$ to be undefined or there are infinitely many points (k, a) such that each is unique point of difference in the first coordinates of a pair of conditions forcing contradictory values for $\{e_1\}(G_1 \oplus Y)$.

Note, R' cannot recursively determine whether there is an n and there is a condition q extending p forcing $\{e_1\}(n, G_1 \oplus Y)$ to be undefined. It would take R'' to determine this. To make our construction effective, we ensure that either we find such and n and q or we produce infinitely many points (k, a) such that each is unique point of difference in the first coordinates of a pair of conditions forcing contradictory values for $\{e_1\}(G_1 \oplus Y)$. In the proof of Lemma 3.1.7, we started with a condition p and either produced n and q forcing divergence or produced a unique point of difference (k, a) appearing in an extension of p. Further, the argument was recursive in R'. Thus, if we return to the pair e_1 and e_2 infinitely often, then either we will find a condition forcing divergence or we will find infinitely many unique points of difference. Either is sufficient to conclude that the sets produced satisfy the theorem.

We proceed by a finite injury construction, organized in stages. At stage s, we are given a condition $p[s]$ produced at the end of the previous stage.

We take the following steps. Let $p[s-1]$ be the condition produced at the end of the previous stage. Here $p[0]$ can be the empty condition.

- First, let t be the least number such that the tth $\Sigma_1^0(R \oplus C)$ subset S of P has not been considered since the most recent stage that t was injured (if any). Either there is an extension q of $p[s]$ in S (a $\Sigma_1^0(R \oplus C)$ property of p) or $p[s]$ forces $\mathcal{G} \bigcap S$ to be empty. Uniformly recursively in R', we find such a condition, call it $q[0]$.
- Next, we proceed by recursion on $t < s$ to consider the first s pairs (e_1, e_2).
  - ∗ If $q[t]$ already forces a disagreement between convergent values of $\{e_0\}(Y \oplus G_0)$ and $\{e_2\}(Y \oplus G_1)$ or for an n identified during an earlier stage forces that $\{e_1\}(n, G_1 \oplus Y)$ is undefined and t has not been injured since that stage, then no further action is required and we let $q[t+1] = q[t]$.
  - ∗ Otherwise, consider the condition $r[t] = (q[t]_1, q[t]_2, F_t)$, where F_t is the finite set $\{A_1, \dots, A_{t-1}\}$ consisting of the first t many element of the antichain to be coded. As above, we find a unique point of difference (k, a) appearing in an extension of $r[t]$ and we let $q_{t+1} = q_t$, or we find an n and a q extending $r[t]$ forcing $\{e_1\}(n, G_1 \oplus Y)$ to be undefined and we let $q[t+1]$ be this q. In the latter case, we end the recursion on t, say that every number larger than t is injured during this stage, and go to the next step.
- Let $q = (q_1, q_2, F)$ be the condition produced by the subrecursion. Define $p[s]$ to be the condition (q_1, q_2, F_s)

We argue by induction on t that we meet the first t requirements for our construction and that there are at most finitely many stages during which t is injured. Assume that t is injured only finitely often and that s be the last injury stage. First, note that all later conditions extend $(p[s]_1, p[s]_2, F_t)$, so our G does code a generic real into the meet of the joins of the tth element of the antichain with G_1 and G_2. Next, note that we ensure that our construction meets the tth $\Sigma_1^0(R \oplus C)$ subset of P no later than stage $s+t$. This establishes that we meet the basic genericity requirements. Third, we directly ensure for the tth pair (e_1, e_2) either there is an n such that $\{e_0\}(n, Y \oplus G_0) \neg \{e_2\}(n, Y \oplus G_1)$, or there is an n such that $\{e_0\}(n, Y \oplus G_0)$ is not defined, or there are infinitely many unique points of difference above some element of $M_1(Y, e_1, e_2)$ and so Y can compute an element of the antichain being coded.

Finally, the only injury produced by t to higher numbers occurs if we find an n and a condition forcing that $\{e_0\}(n, Y \oplus G_0)$ is not defined. This happens at most once in the construction. Thus, $t + 1$ is injured at most finitely often, at the stages when t is injured and possibly once more.

The theorem follows. ■

By the Effective Coding Theorem, we know that for any presentation R of a countable relation $\mathcal{R}$, there are codes for $\mathcal{R}$ which are close to R. In the next lemma, we give a more specialized calculation to show that for every degree x and representative X of x, there are parameters which code a X and are close to x. The idea of localizing a code for X to within a small neighborhood of x has been widely exploited, especially by Nerode and Shore (1980b).

Theorem 3.2.3 *For any degree x and representative X of x, there are parameters $\boldsymbol{p}$ such that*

1. *$\boldsymbol{p}$ codes an isomorphic copy of $\mathbb{N}$ with a unary predicate for X;*
2. *the elements of $\boldsymbol{p}$ are recursive in $x \vee 0'$.*

Proof: Let x be fixed and let X be a representative of x.

We first fix a coding of the natural numbers, which we will use for every x. Let c be the degree of C, a 1-generic real below $0'$. View C as a countable sequence of mutually 1-generic reals $C_0, C_1, \ldots$. Let $(\{C_i : i \in \mathbb{N}\}, 0, s)$ be an isomorphic copy of the standard model of arithmetic. Denote this model by $\mathbb{N}_C$. $\mathbb{N}_C$ is presented recursively in C and so is coded by parameters which are recursive in C', i.e. by parameters which are recursive in $0'$. Let $\boldsymbol{C}$ denote the degrees represented in $\{C_i : i \in N\}$.

Now, we address the coding of X as a predicate on $\mathbb{N}_C$. Following the proof of the Coding Theorem, we fix c as our uniform upper bound on the elements of $\boldsymbol{C}$. Then, we take the degrees of a set of reals which are recursive in $0'$ and mutually Cohen 1-generic relative to C to produce an antichain $\boldsymbol{A}$. As with $\mathbb{N}_C$, we can use the same antichain $\boldsymbol{A}$ for every x. Further, the sets whose degrees belong to $\boldsymbol{A}$ can be taken to be the columns of a set A below $0'$ which is 1-generic relative to C. Fix A and let a be the degree of A. As above, $\boldsymbol{A}$ is coded by parameters below $0'$. Further, the elements of $\boldsymbol{A}$ are independent over the degrees below c.

In order to code X according to the template of the Coding Theorem, we must produce parameters to define the set $\boldsymbol{A}+\boldsymbol{C}_X$, defined by the following equation.

$$A + C_X = \{a_i \vee c_{k_i} : k_i \text{ is the } i\text{th element of } X\}$$

Then X is represented as a unary predicate on $\mathbb{N}_C$ by

$$c_i \in X \iff (\exists a \in A)[a \vee c_i \in A + C_X].$$

Thus, the problem of coding X into the degrees below $x \vee 0'$ is reduced to problem of coding the antichain $A + C_X$ by means of parameters below $x \vee 0'$. Following the proof of the Coding Theorem, we need only find a sufficiently generic filter $\mathcal{G}$ for the forcing P (introduced in Definition 3.1.5) recursively in $X \oplus 0'$. Recall that P is designed to produce two reals G_0 and G_1 of degree g_0 and g_1. Then, we use the degrees g_0 and g_1 of G_0 and G_1 to define $A + C_X$ as the collection of degrees below $a \vee c$ which are minimal elements of the set

$$\{y : y <_T a \vee c \text{ and } (y \vee g_0) \wedge (y \vee g_1) \neq (y)\}.$$

We follow exactly the construction in the proof of Theorem 3.2.2 and make the following observation. At each step of the construction, we considered a condition $p = (p_1, p_2, F(p))$. The technically involved part of the proof focused on pairs of reductions $\{e_1\}(Y \oplus G_1)$ and $\{e_2\}(Y \oplus G_2)$. This part of the proof did not mention extension to $F(p)$ and hence does not refer to X in the current setting. It can be completely analyzed recursively in $0'$. The only references to X appear when we extend $F(p)$. These can be resolved using X itself, without reference to X'. Hence, we can build the required parameters in $X \oplus 0'$. ■

Thus, a representative of the degree x is coded by parameters which are near to x. The next lemma proves a weak converse.

Theorem 3.2.4 (Decoding Theorem) *Suppose that* $\boldsymbol{p}$ *is a sequence of degrees which lie below* y *and* $\boldsymbol{p}$ *codes the relation* $\mathcal{R}$ *(in the sense of the Coding Theorem 3.1.2). Letting* Y *be a representative of* y*,* $\mathcal{R}$ *has a presentation which is* $\Sigma^0_5(Y)$.

Proof: The natural presentation of the Turing partial ordering on the reals which are recursive in Y is arithmetically presentable relative to Y. Representatives of the parameters $\boldsymbol{p}$ define $\mathcal{R}$ in terms of the Turing partial order and quantifiers over the degrees recursive in y. Hence, $\mathcal{R}$ has a presentation which is arithmetic in Y. The calculation that this presentation is below $Y^{(5)}$ is a routine counting of quantifiers.

Suppose that $\mathcal{R}$ is an n-place relation. We will show that the relation R defined by

$$R(X_0, \ldots, X_{n-1}) \iff \mathcal{R}(degree(X_0), \ldots, degree(X_{n-1}))$$

is recursive in $Y^{(5)}$. The set *Total*(Y), consisting of indices for Turing functionals which are total relative to Y is $\Pi^0_2(Y)$ and hence recursive in $Y^{(5)}$. Thus, it will be sufficient to show that the relation R_0, defined on *Total*(Y) by

$$R_0(e_0, \ldots, e_{n-1}) \iff \mathcal{R}(degree(\{e_0\}(Y)), \ldots, degree(\{e_{n-1}\}(Y))),$$

is $\Sigma^0_5(Y)$.

From this point on in the proof, we will work only with indices which are elements of *Total*(Y). The Turing order between indices is given by $e_0 \leq_T e_1$ if and only if $\{e_0\}(Y)$ is recursive in $\{e_1\}(Y)$. Syntactically, $e_0 \leq_T e_1$ if and only if there is an e, such that for all n, there is a w such that the evaluation of $\{e\}$ on n relative to $\{e_0\}(Y)$ and the evaluation of $\{e_1\}(n, Y)$ take less than w many steps (as computations relative to Y) and the computations return the same values. Thus, the relation $e_0 \leq_T e_1$ is $\Sigma^0_3(Y)$ on *Total*(Y).

The next step is to incorporate the uniformity of the join as a function on the reals. Note that an index relative to Y for $\{e_0\}(Y) \oplus \{e_1\}(Y)$ can be found uniformly from e_0 and e_1. So we can view *Total*(Y) as coming equipped with the binary function $\vee$. Set $e_0 \vee e_1 \geq_T e_2$ if and only if $\{e_0\}(Y) \oplus \{e_1\}(Y) \geq_T \{e_2\}(Y)$. We shall say that a statement is *atomic* if it is phrased as a comparison by $\leq_T$ of two terms generated by $\vee$. An atomic statement on *Total*(Y) is equivalent to one that is $\Sigma^0_3(Y)$.

In the degrees, we use parameters g_0, g_1 and b to define an antichain $\boldsymbol{A}$ as follows. We identify $\boldsymbol{A}$ as the collection of points that are the minimal x such that x satisfies the property

$$x \leq_T b \text{ and } (\exists z)[g_0 \vee x \geq_T z \text{ and } g_1 \vee x \geq_T z \text{ and } x \not\geq_T z].$$

Formally, the statement that a satisfies the property is the conjunction of an atomic statement with existential quantifier followed by a Boolean combination of atomic statements. Thus, when viewed on *Total*(Y), this property of a is $\Sigma^0_4(Y)$. To say that a is a minimal solution to the property is to say that a satisfies the property and for all z, if z satisfies the property and $z \leq_T a$ then $a \leq_T z$. On *Total*(Y), the latter condition is a universal quantifier followed by formula with the form of a $\Sigma^0_4(Y)$ condition implies a Boolean combination of atomic conditions. Thus, the latter condition is Π^0_4. Thus, on *Total*(Y), $a \in \boldsymbol{A}$ is a $\Delta^0_5(Y)$ condition on a.

For each i less than n, let $\mathcal{R}(i)$ be the projection of $\mathcal{R}$ onto its ith coordinate. When $\boldsymbol{p}$ codes $\mathcal{R}$, $\boldsymbol{p}$ includes an upper bound b on the field of $\mathcal{R}$ and parameters to define $2n+1$ antichains $\boldsymbol{C}(0), \ldots, \boldsymbol{C}(n-1)$, $\mathcal{A}(0), \ldots, \mathcal{A}(n-1)$ and $\mathcal{S}$. Further, the definitions $\psi_i(x) = c \iff x \vee c \in \mathcal{A}$ must define bijections between $\mathcal{R}(i)$ and $\boldsymbol{C}(i)$. Finally, $(x_0, \ldots, x_{n-1}) \in \mathcal{R}$ if and only if $(\psi_0(x_0), \ldots, \psi_{n-1}(x_{n-1})) \in \mathcal{S}$.

Relative to Y, we can evaluate whether $\psi_i(x)$ is equal to c as follows. By definition, $\psi_i(x) = c$ if and only if $x \leq_T b$ and $c \in \boldsymbol{C}(i)$ and $x \vee c \in \mathcal{A}(i)$. This condition is a Boolean combination of formulas that are at worst $\Delta^0_5(Y)$ and hence $\psi(x) = c$ is $\Delta^0_5(Y)$. Then, $(x_0, \ldots, x_{n-1}) \in \mathcal{R}$ if and only if there is a sequence $(c_0, \ldots, c_{n-1})$ such that for all i less than n, $\psi_i(x_i) = c_i$ and $(c_0, \ldots, c_{n-1}) \in \mathcal{S}$. So $(x_0, \ldots, x_{n-1}) \in \mathcal{R}$ is $\Sigma^0_5(Y)$. ■

Corollary 3.2.5 *Suppose that $\boldsymbol{p}$ is a sequence of degrees below y, and $\boldsymbol{p}$ codes an isomorphic copy of $\mathbb{N}$ together with a unary predicate U. Then, for any representative Y of y, U is $\Sigma^0_5(Y)$.*

Proof: Fix Y to be a representative of y.

A model of arithmetic $(\mathbb{N}, U)$ comes with a successor function s. Then, for any n the statement $n \in U$ is equivalent to an existential positive statement's being satisfied in $(\mathbb{N}, U)$. Thus, $n \in U$ can be expressed by an existential quantifier over numbers followed by a formula which is $\Sigma^0_5(Y)$. Thus, U is $\Sigma^0_5(Y)$. ■

By changing the details in the representation of structures, we could sharpen the bound we obtain in the Decoding Theorem and in its corollary. In some contexts, sharper bounds are important in making as direct a connection between the set of reals which are coded by parameters below x and the degree of x. For example, Shore (1981) used an ingenious coding scheme to give a syntactically simple method to recognize a class of special codes for isomorphic copies of $\mathbb{N}$ in $\mathfrak{D}(\leq_T 0')$. Thus, he was able to interpret the first order theory of $\mathbb{N}$ in the first order theory of $\mathfrak{D}(\leq_T 0')$.

For the most part, our proofs are not sensitive to the exact bound obtained in the Decoding Theorem. We only need to know that there is some arithmetic bound relative to x on the complexity of the relations coded below x.

3.3 Applications to *Aut*($\mathfrak{D}$)

We can combine the Coding and Decoding Theorems and draw some preliminary conclusions limiting the possible behavior of an automorphism of $\mathfrak{D}$.

Theorem 3.3.1 (Nerode and Shore (1980a)) *Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$. For every degree x, if x is greater than $\pi^{-1}(0')$ then $\pi(x)$ is arithmetic in x.*

Proof: Let x be fixed so that x is above $\pi^{-1}(0')$. Then, $\pi(x)$ is above $0'$. By Theorem 3.2.3, there are parameters $\boldsymbol{p}$ below $\pi(x)$ which code a model of arithmetic and a unary predicate for a representative Y of $\pi(x)$. The preimage $\pi^{-1}(\boldsymbol{p})$ of these parameters code the same structure and are below x. By the Decoding Theorem 3.2.4, the structure coded by $\pi^{-1}(\boldsymbol{p})$ has a presentation which is arithmetic in x. Thus, $\pi(x)$ has a representative which is arithmetic in x and so $\pi(x)$ is arithmetic in x. ■

Corollary 3.3.2 (Nerode and Shore (1980a)) *Suppose that π is an automorphism of $\mathfrak{D}$ and π restricts to an automorphism of the arithmetic degrees. Then for every x, $\pi(x)$ is arithmetic in x.*

Proof: In the case that π restricts to an automorphism of the arithmetic degrees, $\pi^{-1}(0')$ is an arithmetic degree. Then, Theorem 3.3.1 implies that for all x in the cone above an arithmetic degree $\pi^{-1}(0')$, $\pi(x)$ is arithmetic in x. Now, $\pi(x)$ is below $\pi(x \vee \pi^{-1}(0'))$; $\pi(x \vee \pi^{-1}(0'))$ is arithmetic in $x \vee \pi^{-1}(0')$; since $\pi^{-1}(0')$ is arithmetic, $x \vee \pi^{-1}(0')$ is arithmetic in x; consequently, $\pi(x)$ is arithmetic in x. ■

We will eventually prove that every automorphism of $\mathfrak{D}$ is the identity above $0''$. (See Theorem 6.2.4.) Consequently, every automorphism of $\mathfrak{D}$ satisfies the conclusion of Corollary 3.3.2.

Theorem 3.3.3 (Nerode and Shore (1980a)) *Suppose $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$ is an automorphism of $\mathfrak{D}$ and $x \geq_T \pi^{-1}(0')^{(5)} \vee \pi^{-1}(\pi(0')^{(5)})$. Then, $\pi(x) = x$.*

Consequently, π is the identity on a cone.

Proof: In the first part of the proof, we will only use the fact that x is above $\pi^{-1}(0')^{(5)}$. Let y_1 and y_2 be fixed so that $y_1 \vee y_2 = x$; $\pi(y_1)$ and $\pi(y_2)$ are greater than $0'$; and $y_1^{(5)}$ and $y_2^{(5)}$ are recursive in x. For example, we could obtain such reals by writing x as the join of two degrees that are 5-generic relative to $\pi^{-1}(0')$. Detailed arguments like this are given in Chapter 5.

Both $\pi(y_1)$ and $\pi(y_2)$ are greater than $0'$. By Theorem 3.2.3, representatives for $\pi(y_1)$ and $\pi(y_2)$ are coded in the degrees below $\pi(y_1)$ and $\pi(y_2)$, respectively. Since the ideals determined by y_1 and y_2 are isomorphic to those determined by $\pi(y_1)$ and $\pi(y_2)$, representatives for $\pi(y_1)$ and $\pi(y_2)$ are coded in the degrees below y_1 and y_2, respectively. By Corollary 3.2.5, the corollary to the Decoding Theorem, $\pi(y_1) \leq_T y_1^{(5)}$ and $\pi(y_2) \leq_T y_2^{(5)}$. Since both $y_1^{(5)}$ and $y_2^{(5)}$ are below x, $\pi(y_1)$ and $\pi(y_2)$ are both below x. Now, x is equal to the join of y_1 and y_2, so $\pi(x)$ is equal to the join of $\pi(y_1)$ and $\pi(y_2)$. In particular, $\pi(x)$ is below any upper bound on the pair $\pi(y_1)$ and $\pi(y_2)$. Hence, $\pi(x)$ is below x.

Since x is above $\pi^{-1}(\pi(0')^{(5)})$, $\pi(x)$ is above $\pi(0')^{(5)}$. The above argument applied to $\pi(x)$ in place of x and the automorphism π^{-1} in place of π shows that $\pi^{-1}(\pi(x))$ is below $\pi(x)$. That is, x is below $\pi(x)$.

Combining the two inequalities, $\pi(x) = x$. ∎

Remark 3.3.4 Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$. We have just shown that π is equal to the identity on a cone, say on the cone above a. If $\mathcal{I}$ is an ideal in $\mathfrak{D}$ which includes a, then every element x of $\mathcal{I}$ is below an element $x \vee a$ of $\mathcal{I}$ such that $x \vee a$ is above a. Then $\pi(x)$ is below $\pi(x \vee a)$ which is equal to $x \vee a$. Thus, π maps $\mathcal{I}$ into $\mathcal{I}$. The same argument shows that π^{-1} maps $\mathcal{I}$ into $\mathcal{I}$. Thus, the restriction of π to $\mathcal{I}$ is an automorphism of $\mathcal{I}$.

The second application of the effective coding and decoding lemmas is due to Odifreddi and Shore.

Theorem 3.3.5 (Odifreddi and Shore (1991)) *Suppose that π is an automorphism of $\mathfrak{D}$ and that $\mathcal{I}$ is an ideal in $\mathfrak{D}$ which includes $0'$ such that π restricts to an automorphism of $\mathcal{I}$. For any real I, if there is a presentation of $\mathcal{I}$ which is recursive in I then the restriction of π to $\mathcal{I}$ has a presentation which is arithmetic in I.*

Proof: Let I be fixed so that the sequence $\boldsymbol{X} = (X_i : i \in \mathbb{N})$ is a presentation of $\mathcal{I}$ which is recursive in I. Let $\mathbb{N}^* = (N^*, 0, s)$ be an isomorphic copy of the standard model of arithmetic so that N^* is a set of Turing degrees and $\mathbb{N}^*$ has a presentation which is arithmetic in I. Let $(\mathbb{N}^*, \psi, \mathcal{I})$ be the expansion of $\mathbb{N}^*$ in which ψ is the bijection between $\mathbb{N}$ and $\mathcal{I}$ mapping i to the degree of X_i. Since $\boldsymbol{X}$ is recursive in I, $(\mathbb{N}^*, \psi, \mathcal{I})$ has a presentation which is arithmetic in I. By the Effective Coding Theorem 3.2.2, let $\boldsymbol{p}$ be a sequence of parameters which codes $(\mathbb{N}^*, \psi, \mathcal{I})$ in $\mathfrak{D}$ and is arithmetically presented relative to I.

We observe that the action of π on $\mathcal{I}$ is determined by its value on $\boldsymbol{p}$. Both $\boldsymbol{p}$ and $\pi(\boldsymbol{p})$ code models of arithmetic and bijections between those models and the ideals $\mathcal{I}$ and $\pi(\mathcal{I})$. By assumption, $\pi(\mathcal{I})$ is equal to $\mathcal{I}$. Thus, both $\boldsymbol{p}$ and $\pi(\boldsymbol{p})$ code models of arithmetic with bijections between those models and $\mathcal{I}$. In other words, $\boldsymbol{p}$ and $\pi(\boldsymbol{p})$ both code enumerations of $\mathcal{I}$. Since π is an automorphism, for each n the nth element enumerated by $\boldsymbol{p}$ must be mapped to the nth element enumerated by $\pi(\boldsymbol{p})$. This determines the values of π on all of $\mathcal{I}$.

In the previous paragraph, we observed that the action of π is determined by the value of π on $\boldsymbol{p}$. Moreover, we gave a description of the point-wise evaluation of π on $\mathcal{I}$ directly in terms of the models coded by $\boldsymbol{p}$ and $\pi(\boldsymbol{p})$. By Corollary 3.2.5 this description is arithmetic in $\boldsymbol{p}$ and $\pi(\boldsymbol{p})$. Now, $0'$ is an element of $\mathcal{I}$ and π restricts to an automorphism of $\mathcal{I}$, so $\pi^{-1}(0')$ also an element of I. By Theorem 3.3.1, $\pi(\boldsymbol{p})$ is arithmetic in $\boldsymbol{p} \vee \pi^{-1}(0')$. Since $\boldsymbol{p}$ is arithmetic in I and every element of $\mathcal{I}$, including $\pi^{-1}(0')$, has a representative which is recursive in I, $\pi(\boldsymbol{p})$ is arithmetic in I. Since both $\boldsymbol{p}$ and $\pi(\boldsymbol{p})$ are arithmetic in I, the description of π is arithmetic in I, as desired. ■

Remark 3.3.6 Just as $\pi(x)$ is close to x, the restriction of π to $\mathcal{I}$ is close to $\mathcal{I}$.

4

Persistent Automorphisms

In this chapter, we will introduce the notion of a *persistent* automorphism of a countable ideal in $\mathfrak{D}$. We will show that many of the properties we observed in Chapter 3 to hold of automorphism of $\mathfrak{D}$ also hold of persistent automorphisms. Ultimately, our goal in this chapter and the next one is to show that any persistent countable automorphism extends to an automorphism of $\mathfrak{D}$.

4.1 Fundamental properties

Definition 4.1.1 An automorphism ρ of a countable ideal $\mathcal{I}$ is *persistent* if for every degree x there is a countable ideal $\mathcal{I}_1$ such that

1. $x \in \mathcal{I}_1$ and $\mathcal{I} \subseteq \mathcal{I}_1$;
2. there is an automorphism ρ_1 of $\mathcal{I}_1$ such that the restriction of ρ_1 to $\mathcal{I}$ is equal to ρ.

Note, there is no restriction in requiring that $\mathcal{I}_1$ be countable. Given an uncountable $\mathcal{I}_1$ and ρ_1, we could obtain a countable one by applying the Lowenheim-Skolem Theorem.

Theorem 4.1.2 *Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$. For any ideal $\mathcal{I}$, if π restricts to an automorphism $\pi \restriction \mathcal{I}$ of $\mathcal{I}$ then $\pi \restriction \mathcal{I}$ is persistent.*

Proof: Let $\mathcal{I}$ be an ideal in $\mathfrak{D}$ such that π restricts to an automorphism of $\mathcal{I}$. Let b be an upper bound on $\mathcal{I}$. By Remark 3.3.4, there is a degree, call it a, such that for any degree $y \geq_T a$, π restricts to an automorphism of the principal ideal (y).

To show that $\pi \restriction \mathcal{I}$ is persistent, suppose that x is given. Let $\mathcal{I}_1$ be the principal ideal $(x \vee a \vee b)$. Then, π restricts to an automorphism $\pi \restriction \mathcal{I}_1$ of $\mathcal{I}$. Of course, $\pi \restriction \mathcal{I}_1$ extends $\pi \restriction \mathcal{I}$. ■

Thus, if $\mathfrak{D}$ is not rigid, then there is a nontrivial persistent automorphism of some countable ideal. As we stated above, we will prove that the converse is also true.

Definition 4.1.3 An ideal $\mathcal{I}$ is a *jump ideal* if $\mathcal{I}$ is closed under application of the Turing jump.

Theorem 4.1.4 *Suppose that $\rho : \mathcal{I} \tilde{\to} \mathcal{I}$, that $\mathcal{J}$ is a jump ideal contained in $\mathcal{I}$ and that $\rho(0') \vee \rho^{-1}(0') \in \mathcal{J}$. Then $\rho \restriction \mathcal{J}$ is an automorphism of $\mathcal{J}$.*

Proof: Our argument is similar to the one used in Chapter 3 to show that if $\pi : \mathfrak{D} \tilde{\to} \mathfrak{D}$ then for every x, $\pi(x)$ is close to x.

We must verify that $\mathcal{J}$ is closed under application of ρ and of ρ^{-1}. In fact, by symmetry, it is enough to show that $\mathcal{J}$ is closed under ρ. Let x be an element of $\mathcal{J}$. By Theorem 3.2.3, a representative X^* of $\rho(x)$ is coded by parameters below $\rho(x) \vee 0'$. Then, X^* is coded by parameters below $\rho^{-1}(\rho(x) \vee 0')$. That is, X^* is coded by parameters below $x \vee \rho^{-1}(0')$. By the Decoding Lemma 3.2.4, the degree of X^*, namely $\rho(x)$, is arithmetic in $x \vee \rho^{-1}(0')$. Since $\rho^{-1}(0') \in \mathcal{J}$ and $\mathcal{J}$ is a jump ideal, this implies that $\rho(x)$ is an element of $\mathcal{J}$. ∎

A nontechnical reading of Theorem 4.1.4 would say that any automorphism of an ideal in $\mathfrak{D}$ acts locally.

Corollary 4.1.5 *Suppose that $\mathcal{I}$ is an ideal such that $0'$ is an element of $\mathcal{I}$ and suppose that ρ is a persistent automorphism of $\mathcal{I}$. For any countable jump ideal $\mathcal{J}$ extending $\mathcal{I}$, ρ extends to an automorphism of $\mathcal{J}$.*

Proof: Let $\mathcal{J}$ be a jump ideal such that $\mathcal{I}$ is contained in $\mathcal{J}$. Let x be an upper bound on the elements of $\mathcal{J}$. By the persistence of ρ, let $\mathcal{I}_1$ be an ideal which includes x and extends $\mathcal{I}$ and let ρ_1 be an automorphism of $\mathcal{I}_1$ which agrees with ρ on $\mathcal{I}$.

Since x is an upper bound on the elements of $\mathcal{J}$, $\mathcal{J}$ is contained in $\mathcal{I}_1$. By Theorem 4.1.4, ρ_1 restricts to an automorphism of $\mathcal{J}$. Then $\rho_1 \restriction \mathcal{J}$ is the desired extension of ρ to $\mathcal{J}$. ∎

Corollary 4.1.6 *If $\pi : \mathfrak{D} \tilde{\to} \mathfrak{D}$ and $\mathcal{J}$ is a jump ideal such that both $\pi(0')$ and $\pi^{-1}(0')$ are elements of $\mathcal{J}$ then $\pi \restriction \mathcal{J}$ is a persistent automorphism of $\mathcal{J}$.*

Proof: By Theorem 4.1.4, the restriction of π to $\mathcal{J}$ is an automorphism of $\mathcal{J}$. By 4.1.2, $\pi \upharpoonright \mathcal{J}$ is persistent. ■

The Odifreddi-Shore Theorem 3.3.5 states that if $\pi : \mathfrak{D} \tilde{\to} \mathfrak{D}$ and π restricts to an automorphism of an ideal $\mathcal{I}$ such that $0' \in \mathcal{I}$ then $\pi \upharpoonright \mathcal{I}$ is arithmetically presented relative to any presentation of $\mathcal{I}$. In the next lemma, we establish a similar upper bound on the possible complexity of a persistent automorphism of an ideal $\mathcal{I}$ in terms of an arbitrary presentation of $\mathcal{I}$.

Theorem 4.1.7 *Suppose that $\mathcal{I}$ is an ideal in $\mathfrak{D}$ such that $0'$ is an element of $\mathcal{I}$. Suppose that there is a presentation of $\mathcal{I}$ which is recursive in I. Finally, suppose that $\mathcal{J}$ is a jump ideal which includes I and ρ is an automorphism of $\mathcal{J}$ that restricts to an automorphism of $\mathcal{I}$. Then, the restriction $\rho \upharpoonright \mathcal{I}$ of ρ to $\mathcal{I}$ has a presentation which is arithmetic in I.*

Proof: Let $(X_i : i \in \mathbb{N})$ be a presentation of $\mathcal{I}$ which is recursive in I. As in the proof of Theorem 3.3.5, there are parameters $\boldsymbol{p}$ which are arithmetic in I and code the structure $(\mathbb{N}^*, \psi, \mathcal{I})$, where $\mathbb{N}^*$ is a model of arithmetic and ψ is the bijection between $\mathbb{N}^*$ and $\mathcal{I}$ which maps the ith integer in $\mathbb{N}^*$ to X_i.

Let $\mathcal{J}(I)$ be the smallest jump ideal which includes the degree of I. Namely, $\mathcal{J}(I)$ is just the collection of degrees of sets which are arithmetic in I. Since $\mathcal{J}$ is a jump ideal, $\mathcal{J}(I)$ is contained in $\mathcal{J}$. Further, since ρ maps $\mathcal{I}$ automorphically to itself and $0'$ is an element of $\mathcal{I}$, $\rho(0')$ and $\rho^{-1}(0')$ are both contained in $\mathcal{J}(I)$. By Theorem 4.1.4, ρ restricts to an automorphism of $\mathcal{J}(I)$. In particular, since we took $\boldsymbol{p}$ to be arithmetic in I, both $\boldsymbol{p}$ and $\rho(\boldsymbol{p})$ are elements of $\mathcal{J}(I)$. Thus, they are both arithmetically presented relative to I.

The action of ρ on $\mathcal{I}$ can be read off from the value of ρ on $\boldsymbol{p}$. Both $\boldsymbol{p}$ and $\rho(\boldsymbol{p})$ code enumerations of $\mathcal{I}$. For ρ to be an automorphism, the ith element of $\mathcal{I}$ in the sense of $\boldsymbol{p}$ must be mapped to the ith element of $\mathcal{I}$ in the sense of $\rho(\boldsymbol{p})$. Thus, ρ is arithmetically presented in any upper bound on $\boldsymbol{p}$ and $\rho(\boldsymbol{p})$. Therefore, ρ is arithmetically presented relative to I, as desired. ■

Corollary 4.1.8 *Suppose that $\mathcal{I}$ is an ideal in $\mathfrak{D}$, $0'$ is an element of $\mathcal{I}$ and ρ is a persistent automorphism of $\mathcal{I}$. Then, for any real I which computes a presentation of $\mathcal{I}$, ρ has a presentation which is arithmetic in I.*

Proof: Let I be a real such that there is a presentation of $\mathcal{I}$ which is recursive in I. Let $\mathcal{J}(I)$ be the jump ideal generated by I. Since ρ is persistent, we may apply Corollary 4.1.5 to conclude that there is an extension ρ^* of ρ to an automorphism of $\mathcal{J}(I)$. Then, Theorem 4.1.7 implies that the restriction of ρ^* to $\mathcal{I}$ is arithmetically presented relative to I. In other words, ρ is arithmetically presented relative to I, as desired. ■

Corollary 4.1.9 *Suppose that $\mathcal{I}$ is an ideal and $0'$ is an element of $\mathcal{I}$. There are at most countably many persistent automorphisms of $\mathcal{I}$.*

Proof: Let I be a real such that there is a presentation of $\mathcal{I}$ which is recursive in I. By Theorem 4.1.7, any persistent automorphism of $\mathcal{I}$ is arithmetically presented relative to I. Since there are only countably many arithmetic definitions relative to I there are at most countably many relations which are arithmetically presented relative to I. Consequently, there are at most countably many persistent automorphisms of $\mathcal{I}$. ■

Theorem 4.1.10 *Suppose that $\mathcal{I}$ is an ideal and $0'$ is an element of $\mathcal{I}$. Suppose that ρ is a persistent automorphism of $\mathcal{I}$. For any jump ideal $\mathcal{J}$ which extends $\mathcal{I}$, ρ extends to a persistent automorphism of $\mathcal{J}$.*

Proof: Let $\mathcal{J}$ be a jump ideal which extends $\mathcal{I}$. For the sake of a contradiction, suppose that there is no persistent automorphism of $\mathcal{J}$ which extends ρ. Let J be a real such that there is a presentation of $\mathcal{J}$ which is recursive in J. Since there is no persistent extension of ρ to $\mathcal{J}$, for every arithmetic definition $\varphi_e(J)$ relative to J, there is a degree x_e such that $\varphi_e(J)$ fails to define an automorphism of $\mathcal{J}$ which extends ρ and can itself be extended to an automorphism of some ideal including x_e. Let x be an upper bound on all of the x_e. There is no automorphism of $\mathcal{J}$ which is arithmetically presented relative to J and can be extended to an automorphism of some ideal which includes x.

Since ρ is persistent, let $\mathcal{I}_1$ be an ideal such that $x \in \mathcal{I}_1$ and $\mathcal{I} \subseteq \mathcal{I}_1$ and let ρ_1 be an automorphism of $\mathcal{I}_1$ which extends ρ. Since $0'$ is in $\mathcal{I}$ and ρ_1 maps $\mathcal{I}$ automorphically to itself, both $\rho_1(0')$ and $(\rho_1)^{-1}(0')$ are elements of $\mathcal{I}$ and hence of $\mathcal{J}$. Now, we can apply Theorem 4.1.4, automorphisms act locally. The restriction of ρ_1 to $\mathcal{J}$ is an automorphism of $\mathcal{J}$. But then $\rho_1 \upharpoonright \mathcal{J}$ is arithmetically presented relative to J, by Theorem 4.1.7. This contradicts the conclusion of the previous paragraph, that no automorphism of $\mathcal{J}$ which is arithmetically presented relative to J can be extended to an automorphism of an ideal which has x as an element. ■

4.2 Absoluteness

So far, we have shown that persistent automorphisms are locally presented and that persistent automorphisms have persistent extensions. Thus, given a persistent automorphism $\rho : \mathcal{I} \tilde{\rightarrow} \mathcal{I}$, a countable jump ideal $\mathcal{J}$ extending $\mathcal{I}$, and a presentation J of $\mathcal{J}$, we can find an extension of ρ to $\mathcal{J}$ which is arithmetic in J.

Now we reexamine Definition 4.1.1, in which we defined a countable function $\rho : \mathcal{I} \tilde{\rightarrow} \mathcal{I}$ to be persistent if for every countable degree x there is an ideal $\mathcal{I}_1$ such that $x \in \mathcal{I}_1$ and there is a countable function $\rho^* : \mathcal{I}_1 \tilde{\rightarrow} \mathcal{I}_1$ such that ρ^* extends ρ. If we equate countable sets with the reals which present them, this formulation presents the persistence of ρ as a Π^1_2-property of ρ. In the next theorem, we apply the above remarks to eliminate a real quantifier.

Theorem 4.2.1 *The property* I is a representation of a countable ideal $\mathcal{I}$, $0' \in \mathcal{I}$, and R is a presentation of a persistent automorphism ρ of $\mathcal{I}$ *is a* Π^1_1*-property.*

Proof: Let I and R be fixed. First, note that the properties $0' \in \mathcal{I}$ and R *is a presentation of an automorphism* ρ *of* $\mathcal{I}$ is an arithmetic property of R and I. Suppose that this property does hold of R.

By Theorem 4.1.10, if ρ is persistent then for any jump ideal $\mathcal{J}$ which contains $\mathcal{I}$, ρ can be extended to a persistent automorphism of $\mathcal{J}$. Thus, if ρ is persistent, then for any real J which computes a presentation of a jump ideal extending $\mathcal{I}$ there is an automorphism of $\mathcal{J}$ which is arithmetically presented relative to J and extends ρ, see Theorem 4.1.7.

The property $\{e\}(J)$ *is a presentation of a jump ideal which extends* $\mathcal{I}$ is an arithmetic property of e, J and R. Similarly, the property *for every* e*, if* $\{e\}(J)$ *is a presentation of a jump ideal* $\mathcal{J}$ *which extends* $\mathcal{I}$ *then there are* f *and* n *such that* $\{f\}(J^{(n)})$ *is a presentation of an automorphism of* $\mathcal{J}$ *which extends* ρ is an arithmetic property relative to $J^{(\omega)}$. Hence, this property is a Δ^1_1 property of J and R. Consequently, if R is a presentation of a persistent automorphism, then R is a presentation of an automorphism ρ of an ideal $\mathcal{I}$ and for every e and J, if $\{e\}(J)$ is a presentation of a jump ideal $\mathcal{J}$ which extends $\mathcal{I}$ then there are integers f and n such that $\{f\}(J^{(n)})$ is a presentation of an automorphism of $\mathcal{J}$ which extends ρ. The latter condition is a conjunction of an arithmetic statement with one expressed by a universal quantifier over reals, followed by a Δ^1_1-formula. Thus, the latter condition is Π^1_1.

Conversely, if the latter condition holds then R is a presentation of a persistent automorphism. R is a presentation of an automorphism of an ideal $\mathcal{I}$ by fiat. If x is a degree, we may take $\mathcal{J}(x)$ to be the jump ideal generated by x and $\mathcal{I}$ and extend ρ to $\mathcal{J}(x)$. This gives an extension of ρ to an ideal which includes x, verifying the persistence of ρ. ■

Corollary 4.2.2 *The properties* R is a presentation of a persistent automorphism of $\mathcal{I}$ and $0' \in \mathcal{I}$ *and* There is a countable map $\rho : \mathcal{I} \xrightarrow{\sim} \mathcal{I}$ such that $0' \in \mathcal{I}$, ρ is persistent and not equal to the identity *are absolute between well-founded models of ZFC.*

Proof: By Theorem 4.2.1, the first property is equivalent to a Π^1_1-condition. The second is equivalent to a Σ^1_2-statement. The Shoenfield (1961) Absoluteness Theorem states that all such statements are absolute between well-founded models of *ZFC*.

In the preceding corollary, we deduced that the notion of persistence is absolute between well-founded models of *ZFC* by showing that when viewed as a property of its presentations, the persistence of an automorphism of a countable ideal is equivalent to a Π^1_1-property. Instead of referring to the presentations of countable sets, the same ingredients can be combined to refer directly to the countable functions and ideals in terms of models $\mathcal{M}$ of fragments of set theory and absoluteness between them.

In Definition 2.2.1, we let T be the fragment of *ZFC* in which we include only the instances of replacement and comprehension in which the defining formula is Σ_1.

In any model $\mathcal{M}$ of T, we can define $\mathbb{N}^{\mathcal{M}}$, the standard model of arithmetic in the sense of $\mathcal{M}$; the power set of $\mathbb{N}^{\mathcal{M}}$; and $\mathfrak{D}^{\mathcal{M}}$, the Turing degrees in $\mathcal{M}$. Similarly, $\mathcal{M}$ has its own notion of a set's being countable and of an automorphism's being persistent.

Definition 4.2.3 Suppose that $\mathcal{M} = (M, \in^{\mathcal{M}})$ is a model of T.

1. $\mathcal{M}$ is an *ω-model* if $\mathbb{N}^{\mathcal{M}}$ is isomorphic to the standard model of arithmetic.
2. $\mathcal{M}$ is *well-founded* if the binary relation $\in^{\mathcal{M}}$ is well-founded. That is to say that there is no infinite sequence $(m_i : i \in \mathbb{N})$ of elements of $\mathcal{M}$ such that for all i, $m_{i+1} \in^{\mathcal{M}} m_i$.

Theorem 4.2.4 *Suppose that $\mathcal{M}$ is an ω-model of T. Let $\mathcal{I}$ be an element of $\mathcal{M}$ such that*

$$\mathcal{M} \models \mathcal{I} \text{ is a countable ideal in } \mathfrak{D} \text{ such that } 0' \in \mathcal{I}.$$

Then, every persistent automorphism of $\mathcal{I}$ is also an element of $\mathcal{M}$.

Proof: Since $\mathcal{I}$ is countable in the sense of $\mathcal{M}$ there is a real I in $\mathcal{M}$ such that some presentation of $\mathcal{I}$ is recursive in I. By Theorem 4.1.7, any persistent automorphism has a presentation which is arithmetic in I.

Since $\mathcal{M}$ is a model of T, any arithmetic definition applied to a real in $\mathcal{M}$ has an interpretation in $\mathcal{M}$. Moreover, since $\mathcal{M}$ is an ω-model arithmetic definitions are correctly interpreted in $\mathcal{M}$. Thus, every set which has an arithmetic presentation relative to an element of $\mathcal{M}$ has a presentation in $\mathcal{M}$. The process of converting a presentation of a set into the set itself is a Σ_1-recursion on the elements of the presentation. Since $\mathcal{M}$ is a model of T(especially of Σ_1-replacement), it is closed under this operation.

Thus, every persistent automorphism of $\mathcal{I}$ has a presentation in $\mathcal{M}$ and so is an element of $\mathcal{M}$. ■

Corollary 4.2.5 *Suppose that $\mathcal{M}$ is an ω-model of T and that ρ and $\mathcal{I}$ are elements of $\mathcal{M}$ such that $0' \in \mathcal{I}$, $\rho : \mathcal{I} \xrightarrow{\sim} \mathcal{I}$, and $\mathcal{I}$ is countable in $\mathcal{M}$. Then,*

$$\rho \text{ is persistent} \Longrightarrow \mathcal{M} \models \rho \text{ is persistent}.$$

Proof: Suppose that ρ is a persistent automorphism of $\mathcal{I}$.

Suppose that x is an element of $\mathfrak{D}^{\mathcal{M}}$. Let $\mathcal{J}(x)$ be the least jump ideal which includes x and contains $\mathcal{I}$. Because $\mathcal{I}$ is countable in $\mathcal{M}$ and x is in $\mathcal{M}$, $\mathcal{J}(x)$ is a countable element of $\mathcal{M}$. By Theorem 4.1.10, for any jump ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$ there is a persistent automorphism of $\mathcal{J}$ which extends ρ. In particular, let ρ^* be a persistent automorphism of $\mathcal{J}(x)$ which extends ρ. Since ρ^* is a persistent automorphism of an ideal which is countable in $\mathcal{M}$, we can apply Theorem 4.2.4 to conclude that ρ^* is an element of $\mathcal{M}$. Thus, in $\mathcal{M}$ there is an extension of ρ to an automorphism of an ideal which includes x.

Since x was arbitrary, for any x in $\mathfrak{D}^{\mathcal{M}}$ there are an ideal $\mathcal{J}(x)$ which includes both $\mathcal{I}$ and x, and a map $\rho^* : \mathcal{J}(x) \xrightarrow{\sim} \mathcal{J}(x)$ which extends ρ such that $\rho^* \in \mathcal{M}$, $\mathcal{J}(x) \in \mathcal{M}$, and both sets are countable in $\mathcal{M}$. Thus, $\mathcal{M}$ satisfies the statement that ρ is persistent. ■

Remark 4.2.6 Corollary 4.2.5 can be used to recast the proof that ρ *is a persistent automorphism of an ideal which includes* $0'$ is a Π^1_1 property of any presentation of ρ. Namely, a real R is a presentation of a persistent automorphism of an ideal $\mathcal{I}$ which includes $0'$ if and only if R is a presentation of an automorphism ρ of an ideal which includes $0'$ and for every countable presentation of an ω-model $\mathcal{M}$ of T such that $R \in \mathcal{M}$, $\mathcal{M} \models \rho$ is persistent. The quantifier over ω-models is a quantifier over reals which code models $\mathcal{M}$ with bijections between $\mathbb{N}$ and $\mathbb{N}^{\mathcal{M}}$; the condition that $\mathcal{M}$ be a model of the persistence of ρ is first order over $\mathcal{M}$ and thus arithmetic in its code; consequently, this characterization of persistence is Π^1_1.

4.3 Generic persistence

We now extend the notion of persistence to uncountable ideals. In what follows, V is the universe of sets and G is a V-generic filter for some partial order in V.

Definition 4.3.1 Suppose that $\mathcal{I}$ is an ideal in $\mathfrak{D}$ and ρ is an automorphism of $\mathcal{I}$. We say that ρ is *generically persistent* if there is a generic extension $V[G]$ of V in which $\mathcal{I}$ is countable and ρ is persistent.

In the definition of the generic persistence of $\rho : \mathcal{I} \tilde{\to} \mathcal{I}$, we only require that ρ be persistent in some generic extension of V in which $\mathcal{I}$ is countable. We show in the next theorem that if ρ is generically persistent, then it is persistent in every generic extension of V in which $\mathcal{I}$ is countable.

Theorem 4.3.2 *Suppose that $\rho : \mathcal{I} \tilde{\to} \mathcal{I}$ is generically persistent. If $V[G]$ is a generic extension of V in which $\mathcal{I}$ is countable then ρ is persistent in $V[G]$.*

Proof: Since ρ is generically persistent, there is a generic extension of V in which ρ is persistent. Let P be a forcing partial order in V such that there is a condition p in P which forces $\mathcal{I}$ to be countable and ρ to be persistent. Suppose, for the sake of a contradiction, that P_1 is a partial order in V, p_1 is a condition in P_1 and p_1 forces $\mathcal{I}$ to be countable and ρ not to be persistent.

Let λ be the supremum of the cardinalities of the power sets of the partial orders P and P_1. Let $Coll(\lambda, \omega)$ be the Levy-collapse of λ to ω. That is, let $Coll(\lambda, \omega)$ be the set of functions whose domain is a finite subset of ω and whose range is contained in λ, ordered under inclusion. The union of a generic filter for $Coll(\lambda, \omega)$ is a function from ω onto λ. Let H be $Coll(\lambda, \omega)$-generic over V. The collection of sets from V which are dense subsets of P is a countable set in $V[H]$. The same holds for the collection of sets from V which are dense subsets of P_1. Thus, in $V[H]$, there is a set G^* which is P-generic over V and includes p and there is a set G_1^* which is P_1-generic over V and includes p_1.

Let R be a countable presentation of ρ in $V[G^*]$. By Corollary 4.2.2, the statement R *is a presentation of a persistent automorphism* is absolute between $V[G^*]$ and $V[H]$. Thus, $V[H]$ satisfies the same statement. Thus, $V[H]$ satisfies the statement that ρ is persistent. Now, $V[G_1^*]$ is an inner model of *ZFC*. Thus, it is an ω-model of the theory T discussed in Section 4.2. By Corollary 4.2.5 $V[G_1^*]$ must also satisfy the statement that ρ is persistent.

On the other hand, G_1^* is P_1-generic over V and includes the condition p_1. Hence, every statement forced by p_1 is true in $V[G_1^*]$. Consequently, ρ is not persistent in $V[G_1^*]$. This is the desired contradiction. ■

4.4 Applications to $Aut(\mathfrak{D})$

We can now begin establishing the connection between persistent countable automorphisms and global automorphisms of $\mathfrak{D}$.

Theorem 4.4.1 *Suppose that $\pi : \mathfrak{D} \tilde{\to} \mathfrak{D}$. Then, π is generically persistent.*

Proof: Suppose, for the sake of a contradiction, that π is not generically persistent.

Let λ be a cardinal such that V_λ, the collection of sets of rank less than λ, is a model of T. Let $\mathcal{H}$ be a countable elementary substructure of V_λ such that π is an element of $\mathcal{H}$. Let $\mathcal{M}$ be the transitive collapse of $\mathcal{H}$ and let $\tau : \mathcal{H} \tilde{\to} \mathcal{M}$ be the collapsing isomorphism. Let $\pi^{\mathcal{M}}$ be the image of π and $\mathfrak{D}^{\mathcal{M}}$ be the image of $\mathfrak{D}$ under τ. By induction on $\mathbb{N}$, for any integer n, $\tau(n)$ is equal to n. Consequently, $\tau(\mathbb{N})$ is equal to $\mathbb{N}$ and for any subset X of $\mathbb{N}$, if $X \in \mathcal{H}$, then $\tau(X)$ is equal to X. Thus, $\mathfrak{D}^{\mathcal{M}}$ is an ideal in $\mathfrak{D}$ and $\pi^{\mathcal{M}}$ is the restriction of π to $\mathfrak{D}^{\mathcal{M}}$.

Let P be the partial order in $\mathcal{M}$ which generically adds a counting of $\mathfrak{D}^{\mathcal{M}}$ to $\mathcal{M}$. Since $\mathcal{M}$ is a countable structure, let G be a fixed set which is P-generic over $\mathcal{M}$. First, $\mathcal{M}[G]$ is an ω-model of T. Secondly, by Theorem 4.3.2, the fact that π is not generically persistent in V implies that generically adding a counting of $\mathfrak{D}$ to V, or to V_λ, results in a model in which π is not persistent. Since $\mathcal{M}$ is elementarily equivalent to V_λ, $\mathcal{M}[G]$ must be a model of *$\pi^{\mathcal{M}}$ is not persistent.*

However, $\pi^{\mathcal{M}}$ is the restriction of a global automorphism of $\mathfrak{D}$. By Theorem 4.1.2, $\pi^{\mathcal{M}}$ is a persistent automorphism of $\mathfrak{D}^{\mathcal{M}}$. $\mathcal{M}[G]$ is a generic extension of a well-founded model of T and thus is a well-founded model of T. By Theorem 4.2.4, the persistence of $\pi^{\mathcal{M}}$ in V is reflected by its persistence in $\mathcal{M}[G]$. Consequently, $\mathcal{M}[G]$ is a model of *$\pi^{\mathcal{M}}$ is persistent.* This contradiction proves the theorem. ■

The next theorem is our first result showing that every automorphism of the Turing degrees is definable. Looking ahead to Theorem 6.3.1, we will eventually strengthen this result to show that every automorphism of $\mathfrak{D}$ is induced by a function on reals which is arithmetically definable.

Theorem 4.4.2 *Suppose that $V[G]$ is a generic extension of V. Suppose that π is an element of $V[G]$ which maps the Turing degrees in V automorphically to itself (that is, $\pi : \mathfrak{D}^V \xrightarrow{\sim} \mathfrak{D}^V$). If π is generically persistent in $V[G]$, then π is an element of $L(\mathbb{R}^V)$. That is, π is constructible from the set of reals in V.*

Proof: Let H be a counting of $\mathfrak{D}^V$ which is generic over $V[G]$ for the partial order $Coll(\mathfrak{D}^V, \omega)$, the partial order of finite maps from ω to $\mathfrak{D}^V$. Since π is generically persistent in $V[G]$, π is persistent in $V[G][H]$. We know that every persistent isomorphism of $\mathfrak{D}^V$ belongs to any model of T which includes a counting of $\mathfrak{D}^V$. Thus, π is an element of $L(\mathbb{R}^V)[H]$.

It is a standard feature of forcing that the intersection of all sufficiently generic extensions of $L(\mathbb{R}^V)$ is equal to $L(\mathbb{R}^V)$. Thus, we may conclude that π is an element of $L(\mathbb{R}^V)$.

Here is the proof in more detail. First note that $Coll(\mathfrak{D}^V, \omega)$ is an element of $L(\mathbb{R}^V)$. Since π is an element of $L(\mathbb{R}^V)[H]$ for every H which is $Coll(\mathfrak{D}^V, \omega)$-generic over $V[G]$, there is a term t, in the language for forcing with $Coll(\mathfrak{D}^V, \omega)$ over $L(\mathbb{R}^V)$, such that it is forced in $V[G]$ that for any H^* which is $Coll(\mathfrak{D}^V, \omega)$-generic over $V[G]$, t denotes π in $L(\mathbb{R}^V)[H^*]$. This implies that all of the values of t are forced by the empty condition to equal those of π. But the forcing relation, $\Vdash_{Coll(\mathfrak{D}^V,\omega)} t(x) = y$, is definable in $L(\mathbb{R}^V)$. As we have already seen, $\pi(x) = y$ if and only if $\Vdash_{Coll(\mathfrak{D}^V,\omega)} t(x) = y$. Thus, π is definable in $L(\mathbb{R}^V)$ and therefore is an element of $L(\mathbb{R}^V)$. ■

Theorem 4.4.2 has some remarkable consequences.

Theorem 4.4.3 *Suppose that* $0'$ *is an element of* $\mathcal{I}$ *and* $\rho : \mathcal{I} \overset{\sim}{\to} \mathcal{I}$ *is persistent. Then* ρ *can be extended to a global automorphism* $\pi : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$.

Proof: Let $V[G]$ be a generic extension of V in which $\mathcal{I}$ is countable. By the absoluteness of persistence (Corollary 4.2.2), ρ is persistent in $V[G]$. By Theorem 4.1.10, for any countable jump ideal $\mathcal{J}$ extending $\mathcal{I}$, we can extend the persistent automorphism ρ to a persistent automorphism of $\mathcal{J}$. Since $\mathfrak{D}^V$ is countable in $V[G]$, let $\pi : \mathfrak{D}^V \overset{\sim}{\to} \mathfrak{D}^V$ be an element of $V[G]$ which in $V[G]$ is a persistent extension of ρ. Since persistence implies generic persistence, π is generically persistent in $V[G]$. By Theorem 4.4.2, π is an element of $L(\mathbb{R}^V)$ and therefore is an element of V.

Therefore, π is the desired extension of ρ to a global automorphism. ■

Corollary 4.4.4 *The statement* There is a non-trivial automorphism of the Turing degrees *is equivalent to a* Σ^1_2 *statement. It is therefore absolute between well-founded models of ZFC.*

Proof: By Theorem 4.4.3, any if there is a non-trivial persistent automorphism of a countable ideal $\mathcal{I}$ for which $0' \in \mathcal{I}$ then there is a non-trivial automorphism of $\mathfrak{D}$. Conversely, Theorem 4.1.4 implies that any automorphism of $\mathfrak{D}$ can be restricted to an automorphism of any sufficiently large jump ideal. Thus, if there is a non-trivial automorphism of $\mathfrak{D}$ then it has a non-trivial restriction to some jump ideal; by Theorem 4.1.2 this restriction would be a non-trivial persistent automorphism of a countable ideal which includes $0'$.

Thus, there is a non-trivial automorphism of the Turing degrees if and only if there is a non-trivial persistent automorphism of a countable ideal $\mathcal{I}$ for which $0' \in \mathcal{I}$. By Corollary 4.2.2, the latter condition is Σ^1_2. ■

Theorem 4.4.5 *Let* π *be an automorphism of* $\mathfrak{D}$. *Suppose that* $V[G]$ *is a generic extension of* V. *Then, there is an extension of* π *in* $V[G]$ *to an automorphism of* $\mathfrak{D}^{V[G]}$, *the Turing degrees in* $V[G]$.

Proof: Since π is an automorphism of D, π is generically persistent. Let $V[G][H]$ be the generic extension of $V[G]$ obtained by generically introducing a counting of $\mathfrak{D}^{V[G]}$. Any two-fold iteration of forcing can be regarded as a single application of forcing; hence, $V[G][H]$ is a generic extension of V in which $\mathfrak{D}^V$ is countable. So, π is persistent in $V[G][H]$.

By Theorem 4.1.10, in $V[G][H]$, there is a persistent extension π_1 of π to the ideal $\mathfrak{D}^{V[G]}$. By Theorem 4.4.2, π_1 is an element of $L(\mathbb{R}^{V[G]})$ and therefore an element of $V[G]$. ■

5

Representing Automorphisms of $\mathfrak{D}$

5.1 Continuous representations on generic sequences

Suppose that π is an automorphism of $\mathfrak{D}$ which belongs to V or even to a generic extension of V. In Theorem 4.4.2, we showed that π must be an element of $L(\mathbb{R})$. Thus, π can be defined by transfinite recursion, from the set of reals and a real parameter. In this chapter, we establish sharper definability bounds on π.

Definition 5.1.1 Given two functions $\tau : \mathfrak{D} \to \mathfrak{D}$ and $t : \mathbb{R} \to \mathbb{R}$, we say that t *represents* τ if and only if for every degree x and every real X in x, the Turing degree of $t(X)$ is equal to $\tau(x)$.

In this chapter, we will show that every automorphism of $\mathfrak{D}$ is represented by a function on $\mathbb{R}$ which can be defined arithmetically in a real parameter. We will conclude that any automorphism of $\mathfrak{D}$ is determined by its action on a finite set of degrees. That is, $\mathfrak{D}$ has a finite automorphism base.

In the previous chapter, we concentrated on generic extensions of V obtained by adding countings of sets which are uncountable in V. Theorem 4.4.5 states that if π is an automorphism of the Turing degrees in V and $V[G]$ is a generic extension of V, then there is an automorphism of the Turing degrees in $V[G]$ which extends π. In this section, we will apply Theorem 4.4.5 to a generic extension obtained by adding ω_1-many Cohen reals to V. We will analyze the behavior of an automorphism of $\mathfrak{D}$ in terms of the action of its extensions on the degrees of the generic reals.

When we add a generic set to V, we use forcing in V to approximate the properties that set will have in the generic extension. By definition, the forcing relation is continuous in the order topology on the forcing partial order. That continuous behavior is evident in the next theorem.

As in Section 2.3, let P_1 be the partial order (isomorphic to $2^{<\omega}$) to add a Cohen (1966) real to V and let P_{ω_1} be the partial order to add ω_1-many of them.

Theorem 5.1.2 *Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$. There is a countable family $\boldsymbol{D}$ of dense open subsets of P_1 such that π is represented by a continuous function f on the set $\boldsymbol{D}$-generic reals.*

Proof: By Theorem 4.4.5, there is an extension of π in $V[\mathcal{G}]$ mapping the Turing degrees $\mathfrak{D}^{V[\mathcal{G}]}$ automorphically to itself. By Theorems 4.4.1 and 4.4.2, any extension of π to an automorphism of $\mathfrak{D}^{V[\mathcal{G}]}$ is generically persistent in $\mathfrak{D}^{V[\mathcal{G}]}$ and therefore an element of $L(\mathbb{R}^{V[\mathcal{G}]})$. Thus, any such extension of π is absolutely definable from a real X in $V[\mathcal{G}]$, an ordinal γ, and the reals $\mathbb{R}^{V[\mathcal{G}]}$ of $V[\mathcal{G}]$. Let τ_X be a term in the forcing language, let γ be an ordinal, let φ be a first order formula in the language of set theory, and let p_0 be a condition in P_{ω_1} such that p_0 forces the following two statements to hold.

1. $\tau_X \in 2^\omega$.
2. The predicate τ_{π^*} on $L_\gamma[\mathbb{R}^{V[\mathcal{G}]}]$ defined by
$$v \in \tau_{\pi^*} \iff L_\gamma[\mathbb{R}^{V[\mathcal{G}]}] \models \varphi(v, \tau_X) \tag{5.1}$$
is an automorphism of $\mathfrak{D}^{V[\mathcal{G}]}$ which extends π.

We can incorporate the description of p_0 in the term τ_X. Hence, we may assume that p_0 is the null condition.

Let π^* be the denotation of τ_{π^*} in $V[\mathcal{G}]$, and let X be the denotation of τ_X.

Consider the situation from the vantage point of $V[X]$. By the homogeneity of Cohen forcing, $V[\mathcal{G}]$ can be factored as $V[X][\mathcal{G}_X]$, in which $\mathcal{G}_X$ is P_{ω_1}-generic over $V[X]$. (See Section 2.3.) The definition of π^* depends only on X, γ, the formula φ, and the set $\mathbb{R}^{V[\mathcal{G}]}$, which is equal to $\mathbb{R}^{V[X][\mathcal{G}_X]}$.

When we force with P_{ω_1} over $V[X]$, X is an element of the ground model. So, in the term for π^*, we can replace τ_X with a constant for X. Let $\tau_{\pi^*,X}$ be the term in the forcing language for P_{ω_1} over $V[X]$ to designate the subset of $L_\gamma[\mathbb{R}^{V[X][\mathcal{G}_X]}]$ defined by $\{v \in L_\gamma[\mathbb{R}^{V[X][\mathcal{G}_X]}] : L_\gamma[\mathbb{R}^{V[X][\mathcal{G}_X]}] \models \varphi(v, X)\}$. Then, in $V[X]$

$$\Vdash_{P_{\omega_1}} \tau_{\pi^*,X} : \mathfrak{D}^{V[\mathcal{G}]} \xrightarrow{\sim} \mathfrak{D}^{V[\mathcal{G}]}.$$

By Theorem 3.3.1 on page 44, for any Turing degree y, $\pi^*(y)$ is an element of the jump ideal generated by $(\pi^*)^{-1}(0')$ and y. Since π^* is an extension of π and π is defined on all of $\mathfrak{D}^V$, $(\pi^*)^{-1}(0')$ is equal to $\pi^{-1}(0')$ and we shall denote it as such. We also fix a representative of $\pi^{-1}(0')$ and let $\Pi^{-1}(\emptyset')$ refer to that real.

Now suppose that G is a a real which is Cohen generic over $V[X]$. Let g be the Turing degree of G. Specializing the above remark to g, $\pi^*(g)$ is an element of the jump ideal generated by $\pi^{-1}(0')$ and g. Say that $\pi^*(g)$ is represented by $\{e\}((G \oplus \Pi^{-1}(\emptyset'))^{(k)})$. Once again, we can factor $V[X][\mathcal{G}_X]$ as $V[X][G][\mathcal{G}_{XG}]$ in which G is obtained by Cohen forcing P_1 to add one Cohen real to $V[X]$ and $\mathcal{G}_{XG}$ is P_{ω_1}-generic over $V[X][G]$. So there must be a condition in $P_1 \times P_{\omega_1}$ which forces that $\pi^*(g)$ is represented by $\{e\}((G \oplus \Pi^{-1}(\emptyset'))^{(k)})$. By the absoluteness of $\{e\}((G \oplus \Pi^{-1}(\emptyset'))^{(k)})$ and the invariance of the interpretation of the term τ_{π^*}, forcing this fact cannot depend upon the value of this condition on P_{ω_1}. Further, we can absorb the finite condition in P_1 into the description of the recursive function $\{e\}$. Consequently, we can assume that the empty condition forces that $\pi^*(g)$ is represented by $\{e\}((G \oplus \Pi^{-1}(\emptyset'))^{(k)})$. Since G is P_1-generic, $(G \oplus \Pi^{-1}(\emptyset'))^{(k)}$ is Turing equivalent to $G \oplus \Pi^{-1}(\emptyset')^{(k)}$, so we may replace $\{e\}((G \oplus \Pi^{-1}(\emptyset'))^{(k)})$ with $\{e\}(G \oplus \Pi^{-1}(\emptyset')^{(k)})$.

Thus, $V[\mathcal{G}]$ satisfies a weak version of our desired result. The automorphism π^* is continuously represented on The set of reals in $V[\mathcal{G}]$ which are P_1-generic over $V[X]$ by the function $G \mapsto \{e\}(G \oplus \Pi^{-1}(\emptyset')^{(k)})$. Our final step is to transfer the representation of π^* in the generic extension to a representation of π in the ground model.

Definition 5.1.3 Given $G \subseteq \omega$, let G_{even} be the set $\{n : 2n \in G\}$ and let G_{odd} be the set $\{n : 2n + 1 \in G\}$.

In Definition 5.1.3, we are decomposing G into its even and odd parts. In Definition 1.2.2, we wrote the join of two reals as the union of its even and odd parts, with each part coding one of the reals to be joined. So, we have the identity $G = G_{even} \oplus G_{odd}$, which is notationally convenient.

Definition 5.1.4 For reals X and G, define the real $\mathbb{C}(X, G)$ as follows.

$$\mathbb{C}(X, G)(n) = \begin{cases} G_{even}(n - m), & \text{if } G_{odd}(n) = 0 \text{ and there are } m \\ & \quad \text{elements of } G_{odd} \text{ less than } n; \\ X(m), & \text{if } G_{odd}(n) = 1 \text{ and } n \text{ is the } m\text{th} \\ & \quad \text{element of } G_{odd}. \end{cases}$$

Let $m_1, m_2, m_3, \ldots$ be an enumeration of the elements of G_{odd} in increasing order. For all n which are strictly less than m_1, $\mathbb{C}(X, G)(n)$ is equal to $G_{even}(n)$; at m_1, $\mathbb{C}(X, G)(m_1)$ is equal to $X(1)$; for all n in the interval $[m_1 + 1, m_2)$, $\mathbb{C}(X, G)(n)$ is equal to $G_{even}(n - 1)$; at m_2, $\mathbb{C}(X, G)(m_2)$ is equal to $X(2)$; for larger values of n, $\mathbb{C}(X, G)$ is defined similarly. Thus, we use G_{odd} to partition $\mathbb{N}$ into intervals; we obtain $\mathbb{C}(X, G)$ by inserting the values of X on the end-points of the intervals and the values of G_{even} on the interior points of the intervals.

Similarly, if we are given finite binary sequences s, and p we can interpret the definition above using s and p in place of X and G until reaching the point at which we require a value at an argument that is greater than the length of its associated finite sequence. We will use $\mathbb{C}(s, p)$ to represent the finite sequence which results. Similarly, $\mathbb{C}(X, p)$ and $\mathbb{C}(s, G)$ represent the finite sequences obtained until a value of p or s is required at a number greater than its length.

We make an immediate observation about $\mathbb{C}(X, G)$.

Lemma 5.1.5 *$\mathbb{C}(X, G)$ is recursive in $X \oplus G$. Additionally, if there are infinitely many m such that $G_{odd}(m)$ is equal to 1, then $X \oplus G$ is recursive in $\mathbb{C}(X, G) \oplus G$.*

Proof: First, by virtue of its definition, $\mathbb{C}(X, G)$ is recursive in $X \oplus G$. Second, if we assume that there are infinitely many m such that $G_{odd}(m)$ is equal to 1 then we can recover X from $\mathbb{C}(X, G)$ and G: G_{odd} recursively determines the partition of $\mathbb{N}$ into infinitely many intervals; the values of X are embedded in those of $\mathbb{C}(X, G)$ on the end-points of those intervals. ■

Definition 5.1.6 Say that a condition q is an *odd-null-extension* of another condition p if for all m, if $q_{odd}(m) = 1$ then m is in the domain of p_{odd}.

Lemma 5.1.7 *Suppose that $s \in P_1$, $p \in P_1$, and the number of arguments on which p_{odd} is nonzero is less than or equal to the length of s. For each dense open subset D of P_1, there is an odd-null-extension q of p such that $\mathbb{C}(s, q) \in D$.*

Proof: Consider p as the join of two conditions p_{even} and p_{odd}. If q is an odd-null-extension of p, then evaluating the term $\mathbb{C}(s, p)$ can only query s at numbers less than or equal to the number of arguments on which p_{odd} is nonzero. Consequently, this evaluation will not terminate by querying s at a point not in its domain. Since the values of $\mathbb{C}(s, q)$ are determined by reference to s only at those arguments on which q_{odd} is nonzero and q is an odd-null-extension of p, $\mathbb{C}(s, q)$ is the extension of $\mathbb{C}(s, p)$ obtained by appending additional values of q_{even}.

Now, we meet D by making an odd-null-extension of p. Let F_{odd} be the function from ω to 2 which is equal to p_{odd} on the domain of p_{odd} and equal to 0, elsewhere. The evaluation of $\mathbb{C}(s, p_{even} \oplus F_{odd})$ terminates at the first query to p_{even} which is not in its domain. Fix $r \in P_1$ so that $\mathbb{C}(s, p_{even} \oplus F_{odd})^\frown r$ belongs to D. Let ℓ be the length of $\mathbb{C}(s, p_{even} \oplus F_{odd})^\frown r$. Then, $\mathbb{C}(s, p_{even}{}^\frown r \oplus F_{odd} \restriction \ell)$ is equal to $\mathbb{C}(s, p_{even} \oplus F_{odd})^\frown r$ and belongs to D, as required. ■

Theorem 5.1.8 *Suppose that $\boldsymbol{D}$ is countable collection of dense open subsets of P_1. Then, there is a countable collection of dense open subsets of P_1, $\boldsymbol{D}^*$ such that for any $\boldsymbol{D}^*$-generic G and any real Y, $\mathbb{C}(Y, G)$ is $\boldsymbol{D}$-generic*

Proof: Suppose that $p \in P_1$ and $D \subseteq P_1$ determines a dense open subset of 2^ω. It is sufficient to show that there is a q extending p such that for every G, if G extends q, then for every Y, there is an $r \in D$ such that $\mathbb{C}(Y, G)$ extends r.

Let k be the number of arguments at which p_{odd} is non-zero. Given any finite sequence s for length k, we can find an odd-null-extension q of p such that $\mathbb{C}(s, q) \in D$.

Since there are only finitely many binary sequences of length k, by a sequence of odd-null-extensions, we can find a q extending p so that for all s of length k, $\mathbb{C}(s, q) \in D$. But then, for all Y and all G extending p, $\mathbb{C}(Y, G)$ extends an element of D, as required. ■

Corollary 5.1.9 *For all Y and G, if G is generic over $V[X]$, then so is $\mathbb{C}(Y, G)$.*

Definition 5.1.10 For $Y \in 2^\omega$, let (Y) denote the set $\{Z : Z \leq_T Y\}$.

Let Y be given with Turing degree y, and let G_1 and G_2 be mutually Cohen generic over $V[X \oplus Y]$. We can write the ideal generated by Y as the meet of joins of generic ideals.

$$\begin{aligned}(\mathbb{C}(Y, G_1) \oplus G_1) \cap (\mathbb{C}(Y, G_2) \oplus G_2) &= (Y \oplus G_1) \cap (Y \oplus G_2) &(5.2)\\ &= (Y) &(5.3)\end{aligned}$$

Equation 5.2 follows from Lemma 5.1.5, and Equation 5.3 follows from Equation 5.2 and Theorem 2.5.6. The equality expressed in Equation 5.2

also applies to the degrees of these sets. Consequently, it is preserved by π^*. Using the continuous representation of π^* on the set of reals which are Cohen generic over $V[X]$, we have the following equality.

$$\begin{aligned}&\{Z : \text{the degree of } Z \text{ belongs to } (\pi^*(y))\} = \\ &\Big(\{e\}(\mathbb{C}(Y, G_1) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_1 \oplus \Pi^{-1}(\emptyset')^{(k)})\Big) \\ &\qquad \bigcap \Big(\{e\}(\mathbb{C}(Y, G_2) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_2 \oplus \Pi^{-1}(\emptyset')^{(k)})\Big)\end{aligned} \tag{5.4}$$

Consider the case when, in addition, Y is generic over $V[X]$. Equation 5.4 becomes

$$\begin{aligned}&\Big(\{e\}(Y \oplus \Pi^{-1}(\emptyset')^{(k)})\Big) = \\ &\Big(\{e\}(\mathbb{C}(Y, G_1) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_1 \oplus \Pi^{-1}(\emptyset')^{(k)})\Big) \\ &\qquad \bigcap \Big(\{e\}(\mathbb{C}(Y, G_2) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_2 \oplus \Pi^{-1}(\emptyset')^{(k)})\Big)\end{aligned} \tag{5.5}$$

Equation 5.5 expresses a relationship between Y, G_1 and G_2 which is arithmetic relative to $\Pi^{-1}(\emptyset')$. Thus, Equation 5.5 is true of any triple which is arithmetically generic relative to $\Pi^{-1}(\emptyset')$.

Now, consider the case when Y is merely arithmetically generic relative to $\Pi^{-1}(\emptyset')$, and G_1 and G_2 are mutually generic over $V[X \oplus Y]$. As indicated in the previous paragraph, since the triple Y, G_1, and G_2 is arithmetically generic relative to $\Pi^{-1}(\emptyset')$, it satisfies Equation 5.5. On the other hand, since G_1 and G_2 are mutually generic over $V[X \oplus Y]$, Equation 5.4 applies. So the right hand side of Equation 5.5 represents $(\pi^*(y))$. Consequently, if Y is arithmetically generic relative to $\Pi^{-1}(\emptyset')$, then the value of π^* on the degree of Y is represented by $\{e\}(Y \oplus \Pi^{-1}(\emptyset')^{(k)})$. Since π^* is an extension of π, if $Y \in V$ is arithmetically generic relative to $\Pi^{-1}(\emptyset')$, then the value of π on the degree of Y is represented by $\{e\}(Y \oplus \Pi^{-1}(\emptyset')^{(k)})$.

Now, we let $\boldsymbol{D}$ be the set of dense open subsets of P_1 which are arithmetically definable relative to $\Pi^{-1}(\emptyset')$. The function $Y \mapsto \{e\}(Y \oplus \Pi^{-1}(\emptyset')^{(k)})$ is a continuous function defined on this set, and Theorem 5.1.2 is verified. ■

Remark 5.1.11 We can refine the proof of Theorem 5.1.2. By counting the quantifiers in Equation 5.5, there is a countable family of dense open sets $\boldsymbol{D}$ which is arithmetically definable relative to $\Pi^{-1}(\emptyset')^{(k)}$ such that π

is represented on the set of $\boldsymbol{D}$-generic reals by a function which is recursive in $\Pi^{-1}(\emptyset')^{(k)}$. Now, we can represent π on 2^ω as follows. Given Y in 2^ω, find reals G_1 and G_2 arithmetically in Y and $\Pi^{-1}(\emptyset')^{(k)}$ so that Equation 5.5 applies and so that π is continuously represented on the degrees of G_1, G_2, $\mathbb{C}(Y, G_1)$, and $\mathbb{C}(Y, G_2)$ by the $\Pi^{-1}(\emptyset')^{(k)}$-recursive function. Then, π's value on the degree of Y is represented by any generator of the ideal specified by the right hand side of Equation 5.5. It is an arithmetically defined operation to find the generator with the least index relative to Y and $\Pi^{-1}(\emptyset')^{(k)}$. Thus, π is represented by a function on 2^ω which is arithmetically definable relative to $\Pi^{-1}(\emptyset')^{(k)}$.

Jockusch and Shore (1984) have shown that any automorphism of $\mathfrak{D}$ preserves the ideal of degrees of arithmetic sets. Consequently, $\Pi^{-1}(\emptyset')^{(k)}$ is an arithmetic set, and so π is represented by a function on 2^ω which is arithmetically definable. In the next chapter, we will give another proof of this conclusion, with some sharp bounds on the arithmetic complexity of the definition.

6

Arithmetically representing automorphisms of $\mathfrak{D}$

6.1 Generic parameters

6.1.1 The image of a generic degree

Theorem 6.1.1 *Suppose that $\pi : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$ and that $\boldsymbol{D}^*$ is a countable collection of dense open subsets of P_1. There is a countable collection of dense open subsets of P_1, $\boldsymbol{D}$ such that for all $\boldsymbol{D}$-generic reals G there is a G^* such that G^* is $\boldsymbol{D}^*$-generic and the degree of G^* is less than or equal to $\pi(degree(G))$*

Proof: We work as follows. First, we chose $\boldsymbol{D}_0$ as in the previous chapter so that we have a continuous representation of π on the collection of $\boldsymbol{D}_0$-generic reals. In Lemma 6.1.2, we show that for any sufficiently generic real G, $\pi(degree(G))$ computes a function which has no *a priori* bound on its rate of growth. Then we will define a real G^* by comparing the values of several of these functions. Hence, for any sufficiently generic G, G^* will be defined at every argument and will be recursive in any representative of $\pi(degree(G))$. We will then show that for any dense open set D^* and any finite condition p, there is an extension q of p such that for any G extending q, G^* meets D^*. At the end of the analysis, we will calculate the genericity needed on G to ensure that G^* is $\boldsymbol{D}$-generic.

By Theorem 5.1.2, fix a countable family of dense open sets $\boldsymbol{D}_0$, a recursive function $\{e\}$, and a real P such that

$$\forall G[\text{if } G \text{ is } \boldsymbol{D}_0\text{-generic, then } \pi(degree(G)) = degree(\{e\}(G \oplus P))].$$

Lemma 6.1.2 *There is a recursive function $\{e^*\}$ such that for all Cohen conditions p, there is an $m \in \mathbb{N}$ and q extending p such that, for all k*

less than m, q decides $\{e^*\}(k, \{e\}(G \oplus P))$ *and q does not force an upper bound on* $\{e^*\}(m, \{e\}(G \oplus P))$.

Proof: We view G as a pair G_1 and G_2 of mutually generic reals. We define H_1 and H_2 by the equations $H_1 = \mathbb{C}(G_1, G_2)$ and $H_2 = \mathbb{C}(\overline{G_1}, G_2)$, where $\overline{G_1}$ is the complement of G_1 in $\mathbb{N}$. (See Definition 5.1.4 and the pages after page 61 for the definition and properties of $\mathbb{C}$.) By Lemma 5.1.5, H_1 and H_2 are recursive in G. By Theorem 5.1.8, there is a countable family of dense open sets $\boldsymbol{D}_1$ such that such that if G is $\boldsymbol{D}_1$-generic, then G_1, H_1 and H_2 are $\boldsymbol{D}_0$-generic. We shall assume henceforth that G is $\boldsymbol{D}_0 \cup \boldsymbol{D}_1$-generic.

H_1 and H_2 are defined by inserting the values of G_1 and $\overline{G_1}$, respectively, into G_{2even} at points designated by G_{2odd}. For each n, let m_n be the nth number at which H_1 and H_2 take different values. Then, for each n, $n \in G_1$ if and only if $m_n \in H_1$. Consequently, G_1 is recursive in $H_1 \oplus H_2$.

The fact that $H_1 \oplus H_2 \geq_T G_1$ is degree theoretic, and therefore it is preserved by π.

$$\pi(degree(H_1)) \vee \pi(degree(H_2)) \geq_T \pi(degree(G_1)) \tag{6.1}$$

Since each of H_1, H_2, and G_1 belong to C, we can write Equation 6.1 in terms of representatives as follows.

$$\{e\}(H_1, P) \oplus \{e\}(H_2, P) \geq_T \{e\}(G_1, P) \tag{6.2}$$

But then there is a recursive function $\{e_1\}$ and a finite condition p on G such that p forces

$$\{e_1\}(\{e\}(H_1, P) \oplus \{e\}(H_2, P)) = \{e\}(G_1, P).$$

By absorbing p into the recursive function, we may assume that p is the null condition. Consequently, there is a countable family of dense open sets $\boldsymbol{D}_3$ such that if G is $\boldsymbol{D}_3$-generic, then

$$\{e_1\}(\{e\}(H_1, P) \oplus \{e\}(H_2, P)) = \{e\}(G_1, P). \tag{6.3}$$

Let $\{e^*\}(G \oplus P)$ be the function that maps x to the supremum of the lengths of the computations of $\{e_1\}(\{e\}(H_1, P) \oplus \{e\}(H_2, P))$ on arguments less than or equal to x.

For the sake of Lemma 6.1.2, we can now forget how it was constructed and work with the fact that that Equation 6.3 is forced. We check that the function $x \mapsto \{e^*\}(x, G \oplus P)$ has the required properties. Let p be a condition on G. We view conditions q extending p on G as pairs of conditions q_1 and q_2 on G_1 and G_2, respectively. It is safe to assume that the length of p_1 is equal to the number of nonzero values of p_{2odd}.

By using null-odd-extensions q_2 of p_2, we can decide arbitrarily much of $\{e\}(H_1, P) \oplus \{e\}(H_2, P)$ without extending p_1. By the definitions of H_1 as $\mathbb{C}(G_1, G_2)$ and H_2 as $\mathbb{C}(\overline{G_1}, G_2)$, q_2 will force that the values of H_1 and H_2 are extended by copying G_{2even} and not by embedding values of G_1. On the other hand, since $\{e\}(G_1 \oplus P)$ depends in a nontrivial way on G_1, there must be values of $\{e_1\}(\{e\}(H_1, P) \oplus \{e\}(H_2, P))$ which cannot be decided without extending p_1.

Let m be the smallest number such that it is not possible to extend p_2 without extending p_1 and force an upper bound on the length of the computation of $\{e_1\}(m, \{e\}(H_1, P) \oplus \{e\}(H_2, P))$. Let q_2 extend p_2 so that for each k less than m, q_2 decides the exact computations used to evaluate $\{e_1\}(\{e\}(H_1, P) \oplus \{e\}(H_2, P))$ at k. Then, for each k less than m the condition on G given by p_1 and q_2 decides the value of $\{e^*\}(k, G, P)$. However, since we can extend q_2 to decide arbitrarily much of $\{e\}(H_1, P) \oplus \{e\}(H_2, P))$ without extending p_1, q_2 does not force any upper bound on $\{e^*\}(m, G, P)$. This verifies Lemma 6.1.2. ■

We define G^* as required by Theorem 6.1.1. View G as a join of four reals G_1, G_2, G_3, and G_4. For each of these G_i, let $f_i = \{e^*\}(G_i, P)$ be a function with the properties guaranteed by Lemma 6.1.2. Without loss of generality, we may assume that each f_i is strictly increasing. Define the following sequence by recursion on i: $m_0 = 0$, $m_{2i+1} = f_1(m_{2i})$, and $m_{2i+2} = f_2(m_{2i+1})$. Then define G^* by cases,

$$G^*(i) = \begin{cases} 0, & \text{if } f_3(m_i) \geq f_4(m_i); \\ 1, & \text{otherwise.} \end{cases}$$

Now we verify that G^* satisfies the conclusion of Theorem 6.1.1. It is enough to show the following. For any $p \in P_1$, a condition on G, and any D^*, a a dense open subset of P_1, there is a q extending p such that for every G extending q, if G^* is defined from G as above, then G^* meets D^*.

We view the condition p on G as a collection of four conditions p_i on G_i, represented in a way so that it is possible to extend them independently.

By the property ensured by Lemma 6.1.2, for i equal to 1 and 2, extend p_i to q_i so that there is an n_i such that q_i decides the values of f_i on all arguments less than n_i, but q_i does not force any upper bound on the value of f_i on n_i. Let j be least such that q_1 and q_2 do not decide the value of m_{j+1}. Our argument is symmetric, so we may assume that j is even.

Hence, q_1 does not decide the value of $f_1(m_j)$. Since the value of each f_i is decided at each argument less than n_i, m_j must be greater than or equal to n_1. Since the f_i's are strictly increasing and since q_1 does not force any upper bound on the value of f_1 at n_1, q_1 does not force any upper bound on the value of f_1 at m_j.

Now, for i equal to 3 and 4, extend p_i to q_i so that the value of f_i is decided on each number less than or equal to m_j. Let p^* be the initial segment of G^* which is computed from the values of the f_i's determined by the q_i's.

Since D^* is a dense open set, let choose q^* extending p^* so that $q^* \in D^*$. We now describe how to extend the q_i's to ensure that G^* extends q^*.

For i equal to 3 and 4, extend q_i to r_i so that there is an n_i such that q_i decides the values of f_i on all arguments less than n_i, but q_i does not force any upper bound on the value of f_i on n_i.

We have created the following situation. G^* is decided on each argument less than or equal to j. No bound is forced on the value of m_{j+1}, which will depend on the value of f_1 at m_j. For each i not equal to 1, we have identified an n_i such that the value of f_i is determined at each argument less than n_i, but no upper bound is forced on the value of f_i on n_i.

We take the following steps to determine another value of G^* and reconstruct the above situation, with the roles of f_1 and f_2 reversed.

1. Extend q_1 to r_1 to decide a value for f_1 at m_j which is greater than or equal to the maximum of n_2, n_3 and n_4.
2. Extend q_3 and q_4 to r_3 and r_4 to decide values for f_3 and f_4 at all arguments less than or equal to $f_1(m_j) = m_{j+1}$. Further, ensure that $f_3(m_j + 1)$ is greater than or equal to $f_4(m_{j+1})$ if and only if $q^*(j + 1) = 0$. (Since no upper bound is forced for either f_3 or f_4 at m_{j+1}, we can decide a value for one of them and then decide a larger value for the other.)
3. For i equal to 1, 3, or 4, extend r_i to s_i so that there is a new n_i such that s_i decides the values of f_i on all arguments less than n_i, but s_i does not force any upper bound on the value of f_i on n_i.

In the course of these three steps, we extended our conditions to decide the value of $m_{j+1} > n_2$, where no bound is forced on $f_2(n_2) \leq m_{j+2}$. We ensured that $G^*(j+1)$ is equal to $q^*(j+1)$. Finally, for each i not equal to 2, we have identified an n_i such that the value of f_i is determined at each argument less than n_i, but no upper bound is forced on the value of f_i on n_i. As promised, our situation after these three steps is the same as it was before them, with the roles of f_1 and f_2 reversed.

By a finite recursion, we can extend p to q to ensure that if G extends q, then G^* extends q^*.

It only remains to tally the dense open sets which appeared in this analysis. They are $\boldsymbol{D}_0$, to ensure that $\{e\}(G \oplus P)$ represents π on G; $\boldsymbol{D}_1$, to ensure the same representation for G_1, G_2, G_3, and G_4; and the collection of dense open sets D needed to ensure that for each D^* in $\boldsymbol{D}^*$, of G meets D then G^* meets D^*. The correspondence from D^* to D is the one obtained from the analysis of going from q^* in D^* to q in D. ■

6.1.2 Evaluating relative to a generic degree

We now obtain a continuous representation of π on the join of an arbitrary degree with a generic one.

Theorem 6.1.3 *There is a family of dense open sets $\boldsymbol{D}$ and a continuous function $F(G, X)$ such that for all $\boldsymbol{D}$-generic G, if $\Pi(G)$ is a representative of $\pi(degree(G))$, then*

$$(\forall X)[degree(F(G, X) \oplus \Pi(G)) = \pi(degree(X \oplus G))]$$

Proof: By Theorem 5.1.2 on page 60, fix $e \in \mathbb{N}$, $P \in 2^\omega$, and a family of dense open sets $\boldsymbol{D}_0$ such that for all $\boldsymbol{D}_0$-generic G, $\pi(degree(G))$ is equal to the degree of $\{e\}(G, P)$. Define $F(G, X)$ as follows.

$$F(G, X) = \{e\}(\mathbb{C}(X, G), P)$$

See Definition 5.1.4 on page 61 for a description of $\mathbb{C}$.

By Theorem 5.1.8, there is a family of dense open sets $\boldsymbol{D}_1$ such that for all $\boldsymbol{D}_1$-generic G and for all X, $\mathbb{C}(X, G) \in C$. Consequently, if G is $\boldsymbol{D}_1$-generic, then $F(G, X)$ is a representative of $\pi(degree(\mathbb{C}(X, G)))$. By Lemma 5.1.5, there is a family of dense open sets $\boldsymbol{D}_2$ such that if G is $\boldsymbol{D}_2$-generic, then $X \oplus G \equiv_T \mathbb{C}(X, G) \oplus G$. Let $\boldsymbol{D}_3$ be the union of $\boldsymbol{D}_0$, $\boldsymbol{D}_1$, and $\boldsymbol{D}_2$. If G is $\boldsymbol{D}_3$-generic, then $\pi(degree(X \oplus G) = \pi(degree(\mathbb{C}(X, G)) \vee G)$. The latter is represented by $F(G, X) \oplus \Pi(G)$, as required. ■

The next theorem provides a sharper version of the previous one: the continuous representation can be done injectively.

Theorem 6.1.4 *There is a family of dense open sets $\boldsymbol{D}$ and a continuous function $F(G, X)$ such that the following conditions hold.*

1. *The function $X \mapsto F(G, X)$ is injective.*
2. *For any $\boldsymbol{D}$-generic G, if $\Pi(G)$ is a representative of $\pi(degree(G))$, then*

$$(\forall X)[degree(F(G, X) \oplus \Pi(G)) = \pi(degree(X \oplus G))]$$

Proof: Given a set G, we now regard G as the join of two sets G_1 and G_2. By Theorem 6.1.3, let $\boldsymbol{D}_0$ be a countable family of dense open sets and let F_0 be a continuous function such that for all $\boldsymbol{D}_0$-generic reals H,

$$(\forall X)[degree(F_0(H, X) \oplus \Pi(H)) = \pi(degree(X \oplus H))]$$

Let $\boldsymbol{D}_1$ be a countable family of dense open sets such that if G is $\boldsymbol{D}_1$-generic, then G_1 is $\boldsymbol{D}_0$-generic. Let $\boldsymbol{D}$ be the countable family of dense open sets which extends $\boldsymbol{D}_0 \cup \boldsymbol{D}_1$ and includes all of the arithmetically definable dense open sets.

Assume that G is $\boldsymbol{D}$-generic.

Now, we work with G_2. Let T be a perfect binary tree such that T is recursive in G_2 and such that for any two infinite distinct paths in T, their joins with G_1 have incomparable Turing degree. See Theorem 2.5.11. For each $X \in 2^\omega$, let $T(X)$ denote the path in T determined by X under the canonical isomorphism between $2^{<\omega}$ and T.

We define $F(G, X)$ in Equation 6.4.

$$F(G, X) = F_0(G_1, \mathbb{C}(T(X), G_1)) \tag{6.4}$$

When G is $\boldsymbol{D}$-generic, we have the following sequence of equalities.

$$\begin{aligned}
&degree(F(G, X)) \vee \pi(degree(G)) = \\
&\quad = degree(F_0(G, X)) \vee \pi(degree(G)) = \\
&\quad = degree(F_0(G_1, \mathbb{C}(T(X), G_1))) \vee [\pi(degree(G_1)) \vee \pi(degree(G))] \\
&\quad = [degree(F_0(G_1, \mathbb{C}(T(X), G_1))) \vee \pi(degree(G_1))] \vee \pi(degree(G)) \\
&\quad = \pi(degree(\mathbb{C}(T(X), G_1) \oplus G_1)) \vee \pi(degree(G)) \\
&\quad = \pi(degree(T(X) \oplus G_1)) \vee \pi(degree(G)) \\
&\quad = \pi(degree(T(X) \oplus G)) \\
&\quad = \pi(degree(X \oplus G))
\end{aligned}$$

Thus, for each X, $degree(F(G, X) \oplus \Pi(G)) = \pi(degree(X \oplus G))$ as required. To verify the required injectivity, suppose that X and Y are distinct. Then, $T(X) \oplus G_1$ and $T(Y) \oplus G_1$ have distinct Turing degree. But then, $\mathbb{C}(T(X), G_1)$ and $\mathbb{C}(T(Y) \oplus G_1)$ have distinct Turing degree, as they have the same degrees as $T(X) \oplus G_1$ and $T(Y) \oplus G_1$, respectively. Then, $F_0(G_1, \mathbb{C}(T(X), G_1))$ and $F_0(G_1, \mathbb{C}(T(Y), G_1))$ have distinct Turing degree, since these sets represent the images of the degrees of $\mathbb{C}(T(X), G_1)$ and $\mathbb{C}(T(Y) \oplus G_1)$, respectively. Consequently, $F(X, G)$ and $F(Y, G)$ are distinct, as they are equal to $F_0(G_1, \mathbb{C}(T(X), G_1))$ and $F_0(G_1, \mathbb{C}(T(Y), G_1))$, respectively. So $X \mapsto F(X, G)$ is injective, as was required. ∎

Definition 6.1.5 For F as defined in Theorem 6.1.4, let F_G be the function $X \mapsto F(G, X)$.

Theorem 6.1.6 *There is a countable family of dense open sets $\boldsymbol{D}$ such that for all $\boldsymbol{D}$-generic reals G the following conditions hold.*

1. *If P is a perfect set with tree T_P, then the range of F_G on P contains a perfect set Q with tree T_Q such that*

$$degree(T_Q) \leq_T \pi(degree(G) \vee degree(T_P)).$$

2. *If Q is a perfect subset of the range of F_G with tree T_Q, then there is a perfect set P contained in the range of F_G^{-1} applied to Q with tree T_P such that*

$$degree(T_P) \leq_T (degree(G) \vee \pi^{-1}(degree(T_Q))).$$

Proof: Let $\boldsymbol{D}$ be the countable family of dense open sets such that F_G behaves as described in the previous lemma for all $\boldsymbol{D}$-generic reals G. Fix a particular $\boldsymbol{D}$-generic G, and let $\Pi(G)$ be an element of $\pi(degree(G))$. Similarly, let $\Pi(T_P)$ be a representative of $\pi(degree(T_P))$.

Fix Z_G so that the injective continuous functions F_G and F_G^{-1} are recursive relative to Z_G. Consider a set H which is arithmetically generic relative to $\Pi(G) \oplus \Pi(T_P) \oplus Z_G$, and let $\Pi^{-1}(H)$ be a representative of $\pi^{-1}(degree(H))$.

As T_P is a perfect subtree of $2^{<\omega}$, it is isomorphic to $2^{<\omega}$ by a function which is recursive in T_P. Let $X(H)$ be the image of $\Pi^{-1}(H)$ under this isomorphism. Then, $X(H) \oplus T_P$ has the same Turing degree as $\Pi^{-1}(H) \oplus T_P$. Consequently, the image of its degree under π is the degree of $H \oplus \Pi(T_P)$. Now, $\pi(degree(X(H) \oplus G))$ is the Turing degree of $F_G(X(H)) \oplus \Pi(G)$, so $F_G(X(H)) \oplus \Pi(G) \oplus \Pi(T_P)$ has the degree as $H \oplus \Pi(G) \oplus \Pi(T_P)$.

Consequently, there must be an integer e and a condition p on H, which we can take to be the null condition, such that

$$\Vdash \left(\begin{array}{c} \{e\}(H \oplus \Pi(T_P) \oplus \Pi(G)) \equiv_T H \oplus \Pi(G) \oplus \Pi(T_P) \text{ and} \\ \{e\}(H \oplus \Pi(T_P) \oplus \Pi(G)) \text{ is image under } F_G \text{ of a path in } T_P. \end{array} \right)$$

Let T_Q be defined by

$$T_Q = \{q : \exists p\, [q \text{ is extended by } \{e\}(p \oplus \Pi(T_P) \oplus \Pi(G))]\}$$

Note that T_Q is recursive in $\Pi(T_P) \oplus \Pi(G)$.

Clearly, T_Q is a tree. Every element of T_Q has a proper extension in T_Q as $\{e\}(p \oplus \Pi(T_P) \oplus \Pi(G))$ is forced to be total. Let Q be the set of paths in T_Q. Since P is compact and F_G is continuous, the range of F_G on P, written $F_G(P)$, is closed. Note that every $q \in T_Q$ can be extended to an element of $F_G(P)$. In fact for every q in T_Q there is an H as above such that q is extended by $\{e\}(H \oplus \Pi(T_P) \oplus \Pi(G))$ and therefore belongs to $F_G(P)$. Then the closed set Q generated by T_Q is a subset of the closed set $F_G(P)$. Finally, since no finite condition on H can determine the Turing degree of $H \oplus \Pi(T_P) \oplus \Pi(G)$, no finite condition on H can determine the Turing degree of $\{e\}(p \oplus \Pi(T_P) \oplus \Pi(G))$. Thus, every element of T_Q has incompatible extensions, and so Q is a perfect set.

Thus, we have proven the first claim of Theorem 6.1.6. The second claim follows by an analogous argument. ■

6.2 Moving information through π

6.2.1 Fixing the cone above $0''$

Theorem 6.2.1 *For every $Z \subseteq \omega$, there is a countable family of dense open sets $\boldsymbol{D}$ such that for all $\boldsymbol{D}$-generic reals G,*

$$\pi(\mathit{degree}(Z \oplus G))'' \geq_T \mathit{degree}(Z)$$

Proof: Let $Z \subseteq \omega$ be given.

We view G as a quadruple of reals G_1, G_2, G_3, and G_4. We will think of G as a generic real, and let the countable family of dense open sets $\boldsymbol{D}$ emerge at the end of our proof, once we know how generic G would have to be in order for our arguments to apply to it.

Step 1. Fix a perfect binary tree T_1 such that T_1 is recursive in G_1 and any finite set of infinite paths in T_1 consists of reals which are mutually generic relative to $G_2 \oplus G_3 \oplus G_4$. Let P_1 be the perfect set of infinite paths in T_1.

Let F_{G_2} be the injective continuous function associated with G_2 in Lemma 6.1.4 such that

$$(\forall X)[degree((F_{G_2}(X)) \oplus \Pi(G_2)) = \pi(degree(X \oplus G_2))],$$

where $\Pi(G_2)$ is a representative of $\pi(degree(G_2))$. By Theorem 6.1.6, fix a perfect set Q with associated perfect tree T_Q such that T_Q is recursive in $\Pi(G_1 \oplus G_2)$, where $\Pi(G_1 \oplus G_2)$ is a representative of $\pi(degree(G_1 \oplus G_2))$ and Q is contained in the image of P_1 under F_{G_2}. Applying Theorem 6.1.6 again, fix a perfect set $P_2 \subseteq P_1$ with associated perfect tree T_2 such that F_{G_2} maps the elements of P_2 into Q and such that T_2 is recursive in $G_1 \oplus G_2$.

Let $(H_i : i \in \omega)$ be the sequence of elements of P_2 given by the leftmost branches in P_2 off of the rightmost branch of P_2. We note that any finite subset of $(H_i : i \in \omega)$ consists of reals which are mutually Cohen generic relative to $G_2 \oplus G_3 \oplus G_4$.

Step 2. We find A_0 and A_1 recursively in $G_1 \oplus G_2 \oplus G_3$ to satisfy the following properties.

$$\forall U \forall j \in \omega \left(\begin{array}{c} (U \in P_1 \text{ and } G_1 \oplus G_2 \geq_T U) \to \\ \left[A_0 \oplus G_2 \oplus H_{2j} \geq U \iff (U = H_{2j} \text{ or } U = H_{2j+1})\right] \end{array} \right) \tag{6.5}$$

$$\forall U \forall j \in \omega \left(\begin{array}{c} (U \in P_1 \text{ and } G_1 \oplus G_2 \geq_T U) \to \\ \left[A_1 \oplus G_2 \oplus H_{2j+1} \geq U \iff (U = H_{2j+1} \text{ or } U = H_{2j+2})\right] \end{array} \right) \tag{6.6}$$

By these properties, we embed the sequence $(H_i : i \in \omega)$ so that it can be recovered from G_1, G_2, A_0, and A_1 by recursion using positive instances of $\leq_T$ and various Δ^0_3 properties relative to these parameters. We can obtain such sets A_0 and A_1 by forcing with the following partially ordered sets.

A condition in P^{A_0} is a pair (p, F) such that p is a finite binary sequences, and F is a finite subset of ω. For such pairs (p_1, F_1) and (p_2, F_2), (p_2, F_2) extends (p_1, F_1) in P^{A_0} if and only if p_2 extends p_1 as a binary sequence, $F_1 \subseteq F_2$, and for all $i \in F_1$ and all (i, x) in $domain(p_2) \setminus domain(p_1)$, $p_2((i, x)) = \mathbb{C}(H_{2i+1}, H_{2i})(x)$. Thus, a condition in P^{A_0} specifies finitely much about A_0 and specifies a finite set of columns on which further values of A_0 must be obtained by copying the appropriate set of the form $\mathbb{C}(H_{2i+1}, H_{2i})$.

We define P^{A_1} similarly, changing the definition of P^{A_0} so that a condition in P^{A_1} specifies finitely much about A_1 and specifies a finite set of columns on which further values of A_1 must be obtained by copying the appropriate set of the form $\mathbb{C}(H_{2i+2}, H_{2i+1})$'s.

By Corollary 2.5.4, let A_0 and A_1 be recursive in $G_1 \oplus G_2 \oplus G_3$ and be P^{A_0} and P^{A_1} generic, respectively.

For each i, the set of conditions (p, F) such that $i \in F$ is dense in P^{A_0} and in P_{A_1}. Consequently, for each i the ith component of A_0 is almost equal to $\mathbb{C}(H_{2i}, H_{2i+1})$, and the ith component of A_1 is almost equal to $\mathbb{C}(H_{2i+1}, H_{2i+2})$. By Lemma 5.1.5 on page 62, the implications from right to left in the conclusions of 6.5 and 6.6 are valid.

For the converse, we will concentrate on the left to right implication in 6.5 and note that 6.6 is similar. Consider the case when the sequence $(H_i : i \in \omega) \cup U$ has the property that for any of its finite subsequences S and for any of its elements $X \notin S$, X is Cohen generic relative to $S \oplus G_2$ and U is a set which is not equal to H_{2j} or H_{2j+1}. Suppose that $(p, F) \in P^{A_0}$, $(q_{2i}, q_{2i+1} : i \in F)$ and q constitute a condition on $(H_{2i}, H_{2i+1} : i \in F)$ and U. We may assume that $j \in F$.

Here we view a Cohen condition as a function from a finite set of natural numbers into the set $\{0, 1\}$. (Note, we do not mean rule out the possibility that U is one of the H_{2i}'s or H_{2i+1}'s for an i unequal to j.) By extending q if necessary, we may assume that it has the following properties.

1. For each i, the length of q_{2i+1} is equal to the number of arguments at which the odd component of q_{2i} is not zero. In other words, q_{2i+1} specifies exactly the amount of H_{2i+1} needed to determine the H_{2i+1} part of $\mathbb{C}(H_{2i+1}, q_{2i})$ up to the length of the odd part of q_{2i}.
2. For each i, the length of the even part of q_{2i} is equal to the number of arguments n at which the odd component of q_{2i} is equal to zero. Similarly to the above, the even part of q_{2i} specifies exactly the amount of H_{2i+1} needed to determine the rest of $\mathbb{C}(H_{2i+1}, q_{2i})$ up to the coding of the first n values of H_{2i+1}.
3. The first element not specified by q_{2i} is an element of its odd part.

Then, suppose that $i \in F$ and c is a finite extension of $\mathbb{C}(q_{2i+1}, q_{2i})$. Since the length of q_{2i+1} is equal to the number of arguments at which the odd component of q_{2i} is not zero, if we extend q_{2i} to r_{2i} so that r_{2i} takes nonzero value at each odd number not in the domain of q_{2i}, then for any r_{2i+1} extending q_{2i+1}, the new values of the term $\mathbb{C}(r_{2i+1}, r_{2i})$ beyond those given in $\mathbb{C}(q_{2i+1}, q_{2i})$ are obtained by copying the values of r_{2i+1} beyond those specified by q_{2i+1}. Similarly, if we extend q_{2i} to r_{2i} so that r_{2i} takes value zero at each odd number not in the domain of q_{2i}, then for any r_{2i+1}, the new values of the term $\mathbb{C}(r_{2i+1}, r_{2i})$ beyond those given in $\mathbb{C}(q_{2i+1}, q_{2i})$ are obtained by copying the values of the even component of r_{2i} beyond those specified by q_{2i}. That is to say that there are very different ways to achieve the same $\mathbb{C}(r_{2i+1}, r_{2i})$. Consequently, there are two pairs of conditions r_{2i} and r_{2i+1}, and r^*_{2i} and r^*_{2i+1} such that the following conditions hold.

1. $\mathbb{C}(r_{2i+1}, r_{2i}) = \mathbb{C}(r^*_{2i+1}, r^*_{2i}) = c$
2. For ℓ_{2i} and ℓ_{2i+1} the least numbers not in the domains of q_{2i} and q_{2i+1}, respectively, $r_{2i}(\ell_{2i}) \neq r^*_{2i}(\ell_{2i})$ and $r_{2i+1}(\ell_{2i+1}) \neq r^*_{2i+1}(\ell_{2i+1})$.

Then, for any finite extension c_i of $\mathbb{C}(q_{2i+1}, q_{2i})$ there are extensions of $\mathbb{C}(H_{2i+1}, H_{2i})$ to extend C but are incompatible with each other on both coordinates at the first places not specified by (q_{2i}, q_{2i+1}).

Now, we can argue that the implication from left to right in 6.5 is valid. Let e be an index for a recursive functional and let $(p, F) \in P^{A_0}$, $(q_{2i}, q_{2i+1} : i \in F)$, and q be given as above. Suppose that there are finite conditions p_1 and $(r^*_{2i}, r^*_{2i+1} : i \in F)$ such that $(r^*_{2i}, r^*_{2i+1} : i \in F)$ forces that (p_1, F) extends (p, F), and r^*_{2j} and p_1 are sufficient to fix the computations of $\{e\}(x, r^*_{2j} \oplus p_1)$ on every x less than or equal to some strict upper bound ℓ on the lengths of the conditions in $(q_{2i}, q_{2i+1} : i \in F)$ and q. Then, we can find an x and an r extending q with $r(x) \neq \{e\}(x, p_1)$ and conditions $(r_{2i}, r_{2i+1} : i \in F)$ compatible with q and r^*_{2j} such that

$$(r_{2i}, r_{2i+1} : i \in F) \Vdash (p_1, F) \text{ extends } (p, F) \text{ in } P^{A_0}.$$

We do so as follows.

If U is not one of the H sets, then we can extend q so that q is incompatible with the value decided by r^* for $\{e\}(H_{2j} \oplus A_0)$. Otherwise, U is one of the H sets other than H_{2j} or H_{2j+1}. Then we can find r^{**} such that r^{**} fixes the computation of $\{e\}(H_{2j} \oplus A_0)$ in the same way that r^* does and such that the conditions in r^{**} on U and the H with which it is paired are incompatible with their associated conditions in r^* at some number less than ℓ. But then, one of r^* or r^{**} forces $\{e\}(x, A_0) \neq U(x)$. Since the statement being forced is arithmetic, if the H_{2i}'s, H_{2i+1}'s, and U are mutually generic with respect to the (countably many) dense sets associated with forcing arithmetic sentences, then the implication from left to right in 6.5 is valid.

Step 3. We find B_0 and B_1 recursively in $G_1 \oplus G_2 \oplus G_3$ to satisfy the following property.

$$(\forall i \in \omega)\,[(B_0 \oplus G_2 \geq_T H_i \iff i \notin Z) \text{ and } (B_1 \oplus G_2 \geq_T H_i \iff i \in Z)] \tag{6.7}$$

By this property, we embed the atomic diagram of Z into the positive instances of $\leq_T$ between $B_0 \oplus G_2$, $B_1 \oplus G_2$ and the elements of $(H_i : i \in \omega)$. We can obtain such sets B_0 and B_1 by forcing with the following partially ordered sets.

A condition in P^{B_0} is a pair (p, F) such that $p \in 2^{<\omega}$ and F is a subset of the complement of Z. For such pairs (p_1, F_1) and (p_2, F_2), (p_2, F_2) extends (p_1, F_1) in P^{B_0} if and only if p_2 extends p_1 as a binary sequence, $F_1 \subseteq F_2$, and for all $i \in F_1$ and all (i, x) in $\mathit{domain}(p_2) \setminus \mathit{domain}(p_1)$, $p_2((i, x)) = H_i(x)$. Thus, a condition in P^{B_0} specifies finitely much about B_0 and specifies a finite set of columns on which further values of B_0 must be obtained by copying the appropriate element from the sequence of H_i's.

We define P^{B_1} similarly, changing the definition of P^{B_0} so that F is required to be finite subset of Z, rather than a finite subset of its complement.

Suppose that B_0 and B_1 are P^{B_0} and P^{B_1} generic, respectively.

Now, we argue thatB_0 and B_1 satisfy 6.7.

For each i, if $i \notin Z$ then the set of conditions (p, F) such that $i \in F$ is dense in P^{B_0}, and if $i \in Z$ then the set of conditions (p, F) such that $i \in F$ is dense in P^{B_1}. Consequently, if $i \notin Z$ then the ith component of B_0 is almost equal to H_i, and if $i \in Z$ then the ith component of B_1 is almost equal to H_i. So the implications from right to left in 6.7 hold.

For the reverse implication, suppose that (p, F) is an element of P^{B_0}, e and m are natural numbers, and $(p, F) \Vdash \{e\}(B_0 \oplus G_2) = H_m$. In particular, (p, F) decides every value of $\{e\}(B_0 \oplus G_2)$ relative to G_2.

For every binary sequence q, if q extends p and for all $i \in F_1$ and all (i, x) in $\mathit{domain}(q) \setminus \mathit{domain}(p)$, $q((i, x)) = H_i(x)$, then (q, F) extends (p, F) in P^{B_0}. The set of such q's is recursive in $\oplus_{i \in F} H_i \oplus G_2$. Further, if (p_1, F_1) extends (p, F) in P^{B_0}, then p_1 is such a q. Thus, for natural numbers x, and y, $(p, F) \Vdash \{e\}(x, B_0 \oplus G_2) = y$ if and only if there is a q as above such that $\{e\}(x, q \oplus G_2) = y$. But then H_m is recursive in $G_2 \oplus_{i \in F} H_i$: given input x, find a q as above and a y such that $\{e\}(x, q) = y$, and then return value y.

By our ongoing assumption that the sets G_i are mutually generic relative to G_2, the Turing degrees of the paths in P_1 form an independent set relative to G_2. That is to say that no one of them is recursive in G_2 together with the join of finitely many others. Consequently, for each pair of natural numbers e and m and for each $(p, F) \in P^{B_0}$ with $m \notin F$, there is an extension of $(p, F) \in P^{B_0}$ which forces $\{e\}(x, B_0 \oplus G_2)$ to be unequal to H_m.

Now, assuming that G_3 is sufficiently generic relative to $G_1 \oplus G_2$, we can apply Theorem 2.5.3 to conclude that there B_0 and B_1 which are recursive in $G_1 \oplus G_2 \oplus G_4 \oplus Z$ which satisfy 6.7. Here, $G_1 \oplus G_2 \oplus Z$ is needed to describe the partial orders P^{B_0} and P^{B_1} and G_4 is needed to construct B_0 and B_1 to be generic for these partial orders.

Genericity. How much genericity did we need for the previous steps? In Step 1, we satisfied arithmetic properties between the G_i's, for which meeting all arithmetic dense open sets is sufficient, and we cited the existence of a countable family of dense open sets sufficient for the conclusions of Theorem 6.1.6. In Step 2, we satisfied only arithmetic properties between the G_i's and the A_j's; arithmetic genericity is sufficient. In Step 3, we satisfied arithmetic properties between the G_i's, B_j's, and Z; arithmetic genericity relative to Z is sufficient. Thus, if the G_i's are generic for the countable family of dense open sets of Theorem 6.1.6 to ensure the continuous representation of π and are mutually arithmetically generic relative to Z, then we can successfully execute Steps 1-3.

Step 4. By Steps 1-3, there is a countable family of dense open sets $\boldsymbol{D}$ such that for any $\boldsymbol{D}$-generic G, viewing G as a quadruple $G_1 \oplus G_2 \oplus G_3 \oplus G_4$, there exist objects as follows:

1. an injective continuous function F as in Lemma 6.1.4

$$(\forall X)[\mathit{degree}(F(X) \oplus \Pi(G_2)) = \pi(\mathit{degree}(X \oplus G_2))],$$

   where we use $\Pi(G_i)$ to denote a representative of $\pi(\mathit{degree}(G_2))$;
2. perfect sets P_1 and P_2, represented by perfect trees T_1 and T_2 recursive in $G_1 \oplus G_2$, and a perfect set Q, represented by T_Q recursive in $\Pi(G_1) \oplus \Pi(G_2)$, such that Q is contained in the range of F on P_1 and P_2 is contained in the range of F^{-1} on Q;
3. a set $(H_i : i \in \omega)$ of paths in P_2 for which there are
   (a) and sets A_0 and A_1 which are recursive in $G_1 \oplus G_2 \oplus G_3$ and satisfy 6.5 and 6.6,

(b) and sets B_0 and B_1 which are recursive in $G_1 \oplus G_2 \oplus G_4 \oplus Z$ and satisfy 6.7.

Let $\Pi(A_0)$, $\Pi(A_1)$, $\Pi(B_0)$, and $\Pi(B_1)$ represent the values of π on the degrees of A_0, A_1, B_0, and B_1, respectively. Now we consider the consequences of properties (1) through (3) for these sets.

For our first observation, suppose that $U^* \in Q$, and fix $U \in P_1$ so that $F(U) = U^*$. Suppose that U^* is recursive in $\Pi(G_1) \oplus \Pi(G_2)$ and also recursive in $\Pi(A_0) \oplus \Pi(G_2) \oplus \Pi(H_{2n})$. Of course, this implies that $U^* \oplus \Pi(G_2)$ is recursive in $\Pi(A_0) \oplus \Pi(G_2) \oplus \Pi(H_{2n})$. By the choice of F, for all X in P_1, $\mathit{degree}(F(X) \oplus \Pi(G_2)) = \pi(\mathit{degree}(X \oplus G_2))$. In particular,

$$\Pi(A_0) \oplus \Pi(G_2) \oplus \Pi(H_{2n}) \geq_T \Pi(U) \oplus \Pi(G_2),$$

where $\Pi(U)$ is a representative of $\pi(\mathit{degree}(U))$. Since π is an automorphism of $\mathfrak{D}$, $A_0 \oplus G_2 \oplus H_{2n} \geq_T U \oplus G_2$. Similarly, $\Pi(G_1) \oplus \Pi(G_2) \geq_T U^*$ implies that $G_1 \oplus G_2 \geq_T U$. By the property 3(a) above, U must be one of H_{2n} or H_{2n+1}. That is to say that either U^* is equal to $F(H_{2n})$ or it is equal to $F(H_{2n+1})$.

Similarly, if U^* is an element of Q, recursive in $\Pi(G_1) \oplus \Pi(G_2)$, and also recursive in $\Pi(A_1) \oplus \Pi(G_2) \oplus \Pi(H_{2n+1})$, then either U^* is equal to $F(H_{2n+1})$ or it is equal to $F(H_{2n+2})$.

We may now conclude that the sequence $(F(H_i) : i \in \omega)$ is uniformly recursive in $\Pi(G)''$. For each i, H_i is recursive in G and $F(H_i) \oplus \Pi(G_2)$ has degree $\pi(\mathit{degree}(H_i \oplus G_2))$, and so $F(H_i)$ is recursive in $\Pi(G)$. We compute the indices e_i to compute $F(H_i)$ from $\Pi(G)$ by recursion from $\Pi(G)''$. Let e_0 be given so that $F(H_0)$ is equal to $\{e_0\}(\Pi(G))$. Knowing e_{2n}, we let e_{2n+1}, e^*_{2n+1}, and e^{**}_{2n+1} be the least triple of numbers such that $\{e_{2n+1}\}(\Pi(G))$ is an element of Q, $\{e_{2n+1}\}(\Pi(G))$ is equal to $\{e^*_{2n+1}\}(\Pi(G_1) \oplus \Pi(G_2))$ and is also equal to $\{e^{**}_{2n+1}\}(\Pi(A_0) \oplus \Pi(G_2) \oplus \{e_{2n}\}(\Pi(G)))$, and $\{e_{2n+1}\}(\Pi(G))$ is not equal to $\{e_{2n}\}(\Pi(G))$. Knowing e_{2n+1}, we define e_{2n+2}, e^*_{2n+2}, and e^{**}_{2n+2} similarly, substituting e_{2n+1}, for e_{2n} and A_1 for A_0. Applying induction, the observations in the previous paragraphs ensure that for each i, e_i is defined and $\{e_i\}(\Pi(G))$ is equal to $F(H_i)$. Further, the triple e_i, e^*_i, and e^{**}_i satisfies a $\Sigma^0_3(\Pi(G))$ property relative to e_i. The search to find the least such triple is recursive in $\Pi(G)''$.

In fact, we have proven something stronger than was claimed. We have shown that there is a sequence $(e_i : i \in \omega)$ which is recursive in $\Pi(G)''$ such that for all i, $\{e_i\}(\Pi(G))$ is equal to $F(H_i)$.

Our second conclusion is that Z is recursive in $(\Pi(Z) \oplus \Pi(G))''$. By the above, the sequence $(F(H_i) : i \in \omega)$ is uniformly recursive in $(\Pi(Z) \oplus \Pi(G))''$. Further, for i equal to 0 or 1, $W_i = \{m : \Pi(B_i) \oplus \Pi(G_2) \geq F(H_m)\}$ is recursively enumerable in $(F(H_i) : i \in \omega)$, $\Pi(B_i)$, and $\Pi(G_2)$, and therefore recursively enumerable in $(\Pi(Z) \oplus \Pi(G))''$.

Consider $W_0 = \{n : \Pi(B_0) \oplus \Pi(G_2) \geq_T F(H_n)\}$. In 6.7 we stated the following: for each m, H_m is recursive in $B_0 \oplus G_2$ if and only if $m \notin Z$. Invoking the implication from right to left in the conclusion, if m is in the complement of Z, then H_m is recursive in $B_0 \oplus G_2$. This relation is preserved by Π, so $\Pi(H_m)$ is recursive in $\Pi(B_0) \oplus \Pi(G_2)$. Since $F(H_m) \oplus \Pi(G_2)$ has the same degree as $\Pi(H_m) \oplus \Pi(G_2)$, $F(H_m)$ is recursive in $\Pi(B_0) \oplus \Pi(G_2)$. Conversely, suppose that $F(H_m)$ is recursive in $\Pi(B_0) \oplus \Pi(G_2)$. Then, $F(H_m) \oplus \Pi(G_2)$ is recursive in $\Pi(B_0) \oplus \Pi(G_2)$, and so $\Pi(H_m) \oplus \Pi(G_2)$ is also recursive in $\Pi(B_0) \oplus \Pi(G_2)$. But then, H_m is recursive in $B_0 \oplus G_2$, as Π represents the automorphism π on these sets. But then m is in the complement of Z by the implication from left to right in the conclusion of 6.7. Consequently, W_0 is the complement of Z. An analogous argument shows that $W_1 = \{n : \Pi(B_1) \oplus \Pi(G_2) \geq_T F(H_n)\}$ is equal to Z.

Now, we can computed Z from $(\Pi(Z) \oplus \Pi(G))''$. Given n, we check search for the least index e such that either $\{e\}(\Pi(B_0) \oplus \Pi(G_2)) = F(H_n)$ or $\{e\}(\Pi(B_1) \oplus \Pi(G_2)) = F(H_n)$. By the above, there is such an e; $n \in Z$ if and only if $\{e\}(\Pi(B_1) \oplus \Pi(G_2)) = F(H_n)$. ■

Theorem 6.2.2 *For every $z \in \mathfrak{D}$, $z'' \geq_T \pi(z)$.*

Proof: Suppose that z is a given Turing degree and that z is a representative of Z. Let $\Pi(Z)$ be a representative of $\pi(degree(Z))$.

Applying Theorem 6.2.1 to the automorphism π^{-1} and the set $\Pi(Z)$, fix a countable family of dense open sets $\boldsymbol{D}^*$ such that for every $\boldsymbol{D}^*$-generic G^*, the following condition holds.

$$\pi^{-1}(degree(\Pi(Z) \oplus G^*))'' \geq_T degree(\Pi(Z)) \tag{6.8}$$

By Theorem 6.1.1, fix a countable family of dense open sets $\boldsymbol{D}$ such that for all $\boldsymbol{D}$-generic G, there is a $\boldsymbol{D}^*$-generic G^* such that the degree of G^* is less than or equal to $\pi(degree(G))$. For the sake of the argument below, we also arrange that $\boldsymbol{D}$ includes all of the dense open sets which are arithmetically defined relative to Z.

Now, suppose that G is $\boldsymbol{D}$-generic, and let G^* be $\boldsymbol{D}^*$-generic degree below $\pi(degree(G))$. By 6.8, $[\pi^{-1}(degree(\Pi(Z) + G^*))]''$ is greater than or equal to $degree(\Pi(Z))$. Thus, $(Z \oplus G)'' \geq_T \Pi(Z)$. By Theorem 2.5.12, $(Z \oplus G)''$ is equivalent to $Z'' \oplus G$, and so $Z'' \oplus G \geq \Pi(Z)$. But then $\Pi(Z)$ is recursive in Z'' and any $\boldsymbol{D}$-generic real. By Theorem 2.5.6, $Z'' \geq \Pi(Z)$ and Theorem 6.2.2 is proven. ∎

Corollary 6.2.3 *For any 2-generic set G,*

$$degree(G) \vee 0'' \geq_T \pi(degree(G)).$$

Proof: Let G be 2-generic. By Theorem 6.2.2, $degree(G)'' \geq_T \pi(degree(G))$ and by Theorem 2.5.12, $G \oplus 0'' \equiv_T G''$. Consequently,

$$degree(G) \vee 0'' \geq_T \pi(degree(G)),$$

as required. ∎

Theorem 6.2.4 *Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$.*

1. *For all $x \in \mathfrak{D}$, $x \vee 0'' \geq_T \pi(x)$.*
2. *For all $x \in \mathfrak{D}$, if $x \geq 0''$ then $x = \pi(x)$.*

Proof: By Theorem 5.1.8, let $\boldsymbol{D}$ be a countable family of dense open sets such that for every $\boldsymbol{D}$-generic G the following conditions hold.

1. G is 2-generic.
2. For every $Y \subseteq \omega$, $\mathbb{C}(Y, G)$ is 2-generic.

Let x be an element of $\mathfrak{D}$, and let X be a representative of x. Let G be $\boldsymbol{D}$-generic. By Lemma 5.1.5, $X \oplus G$ has the same Turing degree as $\mathbb{C}(X, G) \oplus G$. Consequently,

$$\pi(degree(\mathbb{C}(X, G) \oplus G)) \geq_T \pi(degree(X))$$

and hence

$$\pi(degree(\mathbb{C}(X, G))) + \pi(degree(G)) \geq_T \pi(degree(X)).$$

Since each of $\mathbb{C}(X, G)$ and G are 2-generic, $degree(G)+0'' \geq_T \pi(degree(G))$ and $degree(\mathbb{C}(X, G)) + 0'' \geq_T \pi(degree(\mathbb{C}(X, G)))$. Thus,

$$\begin{aligned} degree(\mathbb{C}(X, G)) + degree(G) + 0'' & \\ &\geq_T \pi(degree(\mathbb{C}(X, G))) + \pi(degree(G)) \\ &\geq_T \pi(degree(\mathbb{C}(X, G) \oplus G)) \\ &\geq_T \pi(degree(X)). \end{aligned}$$

Since $X \oplus G \geq_T \mathbb{C}(X, G)$,

$$degree(X) + degree(G) + 0'' \geq_T \pi(degree(X)). \tag{6.9}$$

Since 6.9 holds for every 2-generic G, Theorem 2.5.6 implies that $\pi(degree(X))$ is less than or equal to $degree(X) + 0''$, and we have proven the first claim of Theorem 6.2.4.

The second claim follows easily. Suppose $x \geq_T 0''$, and so $x \geq_T x+0''$. By the above, $x+0'' \geq_T \pi(x)$, and so $x \geq_T \pi(x)$. Since π was an arbitrary automorphism of $\mathfrak{D}$, every automorphism of $\mathfrak{D}$ maps x to a degree below x. Thus, $x \geq_T \pi^{-1}(x)$. By applying π to this inequality, $\pi(x) \geq_T x$. Hence, $x = \pi(x)$, as required. ∎

6.2.2 Invariance of the double-jump

Theorem 6.2.5 *For every $Z \subseteq \omega$, there is a countable family of dense open sets $\boldsymbol{D}$ such that such that for all $\boldsymbol{D}$-generic G,*

$$\pi(degree(Z \oplus G))'' \geq_T degree(Z'')$$

Proof: The proof of Theorem 6.2.5 runs parallel to the proof of Theorem 6.2.1. However, in Theorem 6.2.1 we coded the atomic diagram of Z into the degrees below $Z \oplus G$, and now we will code the atomic diagram of Z'' into the degrees below $Z \oplus G$. This technical improvement should not be too surprising. The restriction of $\geq_T$ to the degrees below $Z \oplus G$ is $\Sigma^0_3(Z \oplus G)$ and should accommodate coding relations that are $\Sigma^0_2(Z)$ or $\Pi^0_2(Z)$.

As before, we view G as a quadruple of reals G_1, G_2, G_3, and G_4. We will think of G as a generic real, and let the countable family of dense open sets $\boldsymbol{D}$ emerge at the end of our proof.

Steps 1 and 2. Define P_1, T_1, F_{G_2}, Q, T_Q,$P_2 \subseteq P_1$, T_2, and $(H_i : i \in \omega)$ as in Step 1 of Theorem 6.2.1. Define A_0 and A_1 as in Step 2 of Theorem 6.2.1.

Step 3 We find B_0 and B_1 recursively in $G_1 \oplus G_2 \oplus G_4$ to satisfy the following property.

$$(\forall i \in \omega)\left[(B_0 \geq_T H_i \iff i \notin Z'') \text{ and } (B_1 \geq_T H_i \iff i \in Z'')\right] \tag{6.10}$$

By this property, we embed the atomic diagram of Z'' into the positive instances of $\leq_T$ between $B_0 \oplus G_2$, $B_1 \oplus G_2$ and the elements of $(H_i : i \in \omega)$.

We can obtain such sets B_0 and B_1 by forcing with the partial orders defined below.

Definition 6.2.6 1. Let φ be a Π^0_0 formula such that for all m,

$$\forall x \exists y\ \varphi(m, x, y, Z \restriction y) \iff m \notin Z''.$$

2. Let Sk be the recursive approximation to the Skolem function for the formula $(\forall x)(\exists y)\varphi(m, x, y, Z \restriction y)$. That is, for each m, let $Sk(m, x)$ be the least y greater than $Sk(m-1, x)$ (if $m > 0$) such that for all $x_1 \leq x$, there is a $y_1 \leq y$ such that $\varphi(m, x_1, y_1, Z \restriction y_1)$, if there is such a y; and let $Sk(m, x)$ be undefined, otherwise.

Substep 3.0. We define P^{B_0} so that for a generic B_0, for all m in the complement of Z'', and for all sufficiently large y, the following conditions hold.

$$B_0((m, 2y)) \neq 0 \iff (\exists x)[y = Sk(m, x)] \tag{6.11}$$

$$B_0((m, 2y+1)) \neq 0 \iff (\exists x)[y = Sk(m, x) \text{ and } x \in H_m] \tag{6.12}$$

If $m \notin Z''$, then Sk is a total injective function. Further, H_m is recursive in B_0 as follows. Fix k_1 and k_2 so that conditions 6.11 and 6.12 hold for all y greater than k_1 and so that k_2 is the greatest x such that $Sk(m, x) < k_1$. Given x, find the $x - k_2$nd y such that $(m, 2y) \in B_0$ and $2y > k_1$, and then $x \in H_m$ if and only if $(m, 2y+1) \in B_0$. For example, to determine whether $k_2 + 1 \in H_m$ find the first y such that $y > k_1$ and $(m, 2y) \in B_0$, and then $k_2 + 1 \in H_m$ if and only if $(m, 2y+1) \in B_0$.

If $m \in Z''$, then $Sk(m, x)$ is defined for only finitely many x's. Consequently, requiring that Conditions 6.11 and 6.12 hold for all sufficiently large y is equivalent to requiring that for all sufficiently large y, $B_0((m, y)) = 0$.

Now, we define P^{B_0}. A condition in P^{B_0} is a pair (p, F) such that p is a finite binary sequence, and F is a finite subset of ω. For such pairs (p_1, F_1) and (p_2, F_2), (p_2, F_2) extends (p_1, F_1) in P^{B_0} if and only if both belong to P^{B_0}, p_2 extends p_1 as a binary sequence, $F_1 \subseteq F_2$, and for all $m \in F_1$ and all y, if $(m, 2y) \in domain(p_2) \setminus domain(p_1)$, then $p_2((m, 2y)) \neq 0$ if and only if $(\exists x)[y = Sk(m, x)]$, and if $(m, 2y+1) \in domain(p_2) \setminus domain(p_1)$, then $p_2((m, 2y+1)) \neq 0$ if and only if $(\exists x)[y = Sk(m, x)$ and $x \in H_m]$.

Thus, a condition in P^{B_0} specifies finitely much about B_0 and specifies a finite set of columns on which further values of B_0 must be obtained so as to satisfy Conditions 6.11 and 6.12.

Suppose that B_0 is P^{B_0} generic. By the remarks above, if $m \notin Z''$, then $B_0 \geq_T H_m$.

For the converse, suppose that $m \in Z''$. Let (p, F) be an element of P^{B_0}, let e be a natural number, and suppose that (p, F) forces $\{e\}(B_0 \oplus G_2) = H_m$. For every binary sequence q, if q extends p and for all $i \in F$ and all (i, x) in $domain(q) \setminus domain(p)$, q is defined so as to satisfy Conditions 6.11 and 6.12, then (q, F) extends (p, F) in P^{B_0}. Notice that the conditions imposed by those i's in $F \cap Z''$ is recursive. Thus, the set of such q's is recursive in $\oplus_{i \in F \setminus Z''} H_i \oplus G_2$. Further, if (p_1, F_1) extends (p, F) in P^{B_0}, then p_1 is such a q. Thus, for natural numbers x, and y, $(p, F) \Vdash \{e\}(x, B_0 \oplus G_2) = y$ if and only if there is a q as above such that $\{e\}(x, q \oplus G_2) = y$. But then H_m would recursive in $\oplus_{i \in F \setminus Z''} H_i \oplus G_2 \oplus Z$: given input x, find a q as above and a y such that $\{e\}(x, q) = y$, and then return value y.

By our ongoing assumption that the sets G_i are mutually generic relative to Z, G_2 and the Turing degrees of the paths in P_1 form an independent set relative to Z. That is to say that no one of them is recursive in Z together with the join of finitely many others. Consequently, if $m \in Z''$, no condition can force H_m to be recursive in B_0.

So, if B_0 is P^{B_0} generic, then B_0 has the property required by Condition 6.10.

Substep 3.1. We define P^{B_1} so that for a generic B_1, for all m in Z'', if x_m is the least witness x to $\exists x \forall y \ \neg\varphi(m, x, y, Z \upharpoonright y)$, the $\Sigma^0_2(Z)$ criterion for membership in Z'', then for all sufficiently large w, $B_1((m, x_m, w)) = H_m(w)$. Conversely, if x_m is not the least such witness, then we will ensure that for all sufficiently large w, $B_1((m, x_m, w)) = 0$.

Now, we define P^{B_1}. A condition in P^{B_1} is a pair (p, F) such that p is a finite binary sequence, and F is a finite subset of $\omega \times \omega \times \omega^{<\omega}$, where each element of F is required to be a sequence of the form $(m, x, (Sk(m, i) : i < x))$. In particular, for each i less than x, $Sk(m, i)$ is defined.

For such pairs (p_1, F_1) and (p_2, F_2), (p_2, F_2) extends (p_1, F_1) in P^{B_0} if and only if both belong to P^{B_1}, p_2 extends p_1 as a binary sequence, $F_1 \subseteq F_2$, and the following condition holds.

$$
\begin{aligned}
&(\forall (m_1, x_1, (Sk(m_1, i) : i < x_1)) \in F_1)(\forall w) \\
&\quad (m_1, x_1, w) \in domain(p_2) \setminus domain(p_1) \to \\
&\qquad \begin{pmatrix} w < Sk(m_1, x_1) \to p_2((m_1, x_1, w)) = H_{m_1}(w) \text{ and} \\ w \geq Sk(m_1, x_1) \to p_2((m_1, x_1, w)) = 0 \end{pmatrix}
\end{aligned} \tag{6.13}
$$

Thus, a condition in P^{B_1} specifies finitely much about B_1 and specifies a finite set of columns indexed by (m, x) on which further values of a generic B_1 are equal to those of H_m up to $Sk(m, x)$, if defined, and then identically 0.

If $m \in Z''$, then let x_m be the least witness x to $\exists x \forall y \neg\varphi(m, x, y, Z \restriction y)$. The set of conditions (p, F) such that $(m, x_m, (Sk(m, i) : i < x_m)) \in F$ is dense in P^{B_1}. But then the extensions of this condition copy H_m into the (m, x_m)th column of B_1. Consequently, if $m \in Z''$ and B_1 is P^{B_1}-generic, then $B_1 \geq_T H_m$.

For the converse, suppose that n is not an element of Z''. Let (p, F) be an element of P^{B_1}, let e be a natural number, and suppose that (p, F) forces $\{e\}(B_1 \oplus G_2) = H_n$. For every binary sequence q, if q extends p and satisfies Condition 6.13, then (q, F) extends (p, F) in P^{B_1}. Notice that the conditions imposed by those m's in such that $m \notin Z''$ and there is an element $(m, x_m, (Sk(m, i) : i < x_m))$ in F is recursive. In fact, for such m and x_m, the (m, x_m)th column is constrained to be finite. Thus, the set of such q's is recursive in the recursive join of the set of H_m's for which $m \in Z''$ and there is an $(m, x_m, (Sk(m, i) : i < x_m)) \in F$. Further, if (p_1, F_1) extends (p, F) in P^{B_1}, then p_1 is such a q. Thus, for natural numbers x, and y, $(p, F) \Vdash \{e\}(x, B_1 \oplus G_2) = y$ if and only if there is a q as above such that $\{e\}(x, q \oplus G_2) = y$. But then H_n would recursive in the above join, a contradiction as in Substep 3.0.

Consequently, for each $n \notin Z''$ and each $e \in \omega$, there is no condition which forces $\{e\}(B_1 \oplus G_2)$ to be equal to H_n.

So, if B_1 is P^{B_1} generic, then B_1 has the property required by Condition 6.10.

Step 4. We can now complete our proof of Theorem 6.2.5. As in the proof of Theorem 6.2.1, there is a countable family of dense open sets $\boldsymbol{D}$ such that for each $\boldsymbol{D}$-generic G, viewing G as a quadruple $G_1 \oplus G_2 \oplus G_3 \oplus G_4$, there exist objects as follows:

1. an injective continuous function F as in Lemma 6.1.4

$$(\forall X)[degree(F(X) \oplus \Pi(G_2)) = \pi(degree(X \oplus G_2))],$$

where we use $\Pi(G_i)$ to denote a representative of $\pi(degree(G_2))$;

2. perfect sets P_1 and P_2, represented by perfect trees T_1 and T_2 recursive in $G_1 \oplus G_2$, and a perfect set Q, represented by T_Q recursive in $\Pi(G_1) \oplus \Pi(G_2)$, such that Q is contained in the range of F on P_1 and P_2 is contained in the range of F^{-1} on Q;
3. a set $(H_i : i \in \omega)$ of paths in P_2 for which there are
   (a) and sets A_0 and A_1 which are recursive in $G_1 \oplus G_2 \oplus G_3$ and satisfy 6.5 and 6.6,
   (b) and sets B_0 and B_1 which are recursive in $G_1 \oplus G_2 \oplus G_4 \oplus Z$ and satisfy 6.10.

Let $\Pi(A_0)$, $\Pi(A_1)$, $\Pi(B_0)$, and $\Pi(B_1)$ represent the values of π on the degrees of A_0, A_1, B_0, and B_1, respectively. Now we consider the consequences of properties (1) through (3) for these sets.

As in the proof of Theorem 6.2.1, the sequence $(F(H_i) : i \in \omega)$ is uniformly recursive in $\Pi(G)''$. Now, let $W_0 = \{n : \Pi(B_0) \oplus \Pi(G_2) \geq_T F(H_n)\}$. Since $G \oplus Z \geq_T B_0 \oplus G_2$, $\Pi(B_0)$ and $\Pi(G_2)$ are recursive in $\Pi(G \oplus Z)$. Thus, W_0 is recursively enumerable relative to $\Pi(G \oplus Z)''$.

By Condition 6.10, for each n, $B_0 \oplus G_2 \geq_T H_n$ if and only if $n \notin Z''$. Consequently, $\Pi(B_0) \oplus \Pi(G_2) \geq_T \Pi(H_n)$ if and only if $n \notin Z''$. Since $F(H_m) \oplus \Pi(G_2)$ has the same degree as $\Pi(H_m) \oplus \Pi(G_2)$, $\Pi(B_0) \oplus \Pi(G_2) \geq_T F(H_n)$ if and only if $n \notin Z''$. Thus, W_0 is equal to the complement of Z'', and so the complement of Z'' is recursively enumerable relative to $(\Pi(Z) \oplus \Pi(G))''$.

By Condition 6.10, similarly applied to B_1, Z'' is recursively enumerable relative to $(\Pi(G) \oplus \Pi(Z))''$. Hence, Z'' is recursive in $(\Pi(G) \oplus \Pi(Z))''$. ■

Genericity. Now, we can calculate the amount of genericity needed above. As in Theorem 6.2.1, arithmetic genericity is sufficient for Steps 1 and 2. In Step 3, we satisfied arithmetic properties between the G_i's, B_j's, and Z; arithmetic genericity relative to Z is sufficient. Thus, as in Theorem 6.2.1 if the G_i's are generic for the countable family of dense open sets of Theorem 6.1.6 to ensure the continuous representation of π and are mutually arithmetically generic relative to Z, then we can successfully execute Steps 1-3.

Theorem 6.2.7 *Suppose that* $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$.

1. *For all* $z \in \mathfrak{D}$, $z'' = \pi(z)''$.
2. *The relation* $R(x, y)$ *defined by* $y = x''$ *is invariant under* π.

Proof: Let $Z \subseteq \omega$ be given of degree z, and let $\Pi(Z)$ be a representative of $\pi(degree(z))$. Let $\boldsymbol{D}^*$ be the countable family of dense open sets obtained in Theorem 6.2.5 for the automorphism π^{-1} and the set $\Pi(Z)$. By Theorem 6.1.1 on page 67, there is a countable family of dense open sets set $\boldsymbol{D}$ such that if G is $\boldsymbol{D}$-generic then G is 2-generic and such that there is a $\boldsymbol{D}^*$ generic set G^* which is recursive in the join of $\Pi(Z)$ with $\Pi(G)$, a representative of $\pi(degree(G))$. Fix such G and and G^*.

Now consider the automorphism $\pi^{-1} : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$. By the previous paragraphs, $\Pi(Z)''$ is recursive in $(Z \oplus \Pi^{-1}(G^*))''$, where $\Pi^{-}(G^*)''$ is a representative of $\pi^{-1}(degree(G^*))$. Since G^* is recursive in $\Pi(Z) \oplus \Pi(G)$, $\Pi^{-1}(G^*)$ is recursive in $Z \oplus G$. Consequently, $\Pi(Z)''$ is recursive in $(Z \oplus G)''$. By Theorem 2.5.12, $(Z \oplus G)'' \equiv_T Z'' \oplus G$, and so $\Pi(Z)''$ is recursive in $Z'' \oplus G$. By Theorem 2.5.6, since for an arbitrary $\boldsymbol{D}$-generic set G, $\Pi(Z)''$ is recursive in $Z'' \oplus G$, it must be the case that $\Pi(Z)''$ is recursive in Z''. Thus, $z'' \geq_T \pi(z)''$.

Now we apply the above paragraphs to the automorphism π^{-1} and the degree $\pi(z)$. We may conclude that $\pi(z)'' \geq_T (\pi^{-1}(\pi(z)))''$, or equivalently that $\pi(z)'' \geq_T z''$. Consequently, $z'' = \pi(z)''$ and we have verified the first claim in Theorem 6.2.7.

For the second claim, let $R(x, y)$ be the relation $y = x''$ and suppose that $y = x''$. Then $y \geq_T 0''$, and by Theorem 6.2.4, $\pi(y) = y$, and so $\pi(y) = x''$. By the previous paragraphs, $x'' = \pi(x)''$, and so $\pi(y) = \pi(x)''$. Thus, $R(x, y)$ implies $R(\pi(x), \pi(y))$. Similarly, $R(\pi(x), \pi(y))$ implies $R(\pi^{-1}(\pi(x)), \pi^{-1}(\pi(y)))$, that is $R(\pi(x), \pi(y))$ implies $R(x, y)$. Thus, R is preserved by any automorphism of $\mathfrak{D}$. ■

6.3 Representing *Aut*($\mathfrak{D}$) by arithmetic functions

Theorem 6.3.1 *Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$.*

1. *There is a recursive function $\{e\}(X, Y)$ such that for all G, if G is 5-generic, then $\pi(degree(G))$ is represented by $\{e\}(G \oplus \emptyset'')$.*
2. *There is an arithmetic function $F : 2^\omega \to 2^\omega$ such that for all $X \in 2^\omega$, $\pi(degree(X))$ is represented by $F(X)$.*

Proof: We prove Theorem 6.3.1 by returning to Theorem 5.1.2 and incorporating the extra information that for all degrees x, $\pi(x)$ is recursive in x''.

By Theorem 5.1.2, there is a real parameter P, a countable family of dense open sets $\boldsymbol{D}_P$, and an index for a recursive functional e_P such that the function $G \mapsto \{e_P\}(G \oplus P)$ represents π on the set of $\boldsymbol{D}_P$-generic reals. By Theorem 6.2.4, for all degrees x, $x \vee 0'' \geq_T \pi(x)$. But then, there must be a Cohen condition p and an index e such that the following condition holds.

$$p \Vdash \{e\}(G \oplus \emptyset'') = \{e_P\}(G \oplus P) \tag{6.14}$$

By replacing e with another index which refers to p as data, we may assume that p is the empty condition. Then, there is a countable family of dense open sets $\boldsymbol{D}$ containing $\boldsymbol{D}_P$ such that for all $\boldsymbol{D}$-generic G, $\{e\}(G \oplus \emptyset'') = \{e_P\}(G \oplus P)$, and so the function $G \mapsto \{e\}(G \oplus \emptyset'')$ represents π on the set of $\boldsymbol{D}$-generic reals.

By Lemma 5.1.5 and Theorem 2.5.6 (see page 63), let G_1 and G_2 be sufficiently generic so that for every set Y, G_1, G_2, $\mathbb{C}(Y, G_1)$, and $\mathbb{C}(Y, G_2)$ are $\boldsymbol{D}$-generic and so that the ideal of degrees below Y is represented as in Equation 6.15.

$$(\mathbb{C}(Y, G_1) \oplus G_1) \cap (\mathbb{C}(Y, G_2) \oplus G_2) = (Y) \tag{6.15}$$

Equation 6.15 expresses a degree theoretic property of these sets. Consequently, it is preserved by π. We consider the case when we replace the variable Y with a particular $\boldsymbol{D}$-generic G and use the representation of π on the set of $\boldsymbol{D}$-generics.

$$\begin{aligned}\big(\{e\}(G \oplus 0'')\big) = \\ \big(\{e\}(\mathbb{C}(G, G_1) \oplus 0'') \oplus \{e\}(G_1 \oplus 0'')\big) \\ \bigcap \big(\{e\}(\mathbb{C}(G, G_2) \oplus 0'') \oplus \{e\}(G_2 \oplus 0'')\big)\end{aligned} \tag{6.16}$$

Equation 6.16 expresses an arithmetic relationship between G, G_1 and G_2, which is true of all $\boldsymbol{D}$-generic reals which are also sufficiently generic so that Equation 6.15 is satisfied. Thus, Equation 6.16 is forced by the empty condition for the partial order to add three Cohen generic reals.

By an elementary counting of the quantifiers, Equation 6.16 is a Π^0_6 statement about G, G_1, and G_2. Consequently, it is satisfied by any triple G, G_1, and G_2 which are mutually 5-generic.

Now, let G be an arbitrary 5-generic set. There are sets G_1 and G_2 such that G, G_1 and G_2 are mutually 5-generic, and also G_1 and G_2 are sufficiently generic so that all of G_1, G_2, $\mathbb{C}(G, G_1)$, and $\mathbb{C}(G, G_2)$ are $\boldsymbol{D}$-generic. For this choice of G_1 and G_2, the right-hand-side of Equation 6.16 represents the ideal of degree below $\pi(degree(G))$. And so, $\{e\}(G \oplus 0'')$ represents $\pi(degree(G))$ as well. Thus, we have verified the first claim in Theorem 6.3.1.

The second claim follows almost immediately. Suppose that X is a given set and let $\Pi(X)$ represent $\pi(degree(X))$.

If G_1 and G_2 are sufficiently generic, then G_1, G_2, $\mathbb{C}(X, G_1)$, and $\mathbb{C}(X, G_2)$ are 5-generic and represent (X) as follows.

$$(\mathbb{C}(X, G_1) \oplus G_1) \cap (\mathbb{C}(X, G_2) \oplus G_2) = (X) \tag{6.17}$$

Now suppose that G_1 and G_2 be 5-generic relative to X. First note, since the transfer of genericity from a generic G to $\mathbb{C}(X, G)$ is direct, $\mathbb{C}(X, G_1)$ and $\mathbb{C}(X, G_2)$ are 5-generic. (For the sake of the claim, we need only that there is a k such that if G is k-generic then $\mathbb{C}(X, G)$ is 5-generic.) Second, since $\mathbb{C}$ is a recursive function, Equation 6.17 is a Π^0_4 statement statement about X which is forced, so it is satisfied by our sets G_1, and G_2. Applying π and representing it by e as above, we obtain Equation 6.18,

$$\begin{aligned}(\{e\}(\mathbb{C}(X, G_1) \oplus \emptyset'') \oplus \{e\}(G_1 \oplus \emptyset'')) \cap (\{e\}(\mathbb{C}(X, G_2) \oplus \emptyset'') \oplus \{e\}(G_2 \oplus \emptyset''))\\ = (\Pi(X))\end{aligned} \tag{6.18}$$

where $\Pi(X)$ is a representative of $\pi(degree(X))$. Let $S(X, G_1, G_2)$ be the set of e^* such that $\{e^*\}(X \oplus \emptyset'')$ represents a maximal element of $(\Pi(X))$, and note that $S(X, G_1, G_2)$ is uniformly arithmetic relative to X, G_1, and G_2.

But now we can represent π by an arithmetic function P on representatives. Given X, we first compute G_1 and G_2 from $X^{(5)}$ so that G_1 and G_2 are mutually 5-generic relative to X; see Theorem 2.5.8. Then we let $P(X)$ be $\{e^*\}(X \oplus 0'')$, where e^* is the least element of $S(X, G_1, G_2)$. The function $X \mapsto \{e^*\}(X \oplus 0'')$ gives the desired arithmetic representation of π. ■

Corollary 6.3.2 *$Aut(\mathfrak{D})$ is countable.*

Proof: Every element of $Aut(\mathfrak{D})$ is represented by an arithmetically defined function, and there are only countably many such definitions.

6.3.1 Generic basis theorem

Definition 6.3.3 A set $A \subseteq \mathfrak{D}$ is an *automorphism base* for $\mathfrak{D}$ if and only if every automorphism of $\mathfrak{D}$ is determined by its action on A. That is, whenever $\pi_1 : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$, $\pi_2 : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$, and π_1 and π_2 agree on A, then $\pi = \pi_2$.

Theorem 6.3.4 (Jockusch and Posner (1981)) *If* $\boldsymbol{D}$ *is a countable family of dense open sets, then the degrees represented by the* $\boldsymbol{D}$*-generic reals form an automorphism base for* $\mathfrak{D}$.

Proof: The Jockusch and Posner (1981) Theorem follows from Equation 6.15 and the surrounding text, which states that for any countable family of dense open sets $\boldsymbol{D}$ and every degree x, x can be written as a meet of joins of degrees of $\boldsymbol{D}$-generic reals. ∎

Theorem 6.3.5 *Suppose that* g *is the degree of a 5-generic subset of* ω *and that* $\pi : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$. *Then* π *is fully determined by its value on* g.

Proof: By Theorem 6.3.1, let e be given so that for all 5-generic sets G, $\{e\}(G \oplus \emptyset'')$ is a representative of $\pi(degree(G))$. Consider the assertion,

$$\{e\}(G \oplus 0'') \not\equiv_T G. \tag{6.19}$$

Since e represents π on all sufficiently generic sets, the satisfaction of Equation 6.19 depends only on the Turing degree the generic, and so it is decided by the null condition. Writing Equation 6.19 out more fully, it says, *For all Turing functionals* Φ *and* Ψ*, there is an argument* x*, such that either* $\Phi(x, G \oplus \emptyset'')$ *is not equal to* $G(x)$ *or* $\Psi(x, G)$ *is not equal to* $\{e\}(x, G \oplus \emptyset'')$. The disjunction of inequalities is a Π^0_3-statement about G. Consequently, Equation 6.19 is a Π^0_5 statement about G.

Now, a Π^0_5 statement about G which is decided by the null condition, is either true of all 5-generic sets or is false of all 5-generic sets. Since e represents π on all 5-generic sets, π is the identity on one 5-generic degree if and only if it is the identity on all 5-generic degrees. By Theorem6.3.4, π is the identity on all 5-generic sets if and only if it is the identity on $\mathfrak{D}$. Therefore, π is the identity on one 5-generic set if and only if it is the identity on $\mathfrak{D}$.

Suppose that g is a 5-generic degree, and π_1 and π_2 are automorphisms of $\mathfrak{D}$ such $\pi_1(g) = \pi_2(g)$. Then, $\pi_1^{-1} \circ \pi_2(g)$ is equal to g. By the above, $\pi_1^{-1} \circ \pi_2$ is the identity function, and so π_1 is equal to π_2. Thus, Theorem 6.3.5 is verified ∎

Corollary 6.3.6 1. *For any 5-generic degree g, the singleton $\{g\}$ is an automorphism base for $\mathfrak{D}$.*
2. *The degrees below* $0^{(5)}$ *form an automorphism base for $\mathfrak{D}$.*

Proof: The first claim is just a restatement of the previous theorem. The second follows from the first by observing that there is a 5-generic degree below $0^{(5)}$. ∎

7

Interpreting $Aut(\mathfrak{D})$ within $\mathfrak{D}$

In this chapter, we interpret the apparatus of persistent functions and generic degrees into the first order theory of $\mathfrak{D}$.

7.1 Assigning representatives to degrees

7.2 Countable assignments

Definition 7.2.1 An *assignment* of reals consists of

1. A countable ω-model $\mathfrak{M}$ of T, the theory of Definition 2.2.1 consisting of the fragment of *ZFC* which includes only the instances of replacement and comprehension in which the defining formula is Σ_1.
2. A function f and a countable ideal $\mathcal{I}$ in $\mathfrak{D}$ such that $f : \mathfrak{D}^{\mathfrak{M}} \to \mathcal{I}$ surjectively and for all x and y in $\mathfrak{D}^{\mathfrak{M}}$, $\mathfrak{M} \models x \geq_T y$ if and only if $f(x) \geq_T f(y)$ in $\mathcal{I}$.

Definition 7.2.2 For assignments $(\mathfrak{M}_0, f_0, \mathcal{I}_0)$ and $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$, $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$ *extends* $(\mathfrak{M}_0, f_0, \mathcal{I}_0)$ if and only if

1. $\mathfrak{D}^{\mathfrak{M}_0} \subseteq \mathfrak{D}^{\mathfrak{M}_1}$,
2. $\mathcal{I}_0 \subseteq \mathcal{I}_1$,
3. and $f_1 \restriction \mathfrak{D}^{\mathfrak{M}_0} = f_0$.

Definition 7.2.3 An assignment $(\mathfrak{M}_0, f_0, \mathcal{I}_0)$ is *extendable* if

$$\forall z_1 \exists (\mathfrak{M}_1, f_1, \mathcal{I}_1) \left[\begin{array}{c} (\mathfrak{M}_1, f_1, \mathcal{I}_1) \text{ extends } (\mathfrak{M}_0, f_0, \mathcal{I}_0)),\ z_1 \in \mathcal{I}_1, \text{ and} \\ \forall z_2 \exists (\mathfrak{M}_2, f_2, \mathcal{I}_2) \left(\begin{array}{c} (\mathfrak{M}_2, f_2, \mathcal{I}_2) \text{ extends } (\mathfrak{M}_1, f_1, \mathcal{I}_1),\ z_2 \in \mathcal{I}_2, \text{ and} \\ \forall z_3 \exists (\mathfrak{M}_3, f_3, \mathcal{I}_3) \left[\begin{array}{c} (\mathfrak{M}_3, f_3, \mathcal{I}_3) \text{ extends} \\ (\mathfrak{M}_2, f_2, \mathcal{I}_2) \text{ and } z_3 \in \mathcal{I}_3 \end{array} \right] \end{array} \right) \end{array} \right]$$

Theorem 7.2.4 *If* $(\mathfrak{M}, f, \mathcal{I})$ *is an extendable assignment, then there is a* $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$ *such that for all* $x \in \mathfrak{D}^{\mathfrak{M}}$, $\pi(x) = f(x)$.

Proof: Let $(\mathfrak{M}, f, \mathcal{I})$ be an extendable assignment. We will show that there is an ideal with a persistent automorphism ρ such that the ideal contains $\mathfrak{D}^{\mathfrak{M}}$ and $\mathcal{I}$ and such that ρ extends f. Consequently, f is persistent. By Theorem 4.4.3, f extends to an automorphism of $\mathfrak{D}$.

$$\begin{tikzcd}
\mathfrak{D}^{\mathfrak{M}_1} \arrow[r, "\sim", "f_1"'] & \mathcal{I}_1 \ni 0' \\
\mathfrak{D}^{\mathfrak{M}} \arrow[u, hook] \arrow[r, "\sim", "f"'] & \mathcal{I} \arrow[u, hook]
\end{tikzcd}$$

Fig. 7.1 Extending to include $0'$ in the range

Let $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$ extend $(\mathfrak{M}, f, \mathcal{I})$, so that $0'$ is an element of $\mathcal{I}_1$ and so that the remaining clauses of Condition 7.2.3 are satisfied. We will show that $\mathfrak{D}^{\mathfrak{M}_1}$ and $\mathcal{I}_1$ are equal and that f_1 is a persistent automorphism of $\mathcal{I}_1$.

To see that $\mathcal{I}_1 \subseteq \mathfrak{D}^{\mathfrak{M}_1}$, suppose that y_1 is an element of $\mathcal{I}_1$. Let Y_1 be a representative of y_1. By Theorem 3.2.3, there is a sequence of parameters $\boldsymbol{p}$ which is recursive in $y_1 \vee 0'$ and which codes an isomorphic copy of $\mathbb{N}$ with a unary predicate for Y_1. Since $0' \in \mathcal{I}_1$, $\boldsymbol{p} \subset \mathcal{I}_1$. Since $\mathfrak{D}^{\mathfrak{M}_1}$ is isomorphic to $\mathcal{I}_1$, there are parameters in $\mathfrak{D}^{\mathfrak{M}_1}$ which code an isomorphic copy of $\mathbb{N}$ with a unary predicate for Y_1. But then, Theorem 3.2.4 implies that Y_1 is arithmetically definable from a sequence of representatives of these parameters. Consequently, $Y_1 \in \mathfrak{M}_1$ and so $y_1 \in \mathfrak{D}^{\mathfrak{M}_1}$.

To see that $\mathfrak{D}^{\mathfrak{M}_1} \subseteq \mathcal{I}_1$, suppose that x_1 is an element of $\mathfrak{D}^{\mathfrak{M}_1}$. By our choice of $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$, there is an $(\mathfrak{M}_2, f_2, \mathcal{I}_2)$ such that $x_1 \in \mathcal{I}_2$ and $(\mathfrak{M}_2, f_2, \mathcal{I}_2)$ extends $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$. This situation is illustrated in Figure 7.2. Now, let $F_2^{-1}(X_1)$ denote a representative of $f_2^{-1}(x_1)$. There are parameters $\boldsymbol{q}$ below $f_2^{-1}(x_1) \vee 0'$ which code $\mathbb{N}$ with a unary predicate for $F_2^{-1}(X_1)$. Consequently, the pointwise image $f_2(\boldsymbol{q})$ of $\boldsymbol{q}$ is a sequence of parameters below $x_1 \vee f_2(0')$ which codes the same information. Of course, $0' \in \mathfrak{M}$ and $f_2(0')$ is equal to $f(0') \in \mathcal{I}$. We have shown that $\mathcal{I}_1 \subseteq \mathfrak{D}^{\mathfrak{M}_1}$, so $f(0') \in \mathfrak{M}_1$. Therefore, $\boldsymbol{q} \in \mathfrak{M}_1$ and, by Theorem 3.2.4, $F_2^{-1}(X_1) \in \mathfrak{M}_1$. But then, $f_2^{-1}(x_1) \in \mathfrak{M}_1$ and so $f_1(f_2^{-1}(x_1)) \in \mathcal{I}_1$. Of

$$\begin{tikzcd}
\mathfrak{D}^{\mathfrak{M}_2} \arrow[r, "\sim", "f_2"'] & \mathcal{I}_2 \ni x_1 \\
\mathfrak{D}^{\mathfrak{M}_1} \arrow[r, "\sim", "f_1"'] \arrow[u, hook] & \mathcal{I}_1 \arrow[u, hook]
\end{tikzcd}$$

Fig. 7.2 Extending $\mathfrak{M}_1$ to include x_1 in the range

course, this means that $x_1 \in \mathcal{I}_1$.

Thus, $\mathfrak{D}^{\mathfrak{M}_1} = \mathcal{I}_1$ and $f_1 : \mathcal{I}_1 \xrightarrow{\sim} \mathcal{I}_1$ is the extension of f.

It remains to show that f_1 is persistent. For this, suppose that $y_2 \in \mathfrak{D}$. It will be sufficient to show that y_2 belongs to an ideal $\mathcal{I}_2$ extending $\mathcal{I}_1$ such that $y_2 \in \mathcal{I}_2$ and f_1 lifts to an automorphism of $\mathcal{I}_2$.

$$\begin{tikzcd}
\mathfrak{D}^{\mathfrak{M}_3} \arrow[r, "\sim", "f_3"'] & \mathcal{I}_3 \ni y_3 \\
\mathfrak{D}^{\mathfrak{M}_2} \arrow[r, "\sim", "f_2"'] \arrow[u, hook, dotted] & \mathcal{I}_2 \ni y_2 \arrow[u, hook, dotted] \\
\mathfrak{D}^{\mathfrak{M}_1} \arrow[r, "\sim", "f_1"'] \arrow[u, hook] & \mathcal{I}_1 \arrow[u, hook]
\end{tikzcd}$$

Fig. 7.3 Extending to include y_2 in the range

We return to the argument of the previous paragraph. By our choice of $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$, there is an $(\mathfrak{M}_2, f_2, \mathcal{I}_2)$ such that $y_2 \in \mathcal{I}_2$ and $(\mathfrak{M}_2, f_2, \mathcal{I}_2)$ extends $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$ so that the remaining clause of Condition 7.2.3 is satisfied. Since $\mathfrak{D}^{\mathfrak{M}_1} = \mathcal{I}_1$, we can identify the two ideals on the bottom level of Figure 7.2 as in Figure 7.3. The remaining clause of Condition 7.2.3 asserts that for any y_3, there is an extension $(\mathfrak{M}_3, f_3, \mathcal{I}_3)$ of $(\mathfrak{M}_2, f_2, \mathcal{I}_2)$ such that $y_3 \in \mathfrak{M}_3$. We have indicated this extension property with dashed arrows.

But now we need only show that $\mathfrak{D}^{\mathfrak{M}_2} = \mathcal{I}_2$ in order to establish our claim. We argue for this equality in the way that we did for $\mathfrak{D}^{\mathfrak{M}_1} = \mathcal{I}_1$. In the previous argument, we applied the assumption that we could incorporate any element of $\mathfrak{D}^{\mathfrak{M}_1}$ into the range of some assignment extending the given one $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$. By the final clause of Condition 7.2.3, this same assumption is available for $(\mathfrak{M}_2, f_2, \mathcal{I}_2)$. So, by the same argument as before, $\mathfrak{D}^{\mathfrak{M}_2} = \mathcal{I}_2$.

We have shown that the function $f : \mathfrak{D}^{\mathfrak{M}} \xrightarrow{\sim} \mathcal{I}$ can be extended to a persistent automorphism f_1 of the ideal $\mathcal{I}_1$, where $\mathfrak{D}^{\mathfrak{M}} \cup \mathcal{I} \subseteq \mathcal{I}_1$. By Theorem 4.4.3, there is an automorphism π of $\mathfrak{D}$ which extends f_1. This automorphism extends f, which verifies Theorem 7.2.4. ■

Remark 7.2.5 By the Coding Theorem 3.1.2, the following properties of $(\boldsymbol{m}, \boldsymbol{f}, \boldsymbol{i})$ are definable in $\mathfrak{D}$. (Here $\boldsymbol{m}$ refers to a finite sequence $(m_1, \ldots, m_k)$ of degrees, and similarly for $\boldsymbol{f}$ and $\boldsymbol{i}$).

1. $\boldsymbol{m}$ codes an ω-model $\mathfrak{M}$ of T
2. $\boldsymbol{i}$ codes a countable ideal $\mathcal{I}$ in $\mathfrak{D}$
3. $\boldsymbol{f}$ codes a function f from $\mathfrak{D}^{\mathfrak{M}}$ onto $\mathcal{I}$.
4. $(\mathfrak{M}, f, \mathcal{I})$ is an extendable assignment.

7.3 Definability in $\mathfrak{D}$

7.3.1 Defining relative to parameters

Theorem 7.3.1 *If g is the Turing degree of an arithmetically definable 5-generic set, then the relation $R(\boldsymbol{c}, d)$ given by*

$$R(\boldsymbol{c}, d) \iff \boldsymbol{c} \textit{ codes a real } D \textit{ and } D \textit{ has degree } d$$

is definable in $\mathfrak{D}$ from g.

Proof: Let G be an arithmetically definable 5-generic subset of ω and let g be the Turing degree of G.

Consider the following property of $\boldsymbol{c}$ and d. There are $\boldsymbol{m}$, $\boldsymbol{f}$, and $\boldsymbol{i}$ such that all of the following conditions are satisfied.

1. $\boldsymbol{c}$ codes $\mathbb{N}$ with a unary predicate for a set D;
2. $\boldsymbol{m}$ codes an ω-model $\mathfrak{M}$ of T;
3. $\boldsymbol{i}$ codes a countable ideal $\mathcal{I}$ in $\mathfrak{D}$;
4. $\boldsymbol{f}$ codes a function f from $\mathfrak{D}^{\mathfrak{M}}$ onto $\mathcal{I}$;

5. $(\mathfrak{M}, f, \mathcal{I})$ is an extendable assignment;
6. $g \in \mathcal{I}$, $degree(G)^{\mathfrak{M}}$ is the Turing degree of G as identified in $\mathfrak{M}$ by G's arithmetic definition, and $f(degree(G)^{\mathfrak{M}}) = g$;
7. the set D coded by $\boldsymbol{c}$ is an element of $\mathfrak{M}$, $degree(D)^{\mathfrak{M}}$ is the Turing degree of D as defined in $\mathfrak{M}$, and $f(degree(D)^{\mathfrak{M}}) = d$.

Suppose that $\boldsymbol{c}$ codes a set D and D has degree d. We form $(\mathfrak{M}, f, \mathcal{I})$ by letting $\mathfrak{M}$ be any countable ω-model of T which contains D as an element. We let $\mathcal{I}$ be $\mathfrak{D}^{\mathfrak{M}}$ and let f be the identity. Then $(\mathfrak{M}, f, \mathcal{I})$ is extendable and any sequence of codes for $(\mathfrak{M}, f, \mathcal{I})$ will satisfy the above property.

Conversely, suppose that $\boldsymbol{c}$ and d are given so that there are $\boldsymbol{m}$, $\boldsymbol{f}$, and $\boldsymbol{i}$ such that all of the above conditions are satisfied. By Theorem 7.2.4, fix an isomorphism $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$ such that for all $x \in \mathfrak{D}^{\mathfrak{M}}$, $\pi(x) = f(x)$. But then $\pi(g) = g$, and by Corollary 6.3.6 π is the identity function. Letting D denote the set coded by $\boldsymbol{c}$, since $f(degree(D)^{\mathfrak{M}}) = d$, D is a representative of d.

By the Coding Theorem 3.1.2, the property written above is definable in $\mathfrak{D}$ relative to the parameter g. This verifies Theorem 7.3.1. ■

Theorem 7.3.2 *Suppose that R is a relation on $\mathfrak{D}$. The following conditions are equivalent.*

1. *R is induced by a projective, degree invariant relation on 2^ω.*
2. *R is definable in $\mathfrak{D}$ using parameters.*

Proof: $\mathfrak{D}$ is defined within the language of second order arithmetic. Consequently, if R is definable in $\mathfrak{D}$ using parameters, then R is induced by a projective, degree invariant relation on 2^ω.

For the converse, suppose that $\mathcal{R}(X_1, \ldots, X_n)$ on 2^ω is a degree invariant relation on 2^ω which is defined in second order arithmetic by the formula φ relative to the real parameters, $P_1, \ldots, P_k$.

$$\mathcal{R}(X_1, \ldots, X_n) \iff \varphi(X_1, \ldots, X_n, P_1, \ldots, P_k)$$

We exhibit a definition of the induced relation R in $\mathfrak{D}$ relative to the parameter g of Theorem 7.3.1 and parameters $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k$ which code $\mathbb{N}$ with unary predicates for $P_1, \ldots, P_k$, respectively. For $x_1, \ldots, x_n$ in $\mathfrak{D}$, $R(x_1, \ldots, x_n)$ if and only if there are $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ such that the following conditions hold.

1. $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ code $\mathbb{N}$ with unary predicates for $X_1, \ldots, X_n$, respectively;

2. $X_1, \ldots, X_n$ are representatives for the degrees $x_1, \ldots, x_n$, respectively;
3. and for $P_1, \ldots, P_k$ the sets coded by $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k$, $\varphi(X_1, \ldots, X_n, P_1, \ldots, P_k)$.

The first and third clauses can be expressed in the language of $\mathfrak{D}$ by applying the Coding Theorem 3.1.2. The second clause can be expressed in the language of $\mathfrak{D}$ relative to the parameter g by applying Theorem 7.3.1. ■

7.3.2 Defining without parameters.

Theorem 7.3.3 *Suppose that R is a relation on $\mathfrak{D}$. The following conditions are equivalent.*

1. *R has the following two properties.*
   (a) *R is induced by a degree invariant relation on 2^ω which is definable in second order arithmetic.*
   (b) *R is invariant under $Aut(\mathfrak{D})$.*
2. *R is definable in $\mathfrak{D}$.*

Proof: The implication from (2) to (1) follows directly from the observation that $\mathfrak{D}$ is defined in a first order way within second order arithmetic.

Suppose that R satisfies the two clauses of (1). Let φ be a formula in the language of second order arithmetic such that for all $X_1, \ldots, X_n$,

$$R(degree X_1, \ldots, degree X_n) \iff \varphi(X_1, \ldots, X_n).$$

Now we exhibit a definition of R in $\mathfrak{D}$. Suppose that $\boldsymbol{x}$ is a length n sequence from $\mathfrak{D}$. Then, since R is invariant under all automorphisms of $\mathfrak{D}$, $R(\boldsymbol{x})$ if and only if there is a $\boldsymbol{z}$ in the orbit of $\boldsymbol{x}$ with $R(\boldsymbol{z})$. The latter condition is equivalent to the statement that there is a sequence of sets $\boldsymbol{Z}$ representing $\boldsymbol{z}$ in the orbit of $\boldsymbol{x}$ such that $\varphi(\boldsymbol{Z})$. By Theorem 7.3.1 this condition is equivalent to the one stating that there is a sequence of sets $\boldsymbol{Z}$ representing $\boldsymbol{z}$ and an extendable assignment $(\mathfrak{M}, f, \mathcal{I})$ such that for all $Z_i \in \boldsymbol{Z}$, $f(degree(Z_i)) = x_i$ and such that $\varphi(\boldsymbol{Z})$. This final condition can be expressed in the language of $\mathfrak{D}$ by applying the Coding Theorem 3.1.2. ■

Corollary 7.3.4 *If R is a relation on $\mathfrak{D}$ and R is contained in the degrees above $0''$, then R is definable in D if and only if R is induced by a degree invariant relation on 2^ω which is definable in second order arithmetic.*

Proof: Corollary 7.3.4 follows from Theorem 7.3.2 by observing that

By Theorem 6.2.4, every automorphism of $\mathfrak{D}$ is the identity on all degrees greater than or equal to $0''$. Consequently, every relation on the degree above $0''$ is invariant under all automorphisms of $\mathfrak{D}$. Thus, if R is a relation on $\mathfrak{D}$ and R is contained in the degrees above $0''$, then R satisfies Condition (1b) of Theorem 7.3.2. Consequently, R is definable in $\mathfrak{D}$ if and only if it satisfies Condition (1a), that it is induced by a degree invariant relation on 2^ω which is definable in second order arithmetic. ∎

7.3.3 Defining the double-jump

Theorem 7.3.5 *The function $x \mapsto x''$ is definable in $\mathfrak{D}$.*

Proof: By Theorem 6.2.7, the relation $y = x''$ is invariant under all automorphisms of $\mathfrak{D}$. It is clearly degree invariant and definable in second order arithmetic. Therefore, by Theorem 7.3.3, it is definable in $\mathfrak{D}$. ∎

7.4 ω-homogeneity

Definition 7.4.1 $\mathfrak{D}$ is *ω-homogeneous* if and only if for all $\boldsymbol{p}$ and $\boldsymbol{q}$ from $\mathfrak{D}$, if $\mathfrak{D}(\boldsymbol{p}) \equiv \mathfrak{D}(\boldsymbol{q})$ then there is an automorphism $\pi : \mathfrak{D} \tilde{\rightarrow} \mathfrak{D}$ such that $\pi(\boldsymbol{p}) = \boldsymbol{q}$.

Theorem 7.4.2 *If there is a wellordering of 2^ω which can be defined in second order arithmetic, then $\mathfrak{D}$ is ω-homogeneous.*

Proof: Suppose that $>$ is a wellordering of 2^ω which can be defined in second order arithmetic. Now suppose that $\boldsymbol{a}$ is a finite sequence from $\mathfrak{D}$.

Let $\boldsymbol{L}(a)$ be the $<$-lexicographically least sequence from 2^ω such that the sequence of degrees represented by $\boldsymbol{A}$ is automorphic to $\boldsymbol{a}$. By Theorem 7.3.3, for each i less than or equal to the length of $\boldsymbol{a}$ and for each n, the property *n is an element of the ith component of $\boldsymbol{L}(a)$* is definable in $\mathfrak{D}$ as a property of $\boldsymbol{a}$. Thus, the type of $\boldsymbol{a}$ determines the sequence $\boldsymbol{L}(a)$. Of course, if for two sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ it is the case that $\boldsymbol{L}(a) = \boldsymbol{L}(b)$, then the orbits of $\boldsymbol{a}$ and $\boldsymbol{b}$ have a common point and so they are automorphic. Consequently, if $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same type, then they are automorphic, as required to verify the ω-homogeneity of $\mathfrak{D}$. ∎

Theorem 7.4.3 *$\mathfrak{D}$ is not ω-homogeneous in the model $V[G_1, G_2]$ obtained by adding two Cohen generic reals to V.*

Proof: We consider Cohen forcing over V to add two mutually generic reals (G_1, G_2). Let (g_1, g_2) denote the Turing degrees of the generic sets.

First observe that the type of (g_1, g_2) in $\mathfrak{D}^{V[G1,G_2]}$ depends only on the Turing degrees of G_1 and G_2. Hence no finite condition on (G_1, G_2) can change the Boolean value for a statement about (g_1, g_2) in the structure $\mathfrak{D}^{V[G_1,G_2]}$. Consequently, the type of (g_1, g_2) in the structure $\mathfrak{D}^{V[G_1,G_2]}$ is equal to the type of (g_2, g_1) in the structure $\mathfrak{D}^{V[G_2,G_1]}$. But $\mathfrak{D}^{V[G_1,G_2]} = \mathfrak{D}^{V[G_2,G_1]}$, so (g_1, g_2) and (g_2, g_1) have the same type in structure $\mathfrak{D}^{V[G_1,G_2]}$. However, by Theorem 6.2.2, there is no automorphism of $\mathfrak{D}^{V[G_1,G_2]}$ which takes g_1 to g_2 as G_2 is not arithmetically definable from G_1. Consequently, the Turing degrees are not ω-homogeneous in $V[G_1, G_2]$. ∎

Corollary 7.4.4 *The statement* $\mathfrak{D}$ is ω-homogeneous *is independent of ZFC.*

7.5 Biinterpretability conjectures

Definition 7.5.1

1. $\mathfrak{D}$ is *biinterpretable with second order arithmetic* if and only if the relation on $\boldsymbol{c}$ and d given by

   $$R(\boldsymbol{c}, d) \iff \boldsymbol{c} \text{ codes a real } D \text{ and } D \text{ has degree } d$$

   is definable in $\mathfrak{D}$.
2. We say that $\mathfrak{D}$ is *biinterpretable with second order arithmetic relative to parameters* if R is first order definable in $\mathfrak{D}$ relative to finitely many parameters from $\mathfrak{D}$.

We can restate Theorem 7.3.1 on page 96 in the language of Definition 7.5.1 .

Theorem 7.5.2 $\mathfrak{D}$ *is biinterpretable with second order arithmetic relative to parameters.*

The question of biinterpretability without parameters is equivalent to the question of rigidity.

Theorem 7.5.3 *The following are equivalent.*

1. $\mathfrak{D}$ *is biinterpretable with second order arithmetic.*
2. $\mathfrak{D}$ *is rigid.*

Proof: Suppose that $\mathfrak{D}$ is biinterpretable with second order arithmetic. Then each degree in $\mathfrak{D}$ is associated with the set of codes for representatives of it by a relation that is defined by a formula in the first order language of $\mathfrak{D}$. By the Coding Theorem 3.1.2, every countable relation on $\mathfrak{D}$ is definable in $\mathfrak{D}$, and so being the code for a subset of $\mathbb{N}$, the standard model of arithmetic, is preserved by any automorphism of $\mathfrak{D}$. In particular, codes for the standard model of arithmetic are mapped to codes for the standard model of arithmetic. It follows that if $\boldsymbol{p}$ codes a set $X \subseteq \mathbb{N}$ and $\pi : \mathfrak{D} \tilde{\rightarrow} \mathfrak{D}$, then $\pi(\boldsymbol{p})$ codes the same set X. Consequently, if $\pi : \mathfrak{D} \rightarrow \mathfrak{D}$ and $x \in \mathfrak{D}$, then $\pi(x)$ and x have the same set of representatives, and so $\pi(x) = x$.

Conversely, suppose that $\mathfrak{D}$ is rigid. Then, by Theorem 7.3.3, for every arithmetically definable set X, the Turing degree of X is definable in $\mathfrak{D}$. In particular, there is a 5-generic set G such that its Turing degree g is definable in $\mathfrak{D}$. Theorem 7.3.1 states that the relation $R(\boldsymbol{c}, d)$ given by

$$R(\boldsymbol{c}, d) \iff \boldsymbol{c} \text{ codes a real } D \text{ and } D \text{ has degree } d$$

is definable in $\mathfrak{D}$ from g. Given that g is definable in $\mathfrak{D}$, we can replace the instances of g in the definition of R and conclude that R is defined by a first order formula in $\mathfrak{D}$. Thus, $\mathfrak{D}$ is biinterpretable with second order arithmetic. ∎

Conjecture 7.5.4 $\mathfrak{D}$ *is biinterpretable with second order arithmetic.*

8
Defining the Turing jump

In this chapter, we prove the Shore and Slaman (1999) theorem that the function $x \mapsto x'$ is definable in $\mathfrak{D}$. The argument given below it taken directly from (Shore and Slaman, 1999).

To establish the definability of the jump, we will first show that the ideal $\mathcal{I}(\Delta_2^0)$ of degrees with Δ_2^0 representatives is definable within $\mathfrak{D}$ in terms of the double jump. By Theorem 7.3.5, the double jump is definable in $\mathfrak{D}$, and so $\mathcal{I}(\Delta_2^0)$ sets is definable in $\mathfrak{D}$. Of course, $0'$ is the greatest element of this ideal, so it is definable in $\mathfrak{D}$, too. Finally, we will observe that the argument works relative to any given degree. Consequently, for each $x \in \mathfrak{D}$, x' is uniformly definable from x within $\mathfrak{D}$. The definability of the Turing jump follows.

Ultimately, our definition of $\mathcal{I}(\Delta_2^0)$ is grounded on a sequence of compactness arguments. In the next section, we present the Jockusch and Soare (1972) Low Basis Theorem, by which we can control the complexity of the sets that we produce. We will give the definition of $\mathcal{I}(\Delta_2^0)$ in the second section, and complete the proof of the definability of the jump in the one after that.

8.1 Jockusch-Soare Low Basis Theorem

Definition 8.1.1 1. A tree T contained in $\omega^{<\omega}$ is *finitely branching* if and only if there is a function $f : \omega \to \omega$ such that for all $\sigma \in T$ and all i in the domain of σ, $f(i) \geq \sigma(i)$.

2. A tree T is *recursively bounded* if and only if there is an f as above such that f is recursive.

Usually one defines a tree to be finitely branching by saying that the tree has finite fan-out. Clearly, our defining property implies this. One proves the converse implication by defining a function f by recursion proving that each level of T is finite and that there is a finite upper bound on the fan-out of finitely many elements of T.

Theorem 8.1.2 (König's Lemma) *Suppose that T is an infinite finitely branching tree. Then T has an infinite path.*

Theorem 8.1.2 is just the assertion that Cantor space is compact, put in the present context.

Proof: We build a path through T by recursion. We let σ_i be the element of T that we choose during step i of the recursion.

Since T is infinite, it is not empty. Let τ be an element of T. Then, every initial segment of τ is also an element of T. Consequently, the null sequence is an element of T, and we take it to be σ_0. Note that since every element of $\omega^{<\omega}$ extends the null sequence and T is an infinite subset of $\omega^{<\omega}$, there are infinitely many extensions of the null sequence in T.

Now, suppose that σ_i is given so that there are infinitely many extensions of σ_i in T. Since T is finitely branching, there are only finitely many immediate successors of σ_i in T. Since there are infinitely many extensions of σ_i in T, at least one of these immediate successors of σ_i must have infinitely many extensions in T. Let σ_{i+1} be $\sigma^\frown(m_{i+1})$, where m_{i+1} is the minimal m such that $\sigma^\frown(m) \in T$ and $\sigma^\frown(m)$ has infinitely many extensions in T.

Clearly, $\{\sigma_i : i \in \omega\}$ is an infinite path in T, as required. ■

Theorem 8.1.3 (Jockusch and Soare (1972)) *Suppose that T is an infinite Π^0_1 recursively bounded tree, and suppose that $A \subseteq \omega$ is not recursive. Then there is an infinite path X through T such that $X \not\geq_T A$.*

Proof: Let f be the recursive function which shows that T is recursively bounded. Let T_f be the subtree of $\omega^{<\omega}$ defined by $\sigma \in T_f$ if and only if for all n less than the length of σ, $f(n) \geq \sigma(n)$. Suppose that T is defined by the Π^0_1 formula, $(\forall n)\varphi(\sigma, n)$, in which φ has only bounded quantifiers. We let T^* be defined as follows.

$$T^* = \{\sigma \in T_f : (\forall \tau \subseteq \sigma)(\forall n \leq \text{length}(\sigma))\varphi(\tau, n)$$

Then T^* is a recursively bounded recursive subtree of $\omega^{<\omega}$ such that T is a subtree of T^*. Further, if X is an infinite path in T^*, then for all ℓ_0, for every $\ell > \ell_0$, $X \upharpoonright \ell \in T^*$ and thus $(\forall n \leq \ell)\varphi(X \upharpoonright \ell_0, n)$. Consequently, X is an infinite path through T. Thus, for every Π^0_1 recursively bounded subtree of $\omega^{<\omega}$, there is a recursively bounded recursive subtree of $\omega^{<\omega}$ which has exactly the same set of infinite paths. Since we are only concerned with the infinite paths in T, we may assume that T is recursive.

Let P be the notion of forcing in which the set of conditions is the set of infinite recursive subtrees of T and these are ordered by inclusion. Suppose that G is an ultrafilter on P which is generic with respect to meeting all the dense subsets of P which are arithmetically definable relative to A. (This is more genericity than we will need, but the arguments which follow do not require that we be subtle here.)

Now, we check that there is a unique infinite path common to all of the trees in G. For $n \in \omega$, consider the set S_n of T^* in P such that there is a $\sigma^* \in T^*$ such that σ^* has length greater than or equal to n and every σ in T_1 is compatible with σ^*. That is, T^* has a stem of length greater than or equal to n. Now, suppose that T_1 is an element of P. Then, T_1 is a recursively bounded, and hence finitely branching, infinite tree. Consequently, T_1 has an infinite path, say X_1. Now, let T_2 be the subtree of T_1 consisting of those elements of T_1 which are compatible with $X_1 \restriction n+1$. Since X_1 is an infinite path in T_2, T_2 is an infinite subtree of T_1. It is recursive since it is determined by a recursive condition on elements of the recursive tree T_1. Finally, T_2 has a stem of length greater than or equal to n. Consequently, S_n is dense in P. It is also arithmetically definable. Since G is generic, for each n there is an element T_n of S_n in G. The elements of G are compatible and so the stems of the T_n belong to every element of G. Hence, the limit of the stems, call it $X(G)$, of the T_n is an infinite path through all of the elements of G. Further, if X is an infinite path in all of the T_n, then every initial segment of X is compatible with the stems of all of the T_n and so $X = X(G)$.

Now, we argue that $X(G)$ has the required property.

Suppose that $T_1 \in P$, that Φ is a Turing functional, and that for each x there is a y such that $T_1 \Vdash \Phi(x, X(G)) = y$. Let x be fixed, let y be the value decided by T_1 for $\Phi(x, X(G))$, and consider the subtree S of T_1 defined as follows.

$$S = \{\sigma : \sigma \in T_1 \text{ and } \Phi(x, \sigma) \neq y\}$$

That is, $\sigma \in S$ if and only if either $\Phi(x, \sigma)$ does not converge by a computation of length less than the length of σ or it does converge by such a computation and the value produced by that computation is different from y. If S were infinite, then it would be an extension of T_1 in P which forced $\Phi(x, X(G))$ not to equal y. Consequently, S is finite. But, then there must be an ℓ such that no element of T_1 of length ℓ belongs to S. It follows that T_1 forces $\Phi(X(G))$ to be recursive. Given x, the value of $\Phi(x, X(G))$ forced by T_1 can be computed by finding the least ℓ such that there is a y, for every σ, if σ belongs to T_1 and σ has length ℓ, then $\Phi(x,\sigma) = y$ by means of a computation of length less than ℓ.

We can complete the proof of Theorem 8.1.3. Given a Turing functional Φ and a tree $T_1 \in P$, either there is an x such that T_1 does not decide $\Phi(x, X(G))$ and so there is a T_2 extending T_1 in P such that $T_2 \Vdash \Phi(x, X(G)) \neq A(x)$, or T_1 forces $\Phi(X(G))$ to be recursive and therefore unequal to A. In either case, T_1 cannot force $\Phi(X(G))$ to be equal to A.

The dense sets associated with making $\Phi(X(G))$ unequal to A are arithmetic in A, so if G is generic with respect to these sets, then $X(G)$ satisfies the claim of Theorem 8.1.3. ∎

We will use Theorem 8.1.3 in the following relativized form.

Theorem 8.1.4 *Suppose that T is an infinite Π^0_n recursively bounded tree, and suppose that $A \subseteq \omega$ is not Δ^0_n. Then there is an infinite path X through T such that $X \not\geq_T A$.*

8.2 Kumabe-Slaman forcing

Our definition of $\mathcal{I}(\Delta^0_2)$ is based on the following technical fact.

Theorem 8.2.1 (Shore and Slaman (1999)) *For $X \in 2^\omega$, the following conditions are equivalent.*

1. *X is not recursive in $0^{(n)}$.*
2. *There is a $G \in 2^\omega$ such that $X \oplus G \geq_T G^{(n+1)}$.*

As stated, Theorem 8.2.1 will be sufficient for our purposes, though it holds in more generality; one can replace $G^{(n+1)}$ by the value of any $n+1$-fold composition of recursively enumerable and degree increasing operations, or prove an analogous theorem for transfinite iterations of such operations. See (Shore and Slaman, 1999).

Proof: The remainder of this section is devoted to the proof of Theorem 8.2.1. The following notion of forcing is due to Kumabe and Slaman, who used it to prove a version of Theorem 8.2.1 in which the nth jump is replaced by the ωth jump and X not recursive in $0^{(n)}$ is replaced by X is not arithmetic.

Definition 8.2.2 Let P be the following partial order.

1. The elements p of P are pairs $(\Phi_p, \boldsymbol{X}_p)$ in which Φ_p is a finite use-monotone Turing functional and $\boldsymbol{X}_p$ is a finite collection of subsets of ω. (See Definition 1.1.3.)
2. If p and q are elements of P, then $p \geq q$ if and only if
   (a) i. $\Phi_p \subseteq \Phi_q$ and
       ii. for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length σ_p,
   (b) $\boldsymbol{X}_p \subseteq \boldsymbol{X}_q$,
   (c) for every x, y, and $X \in \boldsymbol{X}_p$, if $\Phi_q(x, X) = y$ then $\Phi_p(x, X) = y$.

In short, a stronger condition than p can add computations to Φ_p, provided that they are longer than any computation in Φ_p and that they do not apply to any element of $\mathbf{X}_p$.

Definition 8.2.3 If Φ_0 and Φ_1 are finite use-monotone Turing functionals, then $\Phi_0 \geq_0 \Phi_1$ if and only if $(\Phi_0, \emptyset) \geq (\Phi_1, \emptyset)$ in P.

If $G \subseteq P$ is a (sufficiently, or indeed, even slightly) P-generic filter, then G is naturally associated with the functional $\Phi_G = \bigcup\{\Phi_p : p \in G\}$. To prove Theorem 8.2.1, we will construct a G that is sufficiently P-generic so that every Σ^0_n statement about Φ_G is correctly decided by a condition in P that belongs to G. We will also show that it possible meet the relevant dense subsets of P and still arrange that $\Phi_G(A)$ is equal to the characteristic function of the complete Σ^0_n set relative to Φ_G. The total effect will be to ensure that $\Phi_G^{(n)}$ is recursive in the join of Φ_G and A

We will treat Φ_G as if it were a subset of ω and suppress the recursive apparatus needed to represent Φ_G in this way.

Lemma 8.2.4 *Let $p = (\Phi_p, \boldsymbol{X}_p)$ be an element of P.*

1. $p \Vdash a \in \Phi_G$ *if and only if* $a \in \Phi_p$.
2. $p \Vdash a \notin \Phi_G$ *if and only if*
   (a) *either a is not a suitable triple,*
   (b) *or a is equal to (x, y, σ), $a \notin \Phi_p$, and either*
       i. *there is a $(x_0, y_0, \sigma_0) \in \Phi_p$ such that the length of σ_0 is greater than the length of σ, or x_0 is greater than or equal to x and σ_0 is compatible with σ.*
       ii. *or σ is an initial segment of one of the elements of $\boldsymbol{X}_p$.*

Proof: For the first claim, if $a \in \Phi_p$ then $a \in \Phi_G$ whenever $p \in G$. Consequently, if $a \in \Phi_p$ then $p \Vdash a \in \Phi_G$. Conversely, if $a \notin \Phi_p$ then let σ be a sequence such that σ has length greater than the length of any sequence mentioned in p or a and such that σ is incompatible with all of the elements of $\boldsymbol{X}_p$. Let x be the least number such that $\Phi_p(x, \sigma)$ is not defined. Then $q = (\Phi_p \cup \{(x, 0, \sigma)\}, \boldsymbol{X}_p)$ extends p in P, $a \notin \Phi_p \cup \{(x, 0, \sigma)\}$ and no extension r of q can have $a \in \Phi_r$. Consequently, $q \Vdash a \notin \Phi_G$ and so $p \nVdash a \in \Phi_G$. The proof of the second claim is similar. One observes that if conditions 2(a) and (b) do not hold, then it is possible to extend Φ_p to some Φ_q so that $p \geq (\Phi_q, \boldsymbol{X}_p)$ and $a \in \Phi_q$. ∎

Definition 8.2.5 Let Φ_p be a finite use-monotone Turing functional. Let $\psi(\Phi_G)$ be a Π^0_n sentence $(\forall m)\theta(m, \Phi_G)$ about Φ_G in which $\theta(m, \Phi_G)$ is Σ^0_{n-1}. For $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k)$ a sequence of elements of $2^{<\omega}$ all of the same length, we say that $\boldsymbol{\tau}$ is *essential to* $\neg\psi(\Phi_G)$ *over* Φ_p when the following condition holds: For all q and all m, if q is a condition such that $(\Phi_p, \emptyset) > q$ and $q \Vdash \neg\theta(m, \Phi_G)$, then $\Phi_q \setminus \Phi_p$ includes a triple (x, y, σ) such that σ is compatible with at least one component of $\boldsymbol{\tau}$.

Definition 8.2.6 For Φ_0 a finite use-monotone Turing functional, $\psi(\Phi_G)$ a Π^0_n sentence, and k in ω, let $T(\Phi_0, \psi, k)$ be the set of length k vectors $\boldsymbol{\tau}$ which are essential to $\neg\psi(\Phi_G)$ over Φ_0.

We order $T(\Phi_0, \psi, k)$ by extension on all coordinates. That is, $\boldsymbol{\sigma}$ extends $\boldsymbol{\tau}$ if and only if for all i less than or equal to k, the ith coordinate of $\boldsymbol{\sigma}$ extends the ith coordinate of $\boldsymbol{\tau}$. It is immediate that if $\boldsymbol{\sigma}$ extends $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0, then $\boldsymbol{\tau}$ is also essential to $\neg\psi(\Phi_G)$ over Φ_0. Consequently, $T(\Phi_0, \psi, k)$ is a subtree of the tree of length k vectors of binary sequences of equal length ordered as above. That is, $T(\Phi_0, \psi, k)$ is a subtree of a recursively bounded recursive tree.

Lemma 8.2.7 *Suppose that Φ_0 is a finite use monotone functional, $\psi(\Phi_G)$ is a Π^0_n sentence with $n \geq 1$, and k is a natural number.*

1. *If there is a size k set $\boldsymbol{X}$ of subsets of ω such that $(\Phi_0, \boldsymbol{X}) \Vdash \psi(\Phi_G)$, then $T(\Phi_0, \psi, k)$ is infinite.*
2. *If $T(\Phi_0, \psi, k)$ is infinite, then it has an infinite path Y. Further, each such Y is naturally identified with a size k set $\boldsymbol{X}(Y)$ of subsets of ω such that $(\Phi_0, \boldsymbol{X}(Y)) \Vdash \psi(\Phi_G)$.*

Proof: Say that $\psi(\Phi_G)$ is equal to $(\forall m)\theta(m, \Phi_G)$ where $\theta(m, \Phi_G)$ is Σ^0_{n-1}. For the first claim, suppose there is a size k set $\boldsymbol{X} = (X_1, \ldots, X_k)$ of subsets of ω such that $(\Phi_0, \boldsymbol{X}) \Vdash \psi(\Phi_G)$. Fix such an $\boldsymbol{X}$ and consider the set of sequences $\boldsymbol{\tau}_\ell = (X_1 \upharpoonright \ell, \ldots, X_k \upharpoonright \ell)$, as ℓ ranges over ω. For all q extending $(\Phi_0, \emptyset)$ and all m, if $q \Vdash \neg\theta(m, \Phi_G)$, then q is incompatible with $(\Phi_0, \boldsymbol{X})$. In particular, $(\Phi_q, \boldsymbol{X}_q \cup \boldsymbol{X})$ does not extend $(\Phi_0, \boldsymbol{X})$ in P. But then, there must be an i such that $\Phi_q \setminus \Phi_0$ contains an element (x, y, σ) such that X_i extends σ. This σ is compatible with the ith component of each $\boldsymbol{\tau}_\ell$. Consequently, each $\boldsymbol{\tau}_\ell$ is essential to $\neg\psi(\Phi_G)$ over Φ_0 and hence $T(\Phi_0, \psi, k)$ is infinite. This verifies the first claim.

For the second claim of the lemma, suppose that $T(\Phi_0, \psi, k)$ is infinite. By König's Lemma, since $T(\Phi_0, \psi, k)$ is a finitely branching tree, it has at least one infinite path. Now suppose that Y is such an infinite path. Let $\mathbf{X}(Y)$ be the size k set $\{X_1, \ldots, X_k\}$ in which each X_i is the limit of the ith coordinates of the elements of Y. For every extension q of $(\Phi_0, \emptyset)$ and every m, if $q \Vdash \neg\theta(m, \Phi_G)$ then $\Phi_q \setminus \Phi_0$ includes an element (x, y, σ) such that σ is compatible with at least one component of each element of Y. But then, for all sufficiently large elements of Y, σ is extended by a coordinate of Y, and so σ is extended by at least one of the elements of $\boldsymbol{X}(Y)$. Thus, for all m, no extension of $(\Phi_0, \boldsymbol{X}(Y))$ can force $\neg\theta(m, \Phi_G)$. Therefore, $(\Phi_0, \boldsymbol{X}(Y)) \Vdash \psi(\Phi_G)$, as required to verify the second claim. ■

Lemma 8.2.8 *For each finite use monotone functional Φ_0, each Π^0_n sentence $\psi(\Phi_G)$ with $n \geq 1$, and each number k, $T(\Phi_0, \psi, k)$ is Π^0_n, uniformly in Φ_0, ψ, and k.*

Proof: First consider the forcing relation for sentences in which all of the quantifiers are bounded. Suppose $\neg\theta(\Phi_G)$ is a bounded sentence about Φ_G. Applying Lemma 8.2.4, the forcing relation for atomic sentences is defined by a bounded formula. By induction on bounded complexity, whether $(\Phi_0, \boldsymbol{X}) \Vdash \neg\theta(\Phi_G)$ is also defined by a bounded formula, which is given uniformly in terms of Φ_0, $\neg\theta(\Phi_G)$, and $\boldsymbol{X}$. Fix a bound m on the quantifiers in the formula which defines this property. Again, by referring to Lemma 8.2.4, if $\boldsymbol{X}_0$ is a subset of $\boldsymbol{X}$ such that for all $X \in \boldsymbol{X}$, there is an $X_0 \in \boldsymbol{X}_0$ such that X and X_0 agree on the numbers less than m, then $(\Phi_0, \boldsymbol{X}) \Vdash \neg\theta(\Phi_G)$ if and only if $(\Phi_0, \boldsymbol{X}_0) \Vdash \neg\theta(\Phi_G)$. Since there are only finitely many incompatible binary sequences of length m, we can capture the possible behaviors of sets $\boldsymbol{X}$ by quantifying over the possible behaviors of subsets of the set of length m binary sequences.

Consequently, uniformly in $\neg\theta$, whether there is a finite set $\boldsymbol{X}$ such that $(\Phi_0, \boldsymbol{X}) \Vdash \neg\theta(\Phi_G)$ is a bounded property given uniformly in terms of Φ_0 and $\neg\theta(\Phi_G)$.

We now prove Lemma 8.2.8 by induction on n. First consider the base case when n is equal to 1. That is ψ is of the form $(\forall x)\theta(x, \Phi_G)$ and $\theta(x, \Phi_G)$ is bounded. Let k be fixed and suppose that $\boldsymbol{\tau}$ is a length k sequence of elements of $2^{<\omega}$ all of the same length. By Definition 8.2.5, $\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0 if and only if for all $q \in P$ and all $m \in \omega$, if $(\Phi_0, \emptyset) > q$ and $q \Vdash \neg\theta(m, \Phi_G)$, then $\Phi_q \setminus \Phi_0$ includes a triple (x, y, σ) such that σ is compatible with at least one component of $\boldsymbol{\tau}$. By the analysis of the forcing relation for bounded sentences, for each finite use-monotone functional Φ_q, whether there is a finite set $\boldsymbol{X}_q$ such that $(\Phi_0, \emptyset) > (\Phi_q, \boldsymbol{X}_q)$ and $(\Phi_q, \boldsymbol{X}_q) \Vdash \neg\theta(m, \Phi_G)$ is a bounded property of Φ_q and m. Thus, the quantifier over q in P can be replaced by a quantifier over finite use-monotone functionals Φ_q with $\Phi_0 \geq_0 \Phi_q$. (See Definition 8.2.3.) Consequently, $\boldsymbol{\tau}$'s being essential to $\neg\psi(\Phi_G)$ over Φ_0 is a Π^0_1 property of $\boldsymbol{\tau}$, and so $T(\Phi_0, \psi, k)$ is a Π^0_1 tree, verifying the lemma for $n = 1$. Note that the Π^0_1 definition of $T(\Phi_0, \psi, k)$ was obtained uniformly in terms of Φ_0, ψ, and k.

For the inductive argument, we assume that the lemma holds for n. We repeat the argument for the base case, with the inductive assumption used to analyze the forcing relation for Π^0_n sentences. Let k be fixed and suppose that $\boldsymbol{\tau}$ is a length k sequence of elements of $2^{<\omega}$ all of the same length. Again, $\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0 if and only if,

> for all $q \in P$ and all $m \in \omega$, if $(\Phi_0, \emptyset) > q$ and $q \Vdash \neg\theta(m, \Phi_G)$, then $\Phi_q \setminus \Phi_0$ includes a triple (x, y, σ) such that σ is compatible with at least one element of $\boldsymbol{\tau}$.

This condition is equivalent to

> for all Φ_q with $\Phi_0 \geq_0 \Phi_q$, for all k, and all $m \in \omega$, if there is a size k set $\boldsymbol{X}$ such that $(\Phi_q, \boldsymbol{X}) \Vdash \neg\theta(m, \Phi_G)$ then $\Phi_q \setminus \Phi_0$ includes a triple (x, y, σ) such that σ is compatible with at least one element of $\boldsymbol{\tau}$.

By Lemma 8.2.7, "there is a size k set $\boldsymbol{X}$ such that $(\Phi_q, \boldsymbol{X}) \Vdash \neg\theta(m, \Phi_G)$" can be replaced by "$T(\Phi_q, \neg\theta(m), k)$ is infinite". Thus, $\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0 if and only if,

> for all Φ_q such that $\Phi_0 \geq_0 \Phi_q$, for all k, and all $m \in \omega$, if $T(\Phi_q, \neg\theta(m), k)$ is infinite then $\Phi_q \setminus \Phi_0$ includes a triple (x, y, σ)

such that σ is compatible with at least one element of $\boldsymbol{\tau}$.

Since $\neg\theta(m, \Phi_G)$ is a Π^0_n sentence, we can apply induction to conclude that $T(\Phi_q, \neg\theta(m), k)$ is uniformly Π^0_n in terms of Φ_q, ψ, m, and k. As a fact of pure definability, whether a Π^0_n subtree of a recursively bounded recursive tree is infinite is itself Π^0_n: it is Σ^0_n to state that there is a splitting level in the recursive tree which is disjoint from the Π^0_n subtree. So, "$\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0" is equivalent to a condition of the form "for all Φ_q with $\Phi_0 \geq_0 \Phi_q$, for all k, and all $m \in \omega$, if a Π^0_n condition holds, then so does a bounded one". Thus, "$\boldsymbol{\tau}$ is essential to $\neg\psi(\Phi_G)$ over Φ_0" is a Π^0_{n+1} property of $\boldsymbol{\tau}$, Φ_0 and ψ. Consequently, for each k and for each Π^0_{n+1} sentence ψ, $T(\Phi_0, \psi, k)$ is Π^0_{n+1}, uniformly in Φ_0, ψ, and k. This completes the verification of the lemma. ■

Corollary 8.2.9 *Suppose that A is not Δ^0_n. Let Φ_0 be a finite use-monotone functional, $\psi(\Phi_G)$ be a Π^0_n sentence about Φ_G, and k be a positive natural number. If there is a size k set $\boldsymbol{X}$ of subsets of ω such that $(\Phi_0, \boldsymbol{X}) \Vdash \psi(\Phi_G)$, then there is such a set $\boldsymbol{X}$ such that $A \notin \boldsymbol{X}$.*

Proof: Suppose that there is a size k set $\boldsymbol{X}$ of subsets of ω such that

$$(\Phi_0, \boldsymbol{X}) \Vdash \psi(\Phi_G).$$

By Lemmas 8.2.7 and 8.2.8, $T(\Phi_0, \psi, k)$ is a Π^0_n subtree of a recursively bounded recursive tree T which has an infinite path.

By the Jockusch and Soare (1972) Theorem 8.1.4, there is an infinite path Y in $T(\Phi_0, \psi, k)$ in which A is not recursive and so, in particular, $A \notin \mathbf{X}(Y)$. By Lemma 8.2.7, $(\Phi_0, \boldsymbol{X}(Y)) \Vdash \psi(\Phi_G)$ for any such Y. Thus, we have the desired conclusion. ■

Lemma 8.2.10 *Suppose that n is greater than 0, A is not Δ^0_n, and $\psi(\Phi_G)$ is a Π^0_n sentence about Φ_G; say $\psi(\Phi_G) = (\forall x)\theta(x, \Phi_G)$ in which θ is Σ^0_{n-1}. For any condition $p = (\Phi_p, \boldsymbol{X}_p)$ with $A \notin \boldsymbol{X}_p$, there is a stronger condition $q = (\Phi_q, \boldsymbol{X}_q)$ such that the following conditions hold.*

1. *$A \notin \boldsymbol{X}_q$.*
2. *For all x, if $\Phi_q(x, A)$ is defined, then $\Phi_p(x, A)$ is defined. That is, q does not add any new computations to Φ_G which apply to A.*
3. *Either $q \Vdash \psi(\Phi_G)$ or there is an m such that $q \Vdash \neg\theta(m, \Phi_G)$.*

Proof: Fix $p = (\Phi_p, \boldsymbol{X}_p)$ in P. Let $X_1, \ldots, X_k$ be an enumeration of the elements of $\boldsymbol{X}_p$. If $T(\Phi_p, \psi, k+1)$ is infinite, then Corollary 8.2.9 supplies a condition $r = (\Phi_p, \mathbf{X})$ forcing $\psi(\Phi_G)$ with $A \notin \mathbf{X}$. As (2) is trivially satisfied if Φ_p is kept fixed, our desired condition q is $(\Phi_p, \mathbf{X}_p \cup \mathbf{X})$. If $T(\Phi_p, \psi, k+1)$ is not infinite, then $\mathbf{X}_p \cup \{A\}$ does not provide an infinite path through it. Thus, for some ℓ, $\boldsymbol{\tau}(\ell) = (X_1 \restriction \ell, \ldots, X_k \restriction \ell, A \restriction \ell)$ is not essential to $\neg\psi(\Phi_G)$ over Φ_p. Then, there is an number m and a condition $r = (\Phi_r, \boldsymbol{X}_r)$ extending $(\Phi_p, \emptyset)$ such that Φ_r does not add any new computations compatible with any of the components of $\boldsymbol{\tau}(\ell)$ and $r \Vdash \neg\theta(m, \Phi_G)$. In particular, Φ_r does not add any new computations which apply to A or to any element of $\boldsymbol{X}_p$ and, by Lemma 8.2.7 there is a k such that $T(\Phi_r, \neg\theta(m, \Phi_G), k)$ is infinite. We can now apply Corollary 8.2.9 again to get an $\mathbf{X}$ of size k with $A \notin \mathbf{X}$ such that $(\Phi_r, \mathbf{X}) \Vdash \neg\theta(m, \Phi_G)$. Our desired condition q is thus $(\Phi_r, \mathbf{X}_p \cup \mathbf{X})$. ■

Now we can complete the proof of Theorem 8.2.1. Suppose that $A \subseteq \omega$ is not Δ^0_n. Let $(\psi_i(\Phi_G) : i \geq 1)$ be a recursive enumeration of the Π_n sentences about Φ_G. We build a sequence of conditions $(p_i : i \in \omega)$ so that $p_0 = (\emptyset, \emptyset)$, $p_i > p_{i+1}$, and for all i, p_i decides $\psi_i(\Phi_G)$ and $A \notin \mathbf{X}_{p_i}$.

Given p_{i-1}, we obtain p_i in two steps. Suppose that ψ_i is $\forall x \theta_i(x, \Phi_G)$ and $\theta_i(x, \Phi_G)$ is Σ_{n-1}. First, we apply Lemma 8.2.10 to find a condition $q = (\Phi_q, \mathbf{X}_q)$ extending p_{i-1} such that $A \notin \mathbf{X}_q$, $\Phi_q(A)$ is equal to $\Phi_p(A)$, and either $q \Vdash \psi_i(\Phi_G)$ or there is an m in ω such that $q \Vdash \neg\theta_i(m, \Phi_G)$. Let ℓ be so large that it is greater than m, it is greater than the length of any sequence mentioned in Φ_q, and for each X in $\mathbf{X}_q$ there is an x less than ℓ with $X(x) \neq A(x)$. We define p_i to be $(\Phi_q \cup \{(i, 0, A \restriction \ell)\}, \mathbf{X}_q)$, if $q \Vdash \psi_i(\Phi_G)$, and $(\Phi_q \cup \{(i, 1, A \restriction \ell)\}, \mathbf{X}_q)$, if $q \Vdash \neg\psi_i(\Phi_G)$. In other words, we build p_i by first deciding the ith Π_n sentence about Φ_G without extending $\Phi_G(A)$, and then defining $\Phi_G(i, A)$ to record the value decided.

Let Φ_G be the union of the Φ_{p_i}. By induction on the logical complexity of its subformulas, for each Π_n sentence ψ_i about Φ_G, $\psi_i(\Phi_G)$ is true if and only if $p_i \Vdash \psi_i(\Phi_G)$. But then $\Phi_G(A)$ is the characteristic function of a complete Σ_n set relative to Φ_G. So, $\Phi_G \oplus A \geq_T \Phi_G(A) = \Phi_G^{(n)}$, as required. ■

8.3 Defining the jump: Shore-Slaman

Theorem 8.3.1 1. *The ideal $\mathcal{I}(\Delta^0_2)$ is definable in $\mathfrak{D}$.*
2. *$0'$ is definable in $\mathfrak{D}$.*

Proof: It is sufficient to show that $\mathcal{I}(\Delta^0_2)$ is definable in $\mathfrak{D}$, since $0'$ is its greatest element.

Let d be given with $d \in \mathfrak{D}$. If $d \in \mathcal{I}(\Delta^0_2)$, then for all x, $d + x \leq_T x'$. Consequently, for all x, $d + x \not\geq_T x''$. Conversely, if $d \notin \mathcal{I}(\Delta^0_2)$, then by Theorem 8.2.1 with $n = 2$, there is an x such that $d + x \geq x''$. Consequently, we have the following equivalence.

$$(\forall d \in \mathfrak{D})[d \in \mathcal{I}(\Delta^0_2) \text{ if and only if } (\forall x)(d + x \not\geq_T x'')] \tag{8.1}$$

By Theorem 7.3.5, the function $x \mapsto x''$ is definable in $\mathfrak{D}$. Consequently, Equation 8.1 provides a definition of $\mathcal{I}(\Delta^0_2)$ in $\mathfrak{D}$. ■

We can give a similar proof for the definability of the function $x \mapsto x'$.

Theorem 8.3.2 *The functions $a \mapsto \mathcal{I}(\Delta^0_2(a))$ and $a \mapsto a'$ are definable in $\mathfrak{D}$.*

Proof: By relativizing Theorem 8.2.1, for each degree a and each d greater than or equal to a, d is not Δ^0_2 relative to a if and only if there is an x greater than or equal to a such that $d + x \geq_T x''$. Again, the double jump is definable in $\mathfrak{D}$, and this equivalence provides first order definitions as required. ■

Question 8.3.3 Is the relation "y is recursively enumerable relative to x" definable in $\mathfrak{D}$?

A positive answer to Question 8.3.3 would follow from a proof of the Biinterpretability Conjecture.

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