Multilinear Oscillatory Integrals

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Consider multilinear functionals

$$\Lambda_\lambda(f_1, f_2, \cdots, f_n) = \int_{\mathbb{R}^m} \boxed{e^{i\lambda P(x)}} \prod_{j=1}^{n} \boxed{f_j \circ \pi_j(x)}\, \eta(x)\, dx$$

where

- $\lambda \in \mathbb{R}$ is large parameter
- Phase $P : \mathbb{R}^m \to \mathbb{R}$
- Ambient dimension is $m \geq 2$
- η is a smooth cutoff function
- $\pi_j : \mathbb{R}^m \to V_j$ is orthogonal projection to a subspace
- $\kappa \leq m - 1$ is dimension of all V_j
- n is number of functions f_j

Main Question: Does there exist $\varepsilon > 0$ such that

$$|\Lambda_\lambda(f_1, \cdots, f_n)| \leq C|\lambda|^{-\epsilon} \prod_{j=1}^{n} \|f_j\|_{L^\infty}$$

for all $\{f_j\}$?

We then say that P has the $\boxed{\text{power decay property}}$. It is the case where the number n of functions f_j is comparatively large relative to the ambient dimension m $(n\kappa > n)$ which is novel.

All sorts of variant questions can be posed, but this weak version, with strong conditions on the f_j and no attempt to quantify ε, seems to be the most fundamental one. We are also interested in versions which are uniform with respect to various parameters.

The power decay property is fully equivalent to bounds for oscillatory integrals of the so-called first type:

$$\int e^{i\lambda\phi(x)}\eta(x)\,dx = O(\lambda^{-\varepsilon})$$

for all phase functions of the form

$$\phi = P + \sum_j \psi_j \circ \pi_j$$

uniformly for all measurable functions ψ_j.

Obvious **necessary condition**: No such decay is possible if

$$P = \sum_j h_j \circ \pi_j \quad \text{for some functions } h_j$$

(setting $f_j = e^{-i\lambda h_j}$ gives no decay at all).

We are particularly interested in the case where P is a polynomial.

Definition: A polynomial P is $\boxed{\text{degenerate}}$, relative to $\{V_j\}$, if it can be represented as $P = \sum_j q_j \circ \pi_j$ for some *polynomials* q_j.

Conjecture: $\boxed{\text{Nondegeneracy} \Rightarrow \text{power decay.}}$

Let $\{V_j : 1 \le j \le n\}$ be a collection of κ-dimensional subspaces of $\mathbb{R}^m$, lying in general position.

Theorem. *If*

$$n < \frac{2m}{\kappa}$$

then any polynomial P which is nondegenerate relative to $\{V_j\}$ has the power decay property, with appropriate uniformity.

Theorem. *If*

$$n \le \frac{m}{m-\kappa}$$

then any polynomial P which is nondegenerate relative to $\{V_j\}$ has the power decay property, with appropriate uniformity.

In the codimension one case ($\kappa = m - 1$), this holds for all n.

A polynomial P is said to be simply nondegenerate if there exists a differential operator L of the form

$$L = \prod_{j=1}^{r} (w_j \circ \nabla)$$

which annihilates all functions $f_j \circ \pi_j$, but does not annihilate P.

Theorem. *Let m, κ, n be arbitrary. Then any simply nondegenerate polynomial has the power decay property (in every open set).*

Lemma: Nondegeneracy is equivalent to simple nondegeneracy for codimension one ($\kappa = m - 1$), and for higher codimension when $n \leq \frac{m}{m-\kappa}$.

<u>Sublevel Set Bounds</u>:

Consider sublevel sets

$$E_\varepsilon = \left\{ x \in B : |P(x) - \sum_j g_j(\pi_j(x))| < \varepsilon \right\}$$

where $B \subset \mathbb{R}^m$ is bounded.

Theorem. *Let a real-valued polynomial P be nondegenerate with respect to subspaces $\{V_j\}$. Then $\Big|E_\varepsilon\Big| \le C\varepsilon^\delta$ for some $\delta > 0$,*

$$\boxed{\textit{uniformly for all measurable functions } g_j} \; .$$

The power decay property (easily) implies this sublevel set bound. Thus this result provides some evidence supporting the conjecture $\Big[$Nondegeneracy $\Rightarrow$ power decay$\Big]$.

It also rules out the most natural type of counterexample to power decay, making $|P - \sum_j g_j \circ \pi_j|$ constant plus $O(\lambda^{-1})$ on 99% of the support of the integrand.

Elements of proofs of power decay

(1) (for simply nondegenerate case) Iterated TT^*-type arguments, following Carbery, Christ, and Wright (who followed Weyl, among others). These seem to give poor exponents.

(2) (when n is relatively small) Fourier transform/Plancherel plus *uniformity*.

• $f \in L^2(B)$ is λ-uniform if for all polynomials q of degree $\leq d$

$$\int f e^{-iq} = O\Big(|\lambda|^{-\tau} \|f\|_{L^2}\Big)$$

uniformly for all real polynomials of degrees $\leq d$.

This is a version of a notion used by Gowers in his work on Szemerédi's theorem.

Three algebraic aspects of nondegeneracy

• A polynomial P is degenerate with respect to $\{V_j\}$ if and only if there exist distributions h_j such that $P = \sum_j h_j \circ \pi_j$ in some open set.

• A homogeneous polynomial P of degree d is nondegenerate relative to $\{V_j\}$ iff there exists a homogeneous constant-coefficient partial differential operator $\mathcal{L}$ such that

$$\begin{cases} \mathcal{L}(P) \neq 0 & \\ \mathcal{L}(p_j \circ \pi_j) \equiv 0 & \forall \text{ polys } p_j \text{ of degree } d. \end{cases}$$

Difference operators variant

Let e_j denote the j-th coordinate vector in $\mathbb{R}^m$,
$\delta_j f(x) = f(x+e_j) - f(x)$,
$\Delta_\alpha = \delta_1^{\alpha_1} \circ \cdots \circ \delta_m^{\alpha_m}$

• Let $P : \mathbb{R}^m \to \mathbb{R}$ be a nondegenerate homogeneous polynomial of degree D. Then there exist a finite subset $\{y_\beta\}$ of $\mathbb{R}^m$, and coefficients $c_\beta \in \mathbb{R}$, such that for any $1 \le j \le n$ and any continuous function f_j,

$$\begin{cases} \displaystyle\sum_\beta c_\beta P(x + r y_\beta) \equiv r^D \\ \displaystyle\sum_\beta c_\beta f_j \circ \pi_j (x + r y_\beta) = 0 \qquad \forall x, r \end{cases}$$

More robust versions, applying to inhomogeneous polynomials and allowing $x + \sum_\beta r_\beta y_\beta$ with <u>some</u> of the r_β arbitrary, also hold.

The sublevel set result is proved via combinatorics. The difference operator characterizations of nondegneracy is the key to reducing matters to a combinatorial problem which has some (only some) of the flavor of Szemerédi's theorem.

Let $f = P - \sum_j h_j \circ \pi_j$ where the h_j are arbitrary. Let E_ε be a sublevel set for f.

The difference operators criterion says that for certain finite configurations of points y_β, certain linear combinations of values of f at $x+ry_\beta$ are not small (unless r is small) — hence at least one point $x + ry_\beta$ can't belong to E_ε.

The rest of the proof is to show that if a set E contains no such point configurations, then E must have small measure.

The case where the y_β are the coordinate vectors was handled by Carbery, Christ, and Wright and is a continuum variant of a well-known discrete result. The general case easily follows from that one by a lifting argument.

Review of Szemerédi's theorem on arithmetic progressions, in continuum version:

Let $E \subset [0,1]$ be measurable. Suppose that whenever $x, x+r, x+2r, \cdots x+kr$ all belong to E, necessarily $r < \delta$.
Then $|E| < \varepsilon(\delta)$, where $\varepsilon \to 0$ as $\delta \to 0$.
(But ε may tend to zero incredibly slowly.)

Review of a much shallower result: Let $E \subset [0,1]^2$. Consider the vertices of any rectangle with sides parallel to coordinate axes. Suppose that whenever all four vertices lie in E, max sidelength $< \delta$. Then $|E| \le C\delta^\tau$.

Our result is more closely akin to the latter, because we have difference operator identities with sufficiently many multi-parameters like (r_1, r_2), hence $\boxed{\text{many point configurations are excluded}}$ from sublevel sets.