On the Eigenfunctions and Spectra of One-Dimensional chrödinger Operators with Slowly Decaying Potentials

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$$H = -\frac{d^2}{dx^2} + V(x) \qquad \text{on } \mathbb{R}^1$$

Goals

- Presence of absolutely continuous spectrum in perturbative range.
- Asymptotics of <u>generalized</u> eigenfunctions.
- Extend scope of WKB approximation.

Examples:

$$V(x) = \sum_{n \in \mathbb{Z}} a_n h(x - n)$$

where h is Schwartz, and $a_n \to 0$ at some rate.

In this lecture, a **generalized eigenfunction** means **any** solution of $Hu = \lambda^2 u$, $\lambda \in \mathbb{R}$.

Rough trichotomy when $V = O(|x|^{-r})$

• **Short-range** case $r > 1$:

Absolutely continuous spectrum $= \mathbb{R}^+$

Singular continuous spectrum $= \emptyset$

There are finitely many eigenvalues, all < 0.

Every generalized eigenfunction is bounded.

• **Long-range** case $r < \frac{1}{2}$:

There exist such potentials for which spectrum consists entirely of eigenvalues, dense in $[0, \infty)$. For most λ, every solution of the ODE grows rapidly as $x \to +\infty$ or as $x \to -\infty$. (Kotani-Ushiroya)

This happens almost surely in certain classes of *random* potentials.

Our Focus: Intermediate-range case

• There exists $V = O(|x|^{-1+\delta})$ for which eigenvalues are dense in $[0, \infty)$. (Naboko, Simon)
Anomalous behavior can occur for a set of "energies" having positive Hausdorff dimension (Remling).

• If $V(x) = O(|x|^{-r})$ and $r > 1/2$ then the absolutely continuous spectrum is nonempty (and equals $\mathbb{R}^+$). Almost every generalized eigenfunction is globally bounded.
(Kiselev-C and Remling 1997).

• If $V \in L^1 + L^2$ then $\sigma_{\text{ac}} = (0, \infty)$.
(Deift-Killip 1999)

• Spectral properties can be inferred from growth estimates for generalized eigenfunctions.
(see paper of C-Kiselev-Last for recent version)
• Thus we aim to study the operator

$$V \mapsto u(x, \lambda) \ .$$

• We will see that this operator is a (highly) **nonlinear Fourier transform**.

A Sample Result

Theorem. (C-Kiselev) Let

$$V \in L^1 + L^p, \qquad 1 \leq p < 2.$$

Then for <u>almost every</u> energy $\lambda^2 \geq 0$, both eigenfunctions are

globally bounded

and have <u>WKB asymptotics</u>

$$u_\lambda(x) - e^{\imath\lambda x - \imath(2\lambda)^{-1}\int_0^x V(y)\,dy} \to 0$$

as $x \to +\infty$. $\square$

Theorem. (C-Kiselev) If

$$
\begin{aligned}
& V(x) \to 0 \text{ as } |x| \to \infty, \\
\text{and} \quad & V = \sum_j V_j \\
\text{with} \quad & \frac{d^j V_j}{dx^j} \in L^1 + L^p, \quad 1 \leq p < 2
\end{aligned}
$$

then

- The absolutely continuous spectrum is $[0, \infty)$.
- Almost every generalized eigenfunction is bounded.
- Generalized WKB-type asymptotics hold for almost every λ. $\square$

One of our themes is that the map sending potential to generalized eigenfunctions is an analogue of the Fourier transform. For the Fourier transform, dealing with a sum is the same as the same as dealing with the summands. Moreover, differentiation is intertwined with multiplication by ξ, so almost everywhere finiteness of $\widehat{f}$ is equivalent to finiteness of $\widehat{d^j f/dx^j}$.

The theorem of the preceding slide is an analogue of these two facts for our nonlinear transform.

Energy-dependent Potentials

Theorem. (C-Kiselev) Suppose that $1 \le p < 2$ and that $W(x,\lambda)$ is real-valued and satisfies

$$\int_{\mathbb{R}} |\partial^r W(x,\lambda)/\partial\lambda^r|^p \; dx < \infty,$$

for $0 \le r \le 3$, uniformly for all λ in some interval J. Then for almost every $\lambda \in J$, both solutions of

$$u''(x) = W(x,\lambda)u - \lambda^2 u$$

are bounded with WKB asymptotics. $\square$

Variants of this energy-dependent potential result are essential in our analysis of the case where $d^nV/dx^n \in L^p$, $n \ge 2$.

WKB Approximation to Eigenfunctions

Assume $\boxed{u_\lambda(x) = e^{\imath\psi(x)}}$

First approximation:

$$\psi = \lambda x;$$

- $(-\frac{d^2}{dx^2} - \lambda^2 + V)u = \boxed{Vu}$

Second approx (WKB/long range phase shift):

$$\psi = \lambda x - \tfrac{1}{2\lambda}\int_0^x V(y)\,dy$$

- $(-\frac{d^2}{dx^2} - \lambda^2 + V)u = \boxed{(V^2 + V')u}$

Second approx looks worse!

WKB is typically applied in situations where both V^2 and V' decay faster than V itself.

Nonetheless, our analysis proves that the WKB approximation is correct, for $V \in L^p$ and $p < 2$, for almost every λ.

The phase shift $\int_0^x V$ is unbounded but real-valued.

WKB approximations

Form of WKB asymptotics depends on order of differentiability of V. Look for $u(x,\lambda) \sim e^{i\Psi(x,\lambda)}$.

We use

- $\Psi' = \lambda - (2\lambda)^{-1}V$ if V decays

and

- $\Psi' = (\lambda^2 - W)^{1/2} - \frac{V-W}{2\sqrt{\lambda^2-W}}$ if V' decays,

where W is the low-frequency part of V.

General case: Iterate

$$\Psi'_{n+1}(x) = \sqrt{\lambda^2 - W + i\Psi''_n}$$

n times.

For $n \geq 2$, our Ψ is <u>not</u> real-valued; and is constructed as a sum of two terms whose real parts are unbounded; we establish an algebraic identity showing that the sum of these two terms does have bounded real part, although it need not be asymptotically constant. Thus for $n \geq 2$, $|u(x, \lambda)|$ is <u>not</u> in general asymptotic to a constant as $x \to +\infty$.

A Road Not Taken

• Methods based on wave/Schrödinger group are not promising. Spectral measure has dense pure point and probably singular continuous components; we wish to detect its absolutely continuous component amidst this bad stuff.

• It is difficult to detect the absolutely continuous component of a measure from its Fourier transform.

A Well-Trodden Path

• Our goal is almost everywhere convergence for

$$u(x,\lambda)\cdot e^{-\imath\lambda x+\frac{\imath}{2\lambda}\int_0^x V}$$

• Almost everywhere convergence is related to L^p estimate for a maximal operator.

Outline of Method

• Expression of generalized eigenfunctions as sums of infinite series of multilinear operators, applied to potential V.
• A rather general maximal theorem, related to circa 1930 work of Menshov, Paley, Zygmund on convergence of orthogonal series and Fourier integrals.
• General multilinear versions of the maximal theorem, with a very favorable numerical factor.
• Summation of infinite series of multilinear expressions.

Derivation of "Taylor" Expansion

• First-order system: $y = \begin{pmatrix} u_\lambda \\ u'_\lambda \end{pmatrix}$ satisfies

$$y' = \begin{pmatrix} 0 & 1 \\ V(x) - \lambda^2 & 0 \end{pmatrix} y.$$

• Two substitutions exploiting WKB approximation yield

$$z' = \begin{pmatrix} 0 & * \\ \overline{*} & 0 \end{pmatrix} z \text{ where } * = V(x) \cdot e^{-2\imath\lambda x + (\imath/\lambda)\int_0^x V}$$

• Pure Taylor expansion (no WKB Ansatz) does not work; infinitely many terms of the expansion are unbounded.

• Boundedness of $z \Rightarrow$ boundedness of y.

Multilinear operators

• Terms in resulting "Taylor" expansion for generalized eigenfunctions are essentially

multilinear operators of arbitrary degree

applied to $(V, V, \ldots, V)$.

• Basic building blocks (simplified)

$$T_n(f_1, \ldots, f_n)(y, \lambda) = \int \cdots \int_{y \le x_n \le x_{n-1} \le \cdots \le x_1} \prod_{j=1}^{n} f_j(x_j)\, e^{\pm 2i\lambda x_j}\, dx_j.$$

• Actually the exponents $\pm 2i\lambda x_j$ must incorporate the WKB correction.

Features of Problem

• Multilinearity, of arbitrary order.
• Need to sum infinite series.
• Seek supremum bounds.
• These should have an "almost everywhere" character.
• "Superdiagonal" restrictions $x_j \leq x_{j-1}$.
• Symmetrization: operators act on n copies of a single function.

Old Theorem. [Menshov, Paley, Zygmund] Let $1 \leq p < 2$. For any $f \in L^p(\mathbb{R}^1)$,

$$\lim_{y \to \infty} \int_0^y e^{i\lambda x} f(x)\, dx$$

exists for almost every λ. Moreover

$$\sup_y \left| \int_0^y e^{i\lambda x} f(x)\, dx \right| \in L^q(\mathbb{R}), \quad q = \tfrac{p}{p-1}\ .$$

Old Theorem. [Menshov] Let $1 \leq p < 2$. For any orthonormal family $\{\phi_n\}$ of functions in L^2 and any sequence $c_n \in \ell^p$, the series $\sum_n c_n \phi_n(x)$ converges for almost every x.

First theorem for $p = 2$ also true (Carleson plus Plancherel), but much deeper.

General Maximal Theorem

Let $\{Y_n\} \subset Y$ be a sequence of subsets.

Let $T : L^p(Y) \mapsto L^q(X)$.

Associate to T the maximal operator

$$\boxed{T^* f(x) = \sup_n \, |T(f \cdot \chi_{Y_n})(x)|.}$$

Theorem [C-Kiselev] If T is bounded (sublinear) operator and $\{Y_n\}$ is $\boxed{\text{nested}}$ then

T^* is bounded from $L^p(Y)$ to $L^q(X)$,

provided that $\boxed{p < q}$. $\square$

<u>False</u> for $p = q$.

Important Special Case/Corollary

$$Tf(x) = \int_{\mathbb{R}} K(x,y)\, f(y)\, dy$$
$$\tilde{T}f(x) = \int_{y<x} K(x,y)\, f(y)\, dy$$

When $p < q$,

$$\boxed{\text{Boundedness of } T \Rightarrow \text{boundedness of } \tilde{T}.}$$

Proof: Take $Y_n = (-\infty, n]$ for all $n \in \mathbb{R}$ . . .

Miscellaneous application (Tao; Smith-Sogge): The first Strichartz-type estimate always implies the second.

Multilinear Versions

Let $T : L^p(\mathbb{R}, dx) \mapsto L^q(\Lambda, d\mu(\lambda))$ be a bounded linear operator with a locally integrable distribution kernel $K(\lambda, x)$. Define

$$\mathcal{M}_n^*(f_1, f_2, \ldots, f_n)(\lambda) =$$
$$\sup_{y \le y' \in \mathbb{R}} \left| \int \cdots \int_{y \le x_1 \le x_2 \le \cdots \le x_n \le y'} \prod_{i=1}^{n} K(\lambda, x_i) f_i(x_i) dx_i \right| .$$

Theorem. Suppose that $p < q$ and that $2 \le q$. Then for every $n \ge 1$,

$$\mathcal{M}_n^* : \otimes^n L^p(\mathbb{R}) \mapsto L^{q/n}(\Lambda)$$

with

$$\text{operator norm } \le B^n \|T\|_{p,q}^n.$$

$\square$

“Diagonal” Case Improvement

When $f_1 \equiv f_2 \equiv \cdots \equiv f_n$:
Theorem. Suppose that $p < q$ and that $2 \leq q$. Then for every $n \geq 1$

$$\|\mathcal{M}_n^*(f, f, \ldots, f)\|_{L^{q/n}(\Lambda)} \leq \frac{B^n \|T\|_{p,q}^n \|f\|_{L^p}^n}{\sqrt{n!}} \ .$$

$\square$

Gain: numerical factor $1/\sqrt{n!}$.

Idea Underlying Proof for Special Case

$$f \mapsto \int_{y>x} K(x,y) f(y)\,dy$$
$$= \int K(x,y)\,\chi_{\mathbb{R}+}(y-x)\,f(y)\,dy$$

- Main idea: Efficiently expand

$$\chi_{\mathbb{R}+}(y-x) = \sum_\beta \varphi_\beta(x)\psi_\beta(y).$$

- $$\int_{y>x} K(x,y) f(y)\,dy = \sum_\beta \varphi_\beta \cdot T(\psi_\beta f)\ .$$

- Our expansion is not the usual Whitney decomposition of $\{(x,y) : y > x\}$, but instead depends (quite nonlinearly) on f itself.

Some Elements of Proofs

• Given a function $f \in L^p$, construct a martingale structure $\{E_j^m\}$ on $\mathbb{R}$, **adapted to f**; thus $\int_{E_j^m} |f|^p = 2^{-m} \int_{\mathbb{R}} |f|^p$ for all m, j.

• Form the quadratic expression

$$\boxed{G(f,\lambda) = \sum_{m=0}^{\infty} \left(\sum_{1 \le j \le 2^m} |(T f_j^m)(\lambda)|^2 \right)^{1/2}}$$

where $f_j^m = f\Big|_{E_j^m}$.

• G satisfies:

$$\sup_{y \in \mathbb{R}} \left| T\left(f\Big|_{-\infty,y]}\right)(\lambda) \right| \le Gf(\lambda).$$

• If $p < 2 \le q$ and $T : L^p \mapsto L^q$ is bounded, then $G : L^p \mapsto L^q$ is bounded.

Pointwise bound

$$\boxed{\mathcal{M}^*_{\mathbf{n}}(\mathbf{f},\ldots,\mathbf{f})(\lambda) \le \mathbf{B}^{\mathbf{n}}\mathbf{G}(\mathbf{f},\lambda)^{\mathbf{n}}/\sqrt{n!}}$$

Proof based on

- induction on n,
- a <u>recursion</u> relating $\mathcal{M}^*_n$ to $\{\mathcal{M}^*_j : j < n\}$,
- and a numerical inequality: the numbers

$$c_k := A^k k^{-r} k^{-k/2}$$

satisfy for all $x, y \ge 0$:

$$x^n + \sum_{j=2}^{n-2} \left(\frac{c_j c_{n-j}}{c_n}\right) x^{n-j} y^j + y^n \le (x^2 + y^2)^{n/2}$$

for all n (≥ 6), for a certain large constant r.

The pointwise bound in terms of G^n is sharper than $L^{q/n}$ norm bounds.

I've oversimplified throughout the discussion; the Fourier transform should be modified to take the WKB correction into account. Thus the "multilinear" operators are actually fully nonlinear.

Further Remarks

• How to sum the infinite series? Trivial consequence of the pointwise bound $\mathcal{M}_n^*(f,\ldots,f)(\lambda) \le B^n G(f,\lambda)^n/\sqrt{n!}$.

• Proof under hypothesis that $d^nV/dx^n \in L^p$ is similar but uses a different expansion for the eigenfunctions, depending strongly on n.

• Same conclusions hold under locally L^1 / globally ℓ^p hypothesis:

$$\sum_{n=-\infty}^{+\infty} \Big(\int_n^{n+1} |V(y)|\, dy \Big)^p < \infty$$

for some $1 \le p < 2$. (uncertainty principle)

• If $|x|^\gamma V \in L^p$ then WKB asymptotics hold except on a set of energies of Hausdorff dimension $\le 1 - \gamma p'$, where $p' = p/(p-1)$.

Open Problems

• Potentials in L^2?
(= nonlinear Carleson theorem)
• Higher-dimensional analogues?
• Existence of singular continuous spectrum of positive dimension embedded in the a.c. spectrum.

Work in Progress (or near future)

† L^p perturbations of Stark operators ($V_0(x) \equiv x$).

† L^p perturbations of periodic potentials.

† Existence of wave operators.

† Dispersive estimates for time-dependent Schrödinger equations with L^p potentials.

† Nonlinear ODEs.

† Stability for generic orbits of perturbations of integrable dynamical systems.

† Periodic perturbations of a constant electrical field.

References (all joint with Kiselev)

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⊗ (with Y. Last) *Approximate eigenvectors and spectral theory*, in *Differential Equations and Mathematical Physics*, AMS/IP Studies in Advanced Mathematics, edited by R. Weikard and G. Weinstein, Amer. Math. Soc., Providence, RI, June 2000, pp. 85-96.

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