NONUNIQUENESS OF GENERALIZED SOLUTIONS OF THE NAVIER-STOKES EQUATION

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Abstract. Solutions of the Cauchy problem for the Navier-Stokes equation, in a certain generalized sense, are not unique.

1. Introduction

The Cauchy problem for the Navier-Stokes equation governing viscous incompressible two-dimensional fluid flow with spatially periodic boundary conditions can be formulated as

$$\begin{cases} u_t + u \cdot \nabla u = \nu \Delta u + \nabla p \\ \operatorname{div} u = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

where $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, $x \in \mathbb{T}^2$ is the spatial variable, t is the time, $u : \mathbb{T}^2 \to \mathbb{R}^2$ is the velocity field, u_0 is the initial condition, p is the pressure, and $\nabla, \operatorname{div}, \Delta$ denote the spatial gradient, divergence, and Laplacian respectively. $\nu > 0$ is a constant. Here u, p are both unknowns, with the emphasis on the velocity field u, and p is only determined up to a time-dependent additive constant. The Cauchy problem for the Euler equation governing inviscid incompressible two-dimensional fluid flow with spatially periodic boundary conditions is identical, except that $\nu = 0$.

A function $u \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{T}^2))$ is said to be a weak solution if $\operatorname{div}(u) \equiv 0$ in the sense of distributions and

$$\iint -\varphi_t \cdot u - \nabla(\varphi) \cdot u \cdot u - \nu u \Delta \varphi = 0 \tag{1.2}$$

for any $\mathbb{R}^2$-valued test function φ satisfying $\operatorname{div}(\varphi) \equiv 0$. Scheffer [3] and Shnirelman [4] have shown that weak solutions of the Euler equation in the class $L^2([0,1], L^2(\mathbb{T}^2))$ are not unique; there exist nonzero solutions with initial datum $u_0 \equiv 0$. These solutions have (apparently) infinite energy $\int_{\mathbb{T}^2} |u(t,x)|^2\,dx$ for a sequence of times tending to zero. We are not aware of any proof of nonuniqueness of solutions with uniformly bounded energy, that is, within the class $C^0([0,1], L^2(\mathbb{T}))$. Ladyzhenskaya [2] has given two examples of nonuniqueness for the three-dimensional Navier-Stokes equation. Both examples are set in parabolic time-dependent domains, which shrink to a single point as $t \to 0^+$. One example involves solutions with infinite energy for all times, satisfying natural boundary conditions;

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the other involves solutions with uniformly finite energy, satisfying less natural boundary conditions.

In this note we establish nonuniqueness of a certain class of generalized solutions to the (two-dimensional) Navier-Stokes initial value problem (1.1) with periodic boundary conditions. Our generalized solutions are however not even weak solutions; they have infinite energy $\int_{\mathbb{T}^2} |u(t,x)|^2\,dx$ for all time.

Theorem 1.1. *Let $\nu > 0$. For each $s < 0$ there exists $u \in C^0([0,1], H^s(\mathbb{T}^2))$ which is a generalized solution of (1.1) in all of the three senses discussed below, such that u does not vanish identically, but satisfies $u(0,x) \equiv 0$.*

Here H^s denotes the usual Sobolev space. No such result can hold for $s = 1$, for if $u \in C^0(H^1)$ then by the Sobolev embedding theorem, $|u|^2|\nabla u| \in C^0(L^1)$, and the formal integration by parts establishing the conservation law $\int |u(t,x)|^2\,dx \le \int |u_0(t,x)|^2\,dx$ can then be justified. Our construction can readily be refined to produce a solution in $\cap_{s<0} C^0([0,1], H^s(\mathbb{T}^2))$ or other translation-invariant spaces less restrictive than $C^0(H^0)$. This theorem directly implies the corresponding result for $\mathbb{T}^3$, by consideration of $v(t,x',x_3) = (u_1(t,x'), u_2(t,x), 0)$ where $x = (x', x_3) \in \mathbb{T}^2 \times \mathbb{T}$ and u has components (u_1, u_2).

The generalized solutions of the theorem are not weak solutions in the usual sense; $|u|^2$ is not a locally integrable function of (t,x) and hence the usual weak formulation of the differential equation cannot be used. They are instead solutions in three senses:

1. When the differential equation is rewritten in terms of spatial Fourier coefficients $\widehat{u}(t,n)$ (as defined in (1.3)) an infinite coupled system of ordinary differential equations results. Each of these equations involves an infinite sum of nonlinear expressions, which need not converge absolutely when u is not sufficiently regular, that is, when its Fourier coefficients are not square summable. The Fourier coefficients of our solutions are continuously differentiable functions of t, and moreover these Fourier series are sufficiently sparse that for every t and every $n \in \mathbb{Z}^2$, all but finitely many terms in the equation for $\widehat{u}(t,n)$ vanish identically. Convergence is then no issue, and each ordinary differential equation holds in the classical sense.
2. The generalized solutions are limits, in the $C^0([0,1], H^s(\mathbb{T}^2))$ norm, of infinitely differentiable solutions of the inhomogeneous Navier-Stokes equation (see (1.4) below) with initial data $\equiv 0$ and smooth driving forces tending to zero in $C^0([0,1], H^{s-2})$ norm.
3. Let $\{m_k : k = 1,2,3,\cdots\}$ be any sequence of uniformly bounded functions from $\mathbb{Z}^2$ to $\mathbb{C}$, such that $\lim_{k\to\infty} m_k(n) = 1$ for all $n \in \mathbb{Z}^2$, and that m_k has finite support for each k. Let $u_k(t) : \mathbb{T}^2 \to \mathbb{C}$ be defined by $\widehat{u_k}(t,n) = m_k(n)\widehat{u}(t,n)$ for all $n \in \mathbb{Z}^2$, where $\widehat{\cdot}$ denotes the Fourier transform with respect to the spatial variable. Then the sequence of C^∞ functions $u_k \cdot \nabla u_k$ converges, in the sense of distributions, to a limit independent of the choice of truncating sequence $\{m_k\}$, and with $u \cdot \nabla u$ interpreted as this limit, u satisfies (1.1) in the sense of distributions.

That these three conclusions hold is part of the statement of Theorem 1.1.

Define Fourier coefficients by

$$\widehat{u}(t,n) = (2\pi)^{-2} \int_{\mathbb{T}^2} u(t,x) e^{-ix\cdot n}\,dx. \tag{1.3}$$

A function u is said to have finitely supported partial Fourier transform if $\widehat{u}(t,n)$ vanishes identically as a function of $t \in [0,1]$, for all but finitely many $n \in \mathbb{Z}^2$.

The construction, which was employed in [1] to establish a similar result for the nonlinear Schrödinger equation; has little to do with the particular structure of the Navier-Stokes equation. The main step is the following approximation property for smooth solutions of the inhomogeneous equation.

Proposition 1.2. *Let $s < 0$, and let $\nu \in \mathbb{R}$. Let $u \in C^\infty([0,1] \times \mathbb{T}^2)$. Suppose that for each $n \in \mathbb{Z}^2$, $\widehat{u}(t,n)$ vanishes to infinite order as $t \to 0^+$. Suppose also that $\operatorname{div} u \equiv 0$. Then for any $\varepsilon > 0$ and any $\rho < \infty$ there exist $v, F, p \in C^\infty([0,1] \times \mathbb{T}^2)$ with finitely supported partial Fourier transforms, vanishing to infinite order as $t \to 0^+$, such that*

$$\begin{cases} v_t + v \cdot \nabla v = \nu \Delta v + F + \nabla p \\ \operatorname{div} v = 0 \end{cases} \tag{1.4}$$

and

$$\widehat{v - u}(t,n) \equiv 0 \text{ for all } |n| < \rho \tag{1.5}$$

$$\widehat{F}(t,n) \equiv 0 \text{ for all } |n| < \rho \tag{1.6}$$

$$\|v - u\|_{C^0([0,1],H^s(\mathbb{T}^2))} < \varepsilon \tag{1.7}$$

$$\|F\|_{C^0([0,1],H^{s-2}(\mathbb{T}^2))} < \varepsilon. \tag{1.8}$$

2. Reformulation

We begin by reformulating the differential equation. We will work exclusively with velocity fields satisfying

$$\int_{\mathbb{T}^2} u(t,x)\, dx \equiv 0. \tag{2.1}$$

The vorticity $\omega(t,x)$ is defined to be the real-valued function

$$\omega(t,x) = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}, \tag{2.2}$$

where $u = (u_1, u_2)$. At least formally, (1.1) with initial datum $u_0 \equiv 0$ can be reformulated as

$$\begin{cases} \omega_t + u \cdot \nabla \omega = \nu \Delta \omega \\ \omega(0,x) \equiv 0 \\ \int_{\mathbb{T}^2} \omega(t,x)\, dx \equiv 0. \end{cases} \tag{2.3}$$

Here u is defined in terms of ω by (2.2) together with the incompressibility condition $\operatorname{div}(u) \equiv 0$ and (2.1); these together with (2.1) determine u uniquely as an element of $C^0(H^s)$ whenever $\omega \in C^0(H^{s-1})$.

Express ω as a Fourier series

$$\omega(t,x) = \sum_{n \in \mathbb{Z}^2} \omega_n(t) e^{in\cdot x}; \tag{2.4}$$

we write $\omega_n(t) = \widehat{\omega}(t,n)$. These satisfy

$$\omega_{-n} \equiv \overline{\omega_n} \tag{2.5}$$

$$\omega_0 \equiv 0 \tag{2.6}$$

if and only if ω is real-valued, as it is required to be. Then the Fourier coefficients of the associated velocity field u are $\mathbb{R}^2$-valued, and are

$$\widehat{u}(t,n) = -\frac{n^*}{|n|^2}\widehat{\omega}(t,n) \tag{2.7}$$

for all $n \neq 0$, where

$$(n_1,n_2)^* = (-n_2,n_1). \tag{2.8}$$

The Fourier coefficients for ω are to satisfy the system

$$\frac{d\omega_n(t)}{dt} = -\sum_{j+k=n}\frac{j^*\cdot k}{|j|^2}\omega_j\omega_k - \nu|n|^2\omega_n + F_n = \sum_{j+k=n}\tilde{b}(j,k)\omega_j\omega_k - \nu|n|^2\omega_n + F_n \tag{2.9}$$

for some driving sequence F, where the symmetrized coefficients are

$$-2\tilde{b}(j,k) = \frac{j^*\cdot k}{|j|^2} + \frac{k^*\cdot j}{|k|^2} = \frac{(j^*\cdot k)(|k|^2-|j|^2)}{|j|^2|k|^2} \tag{2.10}$$

for all $j,k\neq 0$. If either j,k vanishes then $\omega_j\omega_k \equiv 0$, so this case may be disregarded.

Define $x_n(t)$ by

$$\begin{aligned} x_n(t) &= |n|^{-1}\omega_n(t)\ \forall\, n\neq 0 \\ x_0(t) &\equiv 0 \end{aligned} \tag{2.11}$$

In terms of x our system of ordinary differential equations becomes

$$\frac{dx_n(t)}{dt} = \sum_{j+k=n} b(j,k)x_jx_k - \nu|n|^2x_n + F_n \qquad \forall\, n\neq 0 \tag{2.12}$$

where the sequence F represents some driving force, and

$$b(j,k) = \frac{|j|\cdot|k|}{|j+k|}\tilde{b}(j,k) = -\tfrac{1}{2}\frac{(j^*\cdot k)(|k|^2-|j|^2)}{|j|\cdot|k|\cdot|j+k|} \tag{2.13}$$

whenever $j+k\neq 0$; $b(-k,k)=0$ for all k. Thus we may disregard $n=0$. So long as we work with finitely supported sequence-valued functions, all of whose components are differentiable functions of t, both sides of equation (2.12) are well-defined.

3. The construction

Lemma 3.1. *Suppose that x is a finitely supported sequence-valued function of $t\in[0,1]$, that $x_n(t)$ are infinitely differentiable complex-valued functions of $t\in[0,1]$ for all $n\in\mathbb{Z}^2$, that $x_0(t)\equiv 0$, and that $x_n(t)$ vanishes to infinite order as $t\to 0^+$ for all $n\in\mathbb{Z}^2$. Then there exists $C=C(x)<\infty$ such that for any $\rho<\infty$ there exist finitely supported sequence-valued functions y,g of $t\in[0,1]$ such that*

$$y_n - x_n \text{ and } g_n \text{ are supported in } \{n : |n|\geq\rho\}, \tag{3.1}$$

$y_n(t),g_n(t)$ are infinitely differentiable functions of $t\in[0,1]$ which vanish to infinite order as $t\to 0^+$, and y,g satisfy the driven system

$$\frac{dy_n}{dt} = \sum_{j+k=n} b(j,k)y_jy_k - \nu|n|^2y_n + g_n, \tag{3.2}$$

such that the cardinalities of the supports of y, g are $\le C$, and

$$\|y\|_{C^0(\ell^\infty)} \le C \tag{3.3}$$

$$\|g_n\|_{C^0} \le C|n|^2 \text{ for all } n. \tag{3.4}$$

Let ρ be given. Let $R < \infty$ be some sufficiently large positive number, which will ultimately depend on ρ. y will be constructed inductively in the form

$$y = x + z \tag{3.5}$$

where

$$|n| \ge R \text{ for all } n \in \text{support of } z \cup \text{support of } g, \tag{3.6}$$

where the driving force g is defined to be

$$g_n = \frac{d(x_n + z_n)}{dt} - \sum_{j+k=n} b(j,k)(x_j + z_j)(x_k + z_k) + \nu |n|^2 (x_n + z_n) \tag{3.7}$$

$$= f_n + \frac{dz_n}{dt} - \sum_{j+k=n} b(j,k)\big[z_j x_k + x_j z_k + z_j z_k\big] + \nu |n|^2 z_n. \tag{3.8}$$

Given x, define f by

$$\frac{dx_n}{dt} = \sum_{j+k=n} b(j,k) x_j x_k - \nu |n|^2 x_n + f_n. \tag{3.9}$$

For each n in the support of f, choose two functions $\beta_n(t), \beta_n'(t)$ that are C^∞, vanish to infinite order as $t \to 0^+$, and satisfy

$$\beta_n(t)\beta_n'(t) \equiv \tfrac{1}{2} f_n(t). \tag{3.10}$$

Such functions exist, because f_n itself is C^∞ and vanishes to infinite order as $t \to 0^+$.

We next specify the support of z, as follows. For each n in the support of f, choose $k(n)$ satisfying

$$|k(n)| \ge R \tag{3.11}$$

$$|n \cdot k(n)| \ge \tfrac{1}{2}|n| \cdot |k(n)| \tag{3.12}$$

$$|n^* \cdot k(n)| \ge \tfrac{1}{2}|n| \cdot |k(n)|. \tag{3.13}$$

Do this so that moreover $k(-n) \equiv -k(n)$ for all n in the support of f.

Define $j(n) = n - k(n)$. Then $|k(n)|, |j(n)| \ge \rho$, provided that R is sufficiently large relative to the maximum of $|n|$ for all n in the support of f.

The vectors $k(n)$ can be chosen so that furthermore

$$|k(n) + k(n')|,\ |k(n) + j(n')|,\ |j(n) + j(n')| \text{ are all } \ge \tfrac{1}{2}R \tag{3.14}$$

for all n, n' in the support of f, except for the following exceptions: (i) $k(n) + j(n') = n$ when $n' = n$, (ii) $k(n) + k(n') = j(n) + j(n') = 0$ when $n' = -n$. This can be achieved by choosing one element from each pair $\{n, -n\}$ of elements of the support of f, choosing any ordering $\{n_1, n_2, \cdots\}$ of those, and then choosing each $k(n_{i+1})$ to be sufficiently large relative to $k(n_1), k(n_2), \cdots, k(n_i)$. We can also ensure in the same way that

$$|k(n) + m|, |j(n) + m| > \rho \tag{3.15}$$

for all n in the support of f and m in the support of x.

The coefficients $b(j,k)$ satisfy the upper bound

$$|b(j,k)| \le C \max(|j|,|k|) \ \forall j,k, \tag{3.16}$$

as is seen by a simple case-by-case analysis; the bound for the case $|j+k| \le \frac{1}{10}\max(|j|,|k|)$ uses the identity $j^* \cdot k = (j+k)^* \cdot k$. For our construction a lower bound for $|b(j(n),k(n))|$ is essential:

$$\begin{aligned} |b(j(n),k(n))| &\gtrsim \frac{|j(n)^* \cdot k(n)|||k(n)|^2 - |j(n)|^2|}{|j(n)| \cdot |k(n)| \cdot |n|} \\ &\sim \frac{|n^* \cdot k(n)||k(n) \cdot n|}{|k(n)|^2|n|} \\ &\sim |n|. \end{aligned} \tag{3.17}$$

All that is required for our construction is that this quantity should be bounded below by some strictly positive number. If it were bounded below by $c|k(n)|^\delta$ for some $\delta > 0$, then the construction would produce weak solutions in $C^0([0,1], L^2(\mathbb{T}))$, rather than merely generalized solutions. Terms with this type of growth are formally present in the equation but cancel out when the nonlinearity is symmetrized as in the calculation above of $\tilde{b}(j,k)$.

For each n in the support of f define

$$z_{k(n)}(t) = \frac{\beta_n}{|b(j(n),k(n)|^{1/2}} \tag{3.18}$$

$$z_{j(n)}(t) = \frac{\operatorname{sgn}(b(j(n),k(n))\beta_n}{|b(j(n),k(n)|^{1/2}}. \tag{3.19}$$

Therefore

$$f_n = b(j(n),k(n)) z_{j(n)} z_{k(n)} + b(k(n),j(n)) z_{k(n)} z_{j(n)} \tag{3.20}$$

for all $n \in \mathbb{Z}^2$.

Therefore for any nonzero $n \in \mathbb{Z}^2$,

$$g_n = \frac{dz_n}{dt} - 2\sum_{j+k=n} b(j,k) x_j z_k - \sum_{j+k=n}^{*} b(j,k) z_j z_k + \nu |n|^2 z_n, \tag{3.21}$$

where the summation $\sum^*_{j+k=n}$ extends over those (j,k) satisfying the restriction $(j,k) \ne (j(n),k(n))$ and $(j,k) \ne (k(n),j(n))$. Recall that g_0 automatically vanishes identically because of the form of the equation. In particular, by construction, $g_n(t) \equiv 0$ unless $|n| > \rho$.

Let M,N be the cardinalities of the supports of x,f respectively. Then the cardinality of the support of z is $2N$. The cardinality of the support of g is $\le 2N + 4MN + 4N^2$; in particular, it is independent of ρ. The functions β_n, β_n' and their derivatives satisfy upper bounds that depend only on f, not on ρ. By (3.16), the coefficients $b(\cdot,\cdot)$ contribute factors which are $O(n)$ to g_n. Thus z,g satisfy all conclusions of Lemma 3.1. □

Proposition 1.2 follows from Lemma 3.1 by reversing the substitutions made in §2. Proposition 1.2 has no hypothesis that the partial Fourier transforms are finitely supported, but this can achieved by simply truncating them, at the expense of an arbitrarily small error in any Sobolev norm since u is assumed to be infinitely differentiable. The above construction gives v, F such that

$$\|v - u\|_{C^0([0,1],H^0(\mathbb{T}^2))} \le C(u) \tag{3.22}$$

$$\|F\|_{C^0([0,1],H^{-2})} \le C(u), \tag{3.23}$$

where $C(u)$ is a finite constant that depends on u but is independent of ρ. Therefore

$$\|v - u\|_{C^0([0,1],H^s(\mathbb{T}^2))} \le C(u)\rho^s \tag{3.24}$$

$$\|F\|_{C^0([0,1],H^{s-2})} \le C(u)\rho^s, \tag{3.25}$$

and since s is strictly negative, the right-hand side can be made as small as desired by choosing ρ to be sufficiently large. □

Theorem 1.1 follows from Proposition 1.2 by the same reasoning as given for the nonlinear Schrödinger equation in [1]; this reasoning is purely formal and does not involve the detailed structure of the differential equation. We refer the reader to [1] for further details.

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