# A T(B) THEOREM WITH REMARKS ON ANALYTIC CAPACITY AND THE CAUCHY INTEGRAL

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*Dedicated to Professor Antoni Zygmund on the occasion of his ninetieth birthday.*

## 0. Introduction.

The $T(b)$ theorem of David, Journé and Semmes [DJS] provides a criterion for the $L^2$ boundedness of singular integral operators. Our purpose is to establish a closely related criterion which was suggested by a question about analytic capacity of subsets of the complex plane. Like the $T(b)$ theorem, our criterion applies on arbitrary spaces of homogeneous type, a general setting for singular integral theory introduced by Coifman and Weiss [CW]. Its proof relies on a stopping–time argument; such arguments appear very frequently in real and harmonic analysis on Euclidean space, and often involve dyadic cubes. Spaces of homogeneous type come, in the abstract, with a bare minimum of structure, and so far as we know, no suitable analogue of the system of Euclidean dyadic cubes has yet been constructed in full generality. So our second purpose is to construct them. Lastly we shall give two applications, concerning the relation between positive analytic capacity and $L^2$ boundedness of the Cauchy integral on certain subsets of the complex plane. We shall also indicate the connection between our theorem and work of David [D1] concerning the Cauchy integral on certain curves in the complex plane, which helped to motivate it.

It is a pleasure to thank Guy David and John Garnett for very helpful conversations, and Peter Jones for insisting on the elimination of an extra hypothesis from the main theorem.

A constant with a subscript, such as $A_1$ or $C_3$, will retain its value throughout the article, while constants such as $C$ may change from one occurrence to the next.

## 1. Review.

Let us recall several standard definitions. A quasi–metric $\rho$ on a set $X$ is a function from $X \times X$ to $[0, \infty)$ satisfying the same conditions as a metric, except that the triangle inequality is weakened to

$$\rho(x, y) \le A_0(\rho(x, z) + \rho(z, y)) \qquad \text{for all } x, y, z \in X \tag{1.1}$$

where $A_0 < \infty$ is independent of $x, y, z$. Given a quasi–metric, we set $B(x, r) = \{y \in X : \rho(x, y) < r\}$. Of course, a set $\Omega \subset X$ is defined to be open if for each $x \in \Omega$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \Omega$. Unfortunately, when $A_0 > 1$, it does not follow that the balls $B(x, r)$ are open. However Macias and Segovia [MS] have proved that given any quasi–metric $\rho$, there exists a quasi–metric $\rho'$, equivalent in the sense that there exists $c \in (0, \infty)$ such that for all $x, y \in X$, $c^{-1}\rho(x, y) \le \rho'(x, y) \le c\rho(x, y)$, such that the metric balls defined with respect to $\rho'$ are open. In all our analysis, only the order of magnitude of $\rho(x, y)$ will be significant, so it will be no loss of generality to assume that metric balls are open.

**Definition 1.** *A space of homogeneous type is a set $X$, equipped with first, a quasi–metric $\rho$ for which all the associated balls $B(x, r)$ are open, and second, a nonnegative Borel measure $\mu$ satisfying the doubling condition*

$$\mu(B(x, 2r)) \le A_1 \mu(B(x, r)) \qquad \text{for all } x \in X,\ r > 0 \tag{1.2}$$

*where $A_1$ is finite and independent of $x, r$. It is also required that $\mu(B(x, r)) < \infty$ for all $x, r$.*

Henceforth it is always understood that we are working on a space of homogeneous type $(X, \rho, \mu)$. We denote also by $\mu$ the completion of the original $\mu$. Define for any $x, y \in X$,

$$\lambda(x, y) = \mu(B(x, \rho(x, y))).$$

It follows from (1.1) and (1.2) that $\lambda(y, x)$ is comparable to $\lambda(x, y)$, uniformly in $x, y$.

**Definition 2.** *A standard kernel is a function $K : X \times X \backslash \{x = y\} \mapsto \mathbb{C}$ such that there exist $\varepsilon, \delta > 0$ and $C < \infty$ such that*

$$|K(x, y)| \le \frac{C}{\lambda(x, y)} \qquad \text{for all distinct } x, y \in X$$

*and such that*

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \le C \left( \frac{\rho(x, x')}{\rho(x, y)} \right)^{\varepsilon} \frac{1}{\lambda(x, y)}$$

*whenever $\rho(x, x') \le \delta\rho(x, y)$.*

Alternatively, one says that $K$ satisfies the standard estimates.

Denote by $\Lambda_\alpha$ the class of all bounded functions which are Hölder continuous of order $\alpha \in (0, 1]$, a Banach space under the norm

$$\|f\|_{\Lambda_\alpha} = \|f\|_{L^\infty} + \sup_{x \ne y} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha}.$$

Denote by $\mathcal{D}_\alpha$ the subspace of all functions with compact support. Just as the space of distributions $\mathcal{D}'$ is defined on Euclidean space, we may introduce the dual space $\mathcal{D}'_\alpha$ consisting of all linear functionals $\ell$ from $\mathcal{D}_\alpha$ to $\mathbb{C}$ with the property that for each bounded set $E \subset X$, there exists a finite constant $C_E$ such that for all $f \in \mathcal{D}_\alpha$ with support contained in $E$, $|\ell(f)| \le C_E \|f\|_{\Lambda_\alpha}$.

According to Macias and Segovia [MS], there exists a genuine metric $\rho'$ and exponent $N$ such that $\rho(x,y) \sim \rho'(x,y)^N$ for all $x, y$. Therefore nonconstant $\Lambda_\alpha$ functions exist in profusion for all sufficiently small $\alpha$, but not necessarily for all $\alpha \le 1$. This accounts for the restriction on $\alpha$ in Theorem 8 below.

We denote by $\langle h, g\rangle$ the natural pairing of elements $h \in \mathcal{D}'_\alpha$, $g \in \mathcal{D}_\alpha$. $T^t$ denotes the transpose of a linear operator, with respect to this pairing. By a kernel we shall mean a locally integrable, complex–valued function defined on $X \times X \backslash \{x = y\}$. A linear operator $T : \mathcal{D}_\alpha \mapsto \mathcal{D}'_\alpha$ is said to be associated to a kernel $K$ if for all $f, g \in \mathcal{D}_\alpha$ with disjoint supports,

$$\langle Tf,\, g\rangle = \iint K(x,y)\, f(y)\, g(x)\, d\mu(x)\, d\mu(y).$$

For any $L^\infty$ function $b$ satisfying $|b(x)| \ge \delta > 0$ $a.e.(d\mu)$, one also has the space $b\mathcal{D}_\alpha$ of all functions $b \cdot \varphi$ with $\varphi \in \mathcal{D}_\alpha$. It is naturally isomorphic to $\mathcal{D}_\alpha$, and we assign to it the corresponding topology. There is again the dual space $(b\mathcal{D}_\alpha)'$, and the pairing $\langle h, g\rangle$.

**Definition 3.** *A singular integral operator $T$ on a space of homogeneous type is a continuous linear operator from $b_1\mathcal{D}_\alpha$ to $(b_2\mathcal{D}_\alpha)'$ for some $\alpha \in (0,1]$, and some $L^\infty$ functions $b_1, b_2$ satisfying $|b_1(x)|$, $|b_2(x)| \ge \delta > 0\, a.e.(d\mu)$, which is associated to a standard kernel.*

An important and conveniently simple special case arises when $K$ is an antisymmetric standard kernel, that is, $K(y,x) \equiv -K(x,y)$, $b_1 \equiv b_2 \equiv b$, and an associated operator $T$ is defined by

$$\langle T(b\varphi_1),\, b\varphi_2\rangle = \frac{1}{2}\iint K(x,y)\, b(x)b(y)[\varphi_1(y)\varphi_2(x) - \varphi_2(y)\varphi_1(x)]\, d\mu(y)\, d\mu(x) \tag{1.3}$$

for $\varphi_1, \varphi_2 \in \mathcal{D}_\alpha$. The definition is legitimate because the integral converges absolutely, a consequence of the standard estimates and Hölder continuity of $\varphi_1, \varphi_2$. See [DJS].

**Definition 4.** *For $\alpha \in (0,1]$, $x_0 \in X$, and $r > 0$, $B_{\alpha,x_0,r}$ is the set of all $f \in \mathcal{D}_\alpha$, supported in $\{y : \rho(x_0,y) \le r\}$, for which $\|f\|_{L^\infty} \le 1$ and*

$$|f(x) - f(y)| \le r^{-\alpha}\rho(x,y)^\alpha \qquad \textit{for all } x, y \in X.$$

**Definition 5.** *A continuous linear transformation $T : b_1\mathcal{D}_\alpha \mapsto (b_2\mathcal{D}_\alpha)'$ is said to be weakly bounded (with respect to $b_1$, $b_2$) if there exists $C < \infty$ such that for all $x_0 \in X$, $r > 0$ and all $\varphi_1$, $\varphi_2 \in B_{\alpha,x_0,r}$,*

$$|\langle T(b_1\varphi_1),\, b_2\varphi_2\rangle| \le C\mu(B(x_0,r)). \tag{1.4}$$

Note that if $T$ is bounded on $L^2$ then

$$\begin{aligned} |\langle T(b_1\varphi_1),(b_2\varphi_2)\rangle| &\le \|T\|\cdot\|\varphi_1\|_{L^2}\|\varphi_2\|_{L^2}\|b_1\|_\infty\|b_2\|_\infty \\ &\le C\|T\|\mu(B(x_0,r))^{1/2}\mu(B(x_0,r))^{1/2} \\ &\le C\|T\|\mu(B(x,r)), \end{aligned}$$

so that $L^2$ boundedness implies weak boundedness. In the case where $T$ is associated to an anti–symmetric standard kernel by the procedure (1.3), a straightforward computation shows that it is automatically weakly bounded with respect to $b_1, b_2$ whenever $b_1 \equiv b_2$.

**Definition 6.** *A locally integrable function $f$ belongs to $BMO$ if*

$$\|f\|_{BMO} = \sup_{x\in X, r>0}\ \inf_{c\in\mathbb{C}} \mu(B(x,r))^{-1}\int_{B(x,r)} |f(y)-c|\,d\mu(y)$$

*is finite.*

**Definition 7.** *A function $b \in L^\infty(X)$ is said to be para–accretive if there exists $\delta > 0$ such that for all $x \in X$ and $r > 0$, there exist $x' \in B(x,r)$ and $r' \in [\delta r,\, r]$ such that*

$$\Big|\int_{B(x',r')} b(y)\,d\mu(y)\Big| \ge \delta\mu(B(x',r')).$$

Note that $\mu(B(x',r')) \sim \mu(B(x,r))$, by the doubling property (1.1) of $\mu$. Note also that for any para–accretive $b$, there exists $\varepsilon > 0$ such that $|b| \ge \varepsilon$ almost everywhere, because of the validity of Lebesgue's theorem on differentiation of integrals on spaces of homogeneous type [CW].

The review concludes with the $T(b)$ theorem of David, Journé and Semmes [DJS]:

**Theorem 8.** *Suppose that $b_1$, $b_2$ are para–accretive functions and that $T$ is a singular integral operator, on a space $X$ of homogeneous type. Suppose that $T$ is weakly bounded from $b_1\mathcal{D}_\alpha$ to $(b_2\mathcal{D}_\alpha)'$, that $\alpha$ is sufficiently small, and that $T(b_1)$, $T^t(b_2) \in BMO$. Then $T$ extends to an operator bounded on $L^2(X,\mu)$.*

$\alpha$ is required to be smaller than a parameter which depends only on $A_0$.

## 2. Pseudo–accretive systems and another $T(b)$ theorem.

One last definition:

**Definition 9.** *A pseudo–accretive system is a collection of $L^\infty$ functions $b_B$, one for each ball $B = B(x,r) \subset X$, satisfying for some $C < \infty$, $\delta > 0$,*

$$\|b_B\|_{L^\infty} \le C \qquad \text{for all } B$$
$$\left|\int_B b_B \, d\mu\right| \ge \delta\mu(B) \qquad \text{for all } B.$$

If $T$ is a singular integral operator and $K$ the associated standard kernel, we denote by $T^\varepsilon$ the operator from $\mathcal{D}_\alpha$ to $\mathcal{D}'_\alpha$, for any $\alpha$, defined by

$$T^\varepsilon f(x) = \int_{\rho(x,y)>\varepsilon} K(x,y)\, f(y)\, d\mu(y).$$

$T^\varepsilon$ is called a truncated singular integral operator. Although $T$ need not in any sense be the limit of $T^\varepsilon$ as $\varepsilon \to 0$, nonetheless boundedness of $T$ may often be deduced, once the $T^\varepsilon$ are known to be uniformly bounded, as $\varepsilon \to 0$. In particular, this is true when $K$ is antisymmetric and $T$ is associated to $K$ as in (1.3).

Our principal result is

**Main Theorem 10.** *Let $X$ be a space of homogeneous type, and let $T$ be a truncated singular integral operator. Suppose there exist $C < \infty$ and pseudo–accretive systems $\{b^1_B\}$, $\{b^2_B\}$ on $X$ such that for all $B$,*

$$\|T(b^1_B)\|_{L^\infty} \le C$$

*and*

$$\|T^t(b^2_B)\|_{L^\infty} \le C.$$

*Then $T$ is bounded on $L^2(X,\mu)$, with an operator norm not exceeding a bound which depends only on $A_0, A_1$, on the bounds in the standard estimates of Definition 1 for $K$, on the constants in Definition 9 for $\{b^j_B\}$, and on $C$.*

In particular, the assertion is that for a fixed standard kernel, the operator norm satisfies a bound independent of the radius of truncation, $\varepsilon$ . The hypothesis that $K$ is a truncation of a standard kernel, is made for technical reasons to ensure that $T(f)$ is defined for various functions $f$ which arise in the proof.

The formulation in terms of truncations is admittedly a bit unwieldy. In the important special case of an operator associated to an antisymmetric kernel as in (1.3), the operator is automatically defined on large classes of functions. With some slight additional argument, the proof of Theorem 10 also establishes

**Theorem 10'.** *Let $X$ be a space of homogeneous type and $T$ a singular integral operator on $X$, associated to an antisymmetric kernel by (1.3). Suppose there exists a pseudo-accretive system $\{b_B\}$ on $X$ such that $\|T(b_B)\|_\infty \le C < \infty$ for all balls $B$. Then $T$ is bounded on $L^2(X)$.*

In the case of general operators, one might object that in order to obtain boundedness of a singular integral by applying the theorem to its truncations and then passing to the limit, one would need, for every $\varepsilon$, two pseudo–accretive systems satisfying the hypotheses; it would be preferable to have two families which work simultaneously for all $\varepsilon$. For the applications considered in this article, this will not be a problem. More general considerations suggest that the same may happen if there are to be other applications. For consider the maximal truncated operator

$$T^*f(x) = \sup_{\varepsilon>0} |T^\varepsilon f(x)|.$$

The following inequality is due to Cotlar, and a recent discussion may be found in [J]:

$$T^*f(x) \le CM(Tf)(x) + C'Mf(x) \qquad \text{for all } x$$

where $M$ is the appropriate analogue of the maximal function of Hardy and Littlewood:

$$Mf(x) = \sup_{r>0} \ \mu(B(x,r)^{-1} \int_{B(x,r)} |f(y)|\, d\mu(y),$$

$C$ depends only on $X$, and $C'$ depends also on the $L^2$ operator norm of $T$ and on the constants appearing in the standard estimates for $K$. For us the relevant consequence is that if $T$ is bounded on $L^2$, $b \in L^\infty$ and $T(b) \in L^\infty$, then also $T^\varepsilon(b) \in L^\infty$, uniformly in $\varepsilon$. So if $T$ is bounded and there happens to exist a pseudo–accretive system for which $T(b_B) \in L^\infty$ uniformly in $B$, then the same pseudo–accretive system works for all the truncations $T^\varepsilon$, uniformly in $\varepsilon$.

Theorem 10 is more flexible than the $T(b)$ theorem in that it permits a pseudo–accretive system, rather than a single para–accretive function. It happens that exactly this situation arises in our applications. The existence of a good pseudo–accretive system is a necessary condition for $T$ to be bounded on $L^2$, for given boundedness, the existence of a para–accretive system with $T(b_B)$ uniformly bounded in $L^\infty$ follows from functional–analytic considerations — an application of the Hahn–Banach theorem and weak type $(1,1)$ estimate — which are well–known in the theory of analytic capacity.

Another possible advantage of Theorem 10 is that there is no hypothesis of weak boundedness. On the other hand it is unsatisfactory in at least one respect: it requires that $T(b_B)$ be in $L^\infty$. One might hope that it suffices to have

$$\|T(b_B)\|_{BMO} \le C$$

uniformly in $B$, or even

$$\|T(b_B)\|_{L^1(\hat{B})} \leq C\mu(B)$$

where $\hat{B}$ is the ball with the same center as $B$, but $2A_0$ times as large a radius. The second line would be a consequence of the first. We have not succeeded in modifying the proof to work under either hypothesis, and are mildly skeptical of the validity of the theorem without the $L^\infty$ bound.

## 3. Dyadic cubes.

Let $(X, \rho, \mu)$ be a space of homogeneous type, as defined above. The following sets $Q_\alpha^k$ are our analogues of the Euclidean dyadic cubes; it may help to think of $Q_\alpha^k$ as being essentially a cube or ball of diameter roughly $\delta^k$, with center $z_\alpha^k$.

**Theorem 11.** *There exists a collection of open subsets $\{Q_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$, and constants $\delta \in (0,1)$, $a_0 > 0$, $\eta > 0$ and $C_1, C_2 < \infty$ such that*

$$\mu(X \setminus \bigcup_\alpha Q_\alpha^k) = 0 \qquad \forall k. \tag{3.1}$$

(3.2) *If $\ell \geq k$ then either $Q_\beta^\ell \subset Q_\alpha^k$ or $Q_\beta^\ell \cap Q_\alpha^k = \emptyset$.*

(3.3) *For each $(k, \alpha)$ and each $\ell < k$ there is a unique $\beta$ such that $Q_\alpha^k \subset Q_\beta^\ell$.*

(3.4) *Diameter$(Q_\alpha^k) \leq C_1 \delta^k$.*

(3.5) *Each $Q_\alpha^k$ contains some ball $B(z_\alpha^k, a_0\delta^k)$.*

$$\mu\{x \in Q_\alpha^k : \rho(x, X \setminus Q_\alpha^k) \leq t\delta^k\} \leq C_2 t^\eta \mu(Q_\alpha^k) \qquad \forall k, \alpha, \ \forall t > 0. \tag{3.6}$$

$I_k$ denotes some (possibly finite) index set, depending on $k$. Dyadic cubes have been constructed previously in a little less generality by David [D3], and the formulation of Theorem 11 is based on his work.

We begin by establishing (3.2) through (3.5); these concern only the quasi–metric space structure, and we have nothing at our disposal in the proof but the quasi–triangle inequality. Let $\delta$ be a small positive number to be determined later, and for each $k \in \mathbb{Z}$, fix a maximal collection of points $z_\alpha^k \in X$ satisfying

$$\rho(z_\alpha^k, z_\beta^k) \geq \delta^k \qquad \forall \alpha \neq \beta. \tag{3.7}$$

Of course, by maximality there is the reverse inequality

(3.8) For each $k$, for each $x \in X$ there exists $\alpha$ such that $\rho(x, z_\alpha^k) < \delta^k$.

These points $z_\alpha^k$ remain fixed for the remainder of section 3.

**Definition 12.** *A tree is a partial ordering $\le$ of the set of all ordered pairs $(k,\alpha)$, which satisfies*

(3.9)
$$(k,\alpha) \le (\ell,\beta) \Rightarrow k \ge \ell.$$
(3.10)
*For each $(k,\alpha)$ and $\ell \le k$ there exists a unique $\beta$ such that $(k,\alpha) \le (\ell,\beta)$.*

(3.11)
$$(k,\alpha) \le (k-1,\beta) \Rightarrow \rho(z_\alpha^k, z_\beta^{k-1}) < \delta^{k-1}.$$
(3.12)
$$\rho(z_\alpha^k, z_\beta^{k-1}) < (2A_0)^{-1}\delta^{k-1} \Rightarrow (k,\alpha) \le (k-1,\beta).$$

**Lemma 13.** *There exists at least one tree.*

*Proof.* For each $(k,\alpha)$, there exists at least one $\beta$ for which $\rho(z_\alpha^k, z_\beta^{k-1}) < \delta^{k-1}$, by (3.8). And there exists at most one $\beta$ for which $\rho(z_\alpha^k, z_\beta^{k-1}) < (2A_0)^{-1}\delta^{k-1}$. For if $z_\gamma^{k-1}$ is another such point, then $\rho(z_\beta^{k-1}, z_\gamma^{k-1}) < A_0(2A_0)^{-1}(\delta^{k-1}+\delta^{k-1}) = \delta^{k-1}$, contradicting (3.7).

The partial ordering is constructed according to the following rule: for each $(k,\alpha)$, check whether there exists $\beta$ such that $\rho(z_\alpha^k, z_\beta^{k-1}) \le (2A_0)^{-1}\delta^{k-1}$. If so, decree that $(k,\alpha) \le (k-1,\beta)$, and also that $(k,\alpha)$ is not related to any other $(k-1,\gamma)$. If no such good $\beta$ exists, then select any $\beta$ for which $\rho(z_\alpha^k, {z_\beta}^{k-1}) \le \delta^{k-1}$, and decree that $(k,\alpha) \le (k-1,\beta)$ and is not related to any other $(k-1,\gamma)$.

Finally, extend $\le$ by transitivity to obtain a partial ordering. It is clear that all four clauses of the definition are satisfied.

Now for the dyadic cubes. Fix a tree, and let $a_0 \in (0,1)$ be a small constant to be determined.

**Definition 14.**
$$Q_\alpha^k = \bigcup_{(\ell,\beta)\le(k,\alpha)} B(z_\beta^\ell, a_0\delta^\ell).$$

Certainly each $Q_\alpha^k$ is open, and (3.5) holds. To begin verifying the other conclusions of Theorem 11, note that

(3.13)
$$(\ell,\beta) \le (k,\alpha) \Rightarrow \rho(z_\beta^\ell, z_\alpha^k) \le 2A_0\delta^k.$$

For there exists a chain $(k,\alpha) = (k,\gamma_0) \geq (k+1,\gamma_1) \geq (k+2,\gamma_2) \geq \cdots \geq (\ell,\beta)$. Then

$$\begin{aligned}
\rho(z^k_\alpha, z^\ell_\beta) &\leq A_0\rho(z^k_\alpha, z^{k+1}_{\gamma_1}) + A_0\rho(z^{k+1}_{\gamma_1}, z^\ell_\beta) \\
&\leq A_0\delta^k + A_0\rho(z^{k+1}_{\gamma_1}, z^\ell_\beta) \\
&\leq A_0\delta^k + A_0^2\rho(z^{k+1}_{\gamma_1}, z^{k+2}_{\gamma_2}) + A_0^2\rho(z^{k+2}_{\gamma_2}, z^\ell_\beta) \\
&\leq A_0\delta^k + A_0^2\delta^{k+1} + A_0^2\rho(z^{k+2}_{\gamma_2}, z^\ell_\beta) \\
&\leq A_0\delta^k + A_0^2\delta^{k+1} + A_0^3\delta^{k+2} + \ldots \\
&= A_0\delta^k/(1-A_0\delta) \\
&\leq 2A_0\delta^k,
\end{aligned}$$

requiring $\delta$ to be chosen smaller than $(2A_0)^{-1}$. An immediate consequence is (3.4), with a constant $C_1$ which depends only on $A_0$.

**Lemma 15.** *If $Q^k_\alpha \cap Q^k_\beta \neq \emptyset$ then $\alpha = \beta$.*

*Proof.* Suppose that $x \in Q^k_\alpha \cap Q^k_\beta$. Then there exist $(m,\gamma)$ and $(n,\sigma)$ such that $(m,\gamma) \leq (k,\alpha)$, $(n,\sigma) \leq (k,\beta)$, and $x \in B(z^m_\gamma, a_0\delta^m) \cap B(z^n_\sigma, a_0\delta^n)$. Hence $\rho(z^m_\gamma, z^n_\sigma) \leq A_0a_0\delta^m + A_0a_0\delta^n \leq 2A_0a_0\delta^n$, supposing without loss of generality that $m \geq n$.

Now consider two cases. If $m = n$ then by requiring $a_0$ to be so small that $2A_0a_0 < 1$, we obtain $\rho(z^m_\gamma, z^n_\sigma) < \delta^n$, contradicting (3.7). On the other hand, if $m > n$, there is a unique $z^{n+1}_\lambda$ such that $(m,\gamma) \leq (n+1,\lambda)$. Then, using (3.13) in the second line,

$$\begin{aligned}
\rho(z^{n+1}_\lambda, z^n_\sigma) &\leq A_0\rho(z^{n+1}_\lambda, z^m_\gamma) + A_0\rho(z^m_\gamma, z^n_\sigma) \\
&\leq A_0 2A_0\delta^{n+1} + A_0 2A_0 a_0\delta^n \\
&= 2A_0^2(\delta + a_0)\delta^n \\
&< (2A_0)^{-1}\delta^n
\end{aligned}$$

provided that $\delta$ and $a_0$ are chosen to be sufficiently small. Because of (3.12), this implies that $(n+1,\lambda) \leq (n,\sigma)$. Then $(m,\gamma) \leq (n+1,\lambda) \leq (n,\sigma) \leq (k,\beta)$. Since also $(m,\gamma) \leq (k,\alpha)$, we conclude by virtue of (3.10) that $\alpha = \beta$.

Conclusion (3.2) of the theorem follows at once. For if $\ell \geq k$ and $Q^\ell_\beta \cap Q^k_\alpha \neq \emptyset$, choose $\gamma$ so that $(\ell,\beta) \leq (k,\gamma)$, whence $z^\ell_\beta \subset Q^k_\gamma$. Then $Q^k_\gamma \cap Q^k_\alpha \neq \emptyset$, so $\gamma = \alpha$ by Lemma 15. Thus $Q^\ell_\beta \subset Q^k_\alpha$.

(3.1) is easy to check. Fix $k$ and let $E = \cup_\alpha Q^k_\alpha$. Given any $x \in X$ and any $n$, there exists $z^n_\alpha$ such that $\rho(x, z^n_\alpha) \leq \delta^n$. If $n \geq k$ then $B(z^n_\alpha, a_0\delta^n) \subset E$. Also, by the triangle inequality, $B(z^n_\alpha, a_0\delta^n) \subset B(x, A_0(1+a_0)\delta^n)$, which we call $B$. Note that $\mu(B(z^n_\alpha, a_0\delta^n)) \geq c\mu(B)$ by the triangle inequality and the doubling condition (1.2), where $c \in (0,1]$ is a constant depending only on $A_0, A_1$. In other words

$$\frac{\mu(E \cap B)}{\mu(B)} \geq c > 0.$$

Letting $n \to \infty$ we find that

$$\limsup_{r \to 0} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \geq c > 0 \qquad \forall x \in X.$$

By Lebesgue's theorem on differentiation of the integral, $E$ therefore has full measure, as desired. Let us permanently delete from $X$ the null set $\cup_k (X \setminus \cup_\alpha Q_\alpha^k)$. Some of the $z_\alpha^k$ may conceivably be deleted, but that does no harm, and we may continue to use them in our reasoning.

Condition (3.6) asserts that the mass of a cube is never too strongly concentrated near its boundary, hence that the characteristic function of a cube is in a sense somewhat smooth. It was introduced by David [D3] and plays an essential role in applications of the dyadic cubes to singular integral theory. That it should be attacked by way of Lemma 17 below, was suggested to us by David. First, we shall need a technical improvement on (3.5).

**Lemma 16.** *Set $C_3 = (4A_0^2)^{-1}$. If $\delta, a_0$ are sufficiently small, then for all $(k,\alpha)$, $B(z_\alpha^k, C_3\delta^k) \subset Q_\alpha^k$.*

*Proof.* Suppose $x \in B(z_\alpha^k, C_3\delta^k)$. Suppose that $x \notin Q_\alpha^k$, so that $x$ belongs to some other $Q_\beta^k$. Then there exists $(\ell,\gamma)$, $\ell \geq k$, such that $x \in B(z_\gamma^\ell, a_0\delta^\ell)$ and $(\ell,\gamma)$ is not less than or equal to $(k,\alpha)$. Again let us distinguish two cases. If $\ell = k$, then

$$\begin{aligned}
\rho(z_\gamma^\ell, z_\alpha^k) &\leq A_0(\rho(z_\gamma^\ell, x) + \rho(x, z_\alpha^k)) \\
&\leq A_0(a_0\delta^\ell + C_3\delta^k) \\
&< \delta^k
\end{aligned}$$

since $a_0A_0 < 1/2$, $A_0C_3 = (4A_0)^{-1} < 1/4 < 1/2$. Since $\ell = k$, this contradicts (3.7).

On the other hand if $\ell > k$, then there exists $(k+1,\sigma)$ which is $\geq (\ell,\gamma)$. Hence $\rho(x, z_\sigma^{k+1}) \leq C_1\delta^{k+1}$. Therefore

$$\begin{aligned}
\rho(z_\sigma^{k+1}, z_\alpha^k) &\leq A_0C_1\delta^{k+1} + A_0\rho(x, z_\alpha^k) \\
&\leq A_0C_1\delta^{k+1} + A_0C_3\delta^k \\
&= (A_0C_1\delta + A_0C_3)\delta^k \\
&\leq (A_0C_1\delta + (4A_0)^{-1})\delta^k \\
&< (2A_0)^{-1}\delta^k,
\end{aligned}$$

provided that $\delta$ is small enough. By (3.12) this implies that $(k+1,\sigma) \leq (k,\alpha)$, whence $(\ell,\gamma) \leq (k,\alpha)$, a contradiction.

**Lemma 17.** *For any $\varepsilon > 0$ there exists $\tau \in (0,1]$ such that for every $Q^k_\alpha$,*

$$\mu\{x \in Q^k_\alpha : \rho(x, X\backslash Q^k_\alpha) < \tau\delta^k\} < \varepsilon\mu(Q^k_\alpha).$$

*Proof.* Write $Q = Q^k_\alpha$. Let $N$ be a positive integer, large enough to fulfill a condition to be imposed later, depending on $\varepsilon$. Let $\tau$ be small and for the moment, fix a point $x \in Q$ satisfying $\rho(x, X\backslash Q) < \tau\delta^k$.

We claim that if $\tau$ is sufficiently small, there exists $\sigma$ such that $(k+N, \sigma) \le (k, \alpha)$ and $\rho(x, z^{k+N}_\sigma) \le C_1\delta^{k+N}$. Indeed there exists $(\ell, \beta) \le (k, \alpha)$ with $x \in B(z^\ell_\beta, a_0\delta^\ell)$. The last lemma says that $C_3\delta^\ell \le \rho(z^\ell_\beta, X\backslash Q)$, while on the other hand

$$\rho(z^\ell_\beta, X\backslash Q) \le A_0\rho(z^\ell_\beta, x) + A_0\rho(x, X\backslash Q) \le A_0a_0\delta^\ell + A_0\tau\delta^k.$$

Hence $(C_3 - A_0a_0)\delta^\ell \le A_0\tau\delta^k$, and provided that $a_0$ and $\tau$ are chosen sufficiently small (recall that $C_3$ depends only on $A_0$), this forces $\ell \ge k+N$. Next, choose $\sigma$ so that $(\ell, \beta) \le (k+N, \sigma)$. Then $x \in Q^{k+N}_\sigma$, so $\rho(x, z^{k+N}_\sigma) \le C_1\delta^{k+N}$ by (3.4), as desired. Since $Q^{k+N}_\sigma$ intersects $Q^k_\alpha$, $(k+N, \sigma) \le (k, \alpha)$.

Let $x$ continue to be fixed. There is a unique chain

$$(k+N, \sigma) = (k+N, \sigma_{k+N}) \le (k+N-1, \sigma_{k+N-1}) \le \cdots \le (k, \sigma_k) = (k, \alpha).$$

To keep the notation in check we temporarily write $z^j$ for $z^j_{\sigma_j}$.

We next claim that there exists $\varepsilon_1 > 0$, depending only on $A_0, A_1$, such that $\rho(z^j, z^i) \ge \varepsilon_1\delta^j$ whenever $k \le j < i \le k+N$. For if not,

$$\begin{aligned}\rho(z^j, X\backslash Q) &\le A_0\rho(x, X\backslash Q) + A_0\rho(z^j, x) \\ &\le A_0\tau\delta^k + A_0^2\rho(z^j, z^i) + A_0^2\rho(z^i, x) \\ &\le A_0\tau\delta^k + A_0^2\varepsilon_1\delta^j + A_0^2C_1\delta^i.\end{aligned}$$

This last is $< C_3\delta^j$ if $\tau, \varepsilon_1, \delta$ are chosen to be sufficiently small, depending only on $N$ and on $A_0$. But the last lemma implies that $B(z^j, C_3\delta^j) \subset Q$, so this is a contradiction.

Now we allow $x$ to vary within $E = \{x \in Q : \rho(x, X\backslash Q) < \tau\delta^k\}$. To each such $x$ is associated a chain of pairs $(j, \beta(x,j))$ as above, $k \le j \le k+N$. Let $S_j$ be the collection of all points $z^j_{\beta(x,j)}$ thus obtained, taking the union over all $x \in E$. If $\varepsilon_2$ is a sufficiently small constant, then for any $k \le i, j \le k+N$, for any $z \in S_i$ and $z' \in S_j$, $B(z, \varepsilon_2\delta^i) \cap B(z', \varepsilon_2\delta^j) = \emptyset$, by the last paragraph. Set $G_j = \cup_{z\in S_j} B(z, \varepsilon_2\delta^j)$.

For any $k \le j \le k+N$, we have

$$\begin{aligned}
\mu(E) &\le \mu\Big(\bigcup_{z \in S_{k+N}} B(z, C_1\delta^{k+N})\Big) \\
&\le \sum_{z \in S_{k+N}} \mu(B(z, C_1\delta^{k+N})) \\
&\le C \sum_{z \in S_{k+N}} \mu(B(z, \varepsilon_2\delta^{k+N})) \\
&= \sum_{w \in S_j} \sum_{z \in S_{k+N},\ z \le w} \mu(B(z, \varepsilon_2\delta^{k+N})) \\
&\le C \sum_{w \in S_j} \mu(B(w, C_1\delta^j)) \\
&\le C \sum_{w \in S_j} \mu(B(w, \varepsilon_2\delta^j)) \\
&= C\mu(G_j),
\end{aligned}$$

where we have abused notation by writing $z \le w$ to mean that the corresponding ordered pairs $(i, \gamma)$ are related by $\le$. The first inequality holds by the first paragraph of the proof. The second is trivial. The third is an application of the doubling property (1.2), as is the last. The equality in the fourth line follows from the disjointness of the balls $B(z, \varepsilon_2\delta^j)$, as does the equality in the last line. The fifth line follows from (3.4) and disjointness of the balls.

To finish the proof note that since the $G_j$ are pairwise disjoint,

$$\mu(Q) \ge \sum_{j=k}^{k+N} \mu(G_j) \ge NC^{-1}\mu(E).$$

It suffices to choose $N > C\varepsilon^{-1}$ to obtain $\mu(E) < \varepsilon\mu(Q)$.

*Proof of (3.6).* For any $Q_\alpha^k$ and any integer $j \ge 0$, set

$$E_j(Q_\alpha^k) = \{Q_\beta^{k+j} \subset Q_\alpha^k : \rho(Q_\beta^{k+j}, X\backslash Q_\alpha^k) \le C_4\delta^{k+j}\}$$

where $C_4$ is a large constant. Also let

$$e_j(Q_\alpha^k) = \{x : x \in Q_\beta^{k+j} \text{ for some } Q_\beta^{k+j} \in E_j(Q_\alpha^k)\}$$

be the underlying point set. It was shown in the first paragraph of the proof of Lemma 17 that for $x \in Q_\alpha^k$, $\rho(x, X\backslash Q_\alpha^k) \le \tau\delta^k$ implies $x \in e_j(Q_\alpha^k)$ where $\delta^{j+k} \sim \tau\delta^k$, provided that $C_4$

is taken to be sufficiently large. Conversely, if $x \in e_j(Q^k_\alpha)$ then $\rho(x, X \backslash Q^k_\alpha) \le C\delta^{k+j}$ by the triangle inequality. Thus it suffices to prove that

$$\mu(e_j(Q^k_\alpha)) \le C\delta^{j\eta}\mu(Q^k_\alpha) \qquad \forall \alpha, k, \forall j \ge 0.$$

To accomplish this, fix a large integer $J$ with the property that

$$\mu(e_J(Q^k_\alpha)) \le \frac{1}{2}\mu(Q^k_\alpha) \qquad \forall \alpha, k; \tag{3.14}$$

its existence is assured by Lemma 17. We require also a variant of $E_j(Q^k_\alpha)$: denote by $F_n(Q^k_\alpha)$ the collection of all $Q^{k+nJ}_\beta \subset Q^k_\alpha$ obtained as follows. For $n = 1$, $F_1(Q^k_\alpha) = E_J(Q^k_\alpha)$. Then iterate:

$$F_{n+1}(Q^k_\alpha) = \bigcup_{Q^{k+nJ}_\beta \in F_n(Q^k_\alpha)} E_J(Q^{k+nJ}_\beta).$$

Let $f_n(Q^k_\alpha)$ denote the underlying point set, and observe that by de–obfuscation of notation, $e_{nJ}(Q^k_\alpha) \subset f_n(Q^k_\alpha)$. Roughly speaking, $e_{2J}(Q^k_\alpha)$ is a very small border of $Q^k_\alpha$, while $f_1(Q^k_\alpha) = e_J(Q^k_\alpha)$ is a moderately small border, and $f_2(Q^k_\alpha)$ is the union of the very small borders of all the constituent cubes of $e_J(Q^k_\alpha)$, hence contains points which are not extremely close to the border of $Q^k_\alpha$. We advise drawing a picture, for $\mathbb{R}^1$ or $\mathbb{R}^2$ with the usual Euclidean cubes.

By iterating (3.14) we find that

$$\mu(f_n(Q^k_\alpha)) \le 2^{-n}\mu(Q^k_\alpha),$$

so

$$\mu(e_{nJ}(Q^k_\alpha)) \le 2^{-n}\mu(Q^k_\alpha) = \delta^{\eta nJ}\mu(Q^k_\alpha),$$

with $\delta^{\eta J} = 2^{-1}$. That takes care of (3.6).

Whenever $Q^{k+1}_\beta \subset Q^k_\alpha$, we shall say that $Q^{k+1}_\beta$ is a child of $Q^k_\alpha$, and $Q^k_\alpha$ the parent of $Q^{k+1}_\beta$. Likewise we may speak of ancestors, descendants and siblings of cubes; an ancestor is the parent, or parent of the parent, or so on. A cube $Q^k_\alpha$ is said to be of generation $k$. The number of children which a cube can have is bounded above by a constant depending only on $A_0, A_1$ and on $\delta$; henceforth $\delta$ and $a_0$ will be fixed and taken to depend only on $A_0, A_1$.

There is nothing to prevent the space $X$ from having atoms, points $y$ such that $\mu(\{y\}) > 0$, which possibility will eventually cause us grief. If $y$ is an atom, it must be that for every sufficiently large $k$, the unique cube of generation $k$ which contains $y$ will be simply $\{y\}$. Even if there are no atoms, it can happen that a cube has only one child, and this can occur for an arbitrarily large number of successive generations. Two cubes which are of different generations, but which are identical as sets of points, will always regarded as distinct cubes. Therefore an atom will always be a cube of infinitely many generations.

However for any atom $x$, there exists a finite $k$ such that $\{x\}$ is not a cube of generation $k$ modulo a null set, except in the trivial case where the entire complement of $\{x\}$ is a null set. For suppose the contrary. For any $k$, there exists $\alpha$ such that $x \in Q^k_\alpha$. If $x \neq z^k_\alpha$, choose $0 < \varepsilon < C_3$ such that $x \neq B(z^k_\alpha, \varepsilon)$. Then $\mu(B(z^k_\alpha, \varepsilon)) = 0$, whence $\mu(X) = 0$, a contradiction. Therefore $B(x, C_3\delta^k) \subset Q^k_\alpha$. so that modulo a null set, $x$ is the only point in $B(x, C_3\delta^k)$, for all $k$. Letting $k \to -\infty$, we find that $x$ is the only point of $X$, again modulo a null set.

Observe that if $(X, \rho, \mu)$ is a space of homogeneous type and $Q \subset X$ one of our dyadic cubes, then $(Q, \rho|_Q, \mu|_Q)$ is again a space of homogeneous type. For if $Q$ is of generation $k$, then for all $r \leq \delta^k$, for each $x \in Q$, $\mu(Q \cap B(x, r)) \sim \mu(B(x, r))$. Indeed choose $j$ such that $r \sim \delta^j$. Then $x$ belongs to some cube $Q^j_\beta$. $Q^j_\beta$ intersects $Q$, hence must be a subset of $Q$. But $\mu(Q^j_\beta) \sim \mu(B(x, r))$ by the doubling property, (3.4), and (3.5). If $r > \delta^k$, then $\mu(B(x, r) \cap Q) \sim \mu(Q)$ for the same reason. So the doubling property also holds for $Q$.

Here is a consequence of the "small boundaries" condition (3.6):

**Lemma 18.** *Let $K$ be a standard kernel and $A$ a large constant. Let $Q$ be any dyadic cube of generation $k$ and let $\tilde{Q} = \{x \in X : \rho(x, Q) \leq A\delta^k\}$. Then for any $L^\infty$ function $f$ which vanishes almost everywhere outside of $\tilde{Q} \backslash Q$,*

$$\|T_K f\|_{L^1(Q)} \leq C\|f\|_\infty \cdot \mu(Q).$$

Recall that $T_K f(x) = \int K(x, y) f(y)\, d\mu(y)$; in the lemma we are dealing with $x$ outside of the support of $f$, so no principal–value limit is required to make sense of the integral. The constant $C$ depends only on $A_0, A_1, A$, and the constants in the standard estimates for $K$.

*Proof.* Let $E_j = \{x \in Q : \delta^{k+j-1} < \rho(x, \tilde{Q} \backslash Q) \leq \delta^{k+j}\}$ and for $x \in Q$, $F_j(x) = \{y \in \tilde{Q} \backslash Q : \delta^{k+i-1} < \rho(y, x) \leq \delta^{k+i}\}$. Let us examine, for $x \in E_j$,

$$\int_{\tilde{Q} \backslash Q} \lambda(x, y)^{-1}\, d\mu(y) = \sum_{i=j}^{C} \int_{F_i(x)} \lambda(x, y)^{-1}\, d\mu(y).$$

Recall that $\lambda(x, y) = \mu(B(x, \rho(x, y)))$; it follows from the triangle inequality and doubling property that for all $y \in F_i(x)$, $\lambda(x, y) \sim \mu(B(x, \delta^{k+i}))$ uniformly in $x, y, k, \alpha, i$. Hence

$$\begin{aligned}
\int_{\tilde{Q} \backslash Q} \lambda(x, y)^{-1}\, d\mu(y) &\leq C \sum_{i=j}^{C} \mu(B(x, \delta^{k+i}))^{-1} \int_{B(x, \delta^{k+i}))} d\mu(y) \\
&\leq C \cdot (1 + |j|).
\end{aligned}$$

This is helpful because for $x \in E_j$,

$$\begin{aligned}
|T_K f(x)| &\leq \|f\|_\infty \int_{\tilde{Q} \backslash Q} |K(x, y)|\, d\mu(y) \\
&\leq C\|f\|_\infty \int_{\tilde{Q} \backslash Q} \lambda(x, y)^{-1}\, d\mu(y) \\
&\leq C \cdot (1 + |j|)\|f\|_\infty.
\end{aligned}$$

Summing over $j$ and invoking (3.6) gives

$$\begin{aligned}\int_Q |T_K f(x)|\,d\mu(x) &\le C\|f\|_\infty \sum_{j=-\infty}^{C} (1+|j|)\int_{E_j} d\mu(x)\\ &= C\|f\|_\infty \sum_{j=-\infty}^{C} (1+|j|)\mu(E_j)\\ &\le C\|f\|_\infty \sum_{j=-\infty}^{C} (1+|j|)2^{-\eta j}\mu(Q)\\ &\le C\|f\|_\infty \cdot \mu(Q).\end{aligned}$$

We need one last general fact about the dyadic cubes.

**Lemma 19.** *There exists $C<\infty$ such that for every bounded set $E\subset X$, there exists $k$ such that the number of dyadic cubes of generation $k$ which intersect $E$ is at most $C$.*

*Proof.* Let $r$ be the diameter of $E$ and fix a reference point $y\in E$. Let $k$ be a very large negative integer. Consider the set of all dyadic cybes $Q_\alpha^k$ which meet $E$. For any of them, $\rho(z_\alpha^k, y) \le A_0\rho(z_\alpha^k, E) + A_0 r \le A_0C_1\delta^k + A_0 r \le 2A_0C_1\delta^k$, using (3.4) and choosing $k$ to be sufficiently large. By the doubling property and triangle inequality, if $C'$ is large enough but independent of $r, k$, all the $Q_\alpha^k$ are contained in $B(y, C'\delta^k)$, and all their measures are comparable to that of $B(y, C'\delta^k)$. Since they are disjoint, there are at most a fixed number of them.

## 4. Proof of the main theorem.

Let $X$ be a space of homogeneous type and let $\mathcal{Q} = \{Q_\alpha^k\}$ be a system of "dyadic cubes" on $X$, satisfying the conclusions of Theorem 11. We shall require a slight variant of the $T(b)$ theorem. A function $b\in L^\infty$ is said to be dyadic para–accretive if for every $Q_\alpha^k$, there exists $Q_\beta^\ell \subset Q_\alpha^k$ with $\ell \le k+N$ and

$$|\int_{Q_\beta^\ell} b\,d\mu| \ge \varepsilon\mu(Q_\beta^\ell),$$

for some fixed $\varepsilon>0$, $N<\infty$. Note that $\mu(Q_\beta^\ell)\sim\mu(Q_\alpha^k)$, by the triangle inequality, doubling property, (3.4), and (3.5). $f$ belongs to dyadic $BMO$ if

$$\sup_Q \inf_c \mu(Q)^{-1}\int_Q |f(y)-c|\,d\mu(y) < \infty.$$

**Theorem 20.** *Suppose that $b^1, b^2$ are dyadic para–accretive functions, that $T : b^1\mathcal{D}_\alpha \mapsto (b^2\mathcal{D}_\alpha)'$ is a weakly bounded singular integral operator, and that $T(b^1)$, $T(b^2) \in BMO$ (dyadic). Then $T$ is bounded on $L^2$.*

Given the existence and basic properties of the dyadic cubes, the proof is an exercise (though not a short one) in known technique, and we omit the details. See Coifman–Jones–Semmes [CJS] for the proof in $\mathbb{R}^1$, in the special case $b^1 = b^2$ and $T(b^1) = T(b^2) \equiv 0$. A slightly different version of their argument may be found in [C, Chapter IV], along with a recipe for the paraproducts needed when $T(b^i) \neq 0$.

The strategy for the proof of the main theorem is apparent in the formulation of the next proposition.

**Proposition 21.** *Let $X$ be a space of homogeneous type, let $\mathcal{Q}$ be a system of dyadic cubes on $X$ and suppose that $X$ itself is an element of $\mathcal{Q}$. Let $T$ be a truncated singular integral operator. Suppose there exist pseudo–accretive systems $\{b^1_B\}, \{b^2_B\}$ such that $T(b^1_B), T^t(b^2_B) \in L^\infty$, uniformly in $B$. Then there exist dyadic para–accretive functions $b^1, b^2$ such that $T(b^1), T^t(b^2) \in BMO$ (dyadic) and $T : b^1\mathcal{D}_\alpha \mapsto (b^2\mathcal{D}_\alpha)'$ is weakly bounded, for any $\alpha > 0$. Moreover $b^1, b^2$, $T(b^1), T^t(b^2)$, and the constant in the weak boundedness inequality (1.4) satisfy bounds depending only on $A_0$, $A_1$, on the constants in the standard estimates for $K$, on the constants in the definition of pseudo–accretivity for $\{b^i_B\}$, and on $\sup_B \|T(b^1_B)\|_\infty + \sup_B \|T^t(b^2_B)\|_\infty$.*

For each $Q = Q^k_\alpha$, set $b^i_Q = b^i_B$ where $B = B(z^k_\alpha, a_0\delta^k)$. Then $b^i_Q$ is supported inside $Q$, and $|\int b^i_Q| \geq \varepsilon\mu(Q)$, for another constant $\varepsilon > 0$. Renormalize by multiplying each $b^i_Q$ by a scalar, so that we have for all $Q \in \mathcal{Q}$,

$$\|b^i_Q\|_\infty \leq A_2 \tag{4.1}$$

and

$$\int b^i_Q \, d\mu = \mu(Q). \tag{4.2}$$

$\{b^i_Q\}$ is called a dyadic pseudo–accretive system, for $i = 1, 2$. Henceforth we forget $\{b^i_B\}$, and work with $\{b^i_Q\}$.

Our proof runs into technical difficulty in the presence of atoms, points such that $\mu(\{x\}) > 0$. A cheap trick eliminates all difficulty: Define a new space $(\tilde{X}, \tilde{\rho}, \tilde{\mu})$ where $\tilde{X}$ is obtained from $X$ by retaining each point of $X$ which is not an atom, but replacing each atom $x \in X$ by a pair of points $x_1, x_2$. For each atom $x \in X$ let $r(x) = \inf_{y \neq x} \rho(x, y)$; $r(x)$ must be strictly positive. (Otherwise we could construct infinitely many pairwise disjoint balls $B(y_j, r_j)$ near $x$, such that $x \in B(y_j, Cr_j)$ where $C = C(A_0)$ is a fixed constant. Therefore $\mu(B(y_j, r_j)$ would be comparable to $\mu(B(y_j, Cr_j) \geq \mu(\{x\}) > 0$, whence any ball centered at $x$ would have infinite measure, a contradiction.) Define a quasi–metric $\tilde{\rho}$ on $\tilde{X}$ as follows: $\tilde{\rho} = \rho$ on

$(\tilde{X} \cap X) \times (\tilde{X} \cap X)$. If $x \in X$ is an atom and $y \in X$ is not, then set $\tilde{\rho}(x_i, y) = \tilde{\rho}(y, x_i) = \rho(x, y)$. If $x, y \in X$ are distinct atoms, set $\tilde{\rho}(x_i, y_j) = \rho(x, y)$. Finally if $x \in X$ is an atom, $\tilde{\rho}(x_1, x_2) = r(x)$. It is easy to check that we have introduced no drastic shortcuts between points, so that the quasi–triangle inequality holds for $\tilde{\rho}$. Next define $\tilde{\mu}$ on $\tilde{X}$ by $\tilde{\mu} = \mu$ on $\tilde{X} \backslash X$, and $\tilde{\mu}(\{x_i\}) = \frac{1}{2}\mu(\{x\})$ for every atom. Then the doubling property (1.2) holds for $\tilde{\mu}$, with $A_1$ replaced by a constant not exceeding $2A_1$.

To any function $f$ defined on $X$ may be associated a function $\tilde{f}$ on $\tilde{X}$, defined by $\tilde{f} \equiv f$ on $\tilde{X} \cap X$, and $\tilde{f}(x_i) = f(x)$ for all atoms $x$, for both $i = 1$ and $i = 2$. Similarly, to any standard kernel $K$ on $X$ is associated a standard kernel $\tilde{K}$ on $\tilde{X}$. Denoting by $T$ and $\tilde{T}$ the associated operators (assuming $K$ to be bounded), we have

$$\tilde{T}\tilde{f} \equiv (Tf)^{\sim}.$$

Since $\|\tilde{f}\|_{L^2(\tilde{X},\tilde{\mu})} \equiv \|f\|_{L^2(X,\mu)}$, $L^2$ boundedness of $\tilde{T}$ implies $L^2$ boundedness of $T$.

Out of our system of dyadic cubes on $X$ may be built a corresponding system on $\tilde{X}$. For each $Q \in \mathcal{Q}$ which contains at least two points, construct $\tilde{Q}$ by replacing each atom in $Q$ by the associated pair of points in $\tilde{X}$. For every atom $x \in X$, $\{x\}$ is necessarily a dyadic cube. Recall that we distinguish two dyadic cubes which are of different generations even when they are identical as point sets, so that $\{x\}$ is actually a dyadic cube of infinitely many generations. Let $k$ be the smallest integer such that it is a cube of generation $k$. On $\tilde{X}$, define $\{x_1, x_2\}$ to be a cube of generation $k$, and for every $\ell$ strictly larger than $k$, define $\{x_1\}$ and $\{x_2\}$ to be cubes of generation $\ell$; let's call these *new cubes.*

Under the hypotheses of Proposition 21, the dyadic pseudo–accretive systems $\{b^i_Q\}$ on $X$ give rise to dyadic pseudo–accretive systems on $\tilde{X}$. Each cube $\tilde{Q}$ of $\tilde{X}$ which is not new corresponds to a unique cube $Q$ in $X$, and we define $b^i_{\tilde{Q}} = (b^i_Q)^{\sim}$. If $R = \{x_i\}$ is a new cube of $\tilde{X}$, set $b^i_R(x_i) = b^i_{\{x\}}(x)$. Clearly the resulting two collections of functions on $\tilde{X}$ are dyadic pseudo–accretive systems.

The upshot of all this is that it suffices to prove Proposition 21 on $\tilde{X}$. Once this is done we know by Theorem 20 that $\tilde{T}$ is bounded on $L^2(\tilde{X})$, whence $T$ is bounded on $L^2(X)$, which proves our main theorem.[1]

Change notation: for the remainder of this section, what heretofore was called $(X, \rho, \mu)$ will be called $(Y, \rho_Y, \mu_Y)$, and what was heretofore called $(\tilde{X}, \tilde{\rho}, \tilde{\mu})$ will be called $(X, \rho, \mu)$.

The next result will facilitate cutting and pasting operations. We write $\chi_E$ to denote the characteristic function of an arbitrary set $E$.

**Lemma 22.** *Let $T$ be a truncated singular integral operator, let $\{b_Q\}$ be a dyadic pseudo–accretive system, and suppose that $T^t(b_Q) \in L^\infty$, uniformly in $Q$. For any $\varepsilon > 0$ there exists*

[1] One might object that Proposition 21 has been stated in general, but only proved for spaces $\tilde{X}$. However, once $T$ is known to be bounded on $L^2(X)$, the existence of a suitable dyadic pseudo–accretive system on $X$ follows from functional analysis. See [M, Chapter 3] for this argument, in the context of analytic capacity.

*$C < \infty$ such that for any $Q \in \mathcal{Q}$, for any $f \in L^\infty$ with $Tf \in L^\infty$,*

$$\|T(f\chi_Q)\|_{L^1(Q)} \le C(\|f\|_\infty + \|Tf\|_\infty)\mu(Q).$$

As usual, $C$ is not permitted to depend on the $L^\infty$ norm of $K$.

*Proof.* Normalize the $b_Q$ as in (4.2). Let $Q$ be of generation $k$, let $Q' = \{x : \rho(x,Q) \le C\delta^k\}$ for a large constant $C$, and split $f = f_0 + f_\infty$ where $f_0 = f \cdot \chi_{Q'}$. Look at

$$\begin{aligned} |\langle T(f_0), b_Q\rangle| &= |\langle f_0, T^t(b_Q)\rangle| \\ &\le C\|f_0\|_1 \|T^t(b_Q)\|_\infty \\ &\le C\|f\|_\infty \mu(Q). \end{aligned} \tag{4.3}$$

It follows easily from the standard estimates that for all $x, y \in Q$, if $C$ is chosen to be large enough, then

$$|T(f_\infty)(x) - T(f_\infty)(y)| \le C\|f\|_\infty; \tag{4.4}$$

to prove this write $T$ as integration against the associated kernel, and split the integral into a series and proceed as at the outset of the proof of Lemma 18. Therefore since $\int b_Q\, d\mu = \mu(Q)$, we have for all $x \in Q$,

$$|T(f_\infty)(x) - \langle T(f_\infty), \frac{b_Q}{\mu(Q)}\rangle| \le C\|f\|_\infty.$$

But

$$\begin{aligned} |\langle T(f_\infty), \frac{b_Q}{\mu(Q)}\rangle| &\le |\langle T(f), \frac{b_Q}{\mu(Q)}\rangle| + \langle T(f_0), \frac{b_Q}{\mu(Q)}\rangle| \\ &\le C\|T(f)\|_\infty + C\|f\|_\infty, \end{aligned}$$

using (4.3). Hence $\|T(f_\infty)\|_{L^\infty(Q)} \le C\|f\|_\infty + C\|T(f)\|_\infty$, and the same bound then holds for $T(f_0)$.

It now suffices to verify that $\|T(f_0 - f \cdot \chi_Q)\|_{L^1(Q)} \le C\|f\|_\infty \mu(Q)$. But $f_0 - f\chi_Q$ is in $L^\infty$, and is supported on the union of at most a fixed, finite number of cubes of generation $k$, each at distance not exceeding $C\delta^k$ from $Q$. So we may split $f_0 - f\chi_Q$ into the parts living on each of these disjoint cubes, and apply Lemma 18 to each to obtain the conclusion desired.

The proof of Proposition 21 involves iteration of the following basic algorithm. Let $Q$ be any dyadic cube, and suppose we are given a function $\psi$ which satisfies $\|\psi\|_\infty \le A_2$ and $|\int_Q \psi\, d\mu| \ge \varepsilon_0 \mu(Q)$, where $\varepsilon_0 > 0$ is a small constant, which depends on $A_0, A_1$ and satisfies a condition to be imposed below. Consider all the children $R$ of $Q$. Let $\varepsilon << \varepsilon_0$ be another, even smaller, constant. If for some $R$,

$$|\int_R \psi \, d\mu| < \varepsilon\mu(R), \tag{4.5}$$

and if $R$ has more than one child, we call $R$ a stopping–time cube and put it aside. If $R$ has only one child, we examine all descendants of $R$, and denote by $R'$ the first descendant which has more than one child.

Such an $R'$ must exist, for if not, $R$ must consist of some single point $x$, which is therefore an atom. Then $x = y_1$ or $y_2$ for some atom $y \in Y$. Therefore the most recent ancestor $\hat{R}$ of $R$ which is distinct from $R$ is simply $\{y_1, y_2\}$, and $\psi$ is constant on $\hat{R}$ (for it is obtained by lifting a function on $Y$ by the procedure prescribed above). Thus trivially

$$\mu(\hat{R})^{-1}|\int_{\hat{R}} \psi| = \mu(R)^{-1}|\int_R \psi| < \varepsilon,$$

so that $\hat{R}$ should have already been designated as a stopping–time cube previously in the stopping–time construction. In other words, we should have stopped before ever reaching $R$, a contradiction.

Returning to the paragraph before last, $R'$ is called a stopping–time cube, and is put aside. Note that $R' = R$ as a point set, so that (4.5) also holds for $R'$.

For every child $R$ of $Q$ for which (4.5) fails to hold, we examine in turn each child $S$ of $R$, and ask whether (4.5) holds for $S$, that is, whether $|\int_S \psi \, d\mu| < \varepsilon\mu(S)$. The first descendant of such an $S$ to have more than one child is designated as a stopping–time cube and set aside, and for the remaining $S$, all their children are examined in turn. The process is repeated indefinitely, passing to higher and higher generations.

After infinitely many steps, we obtain a collection $\{P_\gamma\}$ of pairwise disjoint stopping–time cubes contained in $Q$. Each has at least two children, and

$$|\int_{P_\gamma} \psi \, d\mu| < \varepsilon\mu(P_\gamma).$$

Since $\|\psi\|_\infty \leq A_2$ and $|\int_Q \psi| \geq \varepsilon_0\mu(Q)$ by assumption, if $\varepsilon$ is chosen sufficiently small relative to $\varepsilon_0$ then there exists $\eta > 0$, depending only on $A_0, A_1$, such that

$$\sum_\gamma \mu(P_\gamma) \leq (1-\eta)\mu(Q).$$

We also wish to modify $\psi$ on the union of the stopping–time cubes. Fix $\gamma$, and divide the children $R_\beta$ of $P_\gamma$ into two nonempty collections, each with the same number of elements, plus or minus one. Do this so that the sum of $\mu(R_\beta)$ over the cubes of the second collection is greater than or equal to the sum over those in the first. Define a complex number $\omega$ of modulus one so that $\int_{P_\gamma} \psi \, d\mu$ is a nonnegative real number times $\omega$; take $\omega = 1$ if the integral is zero. Define $c_\beta$ to be $\omega$ if $R_\beta$ belongs to the first collection, and $c_\beta = \tilde{c}_\beta \omega$ if $R_\beta$ belongs to the second, where $\tilde{c}_\beta$ is a real number between $-1$ and $-\varepsilon_0$, chosen to satisfy a further constraint below. Define

$$\psi^1 = \sum_\beta c_\beta b^1_{R_\beta} \qquad \text{on } \bigcup_\beta R_\beta = P_\gamma,$$

where $\{b^1_S : S \in \mathcal{Q}\}$ is the dyadic pseudo–accretive system guaranteed by the hypothesis of Proposition 21. Consequently

$$\int_{P_\gamma} \psi^1 \, d\mu = \sum_\beta c_\beta \mu(R_\beta).$$

The number of children $R_\beta$ is between 2 and a fixed upper bound, and they all have comparable measures. Therefore if $\varepsilon_0$ is sufficiently small but positive, it is possible to choose the $\tilde{c}_\beta$, lying between $-1$ and $-\varepsilon_0$, so that

$$\int_{P_\gamma} \psi^1 \, d\mu = \int_{P_\gamma} \psi \, d\mu.$$

Fix such an $\varepsilon_0$, and then fix $\varepsilon$ in the stopping–time condition (4.5).

Recall that our goal is to build a para–accretive function. The stopping–time cubes $P_\gamma$ are where the stronger pseudo–accretivity condition fails (for the first time). $\psi^1$ has the advantage that although $|\int_{P_\gamma} \psi^1 \, d\mu|$ is no better than the corresponding expression for $\psi$, we do have a satisfactory lower bound on the integral over each child of $P_\gamma$. So assuredly, $\psi^1$ satisfies the para–accretivity condition, for $P_\gamma$.

Carry this procedure out for each stopping–time cube. Define $\psi^1 \equiv \psi$ on $Q \setminus \cup_\gamma P_\gamma$, so that $\psi^1$ is defined on all of $Q$. Let $\{R_\beta\}$ be the collection of all children of all the nonatomic stopping–time cubes. This completes the description of the basic algorithm. Its input is a pair $(Q, \psi)$, consisting of a cube $Q$ and a function $\psi$ satisfying $\|\psi\|_\infty \le A_2$ and $|\int_Q \psi \, d\mu| \ge \varepsilon_0 \mu(Q)$. Its output is a function $\psi^1$, and a collection $\{R_\beta\}$ of pairwise disjoint dyadic subcubes of $Q$. Their properties are:

$$\|\psi^1\|_\infty \le A_2, \tag{4.6}$$

$$\int_S \psi^1 \, d\mu = \int_S \psi \, d\mu, \tag{4.7}$$

for any $S \in \mathcal{Q}$ which is contained in $Q$ and is neither a subset nor a parent of any $R_\beta$,

$$|\int_S \psi^1 \, d\mu| \ge \varepsilon \mu(S) \tag{4.8}$$

for every $S$ which is neither a proper subset nor the parent of any $R_\beta$,

$$|\int_{R_\beta} \psi^1 \, d\mu| \geq \varepsilon_0 \mu(R_\beta) \tag{4.8$'$}$$

while

$$\sum_\beta \mu(R_\beta) \leq (1-\eta)\mu(Q) \tag{4.9}$$

and

$$\psi^1 \equiv \psi \text{ on } Q \setminus \cup R_\beta. \tag{4.10}$$

We need to repeat the basic algorithm infinitely many times. Begin with a dyadic cube $R^0$, and a function $\psi^0$ which vanishes almost everywhere outside of $R^0$, and satisfies $\|\psi^0\|_\infty \leq A_2$ and $|\int_{R^0} \psi^0 \, d\mu| \geq \varepsilon_0 \mu(R^0)$. Apply the basic algorithm to obtain a function $\psi^1$ and cubes $\{R^1_\beta\}$ with properties (4.6) through (4.10). For each $\beta$, the pair $(R^1_\beta, \psi^1 \chi_{R^1_\beta})$ is admissible data for the basic algorithm, so we may apply the algorithm on each pair to obtain further subcubes $\{R^2_\beta\}$ and a function $\psi^2$, which is still defined on all of $R^0$ (as described, the basic algorithm only yields a $\psi^2$ defined on $\cup_\beta R^1_\beta$, but we just set $\psi^2 \equiv \psi^1$ on the remainder of $R^0$). An infinite number of repetitions produces functions $\psi^n$ and collections $\{R^n_\beta\}$ of cubes, for every integer $n \geq 1$.

Write $E^0 = R^0$ and $E^n = \cup_\beta R^n_\beta$. Then

$$\mu(E^n) \leq (1-\eta)^n \mu(R^0).$$

Since $E^{n+1} \subset E^n$, almost every point belongs to only finitely many $E^n$. Therefore the sequence $\psi^n(x)$ is eventually constant for almost every $x$, so we may define

$$\varphi(x) = \lim_{n\to\infty} \psi^n(x). \tag{4.11}$$

Surely $\|\varphi\|_\infty \leq A_2$.

Suppose now that $\{b_S : S \in \mathcal{Q}\}$ is a dyadic pseudo–accretive system. Fix a cube $Q$, and apply this whole procedure with $R^0 = Q$ and $\psi^0 = b^1_Q$.

**Lemma 23.** *$\varphi$ is dyadic para–accretive, on $Q$.*

*Proof.* Let $S$ be an arbitrary dyadic cube contained in $Q$. Consider the smallest integer $n \geq 0$ such that $S$ is not contained in any $R^{n+1}_\gamma$. Then $S$ is contained in some $R^n_\beta$. If $S = R^n_\beta$ then $\mu(S)^{-1} \int_S \varphi = \mu(S)^{-1} \int_S \psi^n$, whose absolute value has a fixed lower bound by construction. If $S$ is a proper subset of $R^n_\beta$, then every $R^{n+1}_\gamma$ is either disjoint from $S$, or is properly contained in $S$. Therefore $\int_S \varphi = \int_S \psi^n$. If $S$ is not a stopping–time cube of generation $n+1$ then $|\int_S \psi^n| \geq \varepsilon \mu(S)$ by the stopping–time rule (4.5). If $S$ is a stopping–time cube, then for each of its children $R$, we have just seen that there is a fixed lower bound for the absolute value of the average of $\varphi$ over $R$.

**Lemma 24.** *$T(\varphi)$ belongs to dyadic BMO on $Q$.*

*Proof.* Let $S$ be an arbitrary dyadic subcube of $Q$. Since $\varphi \in L^\infty$ and $T$ is a singular integral operator, by a standard argument it suffices to show that

$$\|T(\varphi\chi_S)\|_{L^1(S)} \le C\mu(S). \tag{4.12}$$

Again let $n$ be the smallest integer for which $S$ is not contained in any $R_\beta^{n+1}$, so that it is contained in some $R_\gamma^n$. Ignore for the rest of the argument those stopping–time cubes $P_\beta^{n+1}$ which are disjoint from $S$. We have $\varphi = \psi^n$ on $S \setminus \cup_\beta P_\beta^{n+1}$, and $\int_{P_\beta^{n+1}} \varphi = \int_{P_\beta^{n+1}} \psi^n$ for all $\beta$.

Recall from Lemma 22 that since $S$ is contained in $R_\gamma^n$ and $\psi^n \equiv c_\gamma \cdot b^1_{R_\gamma^n}$ on $R_\gamma^n$,

$$\|T(\psi^n\chi_S)\|_{L^1(S)} \le C\mu(S). \tag{4.13}$$

Since the $P_\beta^{n+1}$ are pairwise disjoint,

$$T(\varphi\chi_S) = T(\psi^n\chi_S) + \sum_\beta T((\varphi - \psi^n)\chi_{P_\beta^{n+1}}). \tag{4.14}$$

Because $\|\varphi - \psi^n\|_\infty \le 2A_2$ and $\int_{P_\beta^{n+1}}(\varphi - \psi^n) = 0$, it follows readily from the standard estimates that

$$\|T((\varphi - \psi^n)\chi_{P_\beta^{n+1}})\|_{L^1(X\setminus P_\beta^{n+1})} \le C\mu(P_\beta^{n+1}).$$

Combining this with (4.13) and (4.14) yields

$$\|T(\varphi\chi_S)\|_{L^1(S)} \le C\mu(S) + C\sum_{P_\beta^{n+1}\subset S} \mu(P_\beta^{n+1}) + \sum_{P_\beta^{n+1}\subset S} \|T((\varphi - \psi^n)\chi_{P_\beta^{n+1}})\|_{L^1(P_\beta^{n+1})}.$$

But

$$\|T(\psi^n\chi_{P_\beta^{n+1}})\|_{L^1(P_\beta^{n+1})} \le C\mu(P_\beta^{n+1}),$$

so after summing over $\beta$, we've learned that

$$\|T(\varphi\chi_S)\|_{L^1(S)} \le C\mu(S) + \sum_{P_\beta^{n+1}\subset S} \|T(\varphi\chi_{P_\beta^{n+1}})\|_{L^1(P_\beta^{n+1})}. \tag{4.15}$$

Now feed (4.15) into itself, estimating the $\beta$–th term on the right by

$$C\mu(P_\beta^{n+1}) + \sum_{P_\sigma^{n+2}\subset P_\beta^{n+1}} \|T(\varphi\chi_{P_\sigma^{n+2}})\|_{L^1(P_\sigma^{n+2})}.$$

Repeating infinitely many times, we find that

$$\begin{aligned}
(4.16) \qquad \|T(\varphi\chi_S)\|_{L^1(S)} &\le C\mu(S) + C\sum_{j=0}^{\infty}\sum_{P_\sigma^{n+1+j}\subset S}\mu(P_\sigma^{n+1+j})\\
&\le C\mu(S),
\end{aligned}$$

since

$$\sum_{P_\sigma^{n+1+j}\subset S}\mu(P_\sigma^{n+1+j}) \le (1-\eta)^j\mu(S).$$

The passage to the limit implicit in (4.16) is justified by our assumption that $T = T_K$ where $K$ is bounded.

In the same way that $b^1$ has been constructed, we may produce $b^2$ such that $T^t(b^2) \in BMO$ (dyadic). Both functions $b^i$ are defined on some cube $Q$, and in the next lemma we continue to work only on $Q$.

**Lemma 25.** *For any $\alpha > 0$, $T$ is weakly bounded from $b^1\mathcal{D}_\alpha$ to $(b^2\mathcal{D}_\alpha)'$.*

*Proof.* Let $x_0 \in Q$, $r > 0$, and $\alpha > 0$. Suppose that $\varphi_1, \varphi_2 \in B_{\alpha,x_0,r}$. Choose $k$ so that $\delta^{k+1} < r \le \delta^k$. It is no loss of generality to suppose that $k$ is greater or equal to the generation of $Q$. Let $R$ be a cube of generation $k$ such that $x_0 \in R$, and let $\hat{R}$ be the union of all cubes $S \subset Q$ of generation $k$ which intersect $B(x_0, r)$. Then

$$\|T(b^1\chi_{\hat{R}})\|_{L^1(\hat{R})} \le C\mu(\hat{R}) \le C\mu(R)$$

by (4.12) and Lemma 18. For $x \in \hat{R}$, recalling that $T$ is a truncated singular integral,

$$\begin{aligned}
|T(\varphi_1 b^1)(x)| &= |T(\varphi_1 b^1\chi_{\hat{R}})(x)|\\
&= \Big|\int_{\substack{\rho(x,y)>\varepsilon\\ y\in\hat{R}}} K(x,y)\, b^1(y)\ d\mu(y)\Big|\\
&\le |\varphi_1(x)\int_{\substack{\rho(x,y)>\varepsilon\\ y\in\hat{R}}} K(x,y)\, b^1(y)\, d\mu(y)|\\
&\quad + \int_{\substack{\rho(x,y)>\varepsilon\\ y\in\hat{R}}} |K(x,y)|\cdot|\varphi_1(y)-\varphi_1(x)|\cdot|b^1(y)|\, d\mu(y)\\
&\le |T(b^1\chi_{\hat{R}})(x)| + C\|b^1\|_\infty \int_{\substack{\rho(x,y)>\varepsilon\\ y\in\hat{R}}} \lambda(x,y)^{-1}\, r^{-\alpha}\rho(x,y)^\alpha\ d\mu(y)\\
&\le |T(b^1\chi_{\hat{R}})(x)| + C_\alpha\|b^1\|_\infty.
\end{aligned}$$

Thus we obtain an estimate somewhat stronger than the weak boundedness property:

$$\|T(\varphi_1 b^1)\|_{L^1(B(x_0,r))} \le C\mu(B(x_0,r)).$$

Certainly this implies that

$$|\langle T(\varphi_1 b^1),\, \varphi_2 b^2\rangle| \le C\mu(B(x_0,r)).$$

Proposition 21 is now proved, for we are assuming that $X$ is itself a dyadic cube, so it suffices to apply the construction with $Q = X$. It remains only to obtain the main theorem in the case where $X$ itself is not necessarily a dyadic cube. It suffices to prove that $T$ is bounded on $L^2(E)$, for any bounded set $E \subset X$, with a bound independent of $E$. Apply Lemma 19 to cover $E$ by a union of at most $C$ disjoint cubes $Q^k_\alpha$. Then the space $\cup_\alpha Q^k_\alpha$ is again a space of homogeneous type, and a suitable system of dyadic cubes on it may be obtained by defining the whole space to be a cube, its children to be the $Q^k_\alpha$, and their descendants of all generations to be the same as in $X$. The boundedness of $T$ on $\cup_\alpha Q^k_\alpha$ follows immediately from Proposition 21, and the bound is clearly independent of $E$.

## 5. An application.

Our first application is an alternate proof of a theorem of David [D1] concerning the Cauchy integral on certain curves in the complex plane. However, it has much in common with the original proof, and anyway David's theorem may be deduced directly from the $T(b)$ theorem itself. Our purpose is merely to illustrate the relationship of Theorem 10 to an existing circle of ideas.

Let $\Gamma \subset \mathbb{C}$ be a (connected)rectifiable curve, and let $\Lambda_1$ denote one–dimensional Hausdorff measure.

**Definition 26.** *$\Gamma$ is Ahlfors–regular if for every $z \in \mathbb{C}$ and $r > 0$,*

$$\Lambda_1(\Gamma \cap B(z,r)) \le Cr.$$

For $r$ less than half the diameter of $\Gamma$ there is the reverse inequality

$$\Lambda_1(\Gamma \cap B(z,r)) \ge r$$

for all $z \in \Gamma$. Equipped with $\Lambda_1$ and the Euclidean metric, an Ahlfors–regular curve becomes a space of homogeneous type. Then $K(z,w) = (z-w)^{-1}$ is a standard kernel, and the Cauchy integral $\mathcal{C}_\Gamma$ is defined to be the singular integral operator associated to $K$ via the usual procedure (1.3).

The following result is due to David [D1].

**Theorem 27.** *The Cauchy integral $\mathcal{C}_\Gamma$ is bounded on $L^2(\Gamma, \Lambda_1)$ for any Ahlfors–regular curve $\Gamma$.*

In order to bring Theorem 10 to bear, we shall require three facts, the first two of which are already principal ingredients in [D1]. By a Lipschitz curve we shall mean any Lipschitz graph, or any curve obtained by rotating a Lipschitz graph any amount. The first fact is that the Cauchy integral is bounded on $L^2$ on any Lipschitz curve, with a bound depending only on the Lipschitz constant [CMM]. Second, for any Ahlfors–regular $\Gamma$, for any $z \in \Gamma$ and any $r > 0$ not exceeding the diameter of $\Gamma$, there exists a Lipschitz graph $\gamma$ such that

$$\Lambda_1(B(z,r) \cap \Gamma \cap \gamma) \geq Cr,$$

and the Lipschitz constant of $\gamma$ is bounded independent of $z, r$ [D1]. Third, if $\gamma$ is an Ahlfors–regular curve on which $\mathcal{C}_\gamma$ is $L^2$–bounded, then for any closed subset $E \subset \gamma$, there exists an $L^\infty$ function $h$ supported on $E$ such that $\mathcal{C}_\gamma(h) \in L^\infty(\gamma)$, or equivalently, the Cauchy potential $\int h(w)(z-w)^{-1}\,d\Lambda_1(w)$ is uniformly bounded on $\mathcal{C}\backslash E$, and such that $\Lambda_1(E)^{-1}|\int h\,d\Lambda_1|$ is bounded away from zero. Of course the $L^\infty$ norm of $h$ and its Cauchy integral are bounded above by a constant depending only on $\gamma$. See [M, Chapter 3].

Let an Ahlfors–regular curve $\Gamma$ be given. To construct a pseudo–accretive system $\{b_B\}$, for each $B = B(z,r)$ with $z \in \Gamma$ and $r$ not exceeding the diameter of $\Gamma$, choose a Lipschitz curve $\gamma$ as above, and then take $b_B$ to be an $L^\infty$ function supported on $\Gamma \cap \gamma \cap B(z,r)$ and satisfying $|\int b_B\,d\Lambda_1| \geq \varepsilon r$, whose Cauchy integral is in $L^\infty$, uniformly in $z, r$. It follows that $\mathcal{C}_\Gamma(b_B) \in L^\infty(\Gamma)$ uniformly in $B$, and the same for all the operators obtained by truncating the kernel for $\mathcal{C}_\Gamma$. Since the Cauchy integral is antisymmetric, the same pseudo–accretive system works for its transpose, and the main theorem then implies $L^2$ boundedness.

The same argument yields a more general result of David. Consider a $d$–dimensional subset $E$ of $\mathbb{R}^N$, regular in the sense of David, which "contains big pieces of Lipschitz graphs". See [D2] and forthcoming work of David and Semmes for definitions. Consider further an antisymmetric kernel $K$ defined on $\mathbb{R}^N\backslash\{0\}$, which satisfies $|D^\beta K(x)| \leq C_\beta |x|^{-d-|\beta|}$. Then if $\Lambda_d$ denotes $d$–dimensional Hausdorff measure, the operator

$$Tf(x) = \int_E K(x-y)\,f(y)\,d\Lambda_d(y)$$

is bounded on $L^2(E, \Lambda_d)$. This may be proved using Theorem 10, instead of the "good–$\lambda$" method of [D1].

## 6. Analytic capacity and boundedness of the Cauchy integral.

The analytic capacity of a compact set $E \subset \mathbb{C}$ is defined to be

$$\gamma(E) = \sup_{\substack{f \in H^\infty(\bar{\mathbb{C}}\backslash E) \\ \|f\|_\infty \leq 1 \\ f(\infty)=0}} |f'(\infty)|$$

where $f'(\infty) = \lim_{z\to\infty} zf(z)$. The analytic capacity of an arbitrary Borel set is defined as the supremum of the analytic capacities of all its compact subsets. The main question about analytic capacity is which $E$ have positive analytic capacity. It is easy to see that $\gamma(E)$ is zero if $E$ has Hausdorff dimension less than one, and is positive if the dimension is larger than one, so one–sets (Borel sets with $0 < \Lambda_1(E) < \infty$) are of particular interest.

A bounded one-set $\Gamma \subset \mathbb{C}$ is said to be Ahlfors–David regular, or simply regular, if there exists $C < \infty$ such that for all $0 < r \le 1$, for all $z \in \Gamma$,

$$C^{-1}r \le \Lambda_1(\Gamma \cap B(z,r)) \le Cr. \tag{6.1}$$

Again these are spaces of homogeneous type, and the Cauchy integral $\mathcal{C}_\Gamma$ is a singular integral operator. Of course the main question concerning the Cauchy integral is whether it is bounded on $L^2$. The upper bound (6.1) is a necessary condition, while the lower bound ensures that $\Gamma$ is a space of homogeneous type.

Our $T(b)$ theorem has the following implication concerning the relation between these two questions.

**Theorem 28.** *Let $\Gamma$ be a bounded, Ahlfors–David regular one–set, and suppose that*

$$\gamma(\Gamma \cap B(x,r)) \ge Cr \tag{6.2}$$

*for all $z \in \Gamma$ and $r \in (0,1]$. Then $\mathcal{C}_\Gamma$ is bounded on $L^2$.*

Thus positive analytic capacity, uniformly at all places and all scales, implies boundedness of the Cauchy integral; note that for any set $E$, $\gamma(E \cap B(z,r)) \le \gamma(B(z,r)) = Cr$ with $C$ finite, so that the right–hand side of (6.2) is as large as possible, up to a constant factor. Conversely, boundedness of $\mathcal{C}_\Gamma$ implies that $\gamma(E) \ge C\Lambda_1(E)$ for all Borel $E \subset \Gamma$, and the reverse implication was already known to be true. See the book of Murai [M] for this and other information regarding analytic capacity and the Cauchy integral.

For the proof fix $z \in \Gamma$ and $r \in (0,1]$. By hypothesis there exists $f$ holomorphic outside of the closure of $B(z,r) \cap \Gamma$, such that $\|f\|_\infty \le 1$, $f(\infty) = 0$ and $|f'(\infty)| \ge Cr$. The first two facts imply that $f$ can be expressed as a Cauchy potential:

$$f(z) = \int_\Gamma \frac{h(w)}{z-w}\, d\Lambda_1(w)$$

where $h$ is supported on $\Gamma \cap B(z,r)$, and $\|h\|_\infty \le C$, an absolute constant. Furthermore

$$f'(\infty) = \int h(w)\, d\Lambda_1(w)$$

by definition and the dominated convergence theorem. From the upper bound on $\|h\|_\infty$, the regularity of $\Gamma$ and the fact that the Cauchy potential of $h$ is bounded off of $\Gamma \cap B(z,r)$, it follows easily that any truncation of $\mathcal{C}_\Gamma$ maps $h$ to $L^\infty$, uniformly in $z, r$; details may be found in [M].

Thus setting $b_{B(z,r)} = h$ for each $z, r$ yields a pseudo–accretive system, with $\mathcal{C}_\Gamma^\varepsilon(b_B) \in L^\infty(\Gamma)$ for each $B$ and each truncation $\mathcal{C}_\Gamma^\varepsilon$ of the Cauchy integral, uniformly in $B$ and $\varepsilon$. The main theorem then implies that the $\mathcal{C}_\Gamma^\varepsilon$ are bounded on $L^2(\Gamma, \Lambda_1)$, uniformly in $\varepsilon > 0$. Because of its definition (1.3), the Cauchy integral therefore extends to a bounded operator.

The next result is not an application of Theorem 10 *per se*, but is in the same spirit.

**Theorem 29.** *Let $\Gamma$ be a bounded Ahlfors–David regular one–set, and suppose that $E \subset \Gamma$ is compact and satisfies $\gamma(E) > 0$. Then there exists an Ahlfors–David regular set $\Gamma'$ such that $\mathcal{C}_{\Gamma'}$ is bounded on $L^2(\Gamma')$ and such that $\Lambda_1(E \cap \Gamma') > 0$.*

Since $\gamma(A) > 0$ for any subset $A \subset \Gamma'$ with $\Lambda_1(A) > 0$, this says that in a sense, positive analytic capacity, for regular sets, is always due to boundedness of the Cauchy integral.

For the proof, let $\mathcal{Q}$ be a system of dyadic cubes on $\Gamma$, as guaranteed by Theorem 11. Let $f \in H^\infty(\bar{\mathbb{C}} \backslash E)$ satisfy $f(\infty) = 0$ and $f'(\infty) \neq 0$, and write it as the Cauchy potential of an $L^\infty$ function $h$ supported on $E$, as above. Then $\int_\Gamma h\, d\Lambda_1 \neq 0$.

One of the dyadic cubes $Q$ will be all of $\Gamma$. Run a stopping–time procedure on it as in the proof of the main theorem, stopping at a cube $P$ whenever

$$|\int_P h\, d\Lambda_1| < \varepsilon \Lambda_1(P). \tag{6.3}$$

$\Gamma'$ will be constructed by excising from $\Gamma$ the union of all these stopping–time cubes $P_\beta$, and replacing each $P_\beta$ with a certain set $S_\beta$. If $\varepsilon$ is chosen to be sufficiently small in (6.3) then we will have $\Lambda_1(\Gamma \backslash \cup_\beta P_\beta) > 0$, and in fact $\int_{\Gamma \backslash \cup_\beta P_\beta} h \sim \int h \neq 0$, whence $\Lambda_1(E \cap \Gamma') > 0$.

There exists $c > 0$ such that for each $P_\beta$, there exists $z_\beta$ such that $B(z_\beta, cr_\beta) \subset P_\beta$, where $r_\beta$ denotes the diameter of $P_\beta$. Let $S_\beta$ be the union of two circles centered at $z_\beta$, of radii $c/2$ and $c/4$, respectively. Define a function $g_\beta$ on $S_\beta$ to be constant on each of the two circles, and adjust the constants so their absolute values are bounded above and away from zero, and so that

$$\int_{S_\beta} g_\beta\, d\Lambda_1 = \int_{P_\beta} h\, d\Lambda_1,$$

and do this so that the bounds are uniform in $\beta$.

Define

$$\Gamma' = (\Gamma \backslash \bigcup_\beta P_\beta) \bigcup_\beta S_\beta$$

and set

$$b = \begin{cases} h & \text{on } \Gamma \cap \Gamma' \\ g_\beta & \text{on } S_\beta \end{cases}$$

For a system of dyadic cubes on $\Gamma'$ take all $S \in \mathcal{Q}$ which are not contained in any stopping–time cube, together with all $S_\beta$, together with each of the two circles comprising each $S_\beta$, together with subsets of all these circles, obtained by bisecting each circle into two semicircles, then repeatedly bisecting the semicircles and resulting arcs. Then $b$ is dyadic para–accretive with respect to this system of cubes. The proof of Lemma 24 shows that $\mathcal{C}^{\varepsilon}_{\Gamma'}(b)$ belongs to dyadic $BMO(\Gamma')$, uniformly in $\varepsilon$. Therefore the Cauchy integral is bounded on $L^2(\Gamma')$, by the original $T(b)$ theorem.

Finally, $\Gamma'$ is an Ahlfors–David regular set. This follows from the facts that $\Lambda_1(S_\beta) \sim \Lambda_1(P_\beta)$ for all $\beta$, and that the distance from $S_\beta$ to its complement in $\Gamma'$ is comparable to its diameter, and from the proof in section 3 that any dyadic cube is itself a space of homogeneous type.

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