SCATTERING AND WAVE OPERATORS FOR ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SLOWLY DECAYING NONSMOOTH POTENTIALS

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ABSTRACT. We prove existence of modified wave operators for one-dimensional Schrödinger equations with potential in $L^p(\mathbb{R})$, $p < 2$. If in addition the potential is conditionally integrable, then the usual Möller wave operators exist. We also prove asymptotic completeness of these wave operators for some classes of random potentials, and for almost every boundary condition for any given potential.

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1. Introduction

Let H_V be a one-dimensional Schrödinger operator defined by

$$H_V = -\frac{d^2}{dx^2} + V(x). \tag{1.1}$$

Let us discuss the case where the operator is defined on a half-axis, with some self-adjoint boundary condition at zero. We are interested in potentials decaying at infinity, for which we may expect that asymptotically as time tends to infinity, motion of the associated perturbed quantum system resembles the free evolution. What is the critical rate of decay of the potential for which the dynamics remains close to free for large times? The mathematical framework for studying these questions is provided by scattering theory. Recall that the wave operators associated with H_V and H_0 are defined by

$$\Omega_{\pm} f = \lim_{t\to\mp\infty} e^{itH_V} e^{-itH_0} f, \tag{1.2}$$

where the limit is understood in the strong L^2 sense. In particular, existence of the wave operators implies that the absolutely continuous spectrum of H_V fills all of $\mathbb{R}^+$ (see, e.g. [35]) and moreover provides significant information on large time dynamics e^{-itH_V}.

The wave operators will be called asymptotically complete if the range of $\Omega_\pm$ coincides with the orthogonal complement of the subspace spanned by eigenvectors of the operator H_V. An alternative equivalent characterization is that the range of the wave operators is equal to the absolutely continuous subspace $\mathcal{H}_{\mathrm{ac}}(H_V)$ of the operator H_V, and that the singular continuous spectrum $\sigma_{\mathrm{sc}}(H_V)$ is empty. We note that the intended intuitive meaning of asymptotic completeness is that the dynamics of the perturbed operator can be divided into two well-understood parts: scattering states traveling to infinity in a way similar to the free evolution, and bound states which remain confined in a certain sense for all times. We postpone more detailed discussion of the notion of asymptotic completeness to Section 9.

A well-known result, going back to the very first years of the rigorous scattering theory, says that if $V \in L^1$, then the wave operators exist and are asymptotically complete. Moreover, in this case the spectrum on the positive semi-axis is purely absolutely continuous, and there can be only discrete spectrum below zero, possibly accumulating at zero. There has been much work extending the existence of wave operators to wider classes of potentials, with some additional conditions on derivatives, or for potentials of certain particular forms. Generally, for more slowly decaying potentials one needs to modify the free dynamics in the definition of wave operators in order for the limits to exist. The first work of this type was that of Dollard [20], who studied the Coulomb potential. Further developments used computation of asymptotic classical trajectories by means of a Hamilton-Jacobi equation to build an appropriate phase correction to the free dynamics, which was used to prove existence of modified wave operators (in any dimension). See, for instance, Buslaev and Matveev [7], Ahsholm and Kato [1]; the strongest results are contained in Hörmander [23]. For example, one result of Hörmander [23] gives existence of wave operators if

$|V(x)| \le C(1+|x|)^{-1/2-\epsilon}$ and $|D^\alpha V(x)| \le C(1+|x|)^{-3/2-\epsilon}$ for any $|\alpha| = 1$; other results in that paper require stronger conditions on derivatives to compensate for slower decay.

Another series of works (see [3, 4, 19, 44] for further references) studied oscillating potentials of Wigner-von Neumann type, for example $V(x) = \sin(cx^\alpha)/x^\beta$. Interest in this class of potentials was in part fueled by the Wigner-von Neumann example [43], where $V(x) \sim c\sin(2x)/x$ at infinity, which leads to a positive eigenvalue embedded in the absolutely continuous spectrum. For a wide class of potentials of this type (including the original example of [43]), existence and asymptotic completeness of (sometimes modified) wave operators has been shown.

In the opposite direction, Pearson [32] constructed examples of potentials in $\cap_{p>2} L^p(\mathbb{R})$ for which the spectrum is purely singular, and hence wave operators do not exist. Kotani and Ushiroya [28] also provided power decaying examples where the spectrum is purely singular (pure point) for the rate of decay $x^{-\alpha}$, $\alpha < 1/2$.

In the last few years, there has been a series of works studying existence of absolutely continuous spectrum for slowly decaying potentials in full generality, with no additional conditions on derivatives or specific representation. In [8, 36] it was shown that the absolutely continuous spectrum is preserved if $|V(x)| \le C(1+x)^{-\alpha}$, $\alpha > 1/2$, and moreover the generalized eigenfunctions have WKB-type plane wave asymptotics. More recently, we improved the result of [8] to treat potentials in L^p, $p < 2$ [9]. The sharpest result on the stability of the absolutely continuous spectrum is due to Deift and Killip [16], who showed it for $V \in L^2$. This result is optimal in the L^p scale, as Pearson's examples show. The method of [16] is quite different from [8, 36] and yields much less information concerning the nature of the generalized eigenfunctions.

Although the absolute continuity of the spectrum is an important characterization of an operator with direct consequences for the dynamical behavior of the particle, wave operators provide a finer description of long-time dynamics. The main purpose of this paper is to use the additional information contained in the asymptotic behavior of generalized eigenfunctions [9] to prove the existence of (modified) wave operators for potentials $V \in L^p$, $p < 2$. Let

$$W(\lambda, t) = -(2\lambda)^{-1} \int_0^{2\lambda t} V(s)\, ds. \tag{1.3}$$

Define $e^{-iH_0 t \pm iW(H_0^{1/2}, \mp t)}$ to be the Fourier multiplier operator on $L^2(\mathbb{R}^+)$ that maps

$$\int_0^\infty F(\lambda)\sin(\lambda x)\, d\lambda \mapsto \int_0^\infty e^{-i\lambda^2 t \pm iW(\lambda, \mp t)} F(\lambda) \sin(\lambda x)\, d\lambda$$

for all $F \in L^2(\mathbb{R}^+, d\lambda)$. Provisionally define

$$\Omega_\pm^m f = \lim_{t \to \mp\infty} e^{itH_V} e^{-itH_0 \pm iW(H_0^{1/2}, \mp t)} f \tag{1.4}$$

for all $f \in L^2(\mathbb{R}^+)$ for which the limit exists in $L^2(\mathbb{R}^+)$ norm. Among our main conclusions are the following two theorems.

Theorem 1.1. *Let V be a potential in the class $L^1 + L^p(\mathbb{R}^+)$ for some $1 < p < 2$, and let H_V be the associated Schrödinger operator on $L^2(\mathbb{R}^+)$ with any self-adjoint boundary condition at 0. Then for every $f \in L^2(\mathbb{R}^+)$, the limits in (1.4) exist in $L^2(\mathbb{R}^+)$ norm, as $t \to \mp\infty$. The modified wave operators $\Omega^m_\pm$ thus defined are both unitary bijections from $\mathcal{H} = L^2(\mathbb{R}^+)$ to $\mathcal{H}_{ac}(H_V)$.*

Theorem 1.2. *In addition to the hypotheses of the preceding theorem, suppose that the improper integral $\int_0^\infty V(s)\,ds$ exists. Then for every $f \in L^2(\mathbb{R}^+)$, the limits in (1.2) exist in $L^2(\mathbb{R}^+)$ norm, as $t \to \mp\infty$. The wave operators $\Omega_\pm$ thus defined are both unitary bijections from $\mathcal{H} = L^2(\mathbb{R}^+)$ to $\mathcal{H}_{ac}$.*

By the improper integral we mean $\lim_{N\to\infty} \int_0^N V$; we are not assuming that $V \in L^1$. We also prove analogous results for Schrödinger operators on the whole real line $\mathbb{R}^1$, and for some Dirac operators; see Section 8.

Another way to put the conclusion is that for each $f \in L^2(\mathbb{R}^+)$ there exist functions $F_\pm \in \mathcal{H}_{\text{ac}}$ such that $\|e^{itH_V}F_\pm - e^{itH_0}f\|_{L^2(\mathbb{R}^+)} \to 0$ as $t \to \pm\infty$, and each map $f \mapsto F_\pm$ is unitary and surjective onto $\mathcal{H}_{\text{ac}}$. Thus there exists a full family of scattering states, and any state that is asymptotically free at $t = \mp\infty$ is likewise free at $t = \pm\infty$.

The main idea of the proof is to use generalized eigenfunctions to construct the spectral decomposition of the operator H_V, and in particular to derive an explicit expression for the evolution group of the perturbed operator. Generalized eigenfunctions play a role parallel to the Fourier transform in the free case. The existence of wave operators is then proven by direct analysis, comparing the perturbed evolution with modified free dynamics. While the analysis is generally based on results of [8, 9], we need several essential improvements in the estimates to fulfill this plan. In Sections 2, 11 we extend the analysis of multilinear expressions encountered in the series for generalized eigenfunctions to cover the situations arising in applications to wave operators. One new feature is the analysis of a class of multilinear operators on the endpoint domain $L^2 \otimes (L^p)^{n-1}$ for $p < 2$, sharpening our analysis in earlier work for $(L^p)^{n-1}$. In Section 4 we consider solutions of $H_V u = zu$ for complex energies z, and establish certain uniform bounds and asymptotics as $z \to \mathbb{R}^+$. In Section 5 we prove a limiting absorption principle, which allows us to write an explicit formula for the evolution. Sections 6, 7 and 8 contain long-time asymptotic analysis and the rest of the proof of the existence of wave operators. The key step is Lemma 6.2, where the full strength of the multilinear analysis is required to justify discarding all terms with which it is concerned. In Section 9 we discuss the issue of asymptotic completeness for generic potentials within certain classes.

Although all theorems, and many of the intermediate results, are stated for potentials in $L^1 + L^p$ for some $1 < p < 2$, for simplicity we first give the proofs under the simplifying assumption that $V \in L^p$. In Section 10 we review the machinery [9] needed to extend the analysis from L^p to $L^1 + L^p$, and to the more general amalgamated class $\ell^p(L^1)$. Section 11 contains certain inequalities for multilinear operators which follow from the more technical results in Section 2, but are not directly used in our analysis.

We note that the existence of (modified) wave operators for general long-range potentials in higher dimensions remains an open problem. For recent progress in this direction, see [39].

Some of the results of this paper were announced in [12].

2. A bound for iterated multiple integrals

Our analysis relies on certain general inequalities for multilinear operators, which in turn rest on bounds for multiple integrals. Some of the required bounds have been established in earlier work [10], but we also require some refinements not previously encountered. We will begin by reviewing some basic definitions and results from [10], and sketching some elements of the proofs. In the following subsection we will then discuss the needed refinements. Their proofs, and others later in the text, draw heavily on the material reviewed in subsection 2.1.

2.1. **A brief review.** Let $\{f_i\}$ be a collection of integrable functions from $\mathbb{R}$ to $\mathbb{C}$, or more generally, from a subinterval $I \subset \mathbb{R}$ to $\mathbb{C}$.[1] Consider multilinear expressions

$$M_n(f_1, \dots, f_n) = \iint_{x_1 \le \dots \le x_n} \prod_{i=1}^n f_i(x_i)dx_i \tag{2.1}$$

and their maximal variants

$$M_n^*(f_1, \dots, f_n) = \sup_{y \le y'} \Big| \iint_{y \le x_1 \le \dots \le x_n \le y'} \prod_{i=1}^n f_i(x_i)dx_i \Big|, \tag{2.2}$$

the supremum being taken with respect to both variables $y, y' \in \mathbb{R}$. Throughout the paper, we often write double integrals to denote multiple integrals over arbitrarily many variables.

The main results of this subsection are upper bounds for $|M_n|$ and M_n^*, in terms of certain auxiliary functions g_δ of the functions f_i, with particular attention to the dependence on n as $n \to \infty$.

Definition 2.1. A martingale structure $\{E_j^m\}$ on an interval $I \subset \mathbb{R}$ is a collection of subintervals $E_j^m \subset I$, indexed by $m \in \{0, 1, 2, \dots\}$ and $j \in \{1, 2, \dots, 2^m\}$, possessing the following two properties. (i) Except for endpoints, $\{E_j^m : 1 \le j \le 2^m\}$ is a partition of I, for each m. (ii) $E_j^m = E_{2j-1}^{m+1} \cup E_{2j}^{m+1}$ for all m, j.

It is sometimes natural to insist that $\max_j |E_j^m \cap J| \to 0$ as $m \to \infty$ for any bounded interval J, but we will not require this.

Definition 2.2. Let $1 \le p < \infty$. A martingale structure is said to be adapted to a function $f \in L^p(I)$ if $\int_{E_j^m} |f|^p = 2^{-m} \|f\|_{L^p}^p$ for all m, j.

[1]For our purposes the general case may always be reduced to $I = \mathbb{R}$ by defining $f_i(x) \equiv 0$ for all $x \notin I$, so we will restrict most of the discussion to $I = \mathbb{R}$.

Definition 2.3. To any $f \in L^1$, any $\delta \in \mathbb{R}$, and any martingale structure, we associate

$$g_\delta(f) = \sum_{m=1}^{\infty} 2^{\delta m} (\sum_{j=1}^{2^m} |\textstyle\int_{E_j^m} f|^2)^{1/2}. \tag{2.3}$$

More generally,

$$g_\delta(\{f_1, \dots, f_n\}) = \sum_{m=1}^{\infty} 2^{\delta m} (\sum_{j=1}^{2^m} \max_k |\textstyle\int_{E_j^m} f_k|^2)^{1/2}. \tag{2.4}$$

In our applications, the number of *distinct* functions f_k will be at most 3, while n will be arbitrarily large.

Proposition 2.1. [10] *There exists $C < \infty$ such that for any $n \geq 2$, any $f_1, \dots, f_n \in L^1(\mathbb{R})$, and any martingale structure $\{E_j^m\}$ satisfying the auxiliary hypothesis*[2]

$$\lim_{m \to \infty} \max_i \max_j \int_{E_j^m} |f_i| = 0 \tag{2.5}$$

the following conclusions hold:

$$|M_n(f_1, \dots, f_n)| \leq \frac{C^{n+1}}{\sqrt{n!}} g_0(\{f_k : 1 \leq k \leq n\})^n \tag{2.6}$$

and

$$|M_n(f_1, \dots, f_n)| \leq C^n \prod_{k=1}^{n} g_0(f_k). \tag{2.7}$$

Moreover under the same hypotheses, for any $\delta > 0$, there exists $C < \infty$ such that

$$|M_n^*(f_1, \dots, f_n)| \leq \frac{C^{n+1}}{\sqrt{n!}} g_\delta(\{f_k : 1 \leq k \leq n\})^n \tag{2.8}$$

and

$$|M_n^*(f_1, \dots, f_n)| \leq C^n \prod_{k=1}^{n} g_\delta(f_k). \tag{2.9}$$

The symbol χ_E will always denote the characteristic function of a set E. In particular, $\chi_{[y,y']}$ denotes the characteristic function of the interval $[y, y']$.

The conclusions for M_n^* follow from those for M_n by applying the former to $M_n(f_1 \cdot \chi_{[y,y']}, \dots, f_n \cdot \chi_{[y,y']})$, then taking the supremum over y, y' by applying the following very easy lemma.

[2]The results stated in [10] are defective in this regard; an auxiliary hypothesis of this type is implicitly used in the proofs of Propositions 4.1 and 4.2 of that paper but was not stated. We take this opportunity to correct those formulations.

Lemma 2.2. [10] *There exists $C < \infty$ such that for any martingale structure and for any $f \in L^1$,*

$$\sup_{y \le y'} \sum_{m=1}^{\infty} \Big(\sum_j |\textstyle\int_{E_j^m} f \cdot \chi_{[y,y']}|^2 \Big)^{1/2} \le C \sum_{m=1}^{\infty} m \Big(\sum_j |\textstyle\int_{E_j^m} f|^2 \Big)^{1/2}. \tag{2.10}$$

One ingredient in the proof is a variant of the binomial expansion, whose proof will not be discussed here.

Lemma 2.3. [10] *There exist positive constants b_n satisfying $b_n \le C^{n+1}/\sqrt{n!}$ and $n^{1/2} b_{n+1}/b_n \to c_0$ as $n \to \infty$ for some constant $c_0 \in (0, \infty)$, such that for all nonnegative real numbers x, y,*

$$b_n y^n + \sum_{i=2}^{n-2} b_i b_{n-i} x^i y^{n-i} + b_n x^n \le b_n (x^2 + y^2)^{n/2}. \tag{2.11}$$

Lemma 2.4. [10] *The constants b_n of the preceding lemma may also be chosen so that for any functions $f_i \in L^1$,*

$$|M_n(f_1, \dots, f_n)| \le b_n g_0(\{f_i\})^n \tag{2.12}$$

provided that $\max_{i,j} \int_{E_j^m} |f_i| \to 0$ *as $m \to \infty$.*

(2.6) follows directly from this lemma; for the related but simpler proof of (2.7) see [10].

Sketch of proof. We may assume that $n \ge 2$, the case $n = 1$ being trivial. Define $f_{i,j}^m = \int_{E_j^m} f_i$. Let $\Omega = \{t = (t_1, \dots, t_n) : t_1 \le t_2 \le \cdots \le t_n\}$. For $K \ge 1$ define $\Omega_K \subset \Omega$ to be the set of all t for which there does not exist i such that $t_j \in E_i^K$ for all $1 \le j \le n$. Then

$$\begin{aligned}
\Big| \iint_{\Omega \setminus \Omega_K} \prod_j f_j(t_j)\, dt_j \Big| &\le \sum_{j=1}^{2^K} \prod_{i=1}^{n} \int_{E_j^K} |f_i| \\
&\le \Big(\prod_{i=2}^{n} \max_j \int_{E_j^K} |f_i| \Big) \cdot \sum_j \int_{E_j^K} |f_1| \\
&= \|f_1\|_{L^1} \cdot \prod_{i \ge 2} \max_j \int_{E_j^K} |f_i|
\end{aligned}$$

which tends to zero as $K \to \infty$ by the hypothesis (2.5).

Observe that it was only necessary to assume that $\sup_j \int_{E_j^m} |f_i| \to 0$ as $m \to \infty$ for all $i \ge 2$; this refinement will be needed in Proposition 2.5 below, and in its application in Lemma 6.2.

Fix some large K and decompose $\Omega_K = \cup_{k=0}^n S_k$ where

$$S_k = \{t \in \Omega_K : t_1, \dots, t_k \in E_1^1 \text{ and } t_{k+1}, \dots, t_n \in E_2^1\}$$

for $0 < k < n$ and $S_0 = \{t \in \Omega_K : t_1, \dots, t_n \in E_2^1\}$, $S_n = \{t \in \Omega_K : t_1, \dots, t_n \in E_1^1\}$. The sets S_j form a partition of Ω_K, modulo sets of Lebesgue zero. Therefore

$$\iint_{\Omega_K} \prod_j f_j(t_j)\,dt_j = \sum_k \iint_{S_k} \prod_j f_j(t_j)\,dt_j.$$

Moreover, for any $1 \le k \le n-1$,

$$\iint_{S_k} \prod_j f_j(t_j)\,dt_j = M_k(f_{1,1}^1, \dots, f_{k,1}^1) \cdot M_{n-k}(f_{k+1,2}^1, \dots, f_{n,2}^1). \tag{2.13}$$

Consequently when $K \ge 2$,

$$|M_n(f_1, \dots, f_n)| \le \varepsilon_K + \sum_{k=1}^{n-1} |M_k(f_{1,1}^1, \dots, f_{k,1}^1)| \cdot |M_{n-k}(f_{k+1,2}^1, \dots, f_{n,2}^1)| \\ + |\tilde{M}_n(f_{1,1}^1, \dots, f_{n,1}^1)| + |\tilde{M}_n(f_{1,2}^1, \dots, f_{n,2}^1)|$$

where $\varepsilon_K \to 0$ as $K \to \infty$, and where $\tilde{M}_n$ is defined by changing the region of integration in the definition of M_n from Ω to Ω_K. For $K = 1$ the formula is the same, except that the last two terms $\tilde{M}_n$ are absent.

The proof now proceeds by a double induction on n, for those terms with $k \in \{1, \dots, n-1\}$, and on K, for the two terms $\tilde{M}_n(\cdot)$. To do the latter, we note that $\{E_j^m : m \ge 1 \text{ and } j \le 2^{m-1}\}$ forms a martingale structure on E_1^1, and there is a corresponding structure on E_2^1. The effect of this change in perspective is to shift the superscript m by 1, thus transforming K to $K-1$. When $K = 1$ matters reduce to a single induction on n, since the troublesome terms $\tilde{M}_n$ do not arise.

The details of the induction argument based on Lemma 2.3 may be found in [10], and will be reproduced in substantial part in the proof of Proposition 2.5 below, so will not be reviewed here. □

2.2. **A refinement.** In the next result we treat only a more restricted variant of the maximal expression M_n^*, redefining

$$M_n^*(f_1, \dots, f_n) = \sup_{y \in \mathbb{R}} \Big| \iint_{t_1 \le \cdots \le t_n \le y} \prod_{j=1}^n f_j(t_j)\,dt_j \Big|. \tag{2.14}$$

In our applications, $f_i(x)$ will take the form $K_i(x,\lambda)F_i(x)$ where $F_i \in L^p$ and certain bounds will be known for the linear operator mapping $F \in L^p$ to $\int_{\mathbb{R}} K_i(x,\lambda)F(x)\,dx$. Thus $M_n(f_1, \dots, f_n)$ will be a multilinear operator mapping $(F_1, \dots, F_n) \in (L^p)^n$, or in $L^2 \otimes (L^p)^{n-1}$, to a function of another variable λ. For the present, however, we discuss basic inequalities concerning scalar-valued multiple integrals (2.1), (2.14) from which Lebesgue space operator norm inequalities will later be deduced.

Proposition 2.5. *There exists $C < \infty$ such that for any $n \geq 2$, any $\delta \geq 0$, any $f_1, \dots, f_n \in L^1(\mathbb{R})$, and any martingale structure $\{E_j^m\}$ satisfying the auxiliary hypothesis*

$$\lim_{m\to\infty} \max_{2\leq i\leq n} \max_j \int_{E_j^m} |f_i| = 0 \tag{2.15}$$

the multilinear expressions $M_n(f_1, \dots, f_n)$ satisfy

$$|M_n(f_1, \dots, f_n)| \leq \frac{C^{n+1}}{\sqrt{n!}} g_{-\delta}(f_1) \cdot g_\delta(\{f_k : k \geq 2\})^{n-1}. \tag{2.16}$$

Moreover for any $\delta' > \delta \geq 0$, there exists $C < \infty$ such that for all $\{f_i\}$ and all $n \geq 2$,

$$M_n^*(f_1, \dots, f_n) \leq \frac{C^{n+1}}{\sqrt{n!}} g_{-\delta}(f_1) \cdot g_{\delta'}(\{f_k : k \geq 2\})^{n-1}, \tag{2.17}$$

provided that (2.15) *is satisfied.*

Our analysis of the wave group requires (in Lemma 6.2) estimates for multilinear operators acting on $L^2 \otimes (L^p)^{n-1}$ for $1 \leq p < 2$, whereas in earlier work our time-independent analysis had required estimates only on $(L^p)^n$; the endpoint case $p = 2$ is more delicate and certain auxiliary operators employed in the earlier analysis are certainly unbounded for $p = 2$. The above asymmetric majorizations will be used to overcome this problem; $g_{-\delta}(f_1)$ for $\delta > 0$ will ultimately lead to L^2–bounded expressions where the larger quantity $g_0(f_1)$ would not.

Proof. Let $\{b_n\}$ be the sequence of constants appearing in Lemma 2.3. For $n \geq 2$ define $\tilde{b}_n = R^n b_{n-2}$, where R is a sufficiently large positive constant, to be determined later in the proof. The proof will be by induction on n. First we will treat only the case where $n \geq 6$, assuming the result for all $n \leq 5$, and at the end will discuss the modification for small n.

Let $f_{i,j}^m = \chi_{E_j^m} \cdot f_i$.

Lemma 2.6. *If R is chosen to be sufficiently large then for all $n \geq 2$, for any $\delta \geq 0$, and for all L^1 functions $f_1, \dots, f_n$,*

$$|M_n(f_1, f_2, \dots, f_n)| \leq \tilde{b}_n g_{-\delta}(f_1) \cdot g_\delta(\{f_i : i \geq 2\})^{n-1} \tag{2.18}$$

for any martingale structure $\{E_j^m\}$ such that $\lim_{m\to\infty} \max_{i\geq 2} \max_j \int_{E_j^m} |f_i| = 0$.

Proof. Write $\tilde{f} = f_1$ and $\tilde{f}_j^m = \chi_{E_j^m} \cdot \tilde{f}$. As in the proof of Lemma 2.4,

(2.19)

$$\begin{aligned}|M_n(\tilde{f}, f_2, \dots, f_n)| \leq & |M_n(\tilde{f}_1^1, f_{2,1}^1, \dots, f_{n,1}^1)| + |M_{n-1}(\tilde{f}_1^1, f_{2,1}^1, \dots, f_{n-1,1}^1)| \cdot |\textstyle\int f_{n,2}^1| \\ & + \sum_{i=2}^{n-2} |M_{n-i}(\tilde{f}_1^1, f_{2,1}^1, \dots, f_{i,1}^1)| \cdot |M_i(f_{i+1,2}^1, \dots, f_{n,2}^1)| \\ & + |\textstyle\int \tilde{f}_1^1| \cdot M_{n-1}(f_{2,2}^1, \dots, f_{n,2}^1)| + |M_n(\tilde{f}_2^1, f_{2,2}^1, \dots, f_{n,2}^1)|\end{aligned}$$

since $M_1(f^1_{i,j}) = \int f^1_{i,j}$ and likewise for $\tilde f$. This may be rewritten as

(2.20)
$$\begin{aligned}|M_n(\tilde f, f_2, \dots, f_n)| \le & |M_n(2^{-\delta}\tilde f^1_1, 2^\delta f^1_{2,1}, \dots, 2^\delta f^1_{n,1})| \\ & + |M_{n-1}(2^{-\delta}\tilde f^1_1, 2^\delta f^1_{2,1}, \dots, 2^\delta f^1_{n-1,1})| \cdot |\textstyle\int 2^\delta f^1_{n,2}| \\ & + \sum_{i=2}^{n-2} |M_{n-i}(2^{-\delta}\tilde f^1_1, 2^\delta f^1_{2,1}, \dots, 2^\delta f^1_{i,1})| \cdot |M_i(2^\delta f^1_{i+1,2}, \dots, 2^\delta f^1_{n,2})| \\ & + |\textstyle\int 2^{-\delta}\tilde f^1_1| \cdot |M_{n-1}(2^\delta f^1_{2,2}, \dots, 2^\delta f^1_{n,2})| \\ & + |M_n(2^{-\delta}\tilde f^1_2, 2^\delta f^1_{2,2}, \dots, 2^\delta f^1_{n,2})|.\end{aligned}$$

Since $n \ge 2$ and $\delta \ge 0$, at least one factor of 2^δ offsets the factor of $2^{-\delta}$.

The first and last terms in the preceding bound involve M_n itself; we will use as part of our induction hypothesis the desired inequalities for $M_n(\tilde f^1_1, f^1_{2,1}, \cdots, f^1_{n,1})$ and for $M_n(\tilde f^1_2, f^1_{2,2}, \cdots, f^1_{n,2})$. This may be justified by exploiting the auxiliary hypothesis (2.15), by the same argument sketched in the preceding subsection; the details will not be repeated here.

The collection of all those sets E^m_j with $m \ge 1$ and $j \le 2^{m-1}$ forms a martingale structure on E^1_1; however, when it is considered as such, the index m should be replaced by $m-1$. Thus the induction hypothesis, for the first term on the right-hand side of the preceding bound, may be stated as

$$\begin{aligned}(2.21)\quad & |M_n(2^{-\delta}\tilde f^1_1, 2^\delta f^1_{2,1}, \cdots, 2^\delta f^1_{n,1})| \\ & \le \tilde b_n \sum_{m=2}^\infty 2^{-m\delta}\Big(\sum_{j=1}^{2^{m-1}} |\textstyle\int \tilde f^m_j|^2\Big)^{1/2} \cdot \Big[\sum_{m=2}^\infty 2^{m\delta}\Big(\max_{i\ge 2}\sum_{j=1}^{2^{m-1}} |\textstyle\int f^m_{i,j}|^2\Big)^{1/2}\Big]^{n-1}.\end{aligned}$$

There is a corresponding bound for $|M_n(2^{-\delta}\tilde f^1_2, 2^\delta f^1_{2,2}, \cdots, 2^\delta f^1_{n,2})|$, with the inner sum running instead over $2^{m-1} < j \le 2^m$.

To formulate the induction hypothesis for the general term of the preceding expression, introduce

(2.22)
$$\tilde g^1_t = \sum_{m=2}^\infty 2^{-\delta m}\Big(\sum_{j:E^m_j \subset E^1_t} |\textstyle\int \tilde f^m_j|^2\Big)^{1/2} \quad\text{and}\quad g^1_t = \sum_{m=2}^\infty 2^{\delta m}\Big(\max_{i\ge 2}\sum_{j:E^m_j \subset E^1_t} |\textstyle\int f^m_{i,j}|^2\Big)^{1/2}$$

for $t = 1, 2$ and

$$(2.23)\qquad \tilde g^1 = ((\tilde g^1_1)^2 + (\tilde g^1_2)^2)^{1/2}, \qquad g^1 = ((g^1_1)^2 + (g^1_2)^2)^{1/2}.$$

Note that

$$(g^1)^2 + 2^{2\delta}|\int f^1_{i,1}|^2 + 2^{2\delta}|\int f^1_{i,2}|^2 \le g_\delta(\{f_l : l \ge 2\})^2,$$

for all $i \ge 2$, with a corresponding relation between $\tilde g^1$, $g_{-\delta}(\tilde f)$.

From the induction hypothesis and (2.12), (2.20) we obtain

$$\tag{2.24} \begin{aligned} |M_n(\tilde f, f_2, \ldots, f_n)| \leq & \tilde b_n \tilde g_1^1 (g_1^1)^{n-1} + \tilde b_{n-1} \tilde g_1^1 (g_1^1)^{n-2} |\textstyle\int 2^\delta f_{n,2}^1| \\ & + \sum_{i=2}^{n-2} \tilde b_{n-i} b_i \tilde g_1^1 (g_1^1)^{n-1-i} (g_2^1)^i + |\textstyle\int 2^{-\delta} \tilde f_1^1| \cdot b_{n-1} (g_2^1)^{n-1} + \tilde b_n \tilde g_2^1 (g_2^1)^{n-1}. \end{aligned}$$

Consider now the sum of the first term on the right, together with all terms of the summation for which the index i is either in $[2, n-4]$, or equals $n-2$:

$$\begin{aligned} & \tilde b_n \tilde g_1^1 (g_1^1)^{n-1} + \sum_{i=2}^{n-4} \tilde b_{n-i} b_i \tilde g_1^1 (g_1^1)^{n-1-i} (g_2^1)^i + \tilde b_2 b_{n-2} \tilde g_1^1 g_1^1 (g_2^1)^{n-2} \\ & \qquad = R^n \tilde g_1^1 g_1^1 \Big(b_{n-2} (g_1^1)^{n-2} + \sum_{i=2}^{n-4} R^{-i} b_{n-i-2} b_i (g_1^1)^{n-2-i} (g_2^1)^i + R^{-n+2} b_0 b_{n-2} (g_2^1)^{n-2} \Big) \\ & \qquad \leq R^n \tilde g_1^1 g_1^1 \Big(b_{n-2} (g_1^1)^{n-2} + \sum_{i=2}^{n-4} b_{n-i-2} b_i (g_1^1)^{n-2-i} (g_2^1)^i + b_0 b_{n-2} (g_2^1)^{n-2} \Big) \\ & \qquad \leq R^n \tilde g_1^1 g_1^1 b_{n-2} \big((g_1^1)^{2(n-2)} + (g_2^1)^{2(n-2)} \big)^{1/2} \\ & \qquad = \tilde b_n \tilde g_1^1 g_1^1 (g^1)^{n-2}. \end{aligned}$$

To pass from the third line to the fourth we have invoked (2.11).

We now have

$$\tag{2.25} \begin{aligned} |M_n(\tilde f, f_2, \ldots, f_n)| \leq & \tilde b_n \tilde g_1^1 g_1^1 (g^1)^{n-2} + \tilde b_3 b_{n-3} \tilde g_1^1 (g_1^1)^2 (g_2^1)^{n-3} + \tilde b_n \tilde g_2^1 (g_2^1)^{n-1} \\ & + \tilde b_{n-1} \tilde g_1^1 (g_1^1)^{n-2} |\textstyle\int 2^\delta f_{n,2}^1| + |\textstyle\int 2^{-\delta} \tilde f_1^1| b_{n-1} (g_2^1)^{n-1}. \end{aligned}$$

Set $\beta = \tilde b_3 b_{n-3} / \tilde b_n = R^{3-n} b_1 b_{n-3} / b_{n-2}$ and note that $\beta \leq C R^{3-n} n^{1/2} \leq 1$, if R is chosen to be sufficiently large, since we are assuming $n \geq 6$. Applying Cauchy-Schwarz to the sum of the first three terms on the right-hand side of the preceding inequality, we obtain

$$\tag{2.26} \begin{aligned} & \tilde b_n \tilde g_1^1 g_1^1 (g^1)^{n-2} + \tilde b_3 b_{n-3} \tilde g_1^1 (g_1^1)^2 (g_2^1)^{n-3} + \tilde b_n \tilde g_2^1 (g_2^1)^{n-1} \\ & \leq \tilde b_n \tilde g^1 \cdot \Big((g_1^1)^2 (g^1)^{2n-4} + 2\beta (g_1^1)^3 (g_2^1)^{n-3} (g^1)^{n-2} + \beta^2 (g_1^1)^4 (g_2^1)^{2n-6} + (g_2^1)^{2n-2} \Big)^{1/2}. \end{aligned}$$

We claim that

$$\tag{2.27} (g_1^1)^2 (g^1)^{2n-4} + 2\beta (g_1^1)^3 (g_2^1)^{n-3} (g^1)^{n-2} + \beta^2 (g_1^1)^4 (g_2^1)^{2n-6} + (g_2^1)^{2n-2} \leq (g^1)^{2n-2},$$

provided that $n \geq 6$ and β is sufficiently small. Writing $x = g_1^1, y = g_2^1$, the preceding inequality becomes

$$x^2 (x^2 + y^2)^{n-2} + 2\beta x^3 y^{n-3} (x^2 + y^2)^{(n-2)/2} + \beta^2 x^4 y^{2n-6} + y^{2n-2} \leq (x^2 + y^2)^{n-1},$$

which is equivalent to

$$2\beta x^3 y^{n-3} (x^2 + y^2)^{(n-2)/2} + \beta^2 x^4 y^{2n-6} + y^{2n-2} \leq y^2 (x^2 + y^2)^{n-2}.$$

By homogeneity we may assume that $x^2+y^2=1$, so we wish to have

$$2\beta x^3 y^{n-3} + \beta^2 x^4 y^{2n-6} + y^{2n-2} \le y^2.$$

This holds whenever $x^2+y^2=1$ and $x,y\ge 0$, provided β is sufficiently small, provided that each of the exponents on y on the left-hand side is strictly larger than 2; this holds provided $n \ge 6$.

We have thus established that for $n \ge 6$,

(2.28)
$$\begin{aligned}|M_n(\tilde f, f_2,\dots,f_n)| &\le \tilde b_n \tilde g^1 (g^1)^{n-1} + \tilde b_{n-1}\tilde g^1_1 (g^1_1)^{n-2} |\textstyle\int 2^\delta f^1_{n,2}| + |\textstyle\int 2^{-\delta}\tilde f^1_1| b_{n-1}(g^1_2)^{n-1}\\ &= \tilde b_n \tilde x x^{n-1} + \tilde b_{n-1}\tilde x x^{n-2} y + b_{n-1}\tilde y x^{n-1}\end{aligned}$$

where $x = g^1$, $y = |\int 2^\delta f^1_{n,2}|$ and $\tilde x = \tilde g^1$, $\tilde y = |\int 2^{-\delta}\tilde f^1_1|$. We seek to bound this last quantity by[3]

$$\tilde b_n(\tilde x + \tilde y)(x+y)^{n-1} \ge \tilde b_n \tilde x x^{n-1} + \tilde b_n \tilde x (n-1) x^{n-2} y + \tilde b_n \tilde y x^{n-1}. \tag{2.29}$$

Thus by a term-by-term comparison, it suffices to know that $\tilde b_{n-1} \le (n-1)\tilde b_n$ and $b_{n-1} \le \tilde b_n$. The former is equivalent to $R^{n-1} b_{n-3} \le (n-1) R^n b_{n-2}$, whence to $b_{n-3} \le (n-1) R b_{n-2}$. For $R=1$, this is true for all sufficiently large n, since $k^{1/2} b_{k+1}/b_k$ is bounded. By choosing R sufficiently large, we may guarantee that it holds for all n. The latter is equivalent to $b_{n-1} \le R^n b_{n-2}$. Again, since $b_{n-1} \le C n^{-1/2} b_{n-2}$, this holds for all sufficiently large n with $R=1$, hence for all n provided that R is chosen to be sufficiently large. Thus a term-by-term comparison completes the proof of the bound for M_n, for $n \ge 6$.

It remains to treat $n \le 5$. We need only control the first line of the right-hand side of (2.25) by $\tilde b_n \tilde g^1 (g^1)^{n-1}$; then the remainder of the proof applies for any n.

Consider $n=5$, the most complicated case remaining; suppose by induction that the inequality has been proved for smaller values of n. From (2.20) and the induction hypothesis we obtain an upper bound involving some constants C_k of

$$C_5 b_5 \tilde g^1_1 (g^1_1)^4 + C_3 b_2 \tilde g^1_1 (g^1_1)^2 (g^1_2)^2 + C_2 b_3 \tilde g^1_1 g^1_1 (g^1_2)^3 + b_5 C_5 \tilde g^1_2 (g^1_2)^4 \tag{2.30}$$

plus the two special terms involving $\int_{E^1_1} \tilde f$, $\int_{E^1_2} f_n$. By Cauchy-Schwarz we may bound the square of (2.30) by

$$b_5 (C_5 \tilde g^1)^2 \left[(g^1_1)^4 + \beta (g^1_1)^2 (g^1_2)^2 + \beta g^1_1 (g^1_2)^3\right]^2 + (g^1_2)^4,$$

where the constant β may be made as small as desired by choosing C_5 to be sufficiently large relative to C_2, C_3, that is, by choosing R to be sufficiently large. Thus the desired inequality reduces to

$$(x^4 + \beta x^2 y^2 + \beta x y^3)^2 + y^8 \le (x^2+y^2)^4,$$

which holds for small β, as one sees by expanding both sides and comparing terms.

The case $n=4$ is similar but simpler since one fewer term appears in the analogue of (2.30); the cases $n=2,3$ are simpler yet. The details are left to the reader. □

[3] It is in order to be able to assert this last inequality that we incorporate an ℓ^1 norm with respect to m, rather than an ℓ^2 norm, in the definitions of the functionals g_δ.

The bounds of Proposition 2.5 for M_n^* are deduced by applying those just proved to $M_n(\tilde{f}, f_2 \cdot \chi_{(-\infty,y]}, \dots, f_n \cdot \chi_{(-\infty,y]})$, then[4] taking the supremum over y by applying Lemma 2.2. □

3. Nontangential convergence and the maximal function

Throughout the paper we write $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$. Consider the cones

$$\Gamma_\alpha(w) = \{z \in \mathbb{C}^+ : |\,\mathrm{Re}\,(z) - w| < \alpha\,\mathrm{Im}\,(z)\},$$
$$\Gamma_{\alpha,\delta}(w) = \{z \in \mathbb{C}^+ \cap \Gamma_\alpha(w) : \mathrm{Im}\,(z) < \delta\}.$$

A function $f(z)$ defined on $\mathbb{C}^+$ is said to converge to a limit a as $z \to w$ nontangentially if for every $\alpha < \infty$, $f(z) \to a$ as $z \to w$ within the cone $\Gamma_\alpha(w)$. The nontangential maximal function of f is defined by

$$Nf(w) = N_{\alpha,\delta}f(w) = \sup_{z\in\Gamma_{\alpha,\delta}(w)} |f(z)|. \tag{3.1}$$

We will need the following local variant of a standard property of functions holomorphic in the whole half plane $\mathbb{C}^+$.

Lemma 3.1. *Let $\delta > 0$. Let Λ be an open subinterval of $\mathbb{R}$, and let $\Lambda' \Subset \Lambda$ be a relatively compact subinterval. Let $0 < q \le \infty$. Let B be a Banach space. Suppose that f is a holomorphic B-valued function in $\{\lambda + i\varepsilon : 0 < \varepsilon < \delta,\ \lambda \in \Lambda\}$. Then for any $\alpha < \infty$, there exist $C < \infty$ and $\delta' > 0$ depending on $\alpha, \Lambda, \Lambda'$, but not on f, such that*

$$\|N_{\alpha,\delta'}f(\cdot)\|_{L^q(\Lambda')} \le C \sup_{0<\varepsilon<\delta} \|f(\cdot + i\varepsilon)\|_{L^q(\Lambda)}. \tag{3.2}$$

By a B-valued holomorphic function we mean a continuous function f from an open subset of $\mathbb{C}$ to B, such that $z \mapsto \ell(f)(z)$ is holomorphic for every bounded linear functional ℓ on B.

Proof. Suppose first that f is continuous on the closed rectangle $\Lambda + i[0,\delta]$, and that $1 < q < \infty$. Let $0 < \delta'' < \delta$ be a small number to be chosen. Choose a third interval Λ'', so that $\Lambda' \subset \Lambda'' \subset \Lambda$ and the distance from each interval to the boundary of the next larger interval is strictly positive. Consider the rectangle $R = \Lambda'' + i[0,\delta''] = \{w : \mathrm{Re}\,(w) \in R \text{ and } \mathrm{Im}\,(w) \in [0,\delta'']\}$.

For any $z \in R$ with $\mathrm{Re}\,(z) \in \Lambda'$ and $0 < \mathrm{Im}\,(z) < \delta''/2$, we may write $f(z) = \int_{\partial R} f(\zeta)d\omega_z(\zeta)$ where ω_z is harmonic measure on ∂R. Fix a conformal map φ from R to the unit disk; this map is smooth everywhere except at the corners of R, where it behaves locally like $z \mapsto z^2$ in appropriate local coordinates. Harmonic measure

[4] Because the region of integration is $t_1 \le \cdots \le t_n \le y$, and $n \ge 2$, $M_n(\tilde{f}, f_2 \cdot \chi_{(-\infty,y]}, \dots, f_n \cdot \chi_{(-\infty,y]}) = M_n(\tilde{f} \cdot \chi_{(-\infty,y]}, f_2 \cdot \chi_{(-\infty,y]}, \dots, f_n \cdot \chi_{(-\infty,y]})$; there is no need to insert a factor of $\chi_{(-\infty,y]}$ with $\tilde{f}$. This is essential since applying Lemma 2.2 to handle the resulting supremum over y would destroy the decaying factors $2^{-\delta m}$ arising in $g_{-\delta}(\tilde{f})$.

for R may be computed by pulling back harmonic measure from the unit disk under φ. From this we readily deduce that for any $z \in R$ with $\mathrm{Re}\,(z) \in \Lambda$,

$$f(z) = \int_{\Lambda''} k(z,\zeta) f(\zeta)\, d\sigma(\zeta) + O\Big(\int_{\partial R \setminus \Lambda''} |f(\zeta)|\, d\sigma(\zeta) \Big) \tag{3.3}$$

where σ denotes arc length measure on ∂R, and where k satisfies upper bounds of Poisson kernel type: $|k(x+it,y)| \le Ct[|x-y|^2 + t^2]^{-1}$.

Denote by F the restriction of f to Λ'', and by M the Hardy-Littlewood maximal function. We combine these bounds with the usual majorization of the nontangential maximal function of a Poisson integral by M to conclude that for $z \in \Gamma_{\alpha,\delta'}(y)$ with $0 < \mathrm{Im}\,(z) < \delta''$ and $y \in \Lambda'$, the first term on the right-hand side of (3.3) is bounded by $C'MF(y)$.

The second term on the right is not suitably bounded, in general. Therefore we consider all translates $R_s = \{z+s : z \in R\}$ where $s \in \mathbb{R}$ ranges over a small interval I centered at 0. For each R_s we have a variant of (3.3), obtained by conjugating with the translation $z \mapsto z+s$; the integral in the first term now extends over $\Lambda'' + s$. By averaging all these variants over $s \in I$ with respect to normalized Lebesgue measure, we obtain

$$f(z) = \int_{\tilde\Lambda} \tilde k(z,w) d\sigma(w) + O\Big(\int_Q |f(w)|\, dw\Big) + O\Big(\int_{\tilde\Lambda} |f(w+i\delta'')|\, d\sigma(w)\Big)$$

where $\tilde\Lambda = \cup_{s\in I} \Lambda'' + s$ is slightly larger than Λ'', $\tilde k$ satisfies the same bounds as k, dw denotes Lebesgue measure in $\mathbb{C}$, and Q is a certain compact subset of the closed rectangle $\Lambda + i[0,\delta]$. The nontangential maximal function of the first term is majorized by CMF, just as before. The second is majorized, uniformly in z, by $\sup_\varepsilon \|f(\cdot + i\varepsilon)\|_{L^q(\Lambda)}$. The third is already under control, by hypothesis, since $q \ge 1$. Because M is bounded on L^q, we obtain the desired conclusion, under the supplementary hypothesis that f extends continuously to $\mathrm{Im}\,(z) = 0$.

For general f not necessarily continuous up to $\mathrm{Im}\,(z) = 0$, consider $f_\eta(z) = f(z + i\eta)$, for small $\eta \in \mathbb{R}^+$. Apply the result just proved to f_η, and pass to the limit $\eta \to 0^+$ using Fatou's lemma.

In order to extend this argument to $0 < q \le 1$, it suffices to recall that if f is a $\mathbb{C}$-valued holomorphic function, then $|f|^s$ is subharmonic for any $s > 0$. Fixing any $0 < s < q$ along with any bounded linear functional ℓ on B, and setting $F(z) = \ell(f(z))$, we may therefore write instead of (3.3)

$$|F(z)|^s \le \int_{\Lambda''} k(z,\zeta) |F(\zeta)|^s\, d\sigma(\zeta) + O\Big(\int_{\partial R\setminus\Lambda''} |F(\zeta)|^s\, d\sigma(\zeta)\Big).$$

The proof then proceeds as above, using the fact that $f \mapsto [M(|f|^s)]^{1/s}$ is bounded on L^q for all $q > s$. □

4. Generalized eigenfunctions for complex spectral parameters

Consider the generalized eigenfunction equation

$$-u'' + V(x)u = zu, \tag{4.1}$$

where z is permitted to be complex. In earlier work we have analyzed the solutions to this equation for z real and positive, and have shown that for almost every such z there exists a solution with certain asymptotic behavior as $x \to +\infty$. Our present purpose is to analyze solutions for complex z and to show their convergence to the solutions for almost every real z, as z approaches the positive real axis, with suitable uniform bounds.

We restrict attention to parameters $z \in \mathbb{C}^+ \cup \mathbb{R}^+$, and choose a branch of $\sqrt{z}$ which has nonnegative imaginary part for such z. Define the phases

$$\xi(x,z) = \sqrt{z}x - (2\sqrt{z})^{-1} \int_0^x V. \tag{4.2}$$

Theorem 4.1. *Let $1 \le p < 2$ and assume that $V \in L^1 + L^p$. For each $z \in \mathbb{C}^+$ there exists a (unique) solution $u(x,z)$ of the generalized eigenfunction equation* (4.1) *satisfying*

$$\begin{cases} u(x,z) - e^{i\xi(x,z)} \to 0 \\ \partial u(x,z)/\partial x - i\sqrt{z}e^{i\xi(x,z)} \to 0 \end{cases} \qquad \text{as } x \to +\infty. \tag{4.3}$$

$u(x,z)$ is continuous as a function on $\mathbb{R} \times \mathbb{C}^+$, and is holomorphic with respect to z for each fixed x.

Likewise, there exists such a (unique) solution for almost every $z \in \mathbb{R}^+$. For almost every $E \in \mathbb{R}$, $u(x,z)$ converges to $u(x,E)$ uniformly for all x in any bounded interval, as $z \to E$ nontangentially. Moreover, for almost every $E \in \mathbb{R}$,

$$\sup_{z\in\Gamma} \sup_{x\ge 0} |u(x,z)| < \infty$$

for any nontangential cone Γ with vertex at E.

Here "almost every" means with respect to Lebesgue measure. The existence of such solutions for almost every $z \in \mathbb{R}^+$ is proved in [9]. For $z \in \mathbb{C}^+$ it is well known under weaker hypotheses on V. The new points here are the convergence and uniformity of bounds as $z \to \mathbb{R}^+$.

It suffices to prove the theorem for $x \in [0,\infty)$, since the conclusion for $x \in [\rho,\infty)$, for any $\rho > -\infty$, then follows via the eigenfunction equation (4.1).

By rewriting (4.1) as a first-order system, performing a couple of algebraic transformations, reducing to an integral equation, and solving it by iteration, one arrives [9] at a formal series representation for solutions of (4.1):

$$\begin{pmatrix} u(x,z) \\ u'(x,z) \end{pmatrix} = \begin{pmatrix} e^{i\xi(x,z)} & e^{-i\xi(x,z)} \\ i\sqrt{z}e^{i\xi(x,z)} & -i\sqrt{z}e^{-i\xi(x,z)} \end{pmatrix} \cdot \begin{pmatrix} \sum_{n=0}^\infty T_{2n}(V,\dots,V)(x,z) \\ -\sum_{n=0}^\infty T_{2n+1}(V,\dots,V)(x,z) \end{pmatrix} \tag{4.4}$$

where formally

$$T_n(V_1,\dots,V_n)(x,z) = (2\sqrt{z})^{-n} \iint\limits_{x\le t_1\le\cdots\le t_n} \prod_{j=1}^n e^{2i(-1)^{n-j}\xi(t_j,z)} V_j(t_j)\,dt_j, \tag{4.5}$$

with the convention $T_0(\cdot) \equiv 1$. To prove Theorem 4.1 we will show that each multilinear expression T_n is well-defined for all $z \in \mathbb{C}^+$ (see (4.22) and (4.23)), because these multiple integrals are absolutely convergent provided that each $V_j \in L^1 + L^2$, that $T_n(\cdot)(x,z) \to 0$ as $x \to +\infty$ for all $n \geq 1$, that they have the natural limits as $z \to \mathbb{R}^+$ nontangentially, and that these expressions satisfy bounds sufficiently strong to enable us to sum the infinite series to obtain the desired conclusions.

Substitute $\zeta = \sqrt{z}$ and write $\zeta = \lambda + i\varepsilon$, noting that $\varepsilon > 0$. Also write

$$\begin{aligned} \phi(x,\zeta) &= \xi(x,z), \\ S_n(V_1,\dots)(x,\zeta) &= T_n(V_1,\dots)(x,z). \end{aligned} \tag{4.6}$$

Let $\{E_j^m\}$ be a martingale structure on $\mathbb{R}^+$. Denote by $t^{\pm}_{m,j}$ respectively the right $(+)$ and left $(-)$ endpoints of the interval E_j^m.

The real part of the exponent $2i\sum_{j=1}^n (-1)^{n-j}\xi(t_j,z)$ is for large t_j to leading order equal to

$$-2\,\mathrm{Im}\,(\sqrt{z}[(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots]) = -2\varepsilon\cdot[(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots],$$

which is nonnegative (for $x \geq 0$) for all $z \in \mathbb{C}^+$ since $t_1 \leq t_2 \cdots$; the exponential factor decays rapidly as $[(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots] \to \infty$. This accounts for the difference between $z \in \mathbb{C}^+$ and $z \in \mathbb{R}^+$.

Define

$$\begin{aligned} G_m(V)(\zeta) &= \Big(\sum_{j=1}^{2^m} |s_j^{m,-}(V,\zeta)|^2 + |s_j^{m,+}(V,\zeta)|^2\Big)^{1/2} \\ G(V) &= \sum_{m=1}^{\infty} m G_m(V) \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} s_j^{m,-}(V,\zeta) &= \int_{E_j^m} e^{2i[\phi(t,\zeta) - \phi(t^-_{m,j},\zeta)]} V(t)\,dt \\ s_j^{m,+}(V,\zeta) &= \int_{E_j^m} e^{2i[\phi(t^+_{m,j},\zeta) - \phi(t,\zeta)]} V(t)\,dt. \end{aligned} \tag{4.8}$$

When $j = 2^m$, and only then, the right endpoint of E_j^m is infinite. To simplify notation we make the convention that for $j = 2^m$, the second term $|s_j^{m,+}(V,\zeta)|^2$ is always to be omitted, in the definition of G and anywhere else that the quantities $\phi(t^+_{m,j},\zeta)$ arise.

The following definitions will allow us to regard G as a linear operator, and hence to exploit properties of holomorphic functions.

Definition 4.1. $\mathcal{B}$ denotes the Banach space consisting of all sequences $\{\mathbb{C}^2 \ni s_j^m : m \geq 0,\ 1 \leq j \leq 2^m\}$, with the norm $\|s\|_{\mathcal{B}} = \sum_m m\big(\sum_{j=1}^{2^m} |s_j^m|^2\big)^{1/2}$.

$\mathfrak{G} : L^p \mapsto \mathcal{B}$ denotes the operator

$$\mathfrak{G}(V)(\zeta) = \{(s_j^{m,+}(V,\zeta), s_j^{m,-}(V,\zeta)) : 1 \leq m < \infty, 1 \leq j \leq 2^m\}. \tag{4.9}$$

Thus

$$G(V)(\zeta) = \|\mathfrak{G}(V)(\zeta)\|_{\mathcal{B}}. \tag{4.10}$$

Likewise we may write $G_m(V) = \|\mathfrak{G}_m(V)\|_{\mathcal{B}_m}$, where

$$\mathfrak{G}_m(\zeta) = \{(s_j^{m,+}(V,\zeta), s_j^{m,-}(V,\zeta)) : 1 \le j \le 2^m\}$$

and $\mathcal{B}_m$ is defined by omitting the sum over m in the definition of $\mathcal{B}$.

The real parts of $i\phi(t,\zeta) - i\phi(t^-_{m,j},\zeta)$ and $i\phi(t^+_{m,j},\zeta) - i\phi(t,\zeta)$ are bounded above uniformly for $0 < \mathrm{Im}\,(\zeta) \le 1$, and are $\le -c\,\mathrm{Im}\,(\zeta)(t - t^-_{m,j})$ and $\le -c\,\mathrm{Im}\,(\zeta)(t^+_{m,j} - t)$, respectively; see (4.20). Therefore for $V \in L^1 + L^\infty$ and $\zeta \in \mathbb{C}^+$, each of the integrals (4.8) converges absolutely, and each defines a holomorphic scalar-valued function of $\zeta \in \mathbb{C}^+$. Thus $\mathfrak{G}(V)$ may be regarded as a $\mathcal{B}$–valued holomorphic function[5] in any open set where it can be established that the series defining $\|\mathfrak{G}(V)\|_{\mathcal{B}}$ converges uniformly.

We will also use the following variant. Given a collection of functions $(V_1,\dots,V_n)$, we define

$$G(\{V_i\})(\zeta) = \sum_{m=1}^{\infty} m \cdot \Big(\sum_{j=1}^{2^m}\sum_i^{*} |s_j^{m,+}(V_i,\zeta)|^2 + |s_j^{m,-}(V_i,\zeta)|^2\Big)^{1/2}, \tag{4.11}$$

where $\sum^*$ indicates that the sum is taken over a maximal set of indices i for which the functions V_i are all distinct.

Lemma 4.2. *For all $V \in L^1$, all n, and all $\zeta \in \mathbb{C}^+ \cup \mathbb{R}^+$,*

$$\sup_{x\in\mathbb{R}^+} |S_n(V,V,\dots,V)(x,\zeta)| \le \frac{C^{n+1}}{\sqrt{n!}} G(V)(\zeta)^n \tag{4.12}$$

provided that $\max_j \int_{E_j^m} |V| \to 0$ as $m \to \infty$. More generally, for all n and all $\{V_1,\dots,V_n\} \subset L^1$,

$$\sup_{x\in\mathbb{R}^+} |S_n(V_1,V_2,\dots,V_n)(x,\zeta)| \le \frac{C_k^{n+1}}{\sqrt{n!}} G(\{V_i\})(\zeta)^n, \tag{4.13}$$

provided that $\{V_i\}_{i=1}^n$ has cardinality $\le k$ and that $\max_{i,j} \int_{E_j^m} |V_i| \to 0$ as $m \to \infty$.

Write $p' = p/(p-1)$.

Lemma 4.3. *Let V be a fixed real-valued function belong to $L^1 + L^q$ for some $q < \infty$, and let $\phi(x,\zeta) = \zeta x - (2\zeta)^{-1}\int_0^x V$. For any compact interval $\Lambda \Subset (0,\infty)$, there exists $C < \infty$ such that for any $1 \le p \le 2$, for all $t' \in \mathbb{R}$ and $f \in L^p(\mathbb{R})$, for every $\varepsilon \ge 0$,*

$$\| \int_{t\ge t'} e^{2i[\phi(t,\lambda+i\varepsilon)-\phi(t',\lambda+i\varepsilon)]} f(t)\,dt\|_{L^{p'}(\Lambda,d\lambda)} \le C\|f\|_{L^p} \tag{4.14}$$

$$\| \int_{t\le t'} e^{2i[\phi(t',\lambda+i\varepsilon)-\phi(t,\lambda+i\varepsilon)]} f(t)\,dt\|_{L^{p'}(\Lambda,d\lambda)} \le C\|f\|_{L^p}. \tag{4.15}$$

For $\varepsilon = 0$ these integrals need not converge absolutely, hence require interpretation. They are initially well-defined for compactly supported f, and the lemma asserts an *a priori* bound for such functions. Then they are defined for general $f \in L^p$ by

[5]By this we mean simply that it is a continuous mapping into $\mathcal{B}$ with respect to the norm topology, and that each $s_j^{m,\pm}$ is a holomorphic scalar-valued function.

approximating in L^p norm by compactly supported functions, and passing to the limit in the $L^{p'}$ norm; the limit thus obtained is independent of the choice of approximating sequence.

Let an exponent $p < \infty$ be specified. Recall that a martingale structure is said to be adapted to f in L^p if $\int_{E_j^m} |f|^p = 2^{-m} \int |f|^p$ for all m, j. Recall from the preceding section the definitions of the cones $\Gamma_{\alpha,\delta}$ and associated nontangential maximal functions $N_{\alpha,\delta}$.

Corollary 4.4. *Let $\alpha < \infty$, let $1 \le p \le 2$, and let $\Lambda \Subset (0,\infty)$ be any compact subinterval. Then there exist $C < \infty, \delta > 0$ such that for any $f \in L^p(\mathbb{R})$ and for any martingale structure $\{E_j^m\}$ on $\mathbb{R}^+$*

$$\|N_{\alpha,\delta} G_m(f)(\lambda)\|_{L^{p'}(\Lambda,d\lambda)} \le C\|f\|_{L^p}. \tag{4.16}$$

Moreover for each $1 \le p < 2$ there exists $\rho > 0$ such that for any $f \in L^p$ and for any martingale structure adapted to f in L^p,

$$\|N_{\alpha,\delta} G_m(f)(\lambda)\|_{L^{p'}(\Lambda,d\lambda)} \le C 2^{-\rho m}\|f\|_{L^p}. \tag{4.17}$$

Consequently if $1 \le p < 2$ and if the martingale structure $\{E_j^m\}$ is adapted to f in L^p, then

$$\|N_{\alpha,\delta} G(f)(\lambda)\|_{L^{p'}(\Lambda,d\lambda)} \le C\|f\|_{L^p} \tag{4.18}$$

and for almost every $\lambda \in \Lambda$,

$$\|\mathfrak{G}(f)(\zeta) - \mathfrak{G}(f)(\lambda)\|_{\mathcal{B}} \to 0 \tag{4.19}$$

as $\zeta \to \lambda$ nontangentially.

We postpone the proofs of Lemmas 4.2, 4.3 and Corollary 4.4 until the end of the section.

For any $t \ge t'$ and any $\zeta = \lambda + i\varepsilon$ with $\varepsilon \ge 0$,

$$\operatorname{Re}\big(i\phi(t,\lambda+i\varepsilon) - i\phi(t',\lambda+i\varepsilon)\big) = -\varepsilon(t-t') - \varepsilon(\lambda^2+\varepsilon^2)^{-1}\textstyle\int_{t'}^t V, \tag{4.20}$$

and $|\int_{t'}^t V| \le |t-t'|^{1/p'}\|V\|_{L^p}$. Therefore

$$|e^{i[\phi(t,\lambda+i\varepsilon)-\phi(t',\lambda+i\varepsilon)]}| \le Ce^{-c\varepsilon|t-t'|} \tag{4.21}$$

where $C, c \in \mathbb{R}^+$ are constants which depend only on the L^p norm of V. Hence for all $n \ge 1$,

$$\begin{aligned}
|S_{2n}(V,V,\dots,V)(x,\lambda+i\varepsilon)| &\le C^n \iint\limits_{x\le t_1\le\cdots\le t_{2n}} e^{-c\varepsilon(t_{2n}-t_{2n-1}+t_{2n-2}-\cdots)} \prod_j |V(t_j)|\,dt_j \\
&\le \frac{C^n}{n!}\Big(\iint\limits_{x\le t'\le t} e^{-c\varepsilon(t-t')}|V(t')V(t)|\,dt'\,dt\Big)^n \\
&\le \frac{C^n}{n!}\varepsilon^{-2n(1-p^{-1})}\|V\|^{2n}_{L^p([x,\infty))}.
\end{aligned} \tag{4.22}$$

In the same way, for $x \ge 0$, one obtains for $n \ge 0$

$$|S_{2n+1}(V,V,\dots,V)(x,\lambda+i\varepsilon)| \le \frac{C^n}{n!}\varepsilon^{-(2n+1)(1-p^{-1})}\|V\|^{2n+1}_{L^p([x,\infty))}. \tag{4.23}$$

Let z belong to any compact subset of $\mathbb{C}^+$. Then $\varepsilon = \mathrm{Im}(\sqrt{z})$ has a strictly positive lower bound. Therefore the individual terms of the series (4.4) defining a formal solution of (4.1) do define uniformly bounded functions of (x,ζ) for $x \geq 0$ and ζ in any compact subset of $\mathbb{C}^+$, and moreover, the series are uniformly absolutely convergent. As in Lemma 4.2 of [9], it follows that the sums of these series define solutions of the generalized eigenfunction equation (4.1) for all such z. Because the L^p norm of V over $[x,\infty)$ tends to zero as $x \to +\infty$, only $T_0(x)$ contributes in that limit, so these solutions do have the desired WKB asymptotics $\exp(i\xi(x,z))$. Clearly each summand depends holomorphically on ζ, hence so do the sums.

Existence of a (unique) solution for almost every $z \in \mathbb{R}^+$ is proved in [9]. We come now to the main step, where we relate $z \in \mathbb{C}^+$ to $z \in \mathbb{R}^+$. For compactly supported $f \in L^1$, the quantities $s_j^{m,\pm}(f,\zeta)$ are clearly holomorphic functions of ζ where $\mathrm{Im}(\zeta) > 0$, and are continuous at $\varepsilon = 0$ for each $0 \neq \lambda \in \mathbb{R}$. The same holds for $S_n(f,f,\dots,f)(x,\zeta)$, for each $x \in \mathbb{R}$.

$\mathfrak{G}(f)$ is likewise a $\mathcal{B}$–valued holomorphic function of ζ, continuous on $\mathbb{C}^+ \cup \mathbb{R}^+$, for compactly supported $f \in L^1$. This follows by combining the holomorphy of the individual terms (4.8) with the rapid convergence bound (4.17).

Lemma 4.5. *Let $1 \leq p < 2$ and $V \in L^p(\mathbb{R})$. For almost every $E \in \mathbb{R}$, for every $n \geq 1$, $T_n(V,V,\dots,V)(x,z) \to T_n(V,V,\dots,V)(x,E)$ uniformly for all $x \geq 0$ as $\mathbb{C}^+ \ni z \to E$ nontangentially.*

Proof. It is equivalent to show that

$$\sup_{x\geq 0} \left| S_n(V,V,\dots,V)(x,\zeta) - S_n(V,V,\dots,V)(x,\lambda) \right| \to 0 \tag{4.24}$$

as $\zeta \to \lambda$ nontangentially, for almost all $\lambda \in \Lambda$, for any fixed compact interval $\Lambda \subset \mathbb{R}^+$. For $V = W \in L^1$ with compact support, we have already established convergence uniformly in x, λ, as $\mathbb{C}^+ \ni \zeta \to \lambda$ unrestrictedly (rather than merely nontangentially), since the phases $\phi(t,\zeta)$ converge uniformly to $\phi(t,\lambda)$ for t,ζ,λ in any compact set.

Let V be given, and remain fixed for the remainder of this proof. Set $\phi_V(t,\zeta) = \zeta t - (2\zeta)^{-1}\int_0^t V$. Whenever we write $S_n(f_1,f_2,\dots,f_n)$, it is defined in terms of the phases $\phi(t_i,\zeta) = \phi_V(t_i,\zeta)$, independent of $f_1,\dots,f_n$. Thus the S_n are here genuine multilinear operators.

Let $\varepsilon > 0$ be arbitrary, and fix a martingale structure $\{E_j^m\}$ adapted to V in L^p on $\mathbb{R}^+$. Decompose $V = W + (V - W)$ where $W(x) = V(x)\chi_{(0,R]}(x)$, with R chosen so that $\|V - W\|_{L^p} < \varepsilon$ and moreover so that $\|NG(V-W)\|_{L^{p'}(\Lambda)} < \varepsilon$. Such a choice is possible, since

$$\|NG_M(V\chi_{[R,\infty)})\|_{L^{p'}(\Lambda)} \leq C\min(2^{-M\delta}\|V\|_{L^p}, \|V\chi_R\|_{L^p}).$$

Then

$$\begin{aligned}
(4.25)\quad |S_n(V,V,\dots,V)(x,\zeta) &- S_n(V,V,\dots,V)(x,\lambda)| \\
&\leq |S_n(W,W,\dots,W)(x,\zeta) - S_n(W,W,\dots,W)(x,\lambda)| \\
&\quad + |S_n(V,V,\dots,V)(x,\zeta) - S_n(W,W,\dots,W)(x,\zeta)| \\
&\qquad + |S_n(V,V,\dots,V)(x,\lambda) - S_n(W,W,\dots,W)(x,\lambda)|.
\end{aligned}$$

The first term on the right tends to zero, in the sense desired. Majorize the second by

$$
(4.26)\quad |S_n(V,V,\dots,V)(x,\zeta)-S_n(W,W,\dots,W)(x,\zeta)| \\
\le \sum_{i=1}^{n} |S_n(V,\dots,V,V-W,W,\dots,W)(z,\zeta)|
$$

where in the i-th summand, the argument of S_n has $i-1$ copies of V and $n-i$ copies of W. Fix any aperture $\alpha \in \mathbb{R}^+$. By applying conclusion (2.7) of Proposition 2.1 together with Lemma 2.2 to the preceding inequality, we conclude that

$$
\begin{aligned}
(4.27)\quad &\sup_{x\ge 0}\sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)} |S_n(V,V,\dots,V)(x,\zeta)-S_n(W,W,\dots,W)(x,\zeta)| \\
&\le C_n\sum_{i=1}^{n}\sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)} G(V)^{i-1}(\zeta)G(W)^{n-i}(\zeta)G(V-W)(\zeta) \\
&\le C_n\sum_{i=1}^{n} NG(V)^{i-1}(\lambda)NG(W)^{n-i}(\lambda)NG(V-W)(\lambda),
\end{aligned}
$$

where N denotes the nontangential maximal function associated to cones with aperture α, provided that $\max_j \int_{E_j^m}|V|\to 0$ as $m\to\infty$. However, this auxiliary condition need not hold under our present hypotheses. Since this complication is not essential in nature, we defer its discussion to the end of the proof.

Let $q=p'$. By Chebyshev's inequality, for any $\beta>0$,

$$
\begin{aligned}
(4.28)\quad & \\
&|\{\lambda\in\Lambda : \sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)}\sup_{x\ge 0}|S_n(V,V,\dots,V)(x,\zeta)-S_n(W,W,\dots,W)(x,\zeta)|>\beta\}| \\
&\le C_n\beta^{-q/n}\sum_{i=1}^{n}\|NG(V)^{i-1}NG(W)^{n-i}NG(V-W)\|_{L^{q/n}(\Lambda)}^{q/n} \\
&\le C_n\beta^{-q/n}\sum_{i=1}^{n}(\|NG(V)\|_{L^q(\Lambda)}^{i-1}\|NG(W)\|_{L^q(\Lambda)}^{n-i}\|NG(V-W)\|_{L^q(\Lambda)})^{q/n} \\
&\le C_n\beta^{-q/n}\sum_{i=1}^{n}(\|V\|_{L^p}^{n-1}\|V-W\|_{L^p})^{q/n} \\
&\le C_n\beta^{-q/n}\|V\|_{L^p}^{q(n-1)/n}\varepsilon^{q/n}.
\end{aligned}
$$

The same reasoning gives

$$
(4.29)\quad \left|\{\lambda\in\Lambda : \sup_{x\ge 0}\left|S_n(V,V,\dots,V)(x,\lambda)-S_n(W,W,\dots,W)(x,\lambda)\right|>\beta\}\right| \\
\le C_n\beta^{-q/n}\|V\|_{L^p}^{q(n-1)/n}\varepsilon^{q/n}.
$$

Consequently

$$|\{\lambda \in \Lambda : \limsup_{\Gamma_\alpha(\lambda)\ni\zeta\to\lambda} \sup_{x\geq 0} \big|S_n(V,V,\dots,V)(x,\zeta) - S_n(V,V,\dots,V)(x,\lambda)\big| > \beta\} \\ \leq C_n \beta^{-q/n} \|V\|_{L^p}^{q(n-1)/n} \varepsilon^{q/n}, \tag{4.30}$$

for all $\beta, \varepsilon \in \mathbb{R}^+$. Letting $\varepsilon \to 0$, we conclude that the lim sup vanishes for almost every λ.

This completes the proof, except for the issue of justifying the second line of (4.27). Introduce $V_\rho(x) = V(x)\cdot\chi_{[0,\rho]}(x)$. Since the martingale structure is adapted to V in L^p, and since V_ρ has bounded support, it follows from Hölder's inequality that $\max_j \int_{E_j^m} |V_\rho| \to 0$ as $m \to \infty$. Consequently the above reasoning does establish that

$$\sup_{\rho\geq 0}\sup_{x\geq 0} \big|S_n(V_\rho,\dots,V_\rho)(x,\zeta) - S_n(V_\rho,\dots,V_\rho)(x,\lambda)\big| \to 0 \tag{4.31}$$

as $\zeta \to \lambda$ nontangentially, for almost every $\lambda \in \mathbb{R}$. On the other hand, $S_n(V_\rho,\dots,V_\rho)(x,\lambda)$ converges to $S_n(V,\dots,V)(x,\lambda)$ uniformly in $x \in \mathbb{R}^+$ for almost every λ, as proved in Proposition 4.1 of [9]. Likewise $S_n(V_\rho,\dots,V_\rho)(x,\zeta)$ converges to $S_n(V,\dots,V)(x,\zeta)$ for every $\zeta \in \mathbb{C}^+$ with strictly positive real part, uniformly for $x \geq 0$; this is an elementary consequence of the exponential decay of $\exp(2i[\phi(t_n,\zeta) - \phi(t_{n-1},\zeta) + \cdots])$ as $(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots \to +\infty$ and the assumption that $V \in L^1 + L^2$. By passing to the limit as $\rho \to +\infty$ we thus deduce (4.24) from (4.31). □

It is now straightforward to sum the series to obtain the same conclusion regarding convergence of $u(x,z) = u(x,\zeta^2)$ to $u(x,E) = u(x,\lambda^2)$:

$$|u(x,\zeta^2) - u(x,\lambda^2)| \leq \sum_{n=0}^{\infty} |S_n(V,V,\dots,V)(x,\zeta) - S_n(V,V,\dots,V)(x,\lambda)| \tag{4.32}$$

and

(4.33)

$$\begin{aligned}
&\sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)} \sup_{x\geq 0} \sum_{n=M}^{\infty} |S_n(V,V,\dots,V)(x,\zeta) - S_n(V,V,\dots,V)(x,\lambda)| \\
&\qquad \leq \sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)} \sup_{x\geq 0} \sum_{n=M}^{\infty} (|S_n(V,V,\dots,V)(x,\zeta)| + |S_n(V,V,\dots,V)(x,\lambda)|) \\
&\qquad \leq \sum_{n=M}^{\infty} \frac{C^{n+1}}{\sqrt{n!}} \big(NG(V)(\lambda) + G(V)(\lambda)\big)^n \\
&\qquad \leq \frac{C^{M+1}}{\sqrt{M!}} \big(NG(V)(\lambda) + G(V)(\lambda)\big)^M \sum_{k=0}^{\infty} \frac{C^{k+1}}{\sqrt{k!}} \big(NG(V)(\lambda) + G(V)(\lambda)\big)^k \\
&\qquad \leq \frac{C^{M+1}}{\sqrt{M!}} \big(NG(V)(\lambda) + G(V)(\lambda)\big)^M \exp(C\big(NG(V)(\lambda) + G(V)(\lambda)\big)^2).
\end{aligned}$$

For almost every $\lambda \in \mathbb{R}$, $\big(NG(V)(\lambda) + G(V)(\lambda)\big) < \infty$, and hence this expression tends to zero as $M \to \infty$. Coupled with the convergence established for the individual terms S_n in Lemma 4.5, this completes the proof of Theorem 4.1, modulo the proofs of Lemmas 4.3 and 4.2, and Corollary 4.4. □

Proof of Lemma 4.3. The proofs of the two inequalities are essentially the same, so we discuss only the first. The exponent here is

$$i\Phi(t,\lambda+i\varepsilon) = i\phi(t,\lambda+i\varepsilon) - i\phi(t',\lambda+i\varepsilon) = (i\lambda-\varepsilon)(t-t') - (i\lambda+\varepsilon)(\lambda^2+\varepsilon^2)^{-1}(\textstyle\int_{t'}^t V).$$

Since $t \geq t'$ and $|\int_{t'}^t V| \leq C + C|t-t'|^{1/2}$, the real part of Φ is bounded above, uniformly for all λ in any compact subinterval $\Lambda \Subset \mathbb{R}\backslash\{0\}$ and $\varepsilon \geq 0$. Thus L^1 is mapped boundedly to $L^\infty(\Lambda)$, uniformly in ε; by interpolation it suffices to prove the L^2 estimate.

Fix any cutoff function $\eta \in C^\infty(\mathbb{R}\backslash\{0\})$ and consider

$$\int \Big| \int_{t>t'} e^{i\Phi(t,\lambda+i\varepsilon)} f(t)\,dt \Big|^2 \eta(\lambda)\,d\lambda = \iint_{s,t\geq t'} f(t)\bar{f}(s) K(t,s)\,dt\,ds \tag{4.34}$$

where

$$K(t,s) = \int e^{\Psi(t,s,\lambda+i\varepsilon)}\eta(\lambda)\,d\lambda \tag{4.35}$$

with

$$\begin{aligned}\Psi(t,s,\lambda+i\varepsilon) &= 2i\Phi(t,\lambda+i\varepsilon) - 2i\bar{\Phi}(s,\lambda+i\varepsilon)\\ &= 2i\big[\lambda(t-s) - \lambda(\lambda^2+\varepsilon^2)^{-1}\textstyle\int_s^t V\big]\\ &\quad - 2\varepsilon\big[(t-t') + (s-t') + (\lambda^2+\varepsilon^2)^{-1}(\textstyle\int_{t'}^t V) + (\lambda^2+\varepsilon^2)^{-1}(\textstyle\int_{t'}^s V)\big].\end{aligned} \tag{4.36}$$

We claim that $|K(t,s)| \leq C(1+|s-t|)^{-2}$, uniformly in $\varepsilon \geq 0$; this would suffice to imply the L^2 bound. The integrand itself is bounded, uniformly in all parameters, so it suffices to restrict attention to the case where $|s-t| \geq C_0$, where C_0 is a sufficiently large constant. In that case we integrate by parts, integrating $\exp(2i[\lambda(t-s) - \lambda(\lambda^2 + \varepsilon^2)^{-1}\int_s^t V])$, and differentiating $\eta(\lambda)\cdot\exp(-2\varepsilon[(t-t') + (s-t') + (\lambda^2+\varepsilon^2)^{-1}(\int_{t'}^t V) + (\lambda^2+\varepsilon^2)^{-1}(\int_{t'}^s V))$, noting that $\partial[\lambda(t-s) - \lambda(\lambda^2+\varepsilon^2)^{-1}\int_s^t V]/\partial\lambda \geq |s-t|/2$ provided C_0 is chosen to be sufficiently large. Thus we gain a factor of $(s-t)^{-1}$. On the other hand, differentiating the other exponential with respect to λ brings in an unfavorable term $O(\varepsilon\int_{t'}^{\max(s,t)}|V|)$. After two integrations by parts, the integrand is

$$O(|s-t|^{-2})\cdot e^{-2\varepsilon(s-t')-2\varepsilon(t-t')}\cdot O(1+\varepsilon^2(\int_{t'}^{\max(s,t)}|V|)^2) = O(|s-t|^{-2}), \tag{4.37}$$

uniformly in ε. □

Proof of Lemma 4.2. This is a variant of Lemma 2.4, from which it would follow directly, were it not for the factors $\exp(\pm 2i\phi(t^{\pm}_{m,j}))$ in the definitions (4.8) of $s_j^{m,\pm}(V,\zeta)$. We follow the proof of Lemma 2.4, systematically introducing factors of 1 in the form $e^{\pm 2i\phi(t^+_{m,j},\zeta)}e^{\mp 2i\phi(t^-_{m,j+1},\zeta)}$ and redistributing the two constituent subfactors in order to bring $s_j^{m,\pm}(V,\zeta)$ into play.

Recall that

$$S_n(V, V, \ldots, V)(x, \zeta) = \iint\limits_{x \leq t_1 \leq \cdots \leq t_n} e^{2i\Phi(t,\zeta)} \prod_{j=1}^{n} V(t_j)\, dt_j,$$

where

$$\Phi(t, \zeta) = \phi(t_n, \zeta) - \phi(t_{n-1}, \zeta) + \phi(t_{n-2}, \zeta) - \cdots + (-1)^{n-1}\phi(t_1, \zeta),$$

$t = (t_1, \ldots, t_n)$, and $x \geq 0$. To begin, we note that it suffices to prove

$$|S_n(V, V, \ldots, V)(0, \zeta)| \leq \frac{C^{n+1}}{\sqrt{n!}} \left(\sum_{m=0}^{\infty} G_m(V)(\zeta) \right)^n. \tag{4.38}$$

For if $V_x(y) = V(y)\chi_{[x,\infty)}(y)$, then $S_n(V, \ldots, V)(x, \zeta) = S_n(V_x, \ldots, V_x)(0, \zeta)$ for all $x \geq 0$. The preceding inequality thus gives

$$\sup_{x \geq 0} |S_n(V, V, \ldots, V)(x, \zeta)| \leq \sup_{x \geq 0} \frac{C^{n+1}}{\sqrt{n!}} \big(\sum_{m=0}^{\infty} G_m(V_x)(\zeta) \big)^n,$$

which is

$$\leq \frac{C^{n+1}}{\sqrt{n!}} \big(\sum_{m=0}^{\infty} m G_m(V)(\zeta) \big)^n = C_\delta \frac{C^{n+1}}{\sqrt{n!}} G(V)(\zeta)^n.$$

We next reformulate the conclusion in two ways, so as to bring it into a form more convenient for a recapitulation of the proof of Lemma 2.4. Firstly, if n is odd, subtract $0 = \phi(0, \zeta)$ from $\Phi(t, \zeta)$ to obtain the phase $[\phi(t_n, \zeta) - \phi(t_{n-1}, \zeta) \cdots + \phi(t_1, \zeta) - \phi(0, \zeta)]$. Thus regardless of the parity of n, the phase has become an alternating sum with an even number of terms.

Secondly, consider an arbitrary interval $I \subset \mathbb{R}^+$ which may or may not be bounded above, and consider for any $n \geq 1$ expressions

$$S_n^I(V, \ldots, V)(\zeta) = \iint\limits_{t_1 \leq t_2 \leq \cdots \leq t_n} e^{2i\Phi(t,\zeta)} \prod_{j=1}^{n} V(t_j)\, \chi_I(t_j)\, dt_j$$

where $t = (t_1, \ldots, t_n)$ and $\Phi(t, \zeta)$ takes any one of the four forms

$$\begin{aligned} &\sum_{k=1}^{n} (-1)^{k-n} \phi(t_k, \zeta), && \sum_{k=0}^{n+1} (-1)^{k-n+1} \phi(t_k, \zeta), \\ &\sum_{k=1}^{n+1} (-1)^{k-n+1} \phi(t_k, \zeta), && \sum_{k=0}^{n} (-1)^{k-n} \phi(t_k, \zeta), \end{aligned} \tag{4.39}$$

with the restriction that $t_0 \leq s \leq t_{n+1}$ for all $s \in I$; the integral defining S_n^I is in all four cases taken only with respect to $t_1, \ldots, t_n$ while t_0, t_{n+1} remain constant, if they are present at all. Each of these expressions is an alternating sum with an even number of terms, in which the coefficient of the term $\phi(t_k, \zeta)$ is positive for the largest occurring value of k. For even n we consider only phases of the first two types, while for n odd we allow only phases of the third and fourth types. Moreover, for odd n, phases of the third type are of course only allowed if the interval I happens to be bounded above.

Consider any martingale structure $\{E_j^m\}$ on I, and in terms of it define the quantities $s_j^{m,\pm}(V,\zeta)$ as in (4.8), with one clarification: $s_j^{m,+}(V,\zeta)$ is defined whenever a subinterval E_j^m happens to be bounded above; previously $s_{2^m}^{m,+}(V,\zeta)$ had been left undefined since $E_{2^m}^m$ was unbounded. Define $G_m(V)(\zeta)$ in terms of $\{s_j^{m,\pm}(V,\zeta)\}$ as in (4.7).

We claim that for any fixed interval I, for any expression of any of allowed these types, any $n \geq 1$, and any martingale structure E_j^m on I,

$$|S_n^I(V,V,\dots,V)(\zeta)| \leq \frac{C^{n+1}}{\sqrt{n!}} \sum_{m=0}^{\infty} \big(G_m(V)(\zeta)\big)^n, \tag{4.40}$$

provided that $|\operatorname{Re}(\zeta)| \geq c_0 > 0$ and $\lim_{m\to\infty} \int_{E_j^m} |V| \to 0$ as $m \to \infty$. The constant C depends on c_0 but is otherwise independent of ζ, I, n, and the choice of $\{E_j^m\}$, (4.38) is a special case of this claim.

The proof will use the fact that whenever $t \geq t' \in \mathbb{R}$ and $\zeta \in \mathbb{C}^+$, $e^{2i[\phi(t,\zeta)-\phi(t',\zeta)]}$ is bounded by a universal constant, indendent of t, t', ζ. So long as $|\operatorname{Re}(\zeta)| \geq c_0 > 0$, this holds whenever $V \in L^1 + L^p$ for some $1 < p < \infty$, since the real part of $i\phi(t,\lambda+i\varepsilon) - i\phi(t',\lambda+i\varepsilon)$ equals $-\varepsilon(t-t') - \varepsilon(\lambda^2+\varepsilon^2)^{-1}\int_{t'}^t V$.

The proof of the claim (4.40) parallels that of Lemma 2.4. The case $n = 1$ is tautologous when t_0 is the left endpoint of I or t_{n+1} is its right endpoint, since $S_1(V,\zeta)$ is then equal to one of the terms $s_1^{0,\pm}$. For phases (4.39) of the fourth type, when t_0 is strictly less than the left endpoint $t_{0,1}^-$ of I, $\Phi(t,\zeta) = \phi(t_1,\zeta) - \phi(t_0,\zeta)$ may be rewritten as $[\phi(t_1,\zeta) - \phi(t_{0,1}^-,\zeta)] + [\phi(t_{0,1}^-,\zeta) - \phi(t_0,\zeta)]$, and $e^{2i[\phi(t_{0,1}^-,\zeta)-\phi(t_0,\zeta)]}$ is uniformly bounded since $t_{0,1}^- \geq t_0$ by assumption, and is independent of the variables $t_1,\dots,t_n$ of integration. For phases (4.39) of the third type, $\Phi(t,\zeta)$ may be reexpressed as $[\phi(t_2,\zeta) - \phi(t_{0,1}^+,\zeta)] + [\phi(t_{0,1}^+,\zeta) - \phi(t_1,\zeta)]$, and the same reasoning applies since $t_2 \geq t_{0,1}^+$.

We proceed by induction on n, following the proof of Lemma 2.4. Consider the multiple integral defining $S_n^I(V,\dots,V)(\zeta)$; for simplicity of notation we replace V by $V \cdot \chi_I$. Partition the region of integration $\{t = (t_1,\cdots,t_n) \in I^n : t_1 \leq \cdots \leq t_n\}$ as $\cup_{k=0}^n S_k$ where $S_k = \{t \in I^n : t_1 \leq \cdots \leq t_k \leq t_{1,1}^+ = t_{1,2}^- \leq t_{k+1} \leq \cdots \leq t_n\}$.

Consider first the case where n is even, and the phase Φ is of the first of the four types in (4.39). The total integral is then

$$\begin{aligned}
&\sum_{k=0}^n \iint_{S_k} e^{2i[\phi(t_n,\zeta)-\phi(t_{n-1},\zeta)+\cdots\pm\phi(t_{k+1},\zeta)]} \cdot e^{2i[\mp\phi(t_k,\zeta)\pm\phi(t_{k-1},\zeta)\mp\cdots]} \prod_{j=1}^n V(t_j)\,dt_j \\
&= \sum_{k=0}^n \Big(\iint_{t_{1,2}^- \leq t_{k+1} \leq \cdots \leq t_n} e^{2i[\phi(t_n,\zeta)-\phi(t_{n-1},\zeta)+\cdots+(-1)^{n-k-1}\phi(t_{k+1},\zeta)]} \prod_{j=k+1}^n V(t_j)\,dt_j \Big) \\
&\quad \Big(\iint_{t_1 \leq \cdots \leq t_k \leq t_{1,1}^+} e^{2i(-1)^{n-k}[\phi(t_k,\zeta)-\phi(t_{k-1},\zeta)+\cdots-\phi(t_1,\zeta)]} \prod_{j=1}^k V(t_j)\,dt_j \Big),
\end{aligned} \tag{4.41}$$

where the first factor on the right-hand side is replaced by 1 when $k = n$, and the second factor by 1 when $k = 0$.

For each k, each of the two factors on the right-hand side has the same form as the multiple integral with which we began, except that when $n - k$ is odd, an extra factor of -1 appears in the exponent in the second factor on the right-hand side. For each $k \in \{0, \dots, n\}$ for which $n - k$ is odd, we rectify this by introducing a factor of $1 = \exp(2i[-\phi(t_{1,2}^-, \zeta) + \phi(t_{1,1}^+, \zeta)])$ and distributing the two subfactors to reexpress the corresponding k-th term in the preceding sum as the modified product of

$$\Big(\iint_{t_{1,2}^- \le t_{k+1} \le \cdots \le t_n} e^{2i[\phi(t_n,\zeta) - \phi(t_{n-1},\zeta) + \cdots + \phi(t_{k+1},\zeta) - \phi(t_{1,2}^-,\zeta)]} \prod_{j=k+1}^{n} V(t_j)\, dt_j \Big)$$

with

$$\Big(\iint_{x \le t_1 \le \cdots \le t_k \le t_{1,1}^+} e^{2i[\phi(t_{1,1}^+,\zeta) - \phi(t_k,\zeta) + \phi(t_{k-1},\zeta) + \cdots + (-1)^{n-1}\phi(t_1,\zeta)]} \prod_{j=1}^{k} V(t_j)\, dt_j \Big).$$

However when $n - k$ is even, we make no such modification.

If n is even and Φ is of the second type, then we make the same initial decomposition and factorization. We proceed so as to always have phases of the types (4.39), with even numbers of terms, so again if $k, n - k$ are odd we introduce 1 in the above form and reexpress the two factors in the same way, while if $k, n - k$ are even then no modification is made.

If on the other hand n is odd, then Φ is of either the third or fourth types in (4.39), and again, to each $k \in \{0, 1, \dots, n\}$ is associated a splitting of Φ into a sum of two parts. Either both parts have an odd number of terms, in which case we again introduce 1 in the above form and distribute the two factors as above, or both parts have an even number of terms, in which case no new factors are introduced.

In all cases, S_n is expressed as the sum of $n + 1$ terms. Each term is a product of two factors, and each factor is an integral taking the same general form as S_n itself (or equals the constant 1); in particular, the corresponding phase function takes one of the forms (4.39).

The claim (4.40) now follows by induction on n, exactly as in the proof of Lemma 2.4. $\square$

Proof of Corollary 4.4. Let $\Lambda \Subset (0, \infty)$ be a compact interval, and let $1 < p < 2$, the case $p = 1$ being trivial. Set $q = p' = p/(p - 1)$. We discuss only the contributions of terms involving $\phi(t_{m,j}^-, \zeta)$ to G and G_M; those involving $t_{m,j}^+$ are treated in exactly the same way. For any $m \ge 1$ and any $f \in L^p$ we have, since $q/2 \ge 1$,

$$\| \Big(\sum_{j=1}^{2^m} \Big| \int_{E_j^m} e^{2i[\phi(t,\lambda+i\varepsilon) - \phi(t_{m,j}^-,\lambda+i\varepsilon)]} f(t)\, dt \Big|^2 \Big)^{1/2} \|_{L^q(\Lambda,d\lambda)}^q$$
$$\le \Big(\sum_j \Big[\int_\Lambda \Big| \int_{E_j^m} e^{2i[\phi(t,\lambda+i\varepsilon) - \phi(t_{m,j}^-,\lambda+i\varepsilon)]} f(t)\, dt \Big|^q d\lambda \Big]^{2/q} \Big)^{q/2}, \tag{4.42}$$

by Minkowski's integral inequality. By Lemma 4.3, the right-hand side is

$$\leq C\Big(\sum_j \|f\cdot\chi_{E_j^m}\|_{L^p}^2\Big)^{q/2}.$$

Since $p \leq 2$, this is

$$\leq C\Big(\sum_j \|f\|_{L^p}^{2-p}\|f\cdot\chi_{E_j^m}\|_{L^p}^p)^{q/2} = C\|f\|_{L^p}^q.$$

If we assume that the martingale structure is adapted to f in L^p, then for $p<2$ we have the improved majorization

$$\|f\cdot\chi_{E_j^m}\|_{L^p}^2 \leq 2^{-m(2-p)/p}\|f\|_{L^p}^{2-p}\|f\cdot\chi_{E_j^m}\|_{L^p}^p,$$

whence the final bound is $2^{-\rho m}\|f\|_{L^p}^q$ for some $\rho(p)>0$.

Therefore

$$\int_\Lambda \|\mathfrak{G}_M(f)(\lambda+i\varepsilon)\|_{\mathcal{B}}^q\,d\lambda \leq C2^{-\rho M}\|f\|_{L^p}^q, \tag{4.43}$$

uniformly for all $\varepsilon>0$. The first two conclusions of the Corollary now follow from Lemma 3.1, since Λ is an arbitrary compact interval.

That $\mathfrak{G}(f)(\zeta)$ converges almost everywhere to $\mathfrak{G}(f)(\lambda)$ in the $\mathcal{B}$ norm as $\zeta\to\lambda$ nontangentially, is an immediate consequence of the bound $\|NG_M(f)\|_{L^q} \leq C2^{-\varepsilon M}\|f\|_{L^p}$, since

$$\int_{t\geq t'} e^{2i[\phi(t,\zeta)-\phi(t',\zeta)]}f(t)\,dt \to \int_{t\geq t'} e^{2i[\phi(t,\lambda)-\phi(t',\lambda)]}f(t)\,dt \tag{4.44}$$

almost everywhere as $\zeta\to\lambda$ nontangentially, for all $f\in L^p$. This holds for all f in the dense subspace $L^1\cap L^p$, and then follows for general f by Lemma 3.1, Lemma 4.3, and standard reasoning. $\square$

5. Resolvents and spectral projections

5.1. **The half-line case.** Consider the operator

$$H_V = -\frac{d^2}{dx^2} + V(x) \tag{5.1}$$

on $\mathbb{R}^+$, with Dirichlet boundary condition at the origin. Any other selfadjoint boundary condition can be treated in a similar way. For each $z\in\mathbb{C}$, let $u_1(x,z), u_2(x,z)$ be the unique solutions of

$$-u'' + V(x)u = zu \tag{5.2}$$

satisfying the boundary conditions $u_1(0,z) = (0,1)^t, u_2(0,z) = (1,0)^t$; the superscripts t denote transposes. The classical theory of second order differential operators (see e.g. [13, 42]) tells us that if (5.1) is in the limit point case, then for any $z\in\mathbb{C}\setminus\mathbb{R}$ there exists a unique complex number $m(z)$, called the Weyl m-function, such that

$$f(x,z) = u_1(x,z)m(z) + u_2(x,z) \in L^2(\mathbb{R}^+).$$

We will always consider potentials which lead to the limit point situation, as is the case if $V\in L^1+L^p$ for some $1\leq p<\infty$; see for example [34]. The Weyl m-function is a Herglotz function, that is, it is analytic in the upper half-plane and has positive imaginary part there.

By direct computation, the resolvent $(H_V - z)^{-1}g$ for $z \in \mathbb{C}^+$ is given by

$$(H_V - z)^{-1}g(x) = u_1(x,z)\int_x^\infty f(y,z)g(y)\,dy + f(x,z)\int_0^x u_1(y,z)g(y)\,dy. \tag{5.3}$$

Denote by $P_{(a,b)}$ the spectral projection associated to H_V and to the interval (a,b). From the resolvent formula (5.3) we may derive a formula for the projections $P_{(a,b)}$. In doing so, we will use the following three routine facts.

1. The functions $u_1(x,z), u_2(x,z)$ are continuous in x for each z, and are entire holomorphic functions of z for each x.
2. The m function (in fact, any Herglotz function) has a representation

$$m(z) = C_1 + C_2 z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t) \tag{5.4}$$

for some positive Borel measure μ satisfying $\int(1+|t|^2)^{-1}d\mu(t) < \infty$, see e.g. [2]. In the m function context, μ is often called the spectral measure. For Dirichlet boundary conditions, as considered here, it is the spectral measure corresponding to the generalized vector δ_0', defined by $\delta_0'(u) = u'(0)$ for any function u in the domain of H_V. The moment condition above corresponds to the fact that the derivative δ_0' belongs to the Sobolev-like space $H_{-2}(H_V)$ associated to H_V (see e.g. [5] for details on families of Sobolev-like spaces associated with any selfadjoint operator A).
3. Im $m(E+i\epsilon)$ converges weakly to $\pi\mu$ as $\epsilon \to 0^+$. Moreover, Im $m(E+i\epsilon)$ has limiting boundary values for Lebesgue-almost every E, and the density of the absolutely continuous part of μ satisfies

$$d\mu_{\mathrm{ac}}(E) = \frac{1}{\pi}\,\mathrm{Im}\, m(E+i0)\,dE, \tag{5.5}$$

where $m(E+i0) = \lim_{\varepsilon\to 0^+} m(E+i\varepsilon)$. Since the imaginary part of m is simply a Poisson integral of μ, this is straightforward.

Fix functions h and g with compact support. We integrate the resolvent element $\langle (H_V - z)^{-1}g, h\rangle$ over the contour γ_ϵ in the complex plane consisting of two horizontal intervals $(a \pm i\epsilon, b \pm i\epsilon)$, and two vertical intervals at the ends connecting them. In the limit $\epsilon \to 0^+$, the contributions of the vertical intervals disappear unless a or b is an eigenvalue (point mass of μ); we will assume this is not the case. By the spectral theory (see, e.g. [33], Stone formula) we get then the following expression for $\langle P_{(a,b)}g, h\rangle$:

$$\begin{aligned} \langle P_{(a,b)}g, h\rangle &= \frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} \langle (H_V - z)^{-1}g, h\rangle\,dz \\ &= \int_a^b \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} u_1(x,E)u_1(y,E)g(x)\overline{h}(y)\,dx\,dy\,d\mu(E). \end{aligned} \tag{5.6}$$

In passing to the last line, we have taken into account the resolvent formula (5.3), the properties of u_1, u_2 (in particular, u_2 drops out because of analyticity), and the fact that $\pi^{-1}\operatorname{Im} m(E+i\epsilon)$ converges weakly to μ. The compact supports of h, g ensure that the integral is well-defined. Similar formulas can be found in [13, 42].

Since $P_{(a,b)}$ is by its definition an orthogonal projection, an immediate consequence of (5.6) is that the mapping $g \mapsto \int_{\mathbb{R}} u_1(x,E)g(x)\,dx$, initially defined for continuous g having compact support, extends to an orthogonal projection from $L^2(\mathbb{R}^+, dx)$ to $L^2((a,b),\mu)$. Dually, from (5.6) we see that for each $g \in L^2(\mathbb{R}, d\mu)$,

$$U_0 g(x) = \lim_{N\to\infty} \int_0^N u_1(x,E)g(E)d\mu(E) \tag{5.7}$$

exists in $L^2(\mathbb{R}^+, dx)$ norm, and that the linear operator U_0 thus defined is a unitary bijection from $L^2(\mathbb{R}, d\mu)$ to $L^2(\mathbb{R}, dx)$ with inverse

$$U_0^{-1} g(E) = \lim_{N\to\infty} \int_{-N}^N u_1(x,E)g(x)dx, \tag{5.8}$$

where the limit is again taken in L^2 norm.

With the formula (5.6) for the spectral projections in hand, we can invoke general spectral theory to find expressions for the spectral representation and other functions of H_V (see, e.g. [6]). To H_V and any interval (a,b) is associated a maximal closed subspace of $L^2(\mathbb{R}^+)$ on which H_V has purely absolutely continuous spectrum, and spectrum contained in (a,b). By (5.6) and (5.5) as well as by definition of the absolutely continuous part of the spectral projection, the projection $P^{\mathrm{ac}}_{(a,b)}$ of $L^2(\mathbb{R}^+)$ onto this subspace can be written as

$$P^{\mathrm{ac}}_{(a,b)} g(x) = \frac{1}{\pi}\int_a^b u_1(x,E)\Big(\int_{\mathbb{R}^+} u_1(y,E)g(y)\,dy\Big)\operatorname{Im} m(E+i0)\,dE. \tag{5.9}$$

The integral over $\mathbb{R}$ here is generally understood in the L^2-limiting sense; we will omit such explanatory remarks in the future. Consequently the operator U mapping continuous functions with compact support to $L^2(\mathbb{R}, dx)$, defined by

$$Uh(x) = \frac{1}{\pi}\int_{\mathbb{R}} u_1(x,E)h(E)\operatorname{Im} m(E+i0)\,dE \tag{5.10}$$

extends to an isometry of $L^2(\mathbb{R}, \operatorname{Im} m(E+i0)\,dE)$ onto the absolutely continuous subspace associated to H_V.

The unitary evolution operator on the absolutely continuous subspace is given by

$$e^{-iH_V t} g(x) = \frac{1}{\pi}\int_{\mathbb{R}} e^{-iEt} u_1(x,E)\tilde{g}(E)\operatorname{Im} m(E+i0)\,dE, \tag{5.11}$$

where

$$\tilde{g}(E) = \int_{\mathbb{R}^+} u_1(y,E)g(y)\,dy. \tag{5.12}$$

Finally, in the case $V=0$ we can compute explicitly $u_1(x,E) = \sqrt{E}^{-1}\sin\sqrt{E}x$, $m(z) = \sqrt{z}$, and so the evolution operator can be written as

$$e^{-iH_0 t} g(x) = \frac{1}{\pi}\int_{\mathbb{R}} e^{-iEt}\sin(\sqrt{E}x)\hat{g}(E)\,\frac{dE}{\sqrt{E}}, \tag{5.13}$$

where

$$\hat{g}(E) = \int_{\mathbb{R}^+} \sin(\sqrt{E}x) g(x)\, dx.$$

For almost every $E > 0$, define the scattering coefficient $\gamma(E) \in \mathbb{C}$ by

$$\gamma(E) = 1/u(0, E) \tag{5.14}$$

where $u(x, E)$ is the unique generalized eigenfunction asymptotic to $\exp(i\xi(x, E))$ as $x \to +\infty$, whose existence was established in Theorem 4.1. Notice that $\gamma(E)$ is well-defined, since $u(0, E) \neq 0$ for each $E > 0$ for which it is defined. Indeed, $u(x, E)$ satisfies a second order differential equation with real coefficients. If $u(0, E) = 0$ for some E, then by uniqueness of solutions $\overline{u}'(0, E) u(x, E)$ should be real. But this contradicts the asymptotic behavior of u at infinity.[6]

The following proposition connects the formulae of this section with the generalized eigenfunctions analyzed in §4.

Proposition 5.1 (Limiting absorption principle). *Assume that $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. For almost every $E \in \mathbb{R}^+$, the generalized eigenfunction $f(x, E + i0) = u_1(x, E) m(E + i0) + u_2(x, E)$ satisfies*

$$f(x, E + i0) = \gamma(E) e^{i\xi(x,E)}(1 + o(1)) \quad \textit{as } x \to +\infty, \tag{5.15}$$

and

$$|\gamma(E)|^2 = \mathrm{Im}\; m(E + i0)/\sqrt{E}. \tag{5.16}$$

Proof of Proposition 5.1. Denote by $u(x, z)$ the generalized eigenfunctions, with spectral parameter z, whose existence was established in Theorem 4.1. By the uniqueness of L^2 solutions of (4.1) for $z \in \mathbb{C}^+$, $f(x, z) = u(x, z)/u(0, z)$. For a.e. E, $f(x, z)$ converges uniformly as a function of $x \in \mathbb{R}^+$ to $u_1(x, E) m(E + i0) + u_2(x, E)$, as z converges to E nontangentially. The uniform convergence ensures that $f(x, E + i0)$ is indeed a generalized eigenfunction associated to the spectral parameter E. At the same time,

$$\frac{u(x, E + i\epsilon)}{u(0, E + i\epsilon)} \to \frac{u(x, E)}{u(0, E)}$$

for almost every E by Theorem 4.1, and therefore for $E > 0$

$$u_1(x, E) m(E + i0) + u_2(x, E) = \frac{u(x, E)}{u(0, E)} = \gamma(E) e^{i\xi(x,E)}(1 + o(1)) \tag{5.17}$$

as $x \to \infty$ (there is no absolutely continuous spectrum for $E < 0$). The relation between γ and $m(E + i0)$ follows by comparing the Wronskians of the left and right hand sides (taken with their complex conjugates). $\square$

Now we are going to rewrite the spectral representation in a manner convenient for the scattering theory. Notice that

$$u_1(x, E) = \frac{1}{2i\, \mathrm{Im}\; m(E + i0)} \left(\gamma(E) u(x, E) - \overline{\gamma}(E) \overline{u}(x, E) \right);$$

[6] The almost everywhere nonvanishing of $u(0, E)$ can alternatively be deduced from the almost everywhere nonvanishing of boundary values of analytic functions in the Nevanlinna class.

this follows from (5.17) by taking the imaginary parts of both sides, since u_1, u_2 are real-valued. Introduce

$$\psi(x,\lambda) = \lambda\overline{\gamma}(\lambda^2)u_1(x,\lambda^2) = \frac{1}{2i}\Big(u(x,\lambda^2) - \frac{\overline{\gamma}(\lambda^2)}{\gamma(\lambda^2)}\cdot\overline{u}(x,\lambda^2)\Big). \tag{5.18}$$

where $u(x,E)$ continue to denote the generalized eigenfunctions whose existence was established in Theorem 4.1, now for $E \in \mathbb{R}^+$. Then the results of this subsection may be summarized as follows.

Proposition 5.2. *Suppose that $V \in L^1 + L^p(\mathbb{R}^+)$ for some $1 < p < 2$. Then for the associated selfadjoint Schrödinger operator H_V on $L^2(\mathbb{R}^+)$ with Dirichlet boundary conditions, the spectral projection $P^{\mathrm{ac}}_{(a,b)}$ can be expressed as*

$$P^{\mathrm{ac}}_{(a^2,b^2)}g(x) = \frac{2}{\pi}\int_{[a,b]\cap\mathbb{R}^+}\psi(x,\lambda)\tilde{g}(\lambda)\,d\lambda \tag{5.19}$$

for any $g \in L^2(\mathbb{R}^+)$, where the modified Fourier transform $\tilde{g}$ is defined by

$$\tilde{g}(\lambda) = \int_{\mathbb{R}^+}\overline{\psi}(x,\lambda)g(x)\,dx. \tag{5.20}$$

Similarly, the associated wave group is

$$e^{-itH_V}g(x) = \frac{2}{\pi}\int_0^\infty e^{-i\lambda^2 t}\psi(x,\lambda)\tilde{g}(\lambda)\,d\lambda \tag{5.21}$$

for $g \in \mathcal{H}_{ac}$. The operator

$$U_V f(x) = \sqrt{2/\pi}\int_0^\infty f(\lambda)\psi(x,\lambda)\,d\lambda \tag{5.22}$$

is a unitary surjection from $L^2(\mathbb{R}^+, d\lambda)$ onto $\mathcal{H}_{ac}$.

5.2. **Formulae for the case of the whole real line.** Consider $-d^2/dx^2 + V(x)$ as an essentially selfadjoint operator on $L^2(\mathbb{R})$. The asymptotic analysis of the preceding section, in which $x \to +\infty$, works equally well as $x \to -\infty$. Whereas it was necessary in the half-line case to relate asymptotics at $+\infty$ to boundary conditions at $x = 0$, now we must relate asymptotics at $+\infty$ to those at $-\infty$. The absolutely continuous spectrum now has multiplicity two, which complicates some of the formulae.

Let $z \in \mathbb{C}^+$. Solutions $u_1(x,z), u_2(x,z)$ are defined precisely as before, with the same initial conditions at $x = 0$; now they are considered as global solutions on $\mathbb{R}$ rather than merely on $[0,\infty)$. Introduce two solutions $f_\pm(x,z) = u_1(x,z)m_\pm(z) + u_2(x,z)$, so that $f_+ \in L^2(\mathbb{R}^+, dx)$ and $f_- \in L^2(\mathbb{R}^-, dx)$ for each $z \in \mathbb{C}^+$. In terms of these, the resolvent is given by

(5.23)

$$(H_V - z)^{-1}g(x) = \frac{-1}{W[f_+,f_-])}\Big(f_+(x,z)\int_{-\infty}^{x} f_-(y,z)g(y)\,dy + f_-(x,z)\int_x^\infty f_+(y,z)g(y)\,dy\Big).$$

Notice that $W[f_+,f_-] = m_+ - m_-$ (since $\mathrm{Im}\ m_-(z) < 0$ in $\mathbb{C}^+$).

The formula for the spectral projection associated to the absolutely continuous spectrum can be computed from the resolvent in a way similar to the half-line case. In the free case $V = 0$, it simplifies to

$$P^{\mathrm{ac}}_{(a^2,b^2)}g(y) = \frac{1}{2\pi}\int_a^b \chi_{\mathbb{R}^+}(\lambda)\left(e^{i\lambda x}\int_{\mathbb{R}} e^{-i\lambda y}g(y)\,dy + e^{-i\lambda x}\int_{\mathbb{R}} e^{i\lambda y}g(y)\,dy\right)\,d\lambda. \tag{5.24}$$

The evolution operator is given by

$$e^{-iH_0t}g(x) = \frac{1}{2\pi}\int_0^\infty e^{-i\lambda^2 t}\left(e^{i\lambda x}g_+(\lambda) + e^{-i\lambda x}g_-(\lambda)\right)\,d\lambda, \tag{5.25}$$

where

$$g_\pm(\lambda) = \int_{\mathbb{R}} e^{\mp i\lambda y}g(y)dy.$$

In the general case, consider the limits of the functions $f_\pm(x,z)$ as z tends to the real axis, $f_\pm(x,\lambda^2) = (u_1(x,\lambda^2)m_\pm(\lambda^2+i0) + u_2(x,\lambda^2))$. Define

$$\psi_\pm(x,\lambda) = \gamma_\pm^{-1}(\lambda)f_\pm(x,\lambda^2), \tag{5.26}$$

where the coefficents $\gamma_\pm(\lambda)$ are chosen so that for a.e. λ the solutions $\psi_\pm(x,\lambda)$ have the following scattered wave asymptotic behavior

$$\psi_+(x,\lambda) = \begin{cases} t_1(\lambda)e^{i\phi(x,\lambda)}(1+o(1)), & x\to+\infty, \\ \left(e^{i\phi(x,\lambda)} + r_1(\lambda)e^{-i\phi(x,\lambda)}\right) + o(1), & x\to-\infty \end{cases} \tag{5.27}$$

$$\psi_-(x,\lambda) = \begin{cases} t_2(\lambda)e^{-i\phi(x,\lambda)}(1+o(1)), & x\to-\infty, \\ \left(e^{-i\phi(x,\lambda)} + r_2(\lambda)e^{i\phi(x,\lambda)}\right) + o(1), & x\to+\infty, \end{cases} \tag{5.28}$$

where we recall that $\phi(x,\lambda) = \lambda x - (2\lambda)^{-1}\int_0^x V(s)\,ds$. We remark that the fact that $f_\pm(x,\lambda^2)$ have asymptotic behavior $\sim c_\pm e^{\pm i\phi(x,\lambda)}$ at $\pm\infty$ for a.e. λ is established similarly to the half-line case. Normalization as in (5.27) can be done at any λ where $f_\pm(x,\lambda^2)$ have the right asymptotics at $\pm\infty$. The asymptotics in (5.27), (5.28) can be differentiated, in the sense that $d\psi_+(x,\lambda)/dx = i\lambda t_1(\lambda)\exp(i\xi(x,\lambda^2)) + o(1)$ as $x\to+\infty$, with analogous formulae as $x\to-\infty$ and for ψ_-.

Computations of Wronskians, examination of transfer matrices between $\psi_+, \overline{\psi}_+$ and $\psi_-, \overline{\psi}_-$, and exploitation of complex conjugation leads to the following formulae relating the transmission and reflection coefficients t_i, r_i with one another, and with the coefficients $\gamma_\pm, m_\pm$:

$$|r_i|^2 + |t_i|^2 = 1 \text{ for } i = 1,2, \tag{5.29}$$

$$t_1 = t_2, \tag{5.30}$$

$$r_2 = -\frac{t_1}{\overline{t}_1}\overline{r}_1, \tag{5.31}$$

$$|\gamma_\pm|^2|t_1|^2\lambda = \pm\,\mathrm{Im}\ m_\pm(\lambda^2+i0). \tag{5.32}$$

After a computation, we obtain
(5.33)

$$P^{\mathrm{ac}}_{(a^2,b^2)}g(y) = \frac{1}{2\pi}\int_a^b \chi_{\mathbb{R}^+}(\lambda)\Big(\psi_+(x,\lambda)\int_{\mathbb{R}}\overline{\psi}_+(y,\lambda)g(y)\,dy + \psi_-(x,\lambda)\int_{\mathbb{R}}\overline{\psi}_-(y,\lambda)g(y)\,dy\Big)\,d\lambda.$$

Therefore, the evolution operator is given by

$$e^{-iH_V t}g(x) = \frac{1}{2\pi}\int_0^\infty e^{-i\lambda^2 t}\Big(\psi_+(x,\lambda)\tilde{g}_+(\lambda) + \psi_-(x,\lambda)\tilde{g}_-(\lambda)\Big)\,d\lambda, \tag{5.34}$$

where the transforms $\tilde{g}_\pm$ are defined by

$$\tilde{g}_\pm(\lambda) = \int_{\mathbb{R}}\overline{\psi}_\pm(y,\lambda)g(y)dy. \tag{5.35}$$

5.3. **Dirac-type operators on the whole real line.** Here we consider a system of differential equations of second order which plays an important role in analysis of the defocusing nonlinear Schrödinger equation (NLS) by the inverse scattering method. This system is given by [31]

$$y' = iz\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}y + \begin{pmatrix}0 & q\\ \overline{q} & 0\end{pmatrix}y. \tag{5.36}$$

We are going to use two alternative equivalent representations of this system. The first is

$$\begin{pmatrix}-i\partial_x & V\\ \overline{V} & i\partial_x\end{pmatrix}y = zy, \tag{5.37}$$

where $V = iq$. System (5.37) is obtained from (5.36) by multiplying both sides by a matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. The second is obtained by setting $y = Q^{-1}\phi$ in (5.37), where

$$Q = \begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}.$$

Defining

$$D_V = \begin{pmatrix}0 & -\partial_x\\ \partial_x & 0\end{pmatrix} + \begin{pmatrix}\operatorname{Re} V & \operatorname{Im} V\\ \operatorname{Im} V & -\operatorname{Re} V\end{pmatrix}, \tag{5.38}$$

we obtain

$$D_V\phi = z\phi. \tag{5.39}$$

The operator on the left hand side of (5.39) is a particular case of a second order Dirac-type operator (the most general case does not require the second diagonal entry to be minus the first; in the most general case there will be a phase shift in the main term of the solution asymptotics, unlike the situation here).

We digress briefly to comment on the possibility of embedded point spectrum for this operator. Let $z = E \in \mathbb{R}$. The equation (5.37) preserves the analogue of the Wronskian for two solutions $f = (f_1, f_2)$ and $g = (g_1, g_2)$, $W[f,g] = i(f_2g_1 - f_1g_2)$.

Notice that if (g_1, g_2) is a solution, so is $(\overline{g}_2, \overline{g}_1)$. The constancy of the Wronskian for these two solutions implies that $|g_2|^2 - |g_1|^2$ is constant. Consider for simplicity the case where $|g_1| = |g_2|$. Representing $g_1 = R(x)e^{i\theta_1(x)}$, $g_2 = R(x)e^{i\theta_2(x)}$ and using (5.37), we find that $\theta_1 + \theta_2 = c$ is constant, and

$$\begin{aligned} (\log R)' &= -\operatorname{Re} V \sin(2\theta_1 - c) + \operatorname{Im} V \cos(2\theta_1 - c) \\ \theta_1' &= E - \operatorname{Re} V \cos(2\theta_1 - c) - \operatorname{Im} V \sin(2\theta_1 - c). \end{aligned} \tag{5.40}$$

From these Prüfer-like equations one can easily recover certain constructions of embedded eigenvalues available in the Schrödinger case [30, 26, 27]; see [30] for an earlier alternative approach to the Dirac case. In particular, there exist potentials $V = O(|x|^{-1})$ with isolated eigenvalues embedded in the continuous spectrum, and for any function $g(|x|)$ tending to infinity, there exists $|V(x)| \le g(|x|)/|x|$ for which there is a dense set of eigenvalues in $\mathbb{R}^+$.

The form (5.37) will prove useful for studying solutions; the form (5.39) will allow us to set up scattering in a way completely parallel to the setup for Schrödinger operators on the whole axis. First, setting

$$y_1 = \begin{pmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{pmatrix} y$$

we arrive at

$$y_1' = \begin{pmatrix} 0 & -iVe^{-2izx} \\ i\overline{V}e^{2izx} & 0 \end{pmatrix} y_1.$$

Iterating as in the Schrödinger case, we get solutions $\psi_\pm(x, z)$ for any $z \in \mathbb{C}^+$ and almost every $z \in \mathbb{R}$ with the asymptotic behavior

$$\begin{aligned} \psi_+(x, z) &= e^{izx}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1)\right), \qquad x \to +\infty \\ \psi_-(x, z) &= e^{-izx}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)\right), \quad x \to -\infty. \end{aligned} \tag{5.41}$$

An analogue of Proposition 5.1 also holds.

Before proceeding, we record a corollary of the existence of such generalized eigenfunctions for real z.

Corollary 5.3. *For any $1 < p < 2$ and any $V \in L^1 + L^p(\mathbb{R})$, the Dirac operator D_V on $L^2(\mathbb{R})$ has nonempty absolutely continuous spectrum. More precisely, an essential support for the ac spectrum is the entire real line $\mathbb{R}$.*

As remarked above, this hypothesis does not preclude the presence of a dense set of eigenvalues embedded in the continuous spectrum.

Now let us return to (5.39). The existence of solution (5.41) and the relation between (5.39) and (5.37) imply existence of solutions $\tilde{\eta}_\pm(x, z)$ of (5.39), for $z \in \mathbb{C}^+$ and for almost every $z \in \mathbb{R}$, of the form

$$\begin{aligned} \tilde{\eta}_+(x, z) &= e^{izx}\left(\begin{pmatrix} 1 \\ i \end{pmatrix} + o(1)\right), \qquad x \to +\infty \\ \tilde{\eta}_-(x, z) &= e^{-izx}\left(\begin{pmatrix} 1 \\ -i \end{pmatrix} + o(1)\right), \quad x \to -\infty. \end{aligned} \tag{5.42}$$

We proceed very much as in the Schrödinger case. The role of the Wronskian is played by $W[f,g] = f_2 g_1 - f_1 g_2$. As before, we denote by $u_{1,2}(x,z)$ solutions satisfying $u_1(0,z) = (0,1)^T$, $u_2(0,z) = (1,0)^T$. Let $f_\pm(x,z) = u_1(x,z)m_\pm(z) + u_2(x,z)$ be L^2 solutions on $\mathbb{R}^\pm$. The resolvent of D_V is given by

(5.43)

$$(D_V - z)^{-1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \frac{1}{W[f_-, f_+]} \left(f_+ \int_{-\infty}^{x} (f_{-,1} g_1 + f_{-,2} g_2)\, dy + f_- \int_{x}^{\infty} (f_{+,1} g_1 + f_{+,2} g_2)\, dy \right).$$

Let us denote in this section

$$\langle f, g \rangle = \int_{\mathbb{R}} (f_1(x) g_1(x) + f_2(x) g_2(x))\, dx;$$

when necessary, the integral is understood in a limiting sense (in L^2 if f is a solution which also depends on E). An analog of (5.9) can be derived using contour integration. The conclusion is that (one version of) the spectral representation of the absolutely continuous part of the spectral measure is given by the map

$$U : L^2(\mathbb{C}^2, dx) \mapsto L^2(\mathbb{C}^2, M dE), \;\; U \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{pmatrix},$$

where

$$\tilde{g}_i(E) = \langle u_i(x,E), g \rangle$$

and

$$M(E) = \mathrm{Im} \begin{pmatrix} \frac{m_+ m_-}{m_- - m_+} & \frac{m_-}{m_- - m_+} \\ \frac{m_-}{m_- - m_+} & \frac{1}{m_- - m_+} \end{pmatrix} (E + i0).$$

Then U is an isometry, $U D_V U^{-1} = E$ (with U^{-1} given by

$$U^{-1} \tilde{g}(x) = \int_{\mathbb{R}} \begin{pmatrix} u_1(x,E) \\ u_2(x,E) \end{pmatrix} M \tilde{g}\, dE,$$

where (u_1, u_2) is a 2×2 matrix with columns u_1, u_2). The unitary evolution of the absolutely continuous part is then given by

$$e^{-iD_V t} g(x) = \int_{\mathbb{R}} e^{-iEt} \begin{pmatrix} u_1(x,E) \\ u_2(x,E) \end{pmatrix} M \tilde{g}(E)\, dE. \tag{5.44}$$

In the free case, the spectral representation can be simplified. In particular, we get

$$e^{-iD_0 t} g(x) = \frac{1}{4} \int e^{-iEt} \left(\tilde{\eta}_{+,0}(x,E) \langle \overline{\tilde{\eta}}_{+,0}, g \rangle + \tilde{\eta}_{-,0}(x,E) \langle \overline{\tilde{\eta}}_{-,0}, g \rangle \right), \tag{5.45}$$

where

$$\tilde{\eta}_{\pm,0}(x,E) = e^{\pm iEx} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

when $V = 0$. In the perturbed case, for a.e. E we introduce scattering coefficients $r_j(E)$ and $t_j(E)$, defined in terms of the scattered waves

$$\eta_+(x,E) = \begin{cases} t_1(E)e^{iEx}\begin{pmatrix} 1 \\ i \end{pmatrix} + o(1), & x \to +\infty, \\ \left(e^{iEx}\begin{pmatrix} 1 \\ i \end{pmatrix} + r_1(E)e^{-iEx}\begin{pmatrix} 1 \\ -i \end{pmatrix}\right) + o(1), & x \to -\infty \end{cases} \tag{5.46}$$

$$\eta_-(x,E) = \begin{cases} t_2(E)e^{-iEx}\begin{pmatrix} 1 \\ -i \end{pmatrix} + o(1), & x \to -\infty, \\ \left(e^{-iEx}\begin{pmatrix} 1 \\ -i \end{pmatrix} + r_2(E)e^{iEx}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) + o(1), & x \to +\infty \end{cases} \tag{5.47}$$

Existence of such solutions follows from (5.41); these are just renormalized solutions $\tilde{\eta}_\pm$ of (5.42). A computation parallel to the whole axis Schrödinger operator case allows us to rewrite the dynamics (5.44) as

$$e^{-iD_V t}g(x) = \frac{1}{4}\int\limits_{\mathbb{R}} e^{-iEt}\left(\eta_+(x,E)\langle\overline{\eta}_+, g\rangle + \eta_-(x,E)\langle\overline{\eta}_-, g\rangle\right). \tag{5.48}$$

6. Long-time asymptotics

Let $V \in L^1 + L^p(\mathbb{R}^+)$ for some $1 < p < 2$, and consider the selfadjoint Schrödinger operator $H_V = -d^2/dx^2 + V(x)$ on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition. Let $\mathcal{H} = L^2(\mathbb{R}^+)$ and let $\mathcal{H}_{\text{ac}}$ denote the maximal closed subspace of $\mathcal{H}$ on which H_V has purely absolutely continuous spectrum.

Let $\phi(x,\lambda) = \lambda x - (2\lambda)^{-1}\int_0^x V$ be as before, and for almost every $\lambda \in \mathbb{R}^+$ let $u(x,\lambda)$ be the unique generalized eigenfunction associated to the spectral parameter λ^2, satisfying $u(x,\lambda) = \exp(i\phi(x,\lambda)) + o(1)$ as $x \to +\infty$. From the preceding section recall the generalized eigenfunctions $\psi(x,\lambda) = (2i)^{-1}(u - e^{i\omega(\lambda)}\overline{u}))$, where $\exp(i\omega(\lambda)) = \overline{\gamma}(\lambda^2)/\gamma(\lambda^2)$. Recall the unitary bijection $U_V : L^2(\mathbb{R}^+, d\lambda) \mapsto \mathcal{H}_{\text{ac}}$ defined by $U_V f(x) = \sqrt{2/\pi}\int_0^\infty f(\lambda)\psi(x,\lambda)\,d\lambda$. Finally, recall that $e^{-iH_V t}(U_V f)(x) = \sqrt{2/\pi}\int_0^\infty e^{-i\lambda^2 t}f(\lambda)\psi(x,\lambda)\,d\lambda$.

Fix a martingale structure $\{E_j^m\}$ on $\mathbb{R}^+$ that is adapted to V, in the sense that $\int_{E_j^m}|V|^p = 2^{-m}\int_{\mathbb{R}^+}|V|^p$ for all $m \geq 0$ and all j. For any sufficiently small $\delta > 0$, recall the functional

$$g_\delta(f)(\lambda) = \sum_{m=1}^{\infty} 2^{m\delta}\Big(\sum_{j=1}^{2^m}\Big|\int_{E_j^m} e^{2i\phi(x,\lambda)}f(x)\,dx\Big|^2 + \Big|\int_{E_j^m} e^{-2i\phi(x,\lambda)}f(x)\,dx\Big|^2\Big)^{1/2}.$$

We have shown that for all sufficiently small δ_0, $g_{\delta_0}(V)(\lambda) < \infty$ for almost every $\lambda \in \mathbb{R}^+$. Fix some $\delta < \delta_0$.

Definition 6.1. A compact set $\Lambda \Subset (0,\infty)$ is said to be a *set of uniformity* if $g_\delta(V)$ is a bounded function of $\lambda \in \Lambda$, and if $u(x,\lambda) - e^{i\phi(x,\lambda)} \to 0$ as $x \to +\infty$, uniformly for all $\lambda \in \Lambda$.

Lemma 6.1. *For any $f \in L^2(\mathbb{R}^+)$, for any $R < \infty$,*

$$\|e^{-itH_V} U_V f\|_{L^2([0,R])} \to 0 \quad \textit{as } t \to \infty.$$

Proof. Since $\exp(-itH_V)$ is unitary, it suffices to prove this for all f in some dense subspace of $L^2(\mathbb{R}^+)$; we choose the subspace consisting of all f supported on some set of uniformity Λ (which depends on f). The functions $\psi(x,\lambda)$ are uniformly bounded on $[0,\infty) \times \Lambda$, as are their derivatives with respect to x, from which it follows that U_V is a compact mapping from $L^2(\Lambda)$ to $L^2([0,R])$. Thus it suffices to establish weak convergence to zero. Now for any $h \in L^2([0,R])$,

$$\langle e^{-itH_V} U_V f, h\rangle = \int_\Lambda e^{-i\lambda^2 t} f(\lambda) U_V^* h(\lambda)\, d\lambda, \tag{6.1}$$

which converges to zero by the Riemann-Lebesgue lemma, since $U_V^* h \in L^2(\mathbb{R}^+)$. $\square$

Define

$$\psi_0(x,\lambda) = (2i)^{-1}\big(e^{i\phi(x,\lambda)} - e^{i\omega(\lambda)} e^{-i\phi(x,\lambda)}\big) \tag{6.2}$$

$$U_V^\dagger f(x) = \sqrt{2/\pi} \int_0^\infty f(\lambda)\psi_0(x,\lambda)\, d\lambda; \tag{6.3}$$

these are approximations to ψ, U_V, respectively. The next lemma is the key step in showing that only the leading-order approximation ψ_0 to ψ contributes to the long-time asymptotics of the wave group.

Lemma 6.2. *For any set of uniformity Λ and any $R > 0$, there exists $C(R,\Lambda) < \infty$ such that for all $f \in L^\infty(\Lambda)$,*

$$\|(U_V - U_V^\dagger) f\|_{L^2([R,\infty))} \le C(R,\Lambda)\|f\|_{L^\infty}, \quad \textit{where} \;\; C(R,\Lambda) \to 0 \; \textit{as } R \to \infty. \tag{6.4}$$

Proof. It suffices to prove this with $\psi(x,\lambda) - \psi_0(x,\lambda)$ replaced by

$$\begin{aligned} &\Psi(x,\lambda) - e^{i\phi(x,\lambda)} \\ &\quad = e^{i\phi(x,\lambda)} \sum_{n=1}^\infty S_{2n}(V,V,\dots,V)(x,\lambda) - e^{-i\phi(x,\lambda)} \sum_{n=0}^\infty S_{2n+1}(V,V,\dots,V)(x,\lambda), \end{aligned} \tag{6.5}$$

since the conclusion for the other terms follows from this by complex conjugation. We argue by duality; let h be an arbitrary compactly supported function in $L^2([R,\infty))$ of norm 1, and consider

$$\begin{aligned} &\int_R^\infty e^{i\phi(x,\lambda)} h(x) S_{2n}(V,V,\dots,V)(x,\lambda)\, dx \\ &\qquad = \iint_{R \le t_0 \le t_1 \le \cdots \le t_{2n}} e^{i\phi(t_0,\lambda)} h(t_0) \prod_{k=1}^{2n} e^{\pm_k 2i\phi(t_k,\lambda)} V(t_k)\, dt_k\, dt_0. \end{aligned} \tag{6.6}$$

Here $\pm_k$ denotes a certain choice of plus or minus sign depending on k; in fact these signs alternate in our expansion, but the precise choice plays no role in our analysis.

Recall that for $\lambda \in \Lambda$,

$$\sup_\rho \Big| \iint_{x \le t_1 \le \cdots \le t_{2n} \le \rho} \prod_{k=1}^{2n} e^{\pm_k 2i\phi(t_k,\lambda)} V(t_k)\, dt_k \Big|$$

is a uniformly bounded function of $x \in \mathbb{R}^+$, and these multiple integrals converge to $S_{2n}(V, \dots, V)(x, \lambda)$ uniformly in x as $\rho \to +\infty$. As was shown in Proposition 4.1 of [9], this is a consequence of Proposition 2.1 and the reasoning of (4.26), (4.27). Therefore it suffices to obtain a suitable upper bound independent of ρ for the absolutely convergent integrals

$$\iint_{R \le t_0 \le t_1 \le \cdots \le t_{2n} \le \rho} e^{i\phi(t_0,\lambda)} h(t_0) \prod_{k=1}^{2n} e^{\pm_k 2i\phi(t_k,\lambda)} V(t_k)\, dt_k\, dt_0.$$

Since the martingale structure is adapted to V, we have $\max_j \int_{E_j^m \cap [0,\rho]} |V| \to 0$ as $m \to \infty$ for any fixed $\rho < \infty$, since Hölder's inequality may be applied on the finite interval $[0, \rho]$. Thus the auxiliary hypothesis (2.15) of Proposition 2.5 is satisfied by these multiple integrals; the parameter ρ was introduced for that reason alone.

Introduce

$$(6.7)\quad g_{-\delta}(h)(\lambda) = \sum_{m=1}^{\infty} 2^{-m\delta} \Big(\sum_{j=1}^{2^m} \Big| \int_{E_j^m} e^{i\phi(x,\lambda)} h(x)\, dx \Big|^2 + \Big| \int_{E_j^m} e^{-i\phi(x,\lambda)} h(x)\, dx \Big|^2 \Big)^{1/2}.$$

By Proposition 2.5,

(6.8)

$$\Big| \iint_{R \le t_0 \le t_1 \le \cdots \le t_{2n}} e^{i\phi(t_0,\lambda)} h(t_0)\, dt_0 \prod_{k=1}^{2n} e^{\pm_k 2i\phi(t_k,\lambda)} V(t_k)\, dt_k \Big| \le \frac{C^{n+1}}{\sqrt{2n!}} g_{-\delta}(h)(\lambda) g_\delta(V_{R,\rho})^{2n}(\lambda)$$

where $V_{R,\rho}(x) = V(x)\chi_{[R,\rho]}(x)$; there is a corresponding bound for the terms arising from the multilinear expressions S_{2n+1}. We may dominate $\sup_R g_\delta(V_{R,\rho})(\lambda)$ by $C' g_{\delta'}(V)(\lambda)$ for any $\delta' > \delta$; we do this for all $2n-1$ factors of this quantity, but dominate the remaining factor instead by $C' g_{\delta'}(V_R)(\lambda)$ where $V_R = V\chi_{[R,\infty)}$. To simplify notation we replace δ' again by δ. Consequently by summing over n and comparing with the Taylor expansion for the exponential function,

$$(6.9)\quad \Big| \langle \int_\Lambda \int_R^\infty f(\lambda) \left(u(x,\lambda) - e^{i\phi(x,\lambda)} \right) h(x)\, dx\, d\lambda \rangle \Big| \\ \le \int_\Lambda C g_{-\delta}(h)(\lambda) g_\delta(V_R)(\lambda) e^{C g_\delta(V)(\lambda)^2} |f(\lambda)|\, d\lambda.$$

Recall that for any $\delta > 0$, $\|g_{-\delta}(h)\|_{L^2(\Lambda)} \le C_{\Lambda,\delta} \|h\|_{L^2} \le C_{\Lambda,\delta} < \infty$, for any compact $\Lambda \Subset (0, \infty)$, uniformly over all martingale structures, not necessarily adapted to h. The factor $\exp(C g_\delta(V)(\lambda)^2)$ is bounded uniformly for $\lambda \in \Lambda$, by definition of a set of uniformity. Thus it suffices to show that $\|g_\delta(V_R)(\lambda)\|_{L^2(\Lambda, d\lambda)} \to 0$ as $R \to \infty$. Now in the sum (2.3) defining $g_\delta(V)$, the ℓ^2 sum over j for fixed m is $\le C 2^{-\varepsilon m} \|V\|_{L^2}$ in

$L^2(\Lambda)$ for some $\varepsilon > 0$, so it suffices to show that $\int_{E_j^m} e^{2i\phi(x,\lambda)} V_R(\lambda)\,dx \to 0$ in $L^2(\Lambda)$ for any fixed m, j. This holds by Lemma 4.3, since $V_R \to 0$ in $L^1 + L^2$ norm. □

Corollary 6.3. *Let $\rho > 0$. For any $f \in L^2(\mathbb{R}^+, d\lambda)$ supported on $[\rho, \infty)$,*

$$\|e^{-itH_V} U_V f - \sqrt{2/\pi} \int_0^\infty e^{-i\lambda^2 t} f(\lambda)\psi_0(x,\lambda)\,d\lambda\|_{L^2(\mathbb{R}^+)} \to 0 \text{ as } |t| \to \infty. \tag{6.10}$$

The restriction on the support of f comes about because of the factor of λ^{-1} in the definition of ϕ, which causes difficulties as $\lambda \to 0$.

Proof. It is straightforward to show that $\int_0^\infty e^{-i\lambda^2 t} f(\lambda)\psi_0(x,\lambda)\,d\lambda \to 0$ in $L^2([0, R])$ for any finite R.

Let $\varepsilon > 0$. Choose a set of uniformity $\Lambda \subset [\rho, \infty)$ and a function $F \in L^\infty(\Lambda)$ such that $\|f - F\|_{L^2(\mathbb{R}^+)} < \varepsilon$. Then $\|e^{-itH_V}(U_V f - U_V F)\|_{L^2(\mathbb{R}^+)} < \varepsilon$ for every t, as well. By the preceding lemma, there exists $R < \infty$ such that $\|e^{-itH_V} U_V F - \sqrt{2/\pi} \int_0^\infty e^{-i\lambda^2 t} F(\lambda)\psi_0(x,\lambda)\,d\lambda\|_{L^2([R,\infty))} < \varepsilon$ for all $t \in \mathbb{R}$. Fixing such an R, there exists $T < \infty$ such that both $e^{-itH_V} U_V F$ and $\sqrt{2/\pi} \int_0^\infty e^{-i\lambda^2 t} F(E)\psi_0(x,\lambda)\,d\lambda$ have $L^2([0, R])$ norms $< \varepsilon$, for all $|t| \ge T$. Thus

$$\|e^{-itH_V} U_V f - \sqrt{2/\pi} \int_0^\infty e^{-i\lambda^2 t} F(\lambda)\psi_0(x,\lambda)\,d\lambda\|_{L^2(\mathbb{R}^+)} < 4\varepsilon \text{ for all } |t| \ge T. \tag{6.11}$$

Finally, note that $\| \int_\rho^\infty g(\lambda) u_0(x,\lambda)\,d\lambda\|_{L^2(\mathbb{R}^+)} \le C_\rho \|g\|_{L^2([\rho,\infty))}$ for all g, where $C_\rho < \infty$ for all $\rho > 0$; this follows from the proof of Lemma 4.3. Applying this to $g = f - F$ allows us to replace F again by f in the preceding inequality, at the expense of replacing 4ε by $C_\rho \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this completes the proof. □

7. A time-dependent phase correction

The goal of this section is to convert the leading-order asymptotics $\exp(\pm i\lambda x \mp i(2\lambda)^{-1} \int_0^x V)$ to a more standard form, $\exp(\pm i\lambda x \mp i(2\lambda)^{-1} \int_0^{2\lambda|t|} V)$.

Lemma 7.1. *Let $V \in L^1 + L^2$. For any $f \in L^2$ supported in a compact subinterval of $(0, \infty)$,*

$$\begin{aligned} &\Big\| \int_0^\infty e^{-i\lambda^2 t + i\lambda x - i(2\lambda)^{-1} \int_0^x V} f(\lambda)\,d\lambda \Big\|_{L^2(\mathbb{R}^+)} \\ &\qquad + \Big\| \int_0^\infty e^{-i\lambda^2 t + i\lambda x - i(2\lambda)^{-1} \int_0^{2\lambda|t|} V} f(\lambda)\,d\lambda \Big\|_{L^2(\mathbb{R}^+)} \to 0 \text{ as } t \to -\infty. \end{aligned} \tag{7.1}$$

An analogous statement holds as $t \to +\infty$, and follows by taking complex conjugates.

Proof. Since the mapping $f \mapsto \int_0^\infty e^{-i\lambda^2 t + i\lambda x} e^{-i(2\lambda)^{-1} \int_0^x V} f(\lambda)\,d\lambda$ is bounded from $L^2([\rho, \infty))$ to $L^2(\mathbb{R}^+)$ for every $\rho > 0$, and since the variant defined by replacing $\int_0^x V$ by $\int_0^{2\lambda t} V$ is unitary, it suffices to prove this merely for $f \in C_0^\infty$, which we assume henceforth.

Let $\Lambda \Subset \mathbb{R}^+$ be the support of f. By integrating by parts once with respect to λ, integrating $\exp(i[-\lambda^2 t + \lambda x])$ and differentiating the rest, and by moving absolute value signs inside the integral, we obtain a pointwise in (x,t) bound

$$(7.2) \quad |(7.1)| \le C \int_\Lambda |x - 2\lambda t|^{-1}\big(1 + |\textstyle\int_0^x V| + |\int_0^{2\lambda|t|} V| + |tV(2\lambda t)|\big)\, d\lambda$$
$$\le C(|x| + |t|)^{-1}\big(\big(1 + |\textstyle\int_0^x V| + |\int_0^{C|t|} V|\big)$$

for the integrands in (7.1). For $t \le -1$, $|x - 2\lambda t|^{-1} \sim (x + |t|)^{-1}$, which tends to zero in $L^2(\mathbb{R}^+, dx)$. Thus the first term behaves as desired. The second and third terms do also, if $V \in L^1$. For $V \in L^2$, $x^{-1}\int_0^x V \in L^2$ as well, so the term $(x + |t|)^{-1}\int_0^x V$ is an L^2 function times $x/(x + |t|)$, hence $\to 0$ in L^2 norm. Lastly, $\int_0^{C|t|} V = o(|t|^{1/2})$, while $|t|^{1/2}/(x + |t|)$ is $O(1)$ in $L^2(\mathbb{R}^+)$. □

Lemma 7.2. *Let $V \in L^1 + L^2$. For any $f \in L^2$ supported in a compact subinterval of $(0, \infty)$,*

(7.3)
$$\Big\| \int_0^\infty e^{-i\lambda^2 t + i\lambda x}\Big[e^{-i(2\lambda)^{-1}\int_0^x V} - e^{-i(2\lambda)^{-1}\int_0^{2\lambda|t|} V}\Big] f(\lambda)\, d\lambda \Big\|_{L^2(\mathbb{R}^+)} \to 0 \ \textit{as}\ t \to +\infty.$$

Proof. As in the preceding lemma, it suffices to prove this for $f \in C_0^\infty(\mathbb{R}^+)$. Fix a cutoff function $\eta \in C_0^\infty(\mathbb{R})$ supported on $(-2, 2)$ and $\equiv 1$ on $[-1, 1]$, with $0 \le \eta \le 1$. Let ϵ be a strictly positive function, such that $\epsilon(t) \to 0$ as $t \to +\infty$, at a rate to be specified below. Set

$$(7.4) \qquad \eta(\lambda, t, x) = \eta(\epsilon(t) t^{-1/2}(x - 2\lambda t)).$$

Consider the function of (x,t)

$$(7.5) \qquad \int_0^\infty e^{-i\lambda^2 t + i\lambda x}\Big[e^{-i(2\lambda)^{-1}\int_0^x V} - e^{-i(2\lambda)^{-1}\int_0^{2\lambda t} V}\Big] f(\lambda)\, \eta(\lambda, t, x)\, d\lambda\ .$$

We have

$$(7.6) \qquad \begin{aligned} |(7.5)| &\le C \int \int_{|y - 2\lambda t| \le 2\epsilon(t)^{-1} t^{1/2}} |V(y)|\, dy \chi_{|x - 2\lambda t| \le 2\epsilon(t)^{-1} t^{1/2}}\, d\lambda \\ &\le C \int \int_{|y - x| \le 4\epsilon(t)^{-1} t^{1/2}} |V(y)|\, dy \chi_{|x - 2\lambda t| \le 2\epsilon(t)^{-1} t^{1/2}}\, d\lambda \\ &\le C\epsilon(t)^{-1} t^{-1/2} \int_{|y - x| \le 4\epsilon(t)^{-1} t^{1/2}} |V(y)|\, dy \\ &\le C\epsilon(t)^{-2} MV(x), \end{aligned}$$

where M is the maximal function of Hardy and Littlewood. Observe that the restriction $\eta(\lambda, t, x) \ne 0$ for some $\lambda \in \Lambda$ implies that $ct \le x \le Ct$ for some $c, C \in \mathbb{R}^+$ depending only on Λ, provided $\epsilon(t) \ll t^{1/2}$, and moreover that V may be replaced by its restriction to such an interval. If $V \in L^2$ then the L^2 norm of M applied to the restriction of V to such an interval tends to zero as $t \to \infty$, and by choosing $\epsilon(t)$ to tend to zero sufficiently slowly we find that $(7.5) \to 0$ in L^2. For $V \in L^1$ we have the pointwise bound $C\epsilon(t)^{-1}t^{-1/2}\int_{ct}^{Ct}|V|$, which is $o(1)\cdot\epsilon(t)^{-1}$ in $L^2(x\sim t)$; this tends to zero provided $\epsilon(t)$ does so sufficiently slowly. In the general case $V\in L^1+L^2$, we decompose (7.5) into two parts, and estimate them separately.

There remains the contribution of $1-\eta(\lambda,t,x)$:

$$\int_0^\infty e^{-i\lambda^2 t+i\lambda x}f(\lambda)\Big(e^{-i(2\lambda)^{-1}\int_0^x V}-e^{-i(2\lambda)^{-1}\int_0^{2\lambda t}V}\Big)(1-\eta)(\lambda,t,x)\,d\lambda. \tag{7.7}$$

We would like to apply to it the method of stationary phase. However, the phase function $-t\lambda^2+\lambda x-(2\lambda)^{-1}\int_0^{2\lambda t}V$ is not well behaved; its partial derivative with respect to λ is $x-2\lambda t+(2\lambda)^{-2}(\int_0^{2\lambda t}V)-(2\lambda)^{-1}2tV(2\lambda t)$, and the final term is not well under control unless one assumes $V(x)=O(|x|^{-1/2})$. Instead, we integrate by parts, integrating $\exp(i[\lambda x-\lambda^2 t])$ and differentiating the rest, to obtain

$$i\int_0^\infty e^{-i\lambda^2 t+i\lambda x}f(\lambda)\frac{\partial}{\partial\lambda}\Big[(x-2\lambda t)^{-1}\Big(e^{-i(2\lambda)^{-1}\int_0^x V}-e^{-i(2\lambda)^{-1}\int_0^{2\lambda t}V}\Big)(1-\eta)(\lambda,t,x)\Big]\,d\lambda. \tag{7.8}$$

When the derivative is expanded according to Leibniz's rule, various terms result. The main terms are those in which $\partial/\partial\lambda$ acts on either of the two exponentials $\exp(i(2\lambda)^{-1}(\int_0^* V))$, and we discuss these first. One such term is a constant multiple of

$$\begin{aligned}\int_0^\infty & e^{-i\lambda^2 t+i\lambda x-i(2\lambda)^{-1}\int_0^{2\lambda t}V}f(\lambda)(x-2\lambda t)^{-1}(1-\eta)(\lambda,t,x)\lambda^{-1}tV(2\lambda t)\,d\lambda\\ &=c\int_0^\infty e^{-i(y^2/4t)+i(xy/2t)-ity^{-1}\int_0^y V}\tilde f(y/2t)(x-y)^{-1}(1-\eta)(\lambda,t,x)v(y)\,dy\end{aligned} \tag{7.9}$$

where we have substituted $y=2\lambda t$ and written $\tilde f(\lambda)=\lambda^{-1}f(\lambda)\in C_0^\infty$, and where $v=V$ (for the present moment only). The integral operators with kernels $\exp(iAxy)(x-y)^{-1}\eta(\delta(x-y))$ are bounded on $L^2(\mathbb{R})$, uniformly in $A,\delta\in\mathbb{R}^+$. Thus the $L^2(dx)$ norm of this last expression is $O(\|v\|_{L^2})$. Moreover, since f is supported where $\lambda\ge\rho>0$, only the restriction of v to $[2\rho t,\infty)$ comes into play, so we obtain a bound of $C\|v\|_{L^2([2\rho t,\infty))}$. This holds uniformly in all real-valued functions V appearing in the exponent.

There is also an easy alternative bound $O(t^{-1/4}\|v\|_{L^1})$, obtained directly by inserting absolute values inside the integral. Thus the general case $V\in L^1+L^2$ may be treated by decomposing v as a sum, and estimating the two terms separately.

Another term arising from (7.8) differs only in that $tV(2\lambda t)$ is replaced by $c\lambda^{-2}\int_0^{2\lambda t}V$; in terms of the new variable y, $v(y)$ is replaced by $t^{-1}\int_0^y v$. Since λ ranges over a compact interval Λ, y ranges over an interval $[ct,Ct]$ where $0<c<C<\infty$. Therefore $t^{-1}\int_0^y v$ is majorized in $L^q(dy)$ by $\|v\|_{L^q(d\lambda)}$, for every $1\le q\le\infty$. Thus the same reasoning applies to this term.

In the last of the main terms, the derivative falls on $\exp(-i(2\lambda)^{-1}\int_0^x V)$, and we obtain

$$c\big(\textstyle\int_0^x V\big)\displaystyle\int e^{-i\lambda^2 t+i\lambda x-i(2\lambda)^{-1}\int_0^x V}(x-2\lambda t)^{-1}f(\lambda)(1-\eta)(\lambda,t,x)\,d\lambda. \tag{7.10}$$

If $x \geq Ct$ where C is a sufficiently large constant, depending on Λ, this is majorized by $Cx^{-1}\big|\int_0^x V\big|$, which is $o(\|V\|_{L^2})$ in $L^2(|x| \geq Ct)$ as $t \to +\infty$. However, for $x \lesssim t$ and for $V \in L^2$, simple estimates on this quantity seem to miss the desired conclusion by a factor of $\log(t)$, so we integrate by parts again and move absolute values inside the integral to obtain an upper bound

$$(7.11)\quad |(7.10)| \leq C\big|{\textstyle\int_0^x} V\big| \cdot \int_\Lambda |x-2\lambda t|^{-2} \Big(\frac{t|1-\eta|}{|x-2\lambda t|} + |1-\eta| + C\epsilon t^{1/2}|\eta'(\epsilon t^{-1/2}(x-2\lambda t))| + \big|(1-\eta){\textstyle\int_0^x} V\big|\Big)\,d\lambda,$$

where $(1-\eta) \equiv (1-\eta)(\lambda,t,x)$ and $\epsilon \equiv \epsilon(t)$. Bearing in mind that now $x \leq Ct$, one term is

$$(7.12)\quad \leq C\big|{\textstyle\int_0^x} V\big|^2 \int_{|\lambda-(x/2t)|\geq \epsilon^{-1}t^{-1/2}} t^{-2}|\lambda-(x/2t)|^{-2}\,d\lambda \leq C\big|{\textstyle\int_0^x} V\big|^2 t^{-3/2}\epsilon \leq Ct^{-3/2}\epsilon x^{1/2}\big|{\textstyle\int_0^x} V\big| \leq C\epsilon(t)x^{-1}\big|{\textstyle\int_0^x} V\big|,$$

which is $o(\|V\|_{L^2})$ in L^2 norm. Another term is

$$(7.13)\qquad \leq C\big|{\textstyle\int_0^x} V\big| \int_{|\lambda-(x/2t)|\geq \epsilon^{-1}t^{-1/2}} t^{-2}|\lambda-(x/2t)|^{-3}\,d\lambda \leq C\Big|\int_0^x V\Big|\epsilon^2 t^{-1},$$

which again is $o(\|V\|_{L^2})$ in L^2 norm. The two remaining terms are both majorized by the discussion of this last term, thus completing the analysis of the case where $\partial/\partial\lambda$ acts on $\exp(-i(2\lambda)^{-1}\int_0^x V)$, if $V \in L^2$. In the general case $V \in L^1 + L^2$, the contribution of the L^1 part is better behaved and the details are left to the reader.

Another type of term arises when $\partial/\partial\lambda$ acts in (7.8) on $(1-\eta)(\lambda,t,x)$, producing $c\epsilon(t)t^{1/2}\cdot\eta'(\epsilon(t)t^{-1/2}(x-2\lambda t)$. The factor $\eta'(\epsilon(t)t^{-1/2}(x-2\lambda t)$ is supported where $|x-2\lambda t| \leq 2\epsilon(t)^{-1}t^{1/2}$, and $\epsilon(t)t^{1/2}/(x-2\lambda t) = O(1)$ in this region. Therefore the analysis given above for the contribution of $\eta(\lambda,t,x)$ applies equally well to this term.

When the derivative falls on $f(\lambda)$, we have gained a factor of $(x-2\lambda t)^{-1}$, and have an upper bound $C\int_\Lambda |x-2\lambda t|^{-1}\chi_{|x-2\lambda t|\geq\epsilon(t)^{-1}t^{1/2}}\,d\lambda$. For each $\lambda \in \Lambda$, the integrand is $O(t^{-1/4}\epsilon(t)^{1/2})$ in $L^2(\mathbb{R}^+)$.

There remains the term in which the derivative $\partial/\partial\lambda$ falls on $(x-2\lambda t)^{-1}$; this term is a constant multiple of

$$(7.14)\quad \int_0^\infty e^{-i\lambda^2 t+i\lambda x}\Big(e^{-i(2\lambda)^{-1}\int_0^{2\lambda t} V} - e^{-i(2\lambda)^{-1}\int_0^x V}\Big) f(\lambda)\frac{t}{(x-2\lambda t)^2}(1-\eta)(\lambda,t,x)\,d\lambda$$

and hence is majorized by $C\int_\Lambda t|x-2\lambda t|^{-2}\chi_{|\lambda-(x/2t)|\geq\epsilon(t)^{-1}t^{-1/2}}\,d\lambda \leq C\epsilon(t)t^{-1/2}$, which gives the desired L^2 bound for $x \leq C_0 t$, for any fixed C_0. Choosing C_0 to be sufficiently large, we can more simply majorize $|x-2\lambda t|^{-2}$ for $x \geq C_0 t$ by Cx^{-2} in the integrand, obtaining the pointwise bound Ct/x^2 for the integral, giving a contribution $\leq Ct^{-1/2}$ to the $L^2(dx)$ norm. □

8. Modified wave operators

8.1. **The half-line case.** We discuss modified wave operators for Schrödinger operators on the half line.

Recall from the introduction the modified wave operators (1.4)

$$\Omega^m_\pm f = \lim_{t\to\mp\infty} e^{itH_V} e^{-itH_0 \pm iW(H_0^{1/2},\mp t)} f$$

for all $f \in L^2(\mathbb{R}^+)$, where existence of the limit remains to be established. Here W is given by (1.3):

$$W(\lambda,t) = -(2\lambda)^{-1}\int_0^{2\lambda t} V(s)\,dx.$$

Motivating this definition is the heuristic that a wave packet with frequency $\approx \lambda$ should propagate with velocity $\approx 2\lambda$, so that for large t, $\int_0^x V$ should behave like $\int_0^{2\lambda t} V$.

Proof of Theorem 1.1. We write $A(t) \sim B(t)$ to mean that $\|A(t)-B(t)\|_{L^2(\mathbb{R}^+)} \to 0$ as $t \to +\infty$, or as $t\to-\infty$, depending on which limit is presently under discussion. Given $g \in L^2(\mathbb{R}^+)$, write

$$g(x) = \pi^{-1}\int_0^\infty \hat g(\lambda)[e^{i\lambda x} - e^{-i\lambda x}]\,d\lambda, \tag{8.1}$$

where this defines the modified Fourier transform $\hat g$, so that $\|\hat g\|_{L^2(\mathbb{R}^+)} = \sqrt{\pi}\|g\|_{L^2(\mathbb{R}^+)}$. Then

$$e^{-itH_0}g(x) = \pi^{-1}\int_0^\infty \hat g(\lambda)e^{-i\lambda^2 t}[e^{i\lambda x} - e^{-i\lambda x}]\,d\lambda, \tag{8.2}$$

and the modified evolution is
(8.3)

$$e^{-itH_0 - iW(H_0,t)}g(x) = \pi^{-1}\int_0^\infty \hat g(\lambda)e^{-i\lambda^2 t}\left[e^{i\lambda x - i(2\lambda)^{-1}\int_0^{2\lambda t}V} - e^{-i\lambda x - i(2\lambda)^{-1}\int_0^{2\lambda t}V}\right]d\lambda.$$

Define $\tilde g \in \mathcal{H}_{\mathrm{ac}}$ by

$$\tilde g(x) = \pi^{-1}\int_0^\infty \hat g(\lambda)\psi(x,\lambda)\,d\lambda. \tag{8.4}$$

The map $g \mapsto \tilde g$ is a unitary bijection from $L^2(\mathbb{R}^+)$ to $\mathcal{H}_{\mathrm{ac}}$. We aim to prove that $e^{-itH_0 - W(H_0,t)}g \sim e^{-itH_V}\tilde g$, as $t\to+\infty$, for any $g \in L^2(\mathbb{R}^+)$. By unitarity of the evolutions, this implies the existence of the modified wave operator Ω^m_-, and likewise that it is an isometric bijection from $L^2(\mathbb{R}^+)$ to $\mathcal{H}_{\mathrm{ac}}$. Moreover, by unitarity, it suffices to prove this under the assumption that $\hat g \in C_0^\infty(\mathbb{R}^+)$, which we assume henceforth.

Now

$$\begin{aligned} e^{-itH_0 - iW(H_0,t)}g(x) &= \pi^{-1}\int_0^\infty \hat g(\lambda)e^{-i\lambda^2 t}\left[e^{i\lambda x - i(2\lambda)^{-1}\int_0^{2\lambda t}V} - e^{-i\lambda x - i(2\lambda)^{-1}\int_0^{2\lambda t}V}\right]d\lambda \\ &\sim \pi^{-1}\int_0^\infty \hat g(\lambda)e^{-i\lambda^2 t}e^{i\lambda x - i(2\lambda)^{-1}\int_0^{2\lambda t}V}\,d\lambda \end{aligned}$$

by taking complex conjugates in Lemma 7.1. On the other hand,

$$\begin{aligned} e^{-itH_V}\tilde{g} &= \pi^{-1}\int_0^\infty \hat{g}(\lambda)e^{-i\lambda^2 t}\psi(x,\lambda)\,d\lambda \\ &= \pi^{-1}\int_0^\infty \hat{g}(\lambda)e^{-i\lambda^2 t}\big(u(x,\lambda^2) - \frac{\bar{\gamma}(\lambda^2)}{\gamma(\lambda^2)}\overline{u(x,\lambda^2)}\big)\,d\lambda \\ &\sim \pi^{-1}\int_0^\infty \hat{g}(\lambda)e^{-i\lambda^2 t}\big(e^{i\lambda x - i(2\lambda)^{-1}\int_0^x V} - \frac{\bar{\gamma}(\lambda^2)}{\gamma(\lambda^2)}e^{-i\lambda x + i(2\lambda)^{-1}\int_0^x V}\big)\,d\lambda \\ &\sim \pi^{-1}\int_0^\infty \hat{g}(\lambda)e^{-i\lambda^2 t}e^{i\lambda x - i(2\lambda)^{-1}\int_0^x V}\,d\lambda. \\ &\sim \pi^{-1}\int_0^\infty \hat{g}(\lambda)e^{-i\lambda^2 t}e^{i\lambda x - i(2\lambda)^{-1}\int_0^{2\lambda t} V}\,d\lambda. \end{aligned}$$

by first Corollary 6.3, then Lemma 7.1, and then Lemma 7.2. Thus we have the asymptotic relation $e^{-itH_0 - iW(H_0,t)}g \sim e^{-itH_V}\tilde{g}$, as $t \to +\infty$, so $\Omega_-^m = U_V^{-1} \circ U_0$.

The analysis as $t \to -\infty$ is the same, except for the appearance of the unimodular factor $\overline{\gamma(\lambda^2)}/\gamma(\lambda^2)$. The conclusion is therefore that $\Omega_+^m = U_V^{-1} \circ [\overline{\gamma(\lambda^2)}/\gamma(\lambda^2)] \circ U_0$, where the inner factor denotes the operator defined on $L^2(\mathbb{R}^+, d\lambda)$ by multiplication by this function of λ.

Composing, we find that $S^m = (\Omega_-^m)^{-1}\Omega_+^m = U_0^{-1} \circ [\overline{\gamma(\lambda^2)}/\gamma(\lambda^2)] \circ U_0$. □

Proof of Theorem 1.2. Suppose now that the improper integral $\int_0^\infty V(y)\,dy$ exists. The asymptotic behavior of V enters both into the definition of the modified evolution $e^{-itH_0 - W(H_0,t)}$, and into the definition of the scattering coefficients $\gamma(E)$. To sort this out, we change the definition of the phase $\xi(x,\lambda^2)$ to λx. We correspondingly modify the normalization of the generalized eigenfunctions $u(x,\lambda^2)$ to $u(x,\lambda^2) \sim e^{i\lambda x}$. The scattering coefficient $\gamma(\lambda^2)$ is now defined, for almost every $\lambda > 0$, by the relation (5.14) $\gamma(\lambda^2) = 1/\psi(0,\lambda)$. Writing $\gamma(\lambda^2) = |\gamma(\lambda^2)|e^{i\arg\gamma(\lambda^2)}$, we change the definition of $\psi(x,\lambda)$ to

$$\psi(x,\lambda) = e^{i\arg\gamma(\lambda^2)}u(x,\lambda^2) - e^{-i\arg\gamma(\lambda^2)}\overline{u(x,\lambda^2)}. \tag{8.5}$$

In the formal expressions for the spectral projectors and wave group, ψ is now replaced by this new ψ. Retracing the above analysis, we find that the wave operators exist, and

$$\begin{aligned} \Omega_- &= U_V^{-1} \circ e^{-i\arg\gamma(\lambda^2)} \circ U_0, \\ \Omega_+ &= U_V^{-1} \circ e^{i\arg\gamma(\lambda^2)} \circ U_0, \\ S &= U_0{}^{-1} \circ e^{2i\arg\gamma(\lambda^2)} \circ U_0. \end{aligned} \tag{8.6}$$

□

Remark. The exact expressions for the modified wave operators $\Omega_\pm^m$ depend on a choice of normalization of the solutions ψ; see below. These solutions can be modified by factors $e^{i\kappa(\lambda)}$, leading to different U_V and hence different looking expressions for the wave operators, like in (8.6) and in the proof of Theorem 1.1. However, the scattering matrix $S(\lambda)$ is invariant under choice of such normalization.

8.2. **The whole-line case.** In the full-axis Schrödinger case, the results are similar. One defines modified wave operators by

$$\Omega^m_\pm f = \lim_{t\to\mp\infty} e^{itH_V} e^{-iW_a(-i\partial_x,t)} f, \tag{8.7}$$

where ∂_x is the operator of one differentiation in x, and

$$W_a(\lambda,t) = \lambda^2 t + \frac{1}{2\lambda}\int_0^{2\lambda t} V(s)\,ds. \tag{8.8}$$

The modified free evolution operator can be written as (compare with (5.25))

$$e^{-iW_a(-i\partial_x,t)} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t + i\lambda x - \frac{i}{2\lambda}\int_0^{2\lambda t} V(s)\,ds} \hat f(\lambda)\,d\lambda, \tag{8.9}$$

where $\hat f(\lambda) = \int \exp(-i\lambda x) f(x)\,dx$ is the Fourier transform of f. Recall the representation (5.34) for the perturbed evolution (we set here $E = \lambda^2$)

$$e^{-iH_V t} f(x) = \frac{1}{2\pi}\int_0^{\infty} e^{-i\lambda^2 t}(\psi_+(x,\lambda)\tilde f_+(\lambda) + \psi_-(x,\lambda)\tilde f_-(\lambda))\,d\lambda. \tag{8.10}$$

Denote by U_V the operator $f \mapsto (f_+, f_-)$, where $f_\pm(\lambda) = \int_{\mathbb{R}} \overline{\psi}_\pm(x,\lambda) f(x)\,dx$. Denote by $\mathcal{H}_{\mathrm{ac}}$ the maximal closed subspace of $\mathcal{H} = L^2(\mathbb{R})$ on which H_V has purely absolutely continuous spectrum.

Theorem 8.1. *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. Then for every $f \in L^2$, the limits in (8.7) exist in $L^2(\mathbb{R})$ norm as $t \to \mp\infty$. The modified wave operators $\Omega^m_\pm$ thus defined are surjective and unitary from $L^2(\mathbb{R})$ to $\mathcal{H}_{ac}$. One has*

$$\begin{aligned} \Omega^m_+ &= U_V^{-1} U_0, \\ \Omega^m_- &= U_V^{-1} S(\lambda)^{-1} U_0, \\ S^m &= U_0{}^{-1} S(\lambda) U_0 \end{aligned} \tag{8.11}$$

where $S(\lambda)$ denotes multiplication by the scattering matrix

$$S(\lambda) = \begin{pmatrix} t_1(\lambda) & -\overline{r}_1(\lambda)\frac{t_1(\lambda)}{\overline{t}_1(\lambda)} \\ r_1(\lambda) & t_1(\lambda) \end{pmatrix}. \tag{8.12}$$

As in the half-line case, we could make the wave operators look more symmetric by modifying the definition of U_V. Let $A = \sqrt{S}$ be any matrix square root of the unitary operator S. Then if we were to define $\tilde U_V = AU_V$, we would obtain $\Omega^m_+ = \tilde U_V{}^{-1} A^* U_0$, and $\Omega^m_- = \tilde U_V{}^{-1} A U_0$.

We sketch the asymptotic analysis of $e^{-itH_V} g$, as $|t| \to \infty$, for arbitrary $g \in \mathcal{H}_{\mathrm{ac}}$. We proceed formally; all steps are justified as in the preceding subsection. Write $g = (2\pi)^{-1}\int_0^\infty \left[\hat g_+(\lambda)\psi_+(x,\lambda) + \hat g_-(\lambda)\psi_-(x,\lambda)\right] e^{-i\lambda^2 t}\,d\lambda$ with $2\pi\|g\|^2_{L^2} = \|\hat g_+\|^2_{L^2(\mathbb{R}^+)} + |\hat g_-\|^2_{L^2(\mathbb{R}^+)}$. Define

$$\begin{aligned} \Phi_+(\lambda,x,t) &= \quad \lambda x - (2\lambda)^{-1}\textstyle\int_0^{2\lambda t} V, \\ \Phi_-(\lambda,x,t) &= -\lambda x + (2\lambda)^{-1}\textstyle\int_0^{-2\lambda t} V. \end{aligned} \tag{8.13}$$

Formally,

$$e^{-itH_V}\psi_+(x,t) \sim \begin{cases} \big[t_1(\lambda)e^{i\Phi_+(\lambda,x,t)} + r_1(\lambda)e^{i\Phi_-(\lambda,x,t)}\big]e^{-i\lambda^2 t} & \text{as } t\to+\infty, \\ e^{i\Phi_+(\lambda,x,t)}e^{-i\lambda^2 t} & \text{as } t\to-\infty, \end{cases} \tag{8.14}$$

and

$$e^{-itH_V}\psi_-(x,t) \sim \begin{cases} \big[t_2(\lambda)e^{i\Phi_-(\lambda,x,t)} + r_2(\lambda)e^{i\Phi_+(\lambda,x,t)}\big]e^{-i\lambda^2 t} & \text{as } t\to+\infty, \\ e^{i\Phi_-(\lambda,x,t)}e^{-i\lambda^2 t} & \text{as } t\to-\infty, \end{cases} \tag{8.15}$$

Therefore as $t\to+\infty$,

$$\begin{aligned} e^{-itH_V}(2\pi)^{-1}&\int_0^\infty \big[\hat g_+(\lambda)\psi_+(x,\lambda)+\hat g_-(\lambda)\psi_-(x,\lambda)\big]e^{-i\lambda^2 t}\,d\lambda \\ &\sim (2\pi)^{-1}\int_0^\infty \Big(\hat g_+(\lambda)\big[t_1(\lambda)e^{i\Phi_+(\lambda,x,t)}+r_1(\lambda)e^{i\Phi_-(\lambda,x,t)}\big] \\ &\qquad + \hat g_-(\lambda)\big[t_2(\lambda)e^{i\Phi_-(\lambda,x,t)}+r_2(\lambda)e^{i\Phi_+(\lambda,x,t)}\big]\Big)e^{-i\lambda^2 t}\,d\lambda. \end{aligned} \tag{8.16}$$

As $t\to-\infty$, we have instead the asymptotics

$$\sim (2\pi)^{-1}\int_0^\infty \big(\hat g_+(\lambda)e^{i\Phi_+(\lambda,x,t)}+\hat g_-(\lambda)e^{i\Phi_-(\lambda,x,t)}\big)e^{-i\lambda^2 t}\,d\lambda. \tag{8.17}$$

Here "$\sim$" signifies asymptotic equality in $L^2(\mathbb{R},dx)$. Thus for any function $f(\lambda) = (f_+(\lambda), f_-(\lambda))^T$ taking values in $L^2(\mathbb{R}^+,\mathbb{C}^2)$,

$$e^{-itH_V}U_V{}^{-1}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \sim \begin{cases} e^{-iW_a(-i\partial_x,t)}U_0{}^{-1}\begin{pmatrix} t_1(\lambda) & r_2(\lambda) \\ r_1(\lambda) & t_2(\lambda)\end{pmatrix}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} & \text{as } t\to+\infty, \\ e^{-iW_a(-i\partial_x,t)}U_0{}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} & \text{as } t\to-\infty. \end{cases} \tag{8.18}$$

Inverting this and exploiting the identities relating the various scattering coefficients to simplify the resulting formula, we obtain

$$e^{-iW_a(-i\partial_x,t)}U_0{}^{-1}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \sim \begin{cases} e^{-itH_V}U_V{}^{-1}\begin{pmatrix} \bar t_1 & \bar r_1 \\ -(r_1\bar t_1/t_1) & \bar t_1\end{pmatrix}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} & \text{as } t\to+\infty, \\ e^{-itH_V}U_V{}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} & \text{as } t\to-\infty. \end{cases} \tag{8.19}$$

with asymptotic equality holding in $L^2(\mathbb{R}^+,\mathbb{C}^2,d\lambda)$ norm. The conclusions of the theorem follow directly.

As in the half-line case, we obtain a cleaner conclusion under the additional hypothesis that the improper integrals $\int_0^{\pm\infty} V(x)\,dx$ exist. Modify the definitions of the scattering coefficients by redefining

$$\psi_+(x,\lambda) = \begin{cases} t_1(\lambda)e^{i\lambda x} + o(1), & x \to +\infty, \\ \left(e^{i\lambda x} + r_1(\lambda)e^{-i\lambda x}\right) + o(1), & x \to -\infty \end{cases} \tag{8.20}$$

$$\psi_-(x,\lambda) = \begin{cases} t_2(\lambda)e^{-i\lambda x} + o(1), & x \to -\infty, \\ \left(e^{-i\lambda x} + r_2(\lambda)e^{i\lambda x}\right) + o(1), & x \to +\infty. \end{cases} \tag{8.21}$$

Theorem 8.2. *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$, and suppose that both the improper integrals $\int_0^{\pm\infty} V(x)\,dx$ exist. Then for every $f \in L^2(\mathbb{R})$, the two limits*

$$\Omega_\pm f = \lim_{t\to\mp\infty} e^{itH_V}e^{-itH_0}f \tag{8.22}$$

exist in $L^2(\mathbb{R})$ norm, and the operators thus defined are surjective and unitary from $L^2(\mathbb{R})$ to $\mathcal{H}_{ac}$. One has

$$\begin{aligned} \Omega_+ &= U_V^{-1}U_0, \\ \Omega_- &= U_V^{-1}S(\lambda)U_0, \\ S &= U_0{}^{-1}S(\lambda)U_0 \end{aligned} \tag{8.23}$$

where $S(\lambda)$ is defined in terms of the modified scattering coefficients $t_j(\lambda), r_j(\lambda)$ defined in (8.20),(8.21)*, and U_V is defined as before but in terms of $\psi_\pm$ as in* (8.20)*,* (8.21)*.*

8.3. **The Dirac case.** Consider next the operator D_V on $\mathcal{H} = L^2(\mathbb{R})$. Because no phase correction $\int_0^x V$ appears in the generalized eigenfunction asymptotics, there is no need for modification of the standard wave operators in this case.

We calculate

$$\begin{aligned} e^{-itD_V}U_V{}^{-1}G \sim 4^{-1}\int_{\mathbb{R}} e^{-iEt}\Big[G_+(E)\Big(t_1(E)e^{iEx}\begin{pmatrix}1\\ i\end{pmatrix} + r_1(E)e^{-iEx}\begin{pmatrix}1\\ -i\end{pmatrix}\Big) \\ + G_-(E)\Big(t_2(E)e^{-iEx}\begin{pmatrix}1\\ -i\end{pmatrix} + r_2(E)e^{-iEx}\begin{pmatrix}1\\ i\end{pmatrix}\Big)\Big]\,dE \end{aligned} \tag{8.24}$$

as $t \to +\infty$, and

$$\sim 4^{-1}\int_{\mathbb{R}} e^{-iEt}\Big(G_+(E)e^{iEx}\begin{pmatrix}1\\ i\end{pmatrix} + G_-(E)e^{-iEx}\begin{pmatrix}1\\ -i\end{pmatrix}\Big)\,dE \tag{8.25}$$

as $t \to -\infty$.

Theorem 8.3. *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. Then for each $f \in L^2(\mathbb{R})$, both of the limits*

$$\Omega_\pm f = \lim_{t\to\mp\infty} e^{-itD_V}e^{itD_0}f \tag{8.26}$$

exist in $L^2(\mathbb{R})$ norm, and define surjective and unitary operators to $\mathcal{H}_{ac}$. Moreover $\Omega_+ = U_V^{-1}U_0$ and $\Omega_- = U_V^{-1}S(E)^{-1}U_0$, where $S(E)$ is defined as before by (8.12)*. The scattering operator S is equal to $U_0^{-1}S(E)U_0$.*

9. Asymptotic completeness

Let us denote by $\mathcal{H}_{\mathrm{pp}}(A)$ the pure point subspace of the self-adjoint operator A. Recall from [35]

Definition 9.1. The (modified) wave operators $\Omega_\pm$ are called asymptotically complete if their ranges coincide with the orthogonal complement of $\mathcal{H}_{\mathrm{pp}}(H_V)$.

In our context, since the absolutely continuous spectrum is under control, we have

Corollary 9.1. *The wave operators of Theorems 1.1 and 1.2 are asymptotically complete if and only if the singular continuous spectrum of the operator H_V is empty.*

Recently, it has been shown that imbedded singular continuous spectrum (supported on a set of Hausdorff dimension zero) can occur for potentials decaying arbitrarily more slowly than $(1+|x|)^{-1}$ [24]. The condition on the rate of decay is the same as in the earlier examples with embedded dense point spectrum on $\mathbb{R}^+$ [30, 40]. Thus in general asymptotic completeness may fail for potentials in L^p, $p>1$. In a recent work [18], existence of potentials in L^p, $1<p<2$ leading to singular continuous spectrum of positive dimension is proven; L^2 case has been considered earlier in [17]. There also exist estimates on the size of the set where the singular spectrum may be supported for potentials with some power decay [37, 11] and examples of operators which have decaying solutions on a half axis for a set of spectral parameters having exactly the right dimension [38, 29].

Nevertheless, there are some settings where generic (in some sense) asymptotic completeness can be inferred. We discuss two such cases: almost surely for certain random models, and for almost every boundary condition with any fixed potential. In both of these cases, the spectrum is purely absolutely continuous where $E>0$, so asymptotic completeness in the relatively weak sense in which we have defined it implies a stronger form.

Let us say that the half-axis operator H_V^β has boundary condition β at the origin if $u(0)+\beta u'(0)=0$ for any function u in the domain of H_V^β.

Theorem 9.2. *Let H_V^β be a Schrödinger operator defined on a half-axis with the boundary condition β. Assume that $V\in L^1+L^p$ for some $1<p<2$. Then for almost every β, the modified wave operators defined in* (1.4) *are asymptotically complete. Moreover if $\int_0^\infty V(x)\,dx$ exists, then the usual wave operators defined by* (1.2) *are asymptotically complete.*

Proof. The result is a simple corollary of Theorems 1.1, 1.2, and general rank one perturbation theory. Theorem 1.1 and well-known results of scattering theory imply that the absolutely continuous part of the spectral measure of H_V corresponding to some boundary condition β_0, $\mu_{\mathrm{ac}}^{\beta_0}$, fills all of R^+; that is, $D\mu_{\mathrm{ac}}^{\beta_0}(E)>0$ for a.e. E. The variation of the boundary condition can be regarded as rank one perturbation [41], Section I.6. The standard rank one perturbation theory then implies that for any β, the singular part μ_{s}^{β} of the spectral measure can only be supported on a fixed set of energies $S\subset\mathbb{R}^+$ of zero Lebesgue measure [41], Theorem II.2. But then again by rank one theory for almost every β the singular spectrum on $\mathbb{R}^+$ is empty [41], Theorem I.8. Since the potential belongs to $L^1 + L^p$, the spectrum below zero is discrete (with only 0 as a possible accumulation point). $\square$

Next, let us consider the following random model:

$$V(x) = \sum_{n=1}^{\infty} a_n(\omega) g(n) f(x-n), \tag{9.1}$$

where $f(x) \in C_0^\infty(0,1)$, $g \in l^p$ for some $p < 2$, and $a_n(\omega)$ are independent identically distributed bounded random variables with zero expectation. We have

Theorem 9.3. *Let H_V be a one dimensional Schödinger operator with random potential* (9.1)*. Then with probability one, the wave operators $\Omega_\pm$ exist and are asymptotically complete.*

Indeed, Theorem 9.1 of [26] shows that almost surely, the spectrum of the Schrödinger operator with potential (9.1) is purely absolutely continuous on $\mathbb{R}^+$. Notice that our assumptions on the potential easily imply that the improper integral $\int_0^\infty V$ exists almost surely (since the random numerical series $\sum_n a_n(\omega) g_n$) converges almost surely). Theorem 1.1 and decay of the potential then imply asymptotic completeness.

This illustrates another type of situation where asymptotic completeness holds. We remark that the result holds in a variety of more general random models for which (9.1) is just an illustration. For a more general setting, see [26].

Discussion. The above is only one of various possible definitions of asymptotic completeness. The notion of asymptotic completeness is intended to describe a situation where the Hilbert space is split into two orthogonal subspaces [35]: $\mathcal{H}_{\rm pp}(H_V)$ and the range of wave operators, $\mathcal{H}_{\rm ac}(H_V)$. On $\mathcal{H}_{\rm ac}(H_V)$ the perturbed dynamics is close to the modified free evolution at large times, and corresponds to the scattering states. On $\mathcal{H}_{\rm pp}(H_V)$ the dynamics is supposed to be bounded in some sense. However, the intuitive physical assumption that pure point spectrum leads to dynamics which is bounded needs to be clarified, and in recent years there have appeared examples with very non-trivial transport on the pure point component. A widely accepted way to calibrate transport properties is to consider evolution of the averaged moments of coordinate operator:

$$\langle\langle |X|^m \rangle_\phi \rangle_T = \frac{1}{T} \int_0^T |\langle e^{-iHt}\phi, |X|^m e^{-iHt}\phi\rangle| \, dt, \tag{9.2}$$

where ϕ is the initial state, $\langle \phi_1, \phi_2 \rangle$ is the inner product and $|X|^m$ the operator of multiplication by $(|x|+1)^m$ in coordinate representation. The paper [15] contains an example of a (discrete) Schrödinger operator h with pure point spectrum and exponentially decaying eigenfunctions, such that

$$\limsup_{t\to\infty} \langle\langle |X|^2 \rangle_{\delta_0} \rangle_T / T^\alpha = \infty \tag{9.3}$$

for any $\alpha < 2$. Here the initial state is δ_0, the vector localized at the origin. Given that the rate of growth T^2 for the second moment corresponds to ballistic motion, as for the free Laplace operator, this example shows that in some sense the transport associated with point spectrum can be very fast.

However, there is still an important difference between transport associated with point and singular continuous spectrum. Namely, let B_R denote the ball of radius R centered at the origin and B_R^c its complement. Let $P_{\rm pp}$ and $P_{\rm c}$ be the orthogonal projections on $\mathcal{H}_{\rm pp}(H_V)$ and the continuous subspace $\mathcal{H}_{\rm c}(H_V)$ respectively. Then for any $\epsilon > 0$ there exists R_ϵ such that

$$\|e^{-iH_Vt}P_{\rm pp}\phi\|^2_{L^2(B^c_{R_\epsilon})} < \epsilon \tag{9.4}$$

for all t. The growth of the moments in (9.3) is achieved not because of the motion of the whole wavepacket, but because of thin tails escaping to infinity. On the other hand, we have

$$\frac{1}{T}\int\limits_0^T \|e^{-iH_Vt}P_{\rm c}\phi\|^2_{L^2(B_R)}\,dt \stackrel{T\to\infty}{\longrightarrow} 0 \tag{9.5}$$

for any finite R. Equation (9.5) is one of the statements of the RAGE theorem (see, e.g. [14]) and is basically a corollary of Wiener's theorem on Fourier transforms of measures. Moreover, there exist examples of Schrödinger operators [25] in which the dynamics corresponding to the singular continuous subspace is almost ballistic in a sense that the whole wavepacket is moving to infinity at a fast rate: for any $\rho > 0$ there exists C_ρ such that

$$\frac{1}{T}\int\limits_0^T \|e^{-iH_Vt}\phi\|^2_{L^2(B_{C_\rho T^{1-\epsilon}})}\,dt < \rho$$

for all T and ϕ lying in the singular continuous subspace of H_V.

10. Potentials in $\ell^p(L^1)$

Following [9], we sketch here the small modifications needed to extend the analysis of potentials in L^p to those in $L^1 + L^p$, and indeed those in the larger class $\ell^p(L^1)$. A locally integrable function f is said to belong to the amalgamated space $\ell^p(L^1)$ if

$$\sum_{n\in\mathbb{Z}} \Big(\int_n^{n+1} |f(x)|dx\Big)^p < \infty.$$

The norm $\|f\|_{\ell^p(L^1)}$ is the p-th root of this expression. For any $1 \le p < \infty$, this defines the Banach space $\ell^p(L^1)$, which contains $L^1 + L^p$.

A martingale structure $\{E_j^m\}$ is said to be adapted to f in $\ell^p(L^1)$ if

$$\|f\cdot\chi_{E_j^m}\|^p_{\ell^p(L^1)} \le 2^{-m}\|f\|^p_{\ell^p(L^1)} \tag{10.1}$$

for all m, j. For any $f \in \ell^p(L^1)$, there does exist an adapted martingale structure [9].

Lemma 4.3 extends to $\ell^p(L^1)$: this Banach space is mapped boundedly to $L^{p'}(\Lambda)$ for any compact interval $\Lambda \Subset (0,\infty)$, for any $1 \le p \le 2$. The proof is essentially unchanged; see the analogous proof of Proposition 3.5 of [9].

Corollary 4.4 may now be refined by replacing $\|f\|_{L^p}$ by $\|f\|_{\ell^p(L^1)}$ on the right-hand side of each inequality. With these bounds for the operator G in hand, the remainder of the proof is unchanged.

11. Multilinear operators

The bounds derived for multiple integrals in Section 2 can be used to obtain Lebesgue space norm bounds for certain multilinear operators. Such bounds were not directly needed in our analysis of Schrödinger operators, but may be of some independent interest.

Consider a family of integral operators

$$T_i f(\lambda) = \int_{\mathbb{R}} K_i(\lambda, x) f(x)\, dx. \tag{11.1}$$

We assume throughout the discussion that each K_i is a locally integrable function of (λ, x), thus ensuring the convergence of these integrals for any f belonging to the L^p–dense class of all L^∞ functions having bounded essential supports.

Denote by $\|T\|_{p,q}$ the norm of T as an operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$. Consider associated multilinear operators

$$\mathcal{T}_n(f_1, f_2, \dots, f_n)(\lambda) = \iint_{x_1 \le x_2 \le \cdots \le x_n} \prod_{i=1}^n K_i(\lambda, x_i) f_i(x_i) dx_i \tag{11.2}$$

and their maximal variants

$$\mathcal{T}_n^*(f_1, f_2, \dots, f_n)(\lambda) = \sup_y \Big| \iint_{x_1 \le x_2 \le \cdots \le x_n \le y} \prod_{i=1}^n K_i(\lambda, x_i) f_i(x_i) dx_i \Big| . \tag{11.3}$$

Our assumptions ensure that these expressions define locally integrable functions of λ, whenever $f_1, \dots, f_n \in L^\infty$ have bounded supports.

Theorem 11.1. *For any $N < \infty$, $p < q$, and $2 \le q$, there exists $C_N < \infty$ such that for any $n \ge 2$ and any collection of operators $\{T_i : 1 \le i \le n\}$ of cardinality $\le N$, for any collection of functions $\{f_1, f_2 \dots, f_n\} \subset L^\infty$ of cardinality $\le N$ all having bounded supports,*

$$\|\mathcal{T}_n^*(f_1, \dots, f_n)\|_{L^r} \le \frac{C_N^{n+1}}{\sqrt{n!}} \|T_1\|_{2,2} \|f_1\|_{L^2} \prod_{i=2}^n \left(\|T_i\|_{p,q} \|f_i\|_{L^p} \right) \tag{11.4}$$

where

$$\frac{1}{r} = \frac{1}{2} + \frac{n-1}{q} . \tag{11.5}$$

Proof. It suffices to prove this under the assumption that $\|f_1\|_{L^2} = 1 = \|f_i\|_{L^p}$ for every $i \ge 2$. Construct a martingale structure $\{E_j^m\}$ for $\mathbb{R}$ so that

$$\int_{E_j^m} |f_1|^2 + \sum_{i\ge 2}^{*} \int_{E_j^m} |f_i|^p = 2^{-m} \int_{\mathbb{R}} \left(|f_1|^2 + \sum_{i \ge 2}^{*} |f_i|^p \right) \tag{11.6}$$

where the notation $\sum^{*}$ indicates that the sum is to be taken over a maximal set of indices i for which the functions f_i are distinct. Since all these functions have bounded supports, (11.6) guarantees that $\max_i \sup_j \int_{E_j^m} |f_i| \to 0$ as $m \to \infty$, by Hölder's inequality.

Denote by χ_j^m the characteristic function of E_j^m. Let $\delta > 0$ be a constant to be specified. By Proposition 2.5,

$$\mathcal{T}_n^*(f_1, \ldots, f_n)(\lambda) \leq \frac{C^{n+1}}{\sqrt{n!}} \tilde{G}(\lambda) G(\lambda)^{n-1} \tag{11.7}$$

where

$$\tilde{G}(\lambda) = \sum_{m=1}^{\infty} 2^{-\delta m} (\sum_j |T_1(f_1 \cdot \chi_j^m)(\lambda)|^2)^{1/2} \tag{11.8}$$

$$G(\lambda) = \sum_{m=1}^{\infty} 2^{\delta m} (\sum_j \max_i |T_i(f_i \cdot \chi_j^m)(\lambda)|^2)^{1/2}. \tag{11.9}$$

Now

$$\begin{aligned} \|\tilde{G}\|_{L^2} &\leq \sum_m 2^{-\delta m} (\sum_j \|T_1(f_1 \cdot \chi_j^m)\|_{L^2}^2)^{1/2} \\ &\leq \sum_m 2^{-\delta m} \|T_1\|_{2,2} (\sum_j \|f_1 \cdot \chi_j^m\|_{L^2}^2)^{1/2} \leq C_\delta \|T_1\|_{2,2} \|f_1\|_{L^2} \end{aligned} \tag{11.10}$$

provided that $\delta > 0$. As for G, we may majorize

$$G(\lambda) \leq \sum_i^{*} \sum_{m=1}^{\infty} 2^{\delta m} (\sum_j |T_i(f_i \cdot \chi_j^m)(\lambda)|^2)^{1/2}. \tag{11.11}$$

A simple computation parallel to the one on page 413 of [10] (see also the proof of Corollary 4.4 below) shows that since $p < 2 \leq q$, there exists $\varepsilon > 0$ such that

$$\|(\sum_j |T_i(f_i \cdot \chi_j^m)|^2)^{1/2}\|_{L^q} \leq C \|T_i\|_{p,q} 2^{-\varepsilon m}, \tag{11.12}$$

under the condition that $\|f_i \chi_j^m\|_{L^p}^p \leq C' 2^{-m}$ for all j, m. Choosing $\delta < \varepsilon$ and summing over m and i gives

$$\|G\|_{L^q} \leq C \sum_{i=1}^{N} \|T_i\|_{p,q} < \infty, \tag{11.13}$$

of course under the hypothesis that $\|f_i\|_{L^p} = 1$ for all $i \geq 2$. An application of Hölder's inequality concludes the proof. □

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