ONE-DIMENSIONAL SCHRÖDINGER OPERATORS AND NONLINEAR FOURIER ANALYSIS

MICHAEL CHRIST AND ALEXANDER KISELEV

1. Introduction

Consider a one-dimensional Schrödinger operator $H = H_V$ defined on the space $L^2(\mathbb{R}, dx)$ of square integrable functions on $\mathbb{R}$

$$H_V = -\frac{d^2}{dx^2} + V(x). \tag{1.1}$$

The potential V is a real valued, measurable, locally integrable function. H is then a self-adjoint operator whose domain is an appropriate Sobolev space. Considered as an operator on $L^2([0,\infty))$, H is likewise self-adjoint if an appropriate boundary condition at zero, e.g. Dirichlet or Neumann, is specified. The quantum mechanical interpretation is that $H_0 = -\frac{d^2}{dx^2}$ governs the behavior of a free electron, while H_V describes one electron interacting with an external electrical field. The evolution of a particle with initial state $\psi_0(x) \in L^2$ is described by the time-dependent Schrödinger equation $i\psi_t(x,t) = H\psi(x,t)$, with initial condition $\psi(x,0) = \psi_0(x)$, whose solution is denoted by $e^{-iHt}\psi_0$.

(1.1) is perhaps the simplest quantum mechanical model describing unbounded motion — though it is by no means simple. The free case $V = 0$ can be analyzed directly using the Fourier transform

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixk} f(x)\, dx, \tag{1.2}$$

which conjugates H_0 to multiplication by k^2 on $L^2([0,\infty), dk)$. The unitary evolution group is

$$\psi(x,t) = e^{-iH_0 t}\psi_0(x) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx - ik^2 t}\widehat{\psi_0}(k)\, dk.$$

If $\widehat{\psi_0}(k)$ is sufficiently smooth, the stationary phase method can be used to study the dynamics for large times. The probability density to find a particle at x is asymptotically

$$|\psi(x,t)|^2 = \frac{1}{2t} \left|\widehat{\psi_0}(\frac{x}{2t})\right|^2 + O(t^{-3/2}); \tag{1.3}$$

thus the probability of finding it in any particular compact set tends to zero as $t \to \infty$. There are no bound states; mathematically this means that H_0 has no eigenvalues.

How are the properties of H_0 affected by an electrical potential? If $V(x)$ is in some sense small, and in particular decays at infinity, then H should be close to H_0. The most basic result of this type says that the essential spectrum (all points in the spectrum except isolated eigenvalues) of H coincides with $\mathbb{R}^+$ if $|V(x)| \to 0$ as $x \to \infty$. Questions regarding the dynamical behavior, and finer properties of the spectrum, are more subtle.

Date: Printed January 9, 2003.

The first author was supported by NSF grant DMS-9970660.

The second author was supported in part by NSF grant DMS-0129470 and an Alfred P. Sloan Research Fellowship.

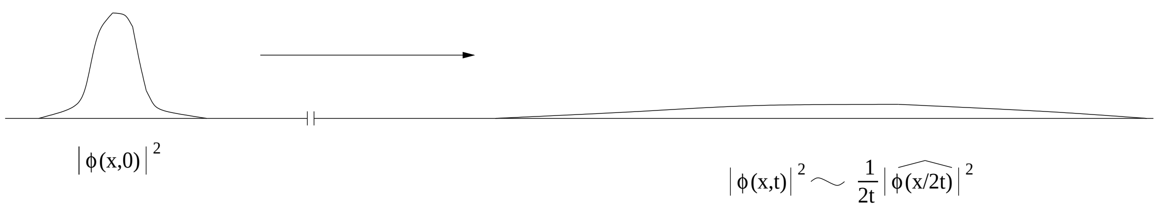

FIGURE 1. Free evolution

This article describes some recent progress in the analysis of decaying long-range perturbations of H_0. We will not discuss the extensive literature concerning larger perturbations, such as quasiperiodic potentials. Traditionally, techniques from functional analysis and ODE have been dominant tools in the area; the results to be described here rely more heavily on Fourier analysis. In particular, we will outline a connection between scattering for long range potentials, almost everywhere convergence of Fourier integrals, inequalities for multilinear operators, and a conjectural nonlinear variant of the celebrated theorem of Carleson on almost everywhere convergence of Fourier series.

2. Spectral measure and generalized eigenfunctions

To set the stage, we introduce certain basic objects of spectral analysis. For simplicity of exposition, we focus mainly on the case of Schrödinger operators on the half axis $\mathbb{R}^+$ with Dirichlet boundary condition $u(0) = 0$, although all results discussed extend readily to other boundary conditions and to the whole axis case. There are also variants for perturbations of a constant electrical field, for Dirac-type operators, and for decaying perturbations of periodic potentials.

Weyl-Titchmarsch theory [23] guarantees that for all potentials discussed in this article, for any $z \in \mathbb{C} \setminus \mathbb{R}$ there exists a unique solution $u_+(x,z)$ of the equation $-u'' + Vu = zu$ which belongs to $L^2(\mathbb{R}^+, dx)$ and satisfies $u_+(0,z) = 1$. The Weyl-Titchmarsch m-function is defined by $m(z) = u_+'(0,z)$. This function is analytic, and has positive imaginary part in the upper half-plane $\mathbb{C}^+$; m is related to the resolvent operator $(H-z)^{-1}$ by $m(z) = \partial^2_{x,y} G(x,y,z)$ evaluated at $(0,0,z)$, where $G(x,y,z)$ is Green's function, the kernel associated to the resolvent. The weak limit

$$\pi^{-1}\,\mathrm{Im}\,(m(E+i\varepsilon))\,dE \to d\mu(E) \quad \text{as } \varepsilon \to 0^+ \tag{2.1}$$

exists and defines a canonical spectral measure μ. As an operator on $L^2(\mathbb{R}^+)$, H is cyclic, that is, unitarily equivalent to multiplication by E on $L^2(\mathbb{R}, d\mu(E))$. The unitary operator $U : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}, d\mu)$ implementing the equivalence is defined by

$$(Uf)(E) = \lim_{N\to\infty} \int_0^N u_1(x,E) f(x)\,dx. \tag{2.2}$$

Here the limit is taken in L^2 norm, and $u_1(x,E)$ satisfies $-u'' + Vu = Eu$ with boundary conditions $u_1(0,E) = 0$ and $u_1'(0,E) = 1$. Such solutions $u_1(x,E)$, for energies E in the support of the spectral measure, are called *generalized* eigenfunctions.

Any locally finite measure μ decomposes as $\mu_{\text{pp}} + \mu_{\text{sc}} + \mu_{\text{ac}}$ where μ_{ac} is absolutely continuous with respect to Lebesgue measure, μ_{pp} is a countable linear combination of Dirac masses, and μ_{sc} is purely singular but nonatomic. H is said to have purely absolutely continuous spectrum if $\mu = \mu_{\text{ac}}$. Similarly one speaks of pure point (pp) and purely sc spectrum. $\mathcal{H}_{\text{ac}}$ denotes the maximal closed subspace of $\mathcal{H}$ on which H has purely ac spectrum.

The global behavior of solutions $u_1(x,E)$ is closely tied to local properties of the spectral measure near the energy E. Two instances of this connection are that (i) point spectrum is dictated by true eigenfunctions, that is, solutions $u_1(x,E)$ which belong to $L^2(\mathbb{R}^+)$ and (ii) global boundedness of $u_1(x,E)$ implies that the Lebesgue derivative $d\mu(E)/dE$ is strictly positive (in an almost everywhere dE sense).

To illustrate, for $V = 0$, $u_+^{(0)}(x,z) \equiv e^{i\sqrt{z}x}$, where the square root branch has positive imaginary part for $z \in \mathbb{C}^+$. The m-function is $\sqrt{z}$, and the spectral measure is $d\mu^{(0)}(E) = \frac{1}{\pi}\sqrt{E}\chi_{[0,\infty)}dE$, which is purely ac and supported on all of $\mathbb{R}^+$. The generalized eigenfunctions are $u_1^{(0)}(x,E) = E^{-1/2}\sin(\sqrt{E}x)$.

In contrast, if V is periodic and locally square integrable then the spectrum of H_V is purely ac, but has *band structure*; it typically consists of an infinite sequence of closed intervals, separated by gaps.

A bounded Borel function F of H, such as e^{-iHt}, can be computed via

$$F(H)\psi(x) = \lim_{N\to\infty}\int_{-N}^{N} u_1(x,E)F(E)U\psi(E)d\mu(E), \tag{2.3}$$

the limit taken in L^2 norm. Thus, knowledge of μ and U gives in principle control of functions of H; even partial information about μ has important consequences for the properties of the quantum system.

3. Asymptotics

3.1. Temporal infinity: Wave operators.

Wave operators, when they exist, provide a convenient description of the long time dynamics for a perturbed Schrödinger operator.

Definition. The *wave operators* $\Omega_\pm$ associated to a perturbed Hamiltonian H_V are

$$\Omega_\pm f = \lim_{t\to\mp\infty} e^{itH}e^{-itH_0}f, \tag{3.1}$$

where the limit is taken in the strong operator topology, *provided* it exists.

The wave operators map $\mathcal{H}$ isometrically to certain closed subspaces $\operatorname{Range}\Omega_\pm$. Assume that $g = \Omega_- f \in \operatorname{Range}\Omega_-$. Then, as $t\to\infty$, $\|e^{-iHt}g - e^{-iH_0t}f\|_{L^2(\mathbb{R}^+)} \to 0$. Thus the perturbed dynamics for vectors in $\operatorname{Range}\Omega_-$ looks asymptotically like free dynamics, but with a different initial datum. This corresponds to a scattering picture: the state of a system is influenced by a potential, but a particle looks increasingly like a free one as time elapses and it escapes to spatial infinity.

Existence of wave operators forces H restricted to $\operatorname{Range}\Omega_\pm$ to be unitarily equivalent to H_0, and in particular implies that the ac spectrum of H has essential support equal to $\mathbb{R}^+$, in the sense that $\mu_{\rm ac}(S) > 0$ if and only if $S\cap\mathbb{R}^+$ has positive Lebesgue measure. Their existence does not exclude singular or pure point spectrum, associated to the orthogonal complement of $\operatorname{Range}\Omega_\pm$.

The wave operators are sometimes called *asymptotically complete* [19] if $\operatorname{Range}\Omega_+ = \operatorname{Range}\Omega_- = \mathcal{H}_{\rm ac}$ and $\mathcal{H}_{\rm ac}\bigoplus\mathcal{H}_{\rm pp} = \mathcal{H}$, where $\mathcal{H}_{\rm pp}$ is the subspace corresponding to pure point spectrum, spanned by L^2 eigenfunctions of H. Physically, asymptotic completeness corresponds to the situation where every particle free at $t\to-\infty$ also becomes free as $t\to+\infty$, and where the only states which do not approach free dynamics at large times are linear combinations of bound states. In particular, the singular continuous spectrum, which is often difficult to interpret and analyze, is empty.

3.2. **Spatial infinity: Scattering coefficients.** Suppose temporarily that V has compact support, and consider the problem on the whole real line. Then for each $k \in \mathbb{R}$ there exists a unique solution $w_+(x,k)$ of $-w'' + Vw = k^2 w$ of the form

$$w_+(x,k) = \begin{cases} t(k)e^{ikx} & \text{for } x \text{ near } +\infty, \\ e^{ikx} + r(k)e^{-ikx} & \text{for } x \text{ near } -\infty. \end{cases} \tag{3.2}$$

$w_+(x,k)$ differs from the solution $u_+(x,k^2)$ introduced above by a factor $t(k)$. The coefficients t and r are called respectively the *transmission* and *reflection coefficients*, and satisfy the conservation law $|t(k)|^2 + |r(k)|^2 = 1$ for all k.

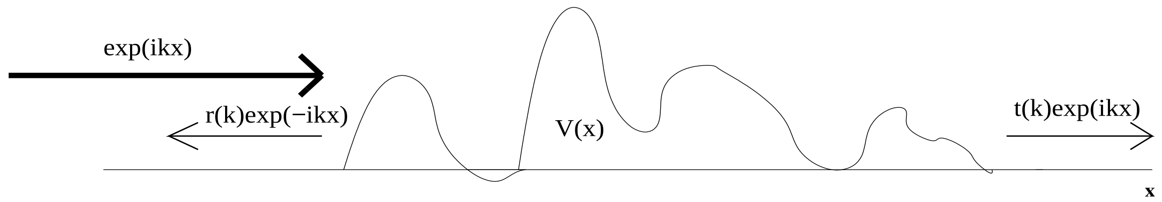

FIGURE 2. Transmission and reflection coefficients $t(k), r(k)$

Their interpretation is that an incoming wave $e^{i(kx-k^2t)}$ from $-\infty$ interacts with the potential and splits into a reflected wave $r(k)e^{i(-kx-k^2t)}$ plus a transmitted wave $t(k)e^{i(kx-k^2t)}$. The scattering coefficients are related to the Weyl m-function of the half-axis operator by

$$m(k^2) = ik\frac{1-r(k)}{1+r(k)}. \tag{3.3}$$

It follows that $\mathrm{Im}\,(m(k^2)) \geq k|t(k)|^2/4$, so lower bounds for t lead to lower bounds for $\mathrm{Im}\,(m)$. Since $\mathrm{Im}\,(m)$ is a Poisson integral of the spectral measure according to (2.1), having such bounds on a set of values of k with positive Lebesgue measure would imply existence of ac spectrum.

If $V \in L^1$ then (3.2) remains valid in modified form: $w_+(x,k) - t(k)e^{ikx} \to 0$ as $x \to +\infty$, and $w_+ - (e^{ikx} + r(k)e^{-ikx}) \to 0$ as $x \to -\infty$; the scattering coefficients remain well-defined. For general potentials, scattering coefficients $t(x,x',k), r(x,x',k)$ may be defined for any $x \leq x' \in \mathbb{R}$ by truncating V to vanish identically outside the interval $[x,x']$.

The dependence of scattering coefficients upon potential has a certain multiplicative structure, illustrated schematically in Figure 3.2.

4. SHORT AND LONG RANGE POTENTIALS

It is well-known that in the *short-range* case $V \in L^1(\mathbb{R}^1)$, $\Omega_\pm$ exist and are asymptotically complete; the singular spectrum consists only of negative eigenvalues, possibly accumulating at zero; the spectrum on $\mathbb{R}^+$ is purely ac, and for every $k^2 \equiv E \in \mathbb{R}^+$, there exists a solution[1] $\mathfrak{u}_+(x,k)$ of the generalized eigenfunction equation

$$-u'' + Vu = k^2 u \tag{4.1}$$

satisfying

$$\mathfrak{u}_+(x,k) = e^{ikx}(1+o(1)) \quad \text{as } x \to \infty. \tag{4.2}$$

[1]For $k > 0$, $\mathfrak{u}_+(x,k) \equiv u_+(x,k^2) = \lim_{z\to k^2} u_+(x,z)$, where $\mathrm{Im}\,(z) > 0$.

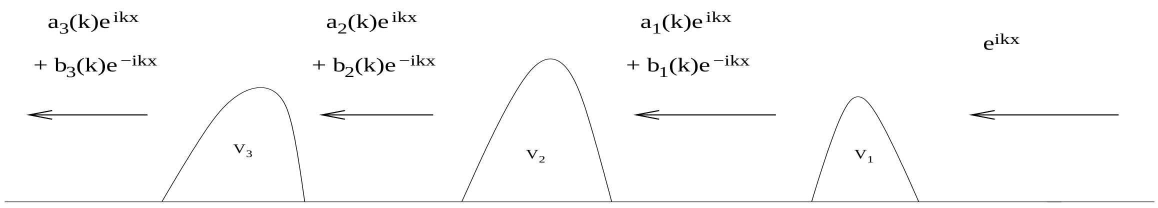

FIGURE 3. Transfer matrices and multiplicative structure. The potential is a sum of three bumps. A solution equal to $ae^{ikx} + be^{-ikx}$ immediately to the right of the support of a bump equals $\tilde{a}e^{ikx} + \tilde{b}e^{-ikx}$ immediately to the left; the linear mapping $(a, b) \mapsto (\tilde{a}, \tilde{b})$ is implemented by a 2×2 *transfer matrix* belonging to $\mathrm{SU}(1, 1)$. The coefficients associated to the whole potential are obtained by multiplying these matrices in sequence.

On the other hand, Wigner and von Neumann constructed a potential of the form $V(x) = -8\sin(2x)/x + o(1/x)$ as $x \to \infty$, with arbitrarily small supremum norm, for which H_V has a positive eigenvalue $k^2 = 1$. This is a quantum effect: in classical mechanics, a potential with supremum norm < 1 cannot bound a particle with total energy 1. Since the set of all solutions of (4.1) is two-dimensional, this excludes the asymptotic behavior (4.2) for $k^2 = 1$. Naboko and Simon later constructed potentials bounded by $(1 + |x|)^{-r}$ for any $r < 1$ for which the eigenvalues of H_V are *dense* in $[0, \infty)$.

Insight into the creation of positive eigenvalues may be gleaned from the *modified Prüfer transform.* To a solution u of (4.1) associate Prüfer amplitude $R(x, k)^2 = u'(x, k)^2 + k^2u(x, k)^2$ and angle $\theta(x, k) = \arctan(ku/u')$. These satisfy

$$(\log R(x, k))' = \frac{1}{2k}V(x)\sin 2\theta(x, k); \quad \theta(x, k)' = k - \frac{1}{k}V(x)(\sin\theta)^2. \tag{4.3}$$

Fix k, and substitute $V(x) = -\frac{C}{1+x}\sin 2\theta(x, k)$ into the second equation. The resulting nonlinear equation for θ has a global solution; define $V(x)$ as above with this θ. The first equation then forces R to decay rapidly to zero, if C is large.

On the other hand, it is evident from the first equation (4.3) that when $V \in L^1$, all solutions are bounded. L^1 perturbations are also a natural class for wave operators: Dollard [11] showed that even for the Coulomb potential $V(x) = 1/x$, wave operators fail to exist. The Coulomb case may be rectified through the introduction of *modified wave operators*; see (6.4).

Pearson [18] established a natural threshold for the existence of ac spectrum: there exist $V \in \cap_{p>2}L^p(\mathbb{R}^+)$ such that H_V has purely singular spectrum. Pearson's *sparse* potentials take the form $\sum_{n=1}^{\infty} a_nW(x - x_n)$, where W is any compactly supported function not $\equiv 0$, and $x_n \to \infty$ rapidly. The spectrum on $\mathbb{R}^+$ is purely ac if $\sum a_n^2 < \infty$, and purely singular continuous if $\sum a_n^2 = \infty$.

Fundamental insight [16, 9] may also be gleaned from families of *random* potentials $V_\omega(x) = \sum_{n=1}^{\infty} r_n(\omega)a_nW(x-x_n)$, where $\{r_n\}$ are independent random variables with expectation zero, taking for instance values ± 1 each with probability 1/2. Then with probability one, the spectrum is purely ac if $a \in l^2$, but (for almost every boundary condition at zero) consists entirely of eigenvalues (dense in $\mathbb{R}^+$) if $|a_n| \sim |n|^{-r}$ for some $r < 1/2$. This parallels well-known results for random Fourier and Rademacher series. Indeed, $\sum r_n(\omega)a_ne^{inx}$ converges almost surely for any x if $\sum |a_n|^2 < \infty$, and otherwise almost surely diverges everywhere.

5. Some maximal inequalities and a.e. limits of Fourier integrals

The integral (1.2) defining the Fourier transform of an L^1 function f converges absolutely for all k. For $f \in L^1 \cap L^2$ one has Plancherel's identity $\|\widehat{f}\|_{L^2}^2 = \|f\|_{L^2}^2$, which permits an extension of the definition to arbitrary $f \in L^2$, via the L^2 norm limit of a sequence of approximations. The existence of the integral for $f \in L^2$, in this indirect sense, is due to subtle cancellation, and for the extension beyond L^1 one pays a price: the transform is defined only modulo redefinition on sets of Lebesgue measure zero.

Perhaps less well known is a more direct alternative definition as

$$\widehat{f}(\xi) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{|x|<R} f(x) e^{-ix\xi}\, dx. \tag{5.1}$$

Existence of this limit for almost every ξ whenever $f \in L^p$ for some $1 \le p < 2$ is a consequence of theorems of Menshov, Paley, and Zygmund, dating roughly to the 1930s. In 1966 Carleson [5] proved that for any $g \in L^2$, $\lim_{R\to\infty} \int_{|\eta|<R} \widehat{g}(\eta) e^{iy\eta}\, d\eta \to 2\pi g(y)$ for a.e. y; prior to this it was not even known whether the Fourier series of a continuous function need converge anywhere. Substituting $(\eta, y) = (x, -\xi)$ and $\widehat{g} = f$ reveals that the limit (5.1) continues to exist, almost everywhere, for all $f \in L^2$.

$p = 2$ is a genuine threshold for these results. The Fourier transform of a general function $f \in L^p$, $p > 2$, is a distribution rather than a function, and the limit (5.1) need not exist almost everywhere.

It is a general principle that the existence of an almost-everywhere limit, for all functions in some Banach space, is due to a maximal inequality; rigorous results justifying this principle under various conditions are due to Calderón, Stein, Sawyer, Nikišin and others. The maximal inequality associated with (5.1) asserts that the maximal operator $\widehat{f}^*(\xi) = \sup_{R>0} \left| \int_{|x|<R} f(x) e^{-ix\xi}\, dx \right|$ maps $L^p(\mathbb{R})$ boundedly to $L^{p'}$, where $p' = p/(p-1)$ is the exponent conjugate to p. The weaker Hausdorff-Young inequality asserts that $f \mapsto \widehat{f}$ maps L^p to $L^{p'}$ boundedly.

A variant due to Menshov holds that for any orthonormal family $\{\phi_n\}$ in L^2 of an arbitrary measure space and any sequence of coefficients $a_n \in l^p$ for some $1 \le p < 2$, the series $\sum a_n \varphi_n(x)$ converges for almost every x. On the other hand, a.e. convergence fails in this generality for l^2 coefficients, for which Menshov established a weaker bound $\sum_{k=1}^N a_k \varphi_k(x) = O(\log N)$ for almost every x.

6. Recent progress

6.1. **Absolutely continuous spectrum and WKB asymptotics.** Recent research has in several respects closed the gap between L^1 and L^2 potentials.

Theorem 6.1. [6] *Assume that $V \in L^p + L^1(\mathbb{R}^+)$ for some $p < 2$. Then for almost every $k \in \mathbb{R}$, there exists a solution $\mathfrak{u}_+(x,k)$ of equation (4.1) such that*

$$\mathfrak{u}_+(x,k) = e^{ikx - \frac{i}{2k}\int_0^x V(s)\,ds}(1 + o(1)) \quad \text{as } x \to \infty. \tag{6.1}$$

The ac spectrum of H has essential support $\mathbb{R}^+$.

Therefore, solutions of (4.1) are bounded for a.e. k^2 if $V \in L^p$, and have asymptotics similar to the free solutions with a correction given by the integral of potential. In particular, the generalized eigenfunction $u_1(x,k^2)$ for a.e. k is a linear combination of $\mathfrak{u}_+(x,k)$ and $\overline{\mathfrak{u}_+}(x,k)$ satisfying the boundary condition, and is bounded. The "almost every" condition is essential: the result can fail for a dense set of positive energies, e.g. all positive eigenvalues.

As will be made clear in (7.2),(7.3), this should be viewed as a nonlinear analogue of the Fourier analysis result of Menshov, Paley, and Zygmund. The formal analogy with the Fourier transform is enhanced by the following two results.

Theorem 6.2. [20],[7] *Suppose that $1 \le p \le 2$, $\gamma > 0$, and $\gamma p' \le 1$. Assume that $(1+|x|)^\gamma V \in L^p$. Then the Hausdorff dimension of the set of energies $k^2 \in \mathbb{R}^+$ for which the asymptotic behavior* (6.1) *may fail is $\le 1 - \gamma p'$.*

The well-known corresponding result on the dimension of a possible divergence set for Fourier integrals takes the same form. Kriecherbauer and Remling have shown that the bound $1-\gamma p'$ is optimal.

Theorem 6.3. [7] *Let $p \in [1, 2)$. Suppose that V can be decomposed as a finite sum $\sum_j V_j$ where each $d^j V_j/dx^j \in L^1 + L^p$, and that $V_j(x) \to 0$ in an appropriate sense as $x \to \infty$. Then for a.e. $k \in \mathbb{R}^+$, all solutions of* (4.1) *are bounded. The ac spectrum of H_V has essential support $\mathbb{R}^+$.*

The Fourier transform analogue is a direct consequence of the case $n = 0$ by the relation $(d^nV/dx^n)\widehat{\ }(k) = i^n k^n \widehat{V}(k)$. For the nonlinear analogue, we know of no such simple way to deduce the case $n > 0$ from $n = 0$, nor to conclude that $\sum V_j$ can be handled if V_j can be individually treated.

The asymptotic behavior (6.1) is called WKB, after 1920's works of Wentzel, Kramers and Brilloin. Suppose temporarily that V satisfies symbol-type hypotheses:

$$|\partial_x^m V(x)| \le C_m |x|^{-\delta - m\rho} \quad \text{for some } \delta, \rho > 0, \text{ for all } m, \text{ for large } x. \tag{6.2}$$

Seeking merely a formal asymptotic approximation to a solution $u(x,k)$ of $H_V u = 0$ as $x \to +\infty$, set $u(x,k) = e^{i\phi(x,k)}$ where ϕ is real-valued, expand $\phi \sim \sum_{n=0}^\infty \phi_n$, and set $\phi_0 = kx$. We seek an expansion in which each ϕ_{n+1} decays more rapidly than ϕ_n. The equation for ϕ is $(\phi')^2 - i\phi'' = k^2 - V$, so $(k + \phi_1')^2 - i\phi_1'' \approx k^2 - V$. Dropping the terms ϕ_1'' and $(\phi_1')^2$ which ought to decay more rapidly than ϕ_1' itself, we arrive at $\phi_1' = -(2k)^{-1}V$, whence the asymptotic behavior (6.1). A novel assertion of Theorem 6.1 is that without differentiability hypotheses, the approximation (6.1) nonetheless remains accurate for Lebesgue-almost every parameter k. In Theorem 6.3, asymptotic behavior involving more complicated WKB-type corrections likewise occurs.

6.2. **Modified wave operators.** The existence of modified wave operators and the form of the required correction are related to the WKB asymptotics. Let

$$W(k,t) = -(2k)^{-1} \int_0^{2kt} V(s)\, ds. \tag{6.3}$$

Define $e^{-iH_0 t \pm iW(H_0^{1/2}, \mp t)}$ to be the Fourier multiplier operator

$$\int_0^\infty F(k) \sin(kx)\, dk \mapsto \int_0^\infty e^{-ik^2 t \pm iW(k,\mp t)} F(k) \sin(kx)\, dk,$$

on $L^2(\mathbb{R}^+)$. Define the modified wave operators

$$\Omega_\pm^m f = \lim_{t \to \mp\infty} e^{itH_V} e^{-itH_0 \pm iW(H_0^{1/2}, \mp t)} f \tag{6.4}$$

for all $f \in L^2(\mathbb{R}^+)$ for which the limit exists in L^2 norm.

Theorem 6.4. [8] *Let V be a potential in the class $L^1 + L^p(\mathbb{R}^+)$ for some $1 < p < 2$. Then for every $f \in L^2(\mathbb{R}^+)$, the limits in* (6.4) *exist as $t \to \mp\infty$. The modified wave operators*

$\Omega_\pm^m$ *thus defined are both unitary bijections from* $\mathcal{H} = L^2(\mathbb{R}^+)$ *to* $\mathcal{H}_{ac}(H_V)$*. Moreover, if* $\lim_{N\to\infty} \int_0^N V(s)\,ds$ *exists, then the usual wave operators* $\Omega_\pm$ *given by* (3.1) *exist.*

Previously, existence of modified wave operators (6.4) and more general ones incorporating higher order WKB corrections has been known for slowly decaying classes of potentials satisfying additional symbol-like hypotheses on derivatives [1, 4, 13]. There exist [15] potentials belonging to L^p for all $p > 1$ having nonempty singular continuous spectrum; hence the associated modified wave operators exist but fail to be asymptotically complete.

6.3. **The L^2 endpoint case.** The analogy with Fourier analysis makes the urge to venture a conjecture irresistible.

Conjecture 6.5. *The conclusions of Theorems* 6.1,6.4 *hold for all* $V \in L^2$.

As will be explained in (7.2),(7.3) below, this would be a nonlinear variant of Carleson's almost everywhere convergence theorem for Fourier series and integrals. On the other hand, Menshov's counterexample for a.e. convergence of general orthogonal series indicates that if true, the conjecture must rely on specific features of Schrödinger operators rather than the more general considerations underlying the above theorems.

One fundamental result, roughly parallel to Plancherel's theorem and L^2 *norm* convergence for Fourier integrals, has been obtained by Deift and Killip.

Theorem 6.6. [10] *If* $V \in L^2(\mathbb{R}^+)$ *then the ac spectrum of* H_V *has essential support* $\mathbb{R}^+$.

The hypothesis is sharp, as Pearson's examples show. The proof relies not on asymptotics of solutions, but rather on *trace identities*; see (8.1).

Analogous questions remain unanswered for higher dimensions. In 1975 Agmon proved existence and asymptotic completeness of wave operators (and hence stability of ac spectrum) under the hypothesis $|V(x)| \le C(1+|x|)^{-1-\varepsilon}$, but the natural power decay threshold may be $O(|x|^{-1/2-\varepsilon})$, in all dimensions. Barry Simon conjectured that $\int_{\mathbb{R}^d} |V(x)|^2 x^{-d+1}\,dx < \infty$ is a sharp condition for the preservation of absolutely continuous spectrum in any dimension, an extension of Theorem 6.6. Bourgain [2] and Rodnianski-Schlag have recently made progress concerning random potentials. Bourgain considered (for the discrete analogue in dimension two) potentials $V(n) = a_n(\omega)(|n|+1)^{-\alpha}$, where a_n are certain independent random variables with mean zero, and showed that for $\alpha > 1/2$, wave operators exist with probability one. This threshold is sharp in the deterministic case, but is believed nonoptimal in the random setting.

7. Some Analysis

Although the generalized eigenfunction equation (4.1) is linear, the transformation mapping potential V to generalized eigenfunctions $u(x,k)$ is highly nonlinear. The proofs of the first four theorems discussed above rest on a modified Maclaurin series expansion for this transformation, a general maximal estimate for linear operators, and inequalities for a family of multilinear integral operators.

7.1. **Series expansion.** Introduce multilinear operators

$$T_n(f_1,\dots,f_n)(x,k) = (i/2k)^n \int_{x\le t_1\le\cdots\le t_n} \prod_{j=1}^n e^{(-1)^{n-j}2i\phi(t_j,k)} f_j(t_j)\,dt_j \tag{7.1}$$

where $\phi(x,k) = xk - (2k)^{-1}\int_0^x V$. By rewriting the equation $-u'' + Vu = k^2 u$ as a first-order system, manipulating it to incorporate the WKB correction, converting to an integral equation, and feeding the resulting equation into itself infinitely many times, one obtains a formal expansion for $\mathfrak{u}_+$:

$$\mathfrak{u}_+(x,k) = a(x,k)e^{ikx} + b(x,k)e^{-ikx}$$

$$a(x,k) = e^{-\frac{i}{2k}\int_0^x V(t)\,dt} + e^{-\frac{i}{2k}\int_0^x V(t)\,dt}\sum_{n=1}^{\infty}(-1)^n T_{2n}(V,\dots,V)(x,k) \tag{7.2}$$

$$b(x,k) = e^{\frac{i}{2k}\int_0^x V(t)\,dt}\sum_{n=1}^{\infty}(-1)^n T_{2n-1}(V,\dots,V)(x,k).$$

If V has compact support then according to (3.2), for all x to the left of the support of V, $a(x,k) = 1/t(k)$ and $b(x,k) = r(k)/t(k)$. In particular,

$$\big|t(k)\big|^{-1} = \big|1 + \sum_{n=1}^{\infty}(-1)^n T_{2n}(V,\dots,V)(x,k)\big|.$$

The simple fact $|t(k)| \le 1$ is rendered completely opaque by this expansion.

Here (7.2) is not exactly a Maclaurin series, and the operators are not quite multilinear, since V still appears in the exponents. Proceeding similarly without incorporating the WKB correction leads to a similar expansion which includes infinitely many terms which are actually *unbounded* for generic V and k unless $V \in L^1$.

The leading term for $\mathfrak{u}_+$ is the modified exponential $e^{i\phi(x,k)}$. The next term is a unimodular exponential times

$$\int_x^{\infty} e^{2i\phi(y,k)}V(y)\,dy. \tag{7.3}$$

If the phase $\phi(y,k)$ is simplified to its main term yk, then boundedness of (7.3) as a function of $x \in \mathbb{R}$, for a.e. k and arbitrary $V \in L^2$, is equivalent to Carleson's theorem.

7.2. **A general maximal estimate for linear operators.** Consider any integral operator $Tf(x) = \int_{\mathbb{R}^1} K(x,y)f(y)\,dy$, and to it associate the maximal operator

$$T^*f(x) = \sup_{x'}\big|\int_{y<x'} K(x,y)f(y)\,dy\big|. \tag{7.4}$$

Theorem 7.1. *Suppose that T is bounded from L^p to $L^q(\mathbb{R})$ for some exponents $p,q \in [1,\infty]$, satisfying the strict inequality $p < q$. Then T^* is likewise bounded; there exists $C < \infty$ such that $\|T^*f\|_q \le C\|f\|_p$ for all f.*

Such a statement is trivial if K is nonnegative, but otherwise the truncation could conceivably destroy essential cancellation. Applying the theorem to $K(x,y) = e^{-ixy}$ yields the boundedness of $f \mapsto \widehat{f}^*$ from L^p to $L^{p'}$, so long as p is strictly < 2. The conclusion fails for $p = q$, as is shown by the example of the Hilbert transform, $K = \pi^{-1}(x-y)^{-1}$.

To understand the simple proof of Theorem 7.1, it is helpful to review Menshov's proof of the $O(\log N)$ bound for maximal partial sums of general orthonormal series with l^2 coefficients. Suppose that $N = 2^M$ and for each $0 \le m \le M$ decompose the discrete interval $[1,N] = \{1,2,\cdots,N\}$ into 2^m consecutive subintervals E_j^m of cardinality 2^{N-m}. To each E_j^m is associated the function $F_j^m(x) = \sum_{k\in E_j^m} a_k\varphi_k(x)$, which satisfies $\|F_j^m\|_2^2 = \sum_{k\in E_j^m}|a_k|^2$. An arbitrary interval $[1,n]$ can be decomposed as the disjoint union of intervals E_j^m, selecting either one or no values of j for each m; this amounts to expressing n as a sum of powers of 2. Therefore $|\sum_{k=1}^n a_k\varphi_k(x)| \le \sum_m \max_j |F_j^m(x)|$. The latter has the less simple but more tractable majorant $\sum_m \big(\sum_j |F_j^m(x)|^2\big)^{1/2}$. For each m, $\|\big(\sum_j |F_j^m|^2\big)^{1/2}\|_{L^2}^2 \equiv \|a\|_{l^2}^2$, as one sees by squaring, integrating, and invoking orthonormality. Summing over m gives the bound $O(M\|a\|_{l^2}) = O(\log N \|a\|_{l^2})$ for $\max_{n\le N} \big|\sum_{k\le n} a_k\varphi_k(x)\big|$, in L^2 norm. For more on the use of such quadratic functionals see [21].

Theorem 7.1 is proved by employing a family $\{E_j^m\}$ of intervals which depends on f to estimate $T^*f(x)$. These form a *martingale filtration*, by which we mean that for each integer $m \ge 0$, $\{E_j^m\}$ forms a partition of $\mathbb{R}$, each E_j^m is the union of two intervals E_i^{m+1}, and $E^0 = \mathbb{R}$. They are chosen to be *adapted to* f in the sense that $\int_{E_j^m} |f|^p \equiv 2^{-m}\|f\|_p^p$. The maximal operator $T^*f(x)$ is majorized by the functional

$$\sum_{m=0}^{\infty} \big(\sum_j |T(f_j^m)(x)|^q\big)^{1/q}, \tag{7.5}$$

where f_j^m is the truncation of f to E_j^m. By invoking the boundedness of T one finds that the L^q norm of $\big(\sum_j |T(f_j^m)(x)|^q\big)^{1/q}$ is $O(2^{-m\varepsilon})$ where $\varepsilon = p^{-1} - q^{-1} > 0$, so the sum over m is rapidly convergent in norm.

7.3. **Digression: Strichartz inequalities.** There is an application to Strichartz inequalities, a useful tool in the analysis of nonlinear differential equations such as the nonlinear Schrödinger equation $iu_t + \Delta_x u = \pm|u|^2 u$ with initial datum $u(0,x) = f(x)$. One of the most basic of the vast family of such inequalities asserts that $F(s,y) \mapsto \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s)\,ds$ maps $L^{q'}(\mathbb{R}^{1+d})$ to $L^q(\mathbb{R}^{1+d})$, where $q = 2(d+2)/d > 2$. Here $F(s)$ denotes the function $y \mapsto F(s,y)$; F is regarded as an $L^{q'}$ function defined on $\mathbb{R}$ and taking values in $L^{q'}(\mathbb{R}^d)$.

Formally the differential equation may be rewritten as Duhamel's integral equation $u(t,x) = e^{-it\Delta} f + \int_0^t e^{i(t-s)\Delta} F(s)\,ds$ where the nonlinear term $|u|^2 u(s,y)$ is treated as the driving force $F(s)$. Various more general Strichartz-type inequalities are used in solving this via a fixed-point argument. One such variant holds that $\int_0^t e^{i(t-s)\Delta} F(s)\,ds$ maps $L^{q'}(\mathbb{R}^{1+d})$ to L^q. The dependence of the interval of integration on t appears to complicate matters, but as was observed by Tao, this follows directly from the first Strichartz inequality, by applying Theorem 7.1 to functions taking values in appropriate Banach spaces $L^q(\mathbb{R}^d)$.

7.4. **Multilinear variants.** Consider maximal multilinear integrals

$$I^*(F_1, \cdots, F_n) = \sup_{y<y'\in\mathbb{R}} \Big| \int_{y\le x_1\le x_2\le\cdots\le x_n\le y'} \prod_{j=1}^n F_j(x_j)\,dx_j \Big|.$$

Let $\{E_j^m\}$ be any martingale filtration on $\mathbb{R}$. Define the quadratic functionals

$$g(F_1, \cdots, F_n) = \sum_{m=1}^{\infty} m \big(\sum_j \max_i |\textstyle\int_{E_j^m} F_i|^2\big)^{1/2}. \tag{7.6}$$

Proposition 7.2. *There exists a universal finite constant C such that for all n and all locally integrable functions,*

$$|I^*(F_1, \cdots, F_n)| \le \frac{C^{n+1}}{\sqrt{n!}} g(F_1, \cdots, F_n)^n. \tag{7.7}$$

A mild technical hypothesis, which is satisfied in applications, has been omitted from the formulation; see Proposition 2.1 of [8]. A trivial variant arises when the l^2 norm with respect to j is replaced by the l^1 norm; then $|I^*(F_1, \cdots, F_n)| \le \frac{1}{n!}(\int \max_i |F_i|)^n$. The proof of the proposition proceeds by decomposing the region of integration into pairwise disjoint Cartesian products of sets E_j^m. Combinatorial manipulations then lead to the majorization (7.7) of the resulting large sum, as depicted in Figure 7.4.

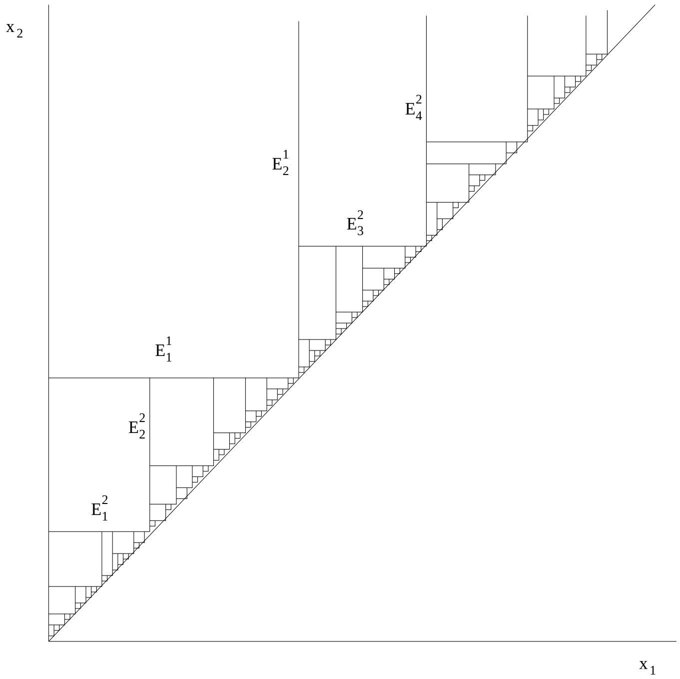

FIGURE 4. Decomposition of the region of integration for the bilinear term into union of the product sets $E_j^m \times E_{j+1}^m$ (j odd). In contrast to the standard Whitney decomposition, the size of the sets is dictated by the structure of functions F_j, not the distance to the boundary, and ratios of sidelengths of rectangles are arbitrary.

This has the following corollary on the operator level. Consider integral operators $T_j f(k) = \int_{\mathbb{R}} K_j(k,x) f(x)\, dx$ and associated multilinear operators

$$\mathcal{T}_n^*(f_1, \cdots f_n)(k) = \sup_{y<y'\in\mathbb{R}} \Big| \int_{y\le x_1 \le x_2 \le \cdots \le x_n \le y'} \prod_{j=1}^n K_j(k, x_j) f_j(x_j)\, dx_j \Big|. \tag{7.8}$$

Corollary 7.3. *Suppose that a single operator T and kernel K are given, that T maps $L^p(\mathbb{R})$ boundedly to L^q with norm $\|T\|_{op}$, and that $p < 2 \le q$. Let $f \in L^p$. Consider maximal multilinear expressions of the form (7.8), where each f_j equals f or $\overline{f}$, and each K_j equals K or $\overline{K}$. Set $r = q/n$. Then*

$$\|\mathcal{T}_n^*(f, \cdots, f)\|_{L^r} \le \frac{C^{n+1}}{\sqrt{n!}} \|T\|_{op}^n \|f\|_{L^p}^n.$$

Here C is some finite constant, which depends only on p, q. The exponent r will be less than 1 for large n.

Another corollary is that for any orthogonal series $\sum a_n \varphi_n(x)$ with $\{\varphi_n\}$ orthonormal and $a \in l^p$, $1 \le p < 2$, $\sum_n \sum_{m<n} a_n a_m \varphi_n(x) \varphi_m(x)$ converges for a.e. x.

To deduce Corollary 7.3, choose a martingale filtration $\{E_j^m\}$ which is adapted to f in L^p, and apply Proposition 7.2 with $F_i(x) = K_i(k,x)f_i(x)$, where K_i, f_i are K, f or their conjugates for each index i. The proof of Theorem 7.1 then yields a good bound for g.

In the application to Schrödinger operators, this is used to control the multilinear expansion (7.2) for scattering coefficients. The underlying linear operator is $Tf(k) = \int e^{2i\phi(k,x)} f(x)\,dx$, where $\phi(k,x) = ikx - (2k)^{-1}\int_0^x V(y)\,dy$. T certainly maps $L^1(dx)$ to $L^\infty(dk)$, and it is straightforward to show that it maps $L^2(dx)$ boundedly to $L^2(dk)$ provided that k is restricted to a compact interval and that $V \in L^p(\mathbb{R})$ for some $1 \le p < \infty$; this implies for instance that the WKB correction term $\int_0^x V$ has sublinear growth. By standard interpolation, it therefore maps L^p to $L^{p'}$ for all $1 < p < 2$. Since p' is strictly greater than p in this range, Proposition 7.2 and Corollary 7.3 can be applied.

8. Trace identities and spectral measure

An alternative approach to the ac spectrum is via the remarkable identity [3, 24]

$$\int_{\mathbb{R}} \log(|t(k)|)^{-1} k^2\,dk + \frac{2\pi}{3}\sum_n |E_n|^{3/2} = \frac{\pi}{8}\int_{\mathbb{R}} V^2(x)\,dx \tag{8.1}$$

where $V \in L^2$ is assumed to be compactly supported and $\{E_n\}$ is the collection of all eigenvalues of H_V. This set is necessarily finite, and each E_n is negative. (8.1) is related to the inverse scattering method for the Korteweg-de Vries equation; the right-hand side is one of the basic quantities invariant under the KdV flow and there are higher-order analogues of (8.1) corresponding to other conserved quantities. (8.1) may be proved by deforming the contour of integration $\mathbb{R}$ through the complex domain. In essence, (8.1) is a consequence of a genuine trace identity $\mathrm{Trace}\,[(H-z)^{-1} - (H_0 - z)^{-1}] = \frac{d}{dz}\log t(\sqrt{z})^{-1}$.

Dropping the first term on the left produces a fundamental bound of Lieb-Thirring type for the negative eigenvalues. If the right hand side is finite, then since $|t(k)| \le 1$, (8.1) effectively provides a lower bound on $|t(k)|$. Via the relation (3.3) with the m-function, a limiting argument establishes existence of the ac spectrum.

For the behavior of generalized eigenfunctions, (8.1) has the following consequence. Let $\Lambda \subset \mathbb{R}\backslash\{0\}$ be a compact interval. Denote by $u(x,k)$ the unique solution of (4.1) with $(u(0,k), u'(0,k))$ equal either to $(1,0)$, or to $(0,1)$. Then

$$\int_\Lambda \log(1 + |u(x,k)|)\,dk \le C + C\int V^2, \tag{8.2}$$

where $C < \infty$ depends only on Λ. In particular, if $V \in L^2(\mathbb{R})$, then the left-hand side is bounded, uniformly in x. The multilinear machinery outlined above, with the factor $1/\sqrt{n!}$, establishes finiteness of the larger quantity $\int_\Lambda \sup_x \log(1 + |u(x,k)|)\,dk$, under the more restrictive hypothesis $V \in L^p$ for some $p < 2$. The appearance of the logarithm on the left-hand side is natural in either version, because of the multiplicative structure of the problem (illustrated in Figure 3.2); this may alternatively be seen from the Prüfer equations.

Following the use of (8.1) to prove Theorem 6.6, there has been much work employing trace identities for spectral analysis. In particular, using higher order trace formulae Molchanov, Novitski and Vainberg proved absolute continuity of the spectrum for decaying potentials with L^2 conditions on derivatives. Laptev, Naboko and Safronov considered decaying perturbations with L^2 conditions on averages the discrete Jacobi matrix case. Denisov gave the first proof of existence of L^2 potentials leading to imbedded singular continuous spectrum. Recently, Killip and Simon [14] obtained a complete description of spectral measures of L^2 perturbations of the free operator, in the Jacobi matrix case. There are interesting connections between these results and questions from the theory of orthogonal polynomials.

9. More on multilinear operators

At the endpoint case $V \in L^2$, these problems make contact with significant recent developments in harmonic analysis. Lacey and Thiele [17] in 1997 answered a longstanding question of Calderón by proving that the *bilinear Hilbert transform*

$$H(f,g)(x) = \int_{\mathbb{R}} f(x+t)g(x-t)\,\frac{dt}{t} = c\iint e^{ix(\xi+\eta)}\,\mathrm{sgn}\,(\xi-\eta)\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi\,d\eta. \tag{9.1}$$

is well-defined for f,g in Lebesgue spaces L^p; in particular it maps $L^2\times L^2$ boundedly to L^1. Here the integral is interpreted in the principal-value sense. The novelty is that the symbol $\mathrm{sgn}\,(\xi-\eta)$ is nonsmooth along the entire diagonal, rather than merely at the origin, as in earlier work of Coifman-Meyer. By taking a linear combination with the product fg, substituting $\widehat{f}=\widehat{g}=V$, and changing the names of variables one arrives at $\iint_{\xi_1<\xi_2} e^{ik\xi_1}e^{ik\xi_2}V(\xi_1)V(\xi_2)\,d\xi\,d\eta$, which closely resembles the bilinear term in the expansion (7.2).

Although groundbreaking work of Calderón, Coifman-Meyer, and David-Journé on the commutator and Cauchy integral operators emphasized the importance of multilinear operators, basic questions remain to be explored. It is unknown whether trilinear Hilbert transforms $\int\prod_{j=1}^3 f_j(x+\alpha_j t)\,t^{-1}\,dt$ are well-defined on Lebesgue spaces. Even simpler variants $\int_0^1\prod_{j=1}^3 f_j(x+\alpha_j t)\,dt$ are subtle; their mapping properties in Lebesgue spaces are partially unresolved but are known to depend on Diophantine-type properties of the coefficient vector (α_j). This is connected with work of Gowers on Szemerédi's theorem and of Bourgain and Katz-Tao on the Kakeya problem.

Muscalu, Tao, and Thiele have obtained two contrasting results concerning trilinear operators of the form $\int_{x_1<x_2<x_3}\prod_{j=1}^3 e^{ika_jx_j}\widehat{f}_j(x_j)\,dx_j$. Firstly, for generic real triples (a_j), these do map $(L^2)^3$ boundedly to $L^{2/3}$. This is closely related to the analysis of the bilinear Hilbert transform, and to the work of Carleson and Fefferman on almost everywhere convergence of Fourier series. Secondly, in the particular case $(a_1,a_2,a_3)=(1,-1,1)$ which arises in the Schrödinger analysis if the WKB correction is disregarded, the triple integral can be infinite for k in a set of positive Lebesgue measure, for certain functions $f_j\in L^2$. This is related to an example of Fefferman [12] concerning the failure of rectangular means of double Fourier series to converge pointwise.

The example does not refute Conjecture 6.5, but certainly indicates that the power series expansion (7.2) approach cannot work as it stands. This is perhaps analogous to the simple fact that $\exp(ix)$ is uniformly bounded for x real, even though the individual terms in its Maclaurin series are not. Indeed, the WKB correction already takes into account the repeated appearance of the Maclaurin expansion for e^{iX} with $X=(i/2k)\int_0^x V$ in a more naive series expansion for $\mathfrak{u}_+(x,k)$.

10. Conclusions

Certain classically known asymptotic properties of one-dimensional Schrödinger operators with short range potentials extend to longer range perturbations, provided that the possibility of a dense set of exceptional energies of Lebesgue measure zero is taken into account. Similarly the WKB approximation remains valid in an extended regime, for families depending suitably on a parameter, if a set of parameter values of measure zero is excluded. These results rely on inequalities for multilinear operators in L^p spaces, and are nonlinear analogues and extensions of old almost everywhere convergence theorems for Fourier integrals and general orthogonal series. Trace identities are a powerful source of spectral information not obtainable from the multilinear analysis. An apparently subtle endpoint problem remains for the natural class of L^2 potentials. Higher-dimensional analogues remain wide open.

References

[1] P.K. Alsholm and T. Kato, *Scattering with long range potentials,* Partial Diff. Eq., Proc. Symp. Pure Math. Vol. **23**, Amer. Math. Soc., Providence, Rhode Island, 1973, 393–399

[2] J. Bourgain, *On random Schrödinger operators on Z^2,* Discrete Contin. Dyn. Syst. **8** (2002), 1–15.

[3] V. S. Buslaev and L. D. Faddeev, *Formulas for traces for a singular Sturm-Liouville differential operator*, Soviet Math. Dokl. 1 (1960) 451–454.

[4] V.S. Buslaev and V.B. Matveev, *Wave operators for Schrödinger equation with slowly decreasing potentials,* Theor. Math. Phys. **2** (1970), 266–274

[5] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135-157

[6] M. Christ and A. Kiselev, *WKB asymptotics of generalized eigenfunctions of one-dimensional Schrödinger oeprators*, J. Funct. Anal. **179** (2001), 426-447.

[7] ———, *WKB and spectral analysis of one-dimensional Schrödinger operators with slowly varying potentials,* Commun. Math. Phys. **218** (2001), 245-262.

[8] ———, *Scattering and wave operators for one-dimensional Schrödinger operators with slowly decaying nonsmooth potentials*, Geom. Funct. Anal., to appear.

[9] F. Delyon, B. Simon and B. Souillard, *From pure point to continuous spectrum in disordered systems,* Ann. Inst. Henri Poincare **42** (1985), 283–309.

[10] P. Deift and R. Killip, *On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials,* Commun. Math. Phys. **203** (1999), 341–347

[11] J. D. Dollard, *Asymptotic convergence and Coulomb interaction,* J. Math. Phys. **5** (1964), 729–738.

[12] C. Fefferman, *On the divergence of multiple Fourier series*, Bull. Amer. Math. Soc. **77** (1971), 191–195

[13] L. Hörmander, *The existence of wave operators in scattering theory*, Math. Z. **146** (1976), 69-91.

[14] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their application to spectral theory*, to appear, Ann. Math.

[15] A. Kiselev, *Imbedded singular continuous spectrum for Schrödinger operators*, preprint.

[16] S. Kotani and N. Ushiroya, *One-dimensional Schrödinger operators with random decaying potentials,* Commun. Math. Phys. **115** (1988), 247–266

[17] M. Lacey and C. Thiele, *L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$*, Ann. of Math. (2) 146 (1997), no. 3, 693–724 and *On Calderón's conjecture*, Ann. of Math. (2) 149 (1999), no. 2, 475–496

[18] D. Pearson, *Singular continuous measures in scattering theory*, Comm. Math. Phys. **60** (1978), 13–36.

[19] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III. Scattering Theory*, Academic Press, London-San Diego, 1979

[20] C. Remling, Bounds on embedded singular spectrum for one-dimensional Schrödinger operators, Proc. Amer. Math. Soc. **128** (2000), 161–171.

[21] E. M. Stein, *The development of square functions in the work of A. Zygmund*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 2, 359–376

[22] T. Tao, *From rotating needles to stability of waves: emerging connections between combinatorics, analysis, and PDE*, Notices Amer. Math. Soc. 48 (2001), no. 3, 294–303

[23] E.C. Titchmarsh, *Eigenfunction Expansions,* 2nd ed., Oxford University Press, Oxford, 1962.

[24] V. E. Zaharov and L. D. Faddeev, *The Korteweg-de Vries equation is a fully integrable Hamiltonian system* (Russian), Funkcional. Anal. i Priložen. 5 (1971), no. 4, 18–27.

Michael Christ, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

E-mail address: mchrist@math.berkeley.edu

Alexander Kiselev, Department of Mathematics, University of Wisconsin, Madison, WI 53706

E-mail address: kiselev@math.wisc.edu