On The Simplest Trilinear Operators

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Talk will be sketchy. Full details available at Math. Research Letters 8 (2001)
or
`www.math.berkeley.edu/~mchrist/preprints.html`

Object of Study – Typical examples:

Trilinear operators:

$$T_\theta(f,g,h)(x) = \int_{|t|\le 1} f(x-t)g(x+t)h(x-\theta t)\,dt\ .$$

θ is any real parameter $\ne \pm 1$; at the outset it is unclear whether θ plays any essential role.
Here x, t both $\in \mathbb{R}^1$.
Simplest possible kernel: $\chi_{|t|\le 1}$.

General case: m-linear operators mapping functions on $\mathbb{R}^d$ to functions on $\mathbb{R}^d$:

$$T(f_1,\cdots,f_m)(x) = \int_{|t|\le 1} \prod_{j=1}^{m} f_j(S_j(x,t))\,dt\ ,$$

where $S_j : \mathbb{R}^{d+d} \mapsto \mathbb{R}^d$ are linear mappings.

Always assume nondegeneracy condition:
For any two indices $0 \le i \ne j$, the mapping $\mathbb{R}^{d+d} \ni (x,t) \mapsto (S_i(x,t), S_j(x,t))$ is invertible.
Here $S_0(x,t) = x$.

Simplest Possible Question

For which p is operator **defined** on $[L^p]^m$? Is there any extension beyond the obvious sufficient condition? (essentially posed by Kenig-Stein)

Positivity considerations $\Rightarrow$ reinterpretation: For which p is $T(f_1, \cdots, f_m)$ a.e. finite for all $f_j \in L^p$?

Many refinements of basic question possible:
- L^q as target space?
- Optimal range of p?
- How does p depend on $\{S_j\}$ (if at all)?
- Analogues in discrete Abelian groups?
- Analogues in noncommutative groups?
- Global analogues (replace $\chi_{|t|\le 1}$ by $\chi_{\mathbb{R}^d}$)
- For which $(p_1, \cdots, p_m)$ does T_m act on $L^{p_1} \otimes \cdots \otimes L^{p_m}$?

But the basic question comes first.

Why Care?

1. Motivation from work of Lacey-Thiele and Muscalu-Tao-Thiele. Trilinear and higher-order SIO not understood. Results obtained here have some small implications there (Thiele).

2. (perhaps) Unexpected subtleties. Diophantine issues. Character of extremals quite different from what might be guessed.

3. Connections with Kakeya problem (via Katz-Tao, Bourgain). Therefore makes link between Kakeya and trilinear (Hilbert) transforms.

4. Connections with additive number theory (via Bourgain, Gowers; cf. Ruzsa, Freiman).

5. Combinatorial aspects.

What's Obvious?

- Smaller $p \Rightarrow$ larger p.
- Positivity: May as well assume all $f_j \geq 0$.
- Real interpolation reduces matters to characteristic functions of sets.

$$\bullet\ L^1 \otimes L^\infty \otimes L^\infty \otimes \cdots \otimes L^\infty \mapsto L^\infty$$
$$\bullet\ L^1 \otimes L^1 \otimes L^\infty \otimes \cdots \otimes L^\infty \mapsto L^1$$

and all possible permutations.

- Multilinear interpolation $\Rightarrow [L^p]^m \mapsto L^q$ for all $p \geq m/2$, where $q = q(p) \geq 1$.

- $\Rightarrow$ Bilinear case trivial: $L^1 \otimes L^1$ is OK.

- Trilinear case: $[L^{3/2}]^3 \mapsto L^1$;
- $p = 3/2$ is the optimal exponent for $T_m(\cdot)$ to be locally L^1.
- More basic Question: Can we go below $L^{3/2}$?
- Thus we seek estimates in L^q for $\boxed{q < 1}$.

More Obvious Stuff

• **Conservation**:
$\|T(f_1, f_2, 1, \cdots, 1)\|_{L^1} = c\|f_1\|_{L^1} \cdot \|f_2\|_{L^1}$
for nonnegative functions (supported near 0).

• **Crucial case** is when $f_j = \chi_{E_j}$ and all the sets E_j have nearly comparable measures.

• A positive result for this case plus multilinear interpolation with trivial estimates would lead to positive result for general case.

• **Rearrangement**:
For $q \geq 1$, among characteristic functions of sets of given measure, L^q norm of $T_m(\chi_{E_1}, \cdots)$ is maximal when each E_j is a ball centered at 0 . . . .

• whereas among functions with given mass ($= L^1$ norm), the L^q norm is maximized for $q < 1$ by <u>dispersion</u>, not <u>concentration</u>.

The 1D Case

$$T_\theta(f,g,h)(x) = \int_{|t|\le 1} f(x-t)g(x+t)h(x-\theta t)\,dt$$

$$T_\infty(f,g,h)(x) = \int_{|t|\le 1} f(x-t)g(x+t)h(t)\,dt$$

• Any T is reducible to one of these by linear changes of coordinates.

• Convention: $\infty \in \mathbb{Q}$.

Main Theorem.

• If $\theta \in \mathbb{Q}$ then there exists $p < 3/2$ such that T_θ maps $[L^p]^3$ boundedly to L^q,
where $1 + q^{-1} = 3p^{-1}$.

• If $\theta \notin \mathbb{Q}$ then for any $p < 3/2$, there exist nonnegative functions $f_j \in L^p$ and a set E of positive measure such that
$T(f_1, f_2, f_3)(x) = +\infty$ for all $x \in E$.

On the Optimal Exponent p

For $\theta \in \mathbb{Q}$, we can go below $p = 3/2$; how far can we go?

• <u>Threshold exponent is always strictly > 1</u>: For any $\theta \in \mathbb{Q}$ there exist $p > 1$, functions $f_j \in L^p$, and a set E as above, such that $T(f_1, f_2, f_3)(x) = +\infty$ for all $x \in E$.

• <u>Threshold can be arbitrarily close to 3/2</u>: Given any $p < 3/2$, there exist rational θ, $0 \leq f_j \in L^p$, and a set E as above, such that for all $x \in E$, $T(f_1, f_2, f_3)(x) = +\infty$.

General Dimensions

Theorem.
Same statements for $d > 1$ as for $d = 1$, with "$\theta \in \mathbb{Q}$" replaced by "$\{S_j\}$ rationally commensurate".

Example. In $\mathbb{R}^2$,

$$\int f_1(x+t)f_2(x-t)f_3(x-\Theta t)\,dt$$

with

$$\Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is <u>not</u> rationally commensurate.

Set $S_0(x,t) \equiv x$.
Definition. Suppose that $\{S_j\}$ is nondegenerate. The collection of linear mappings $\{S_j : 0 \le j \le 3\}$ is <u>rationally commensurate</u> if there exist linear automorphisms h_j of $\mathbb{R}^d$ such that the vector subspace of endomorphisms of $\mathbb{R}^2$ generated by $\mathbb{Q}$-linear combinations of $\{h_j \circ S_j : 0 \le j \le 3\}$ has dimension two over $\mathbb{Q}$.

What I Don't Know

Nearly everything I haven't already told you, including:

- $T_4(f_1, \cdots, f_4)(x) = \int \prod_j f_j(x - \theta_j t)\, dt$

with $\theta_1 = 1, \theta_2 = -1$:
Trilinear analysis $\Rightarrow$ negative result when

$$\left|\theta_j - \frac{p_j}{q}\right| \leq \frac{C}{q^2}$$

for a <u>common sequence</u> of denominators $q \to \infty$.

Trilinear theorem $\Rightarrow$ positive result when **at least one** of θ_3, θ_4 is rational.

Left unresolved: Case where θ_3, θ_4 are both irrational and are not well approximable by rationals with the **same denominators**.

- What is the exact threshold index for trilinear operators? Don't know it in **any** case.

Discrete Analogues of Positive Results

$|S|$ denotes cardinality of any finite set S.

Theorem. Suppose that (m,k), $(m',k') \in \mathbb{Z}^2$ and that no two of the vectors $(1,0)$, $(0,1)$, (m,k), (m',k') are linearly dependent. Then there exists $p < 3/2$, depending only on m,k,m',k', such that for any torsion-free Abelian group G, for any finite subsets $A,B,C \subset G$, the multiplicity function

$$\mu(x) = |\{(a,b,c) \in A \times B \times C : \\ c = m'a + k'b \text{ and } x = ma + kb\}|$$

satisfies

$$\sum_{x \in G} \mu(x)^q \le |A|^{q/p} |B|^{q/p} |C|^{q/p}$$

where $1 + q^{-1} = 3p^{-1}$.

Descendant of results of Bourgain improved by Katz and Tao, descended in turn from Gowers.

Corollary. There exists $\varepsilon > 0$, depending only on m, k, m', k', such that for any torsion-free Abelian group G, any positive integer N, and any finite subsets $A, B, C \subset G$ such that $|A|, |B|, |C| \leq N$,

$$|\{ma + kb : (a, b, m'a + k'b) \in A \times B \times C\}| \leq N^{2-\varepsilon}.$$

The case where $(m, k) = (1, -1)$ and $(m', k') = (1, 1)$ was treated by Bourgain, and an improved estimate for ε was obtained by Katz and Tao.

One can either obtain these discrete analogues from the same proof as in the continuum case, or deduce discrete from continuum, or deduce continuum from discrete.

Elements

Elements of positive results:

• Realize that argument of Katz and Tao gives an $[L^p]^3 \mapsto L^q$ inequality after tiny modification.

• Adapt argument of K and T to general (m,k). This involves Diophantine-type properties of the pair (m,k), and involves passage to an auxiliary space of arbitrarily high dimension.

Elements of counterexamples:

• Three different counterexamples altogether.

• Arithmetic progressions.

• "Dispersion".

• Approximation of irrationals by rationals:

$$|\theta - (p/q)| \le q^{-2}.$$

• Cantor sets (= infinite Cartesian products of arithmetic progressions, projected back onto one dimension with different scaling for each coordinate).

Freiman's Theorem (Ruzsa's extension)

Denote by $A+B$ the sum-set $\{a+b : (a,b) \in A \times B\}$.

If A, B are finite subsets of a torsion-free Abelian group with $|A| \sim |B|$, and if

$$|A+B| \leq K|A|,$$

then A has arithmetic structure: $A \subset P$ for some arithmetic multi-progression P of complexity bounded by $C(K)$ and of cardinality $\leq C'(K)|A|$.

Corollary: the difference set $A-B$ also satisfies $|A-B| \lesssim |A|$.

In simplest discrete version of our problem we are given less:
A **subset** of $A \times B$ whose projection under $+$ has cardinality $\lesssim |A|$,
and we wish to conclude much less:
Its projection under $-$ has cardinality $\ll |A|^{2-\varepsilon}$.

Hence one expects that arithmetic progressions may play an important role or us.

1D Counterexample ($\theta \notin \mathbb{Q}$)
$T(A,B,C)(x) = \int \chi_A(x-t)\chi_B(x+t)\chi_C(x-\theta t)\,dt$: Write

$$\frac{1+\theta}{1-\theta} = \frac{p}{q} + O(q^{-2})$$

for a sequence $q \to \infty$.

Let $\delta \sim q^{-2}$. Set $A, B, C =$

$\boxed{\delta\text{–neighborhoods of arithmetic progressions}}$

of cardinality $\delta^{-1/2}$, with spacings p^{-1}, q^{-1}, and $y^{-1} = (1-\theta)/2q$ respectively.

Then

$$(1+\theta)\frac{i}{p} + (1-\theta)\frac{j}{q} = \frac{k}{y} + O(\delta)$$

where $k \in \mathbb{Z}$ if $i, j \in \mathbb{Z}$, which means that (as in Freiman's hypothesis)

$$\boxed{(1+\theta)A + (1-\theta)B \subset C.}$$

(in other words, if $(x-t, x+t) \in A \times B$ then $x - \theta t \in C$)

Counterexample, continued

On other hand $A+B$ itself is enormous:

$$\Big|\Big(\frac{i}{p}+\frac{j}{q}\Big)-\Big(\frac{i'}{p}+\frac{j'}{q}\Big)\Big| \geq \delta$$

if $(i,j) \neq (i',j')$,

So the $\sim q^2$ "blobs" of which $T_\theta(A,B,C)$ is a sum all have <u>disjoint supports</u> — very strong dispersion.

In fact the union of their supports has measure ~ 1, uniformly as $q \to \infty$.

Thus

$$|A| \sim |B| \sim |C| \sim q^{-1}$$

while

$$T_\theta(A,B,C) \sim q^{-2}$$

on a set of measure ~ 1.

Counterexample, continued

The first fact means that
$T(\chi_A, \chi_B, \chi_C) \equiv T(\chi_A, \chi_B, \chi_{\mathbb{R}})$
and hence
$\|T(\cdot)\|_{L^1}$ is <u>not</u> $\ll |A| \cdot |B| \sim (|A| \cdot |B| \cdot |C|)^{2/3}$;
the trivial $p = 3/2$ inequality is "saturated" in this example. Otherwise we'd have no chance.

The second fact is the dispersion: expanding $T(\cdot)$ in terms of the $\sim q^2$ "blobs" in $A \times B$ gives rise to $\sim q^2$ disjointly supported functions. The union of their supports has measure ~ 1. Hence any L^s norm, no matter how small s is, is $\sim$ the L^1 norm. Hence there is no improvement over the trivial $[L^{3/2}]^3 \mapsto L^1$ estimate.

This can be done for rational θ, but for only <u>finitely many scales</u>; for irrational θ it can be done for an infinite sequence of scales. The rate of approximation $|\tilde{\theta} - (p/q)| = O(q^{-2})$ is what is needed for the argument.

Sketch of Proof of Positive Results
(discrete case)
(strategy follows Katz-Tao improvement of Bourgain's lemma)

Given:

- Finite sets A, B, C
- Parameters $m, k \in \mathbb{Z}$ and λ
- Subset $G \subset A \times B$

- For each $d \in D$, define fibers

$$G_d = \{(a,b) \in G : ma - kb = d \text{ and } a + b \in C\}$$

and to study $\{d : |T(A,B,C)(d)| \sim \lambda\}$ assume

$$|G_d| \sim \lambda \text{ for all } d \in D.$$

• Form product space G^N for a certain N.
• Specify subset $S \subset G^N$ by imposing certain relations of the form

$$a_j + b_j = a_k + b_k, \qquad a_j = a_k, \qquad b_j = b_k$$

for certain pairs of indices $j \neq k$.

If $|A|, |B|, |C|$ are small then equality holds often, hence $|S|$ is large.

• Algebraic part: If the right relations are imposed then there is a key identity

$$ma_N - kb_N = \text{linear combination of } \{a_j, b_j : j < N\}$$

<u>modulo the relations</u> imposed.

Punch Line

If $\{a_j, b_j : j < N\}$ are known then key identity determines $\underline{a_N - b_N}$
and therefore
$\boxed{\text{nearly determines } \textbf{both } a_N \text{ and } b_N,}$
in the sense that both are members of specific sets of cardinality λ,
namely, projection onto A or B of the fiber G_d, where $d = a_N - b_N$.

- This yields an improved upper bound on $|S|$,
- hence by transitivity and bookkeeping an upper bound on $|G| \sim \lambda |D|$,
- hence an upper bound on $|D|$.

General $(\mathbf{m},\mathbf{k})$ aka θ

The scheme outlined is due (for $\lambda = 1$) to Katz and Tao, who also specified which relations are imposed. (They already used the product space G^8.) Bourgain had earlier proved a quantitatively weaker conclusion, by an argument similar in spirit.

I used a generalization of the Katz-Tao algebra as a "black box" to devise systems of relations which (over-simplified) express

$$ma_N - kb_N = \tilde{m}a_{N-M} - \tilde{k}b_{N-M} \text{ mod relations,}$$

where

$$(\tilde{m}, \tilde{k}) = (m-k, k),$$

One applies this recursively (with $m > k$ or with the roles of m, k interchanged) until one of m, k vanishes.

N, M depend on the number of reductions $(m,k) \mapsto (m-k,k)$ needed to achieve this vanishing and can be arbitrarily large. The larger they are, the smaller the improvement over the trivial exponent $p = 3/2$.

There is no reason to believe that the bound thus obtained is ever optimal, but a counterexample does show that the optimal exponent can indeed be arbitrarily close to 3/2.