A REMARK ON SUMS OF SQUARES OF COMPLEX VECTOR FIELDS

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1. Introduction

Let $\{Z_j\}$ be a finite collection of vector fields, with smooth complex-valued coefficients, defined in an open subset U of Euclidean space. Let Z_j^* be the formal adjoint of Z_j, with respect to the Hilbert space structure L^2 associated to some measure with a smooth nonvanishing density. Consider the operator $\mathfrak{L} = \sum_j Z_j^* Z_j$, which we shall refer to as a sum of squares. $\mathfrak{L}$ is said to be hypoelliptic in U if for any open subset $V \subset U$ and any distribution $u \in \mathcal{D}'(V)$ such that $\mathfrak{L}(u) \in C^\infty(V)$, necessarily $u \in C^\infty(V)$.

Assume throughout this paragraph only that all vector fields are real. Then a well-known sufficient condition for hypoellipticity is the bracket condition of Hörmander, that the Lie algebra generated by $\{Z_j\}$ should span the tangent space to U at each of its points. This condition ensures, and is equivalent to, the condition that $\mathfrak{L}$ is subelliptic in the sense that for any relatively compact open subset $V \Subset U$, there exist $\varepsilon > 0$ and $C < \infty$ such that for all $u \in C_0^2(V)$,

$$\|u\|_{H^\varepsilon} \le C\|\mathfrak{L}u\|_{H^0} + C\|u\|_{H^0}. \tag{1.1}$$

This can be equivalently reformulated as

$$\|u\|_{H^\varepsilon}^2 \le CQ(u,u) + C\|u\|_{H^0} \tag{1.2}$$

where $Q(u,u) = \sum_j \|Z_j u\|_{H^0}^2$. Subellipticity in turn implies hypoellipticity for sums of squares operators. However, $\mathfrak{L}$ is sometimes hypoelliptic without satisfying the bracket condition. See for instance [2] and the references cited there.

Henceforth we allow vector fields to be complex. Weaker inequalities than (1.1) are then conceivable.

Definition 1.1. We say that $\mathfrak{L}$ loses at most finitely many derivatives in any open set U if for every $V \Subset U$ there exist $s > -\infty$, $t < +\infty$ and $s' < s$ such that for all $u \in C_0^\infty(V)$,

$$\|u\|_{H^s} \le C\|\mathfrak{L}u\|_{H^t} + C\|u\|_{H^{s'}}. \tag{1.3}$$

We say that $\mathfrak{L}$ genuinely loses derivatives[1] if for any t, no such inequality holds with $s = t$.

This usage is not universally accepted, and will be discussed further in §4 below.

For complex vector fields the bracket condition still makes sense, and subellipticity in the sense (1.2) continues to imply hypoellipticity. Siu has asked whether the bracket condition continues to imply subellipticity in this sense for complex fields. Kohn [5] has answered

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[1] Many authors say that an operator $\mathfrak{L}$ of order m loses at least k derivatives if some inequality related to (1.3) (see §4) does not hold for any $s > t + m - k$. Genuine loss of derivatives, in our language, corresponds essentially to loss of *more than* m derivatives in that alternative language.

this in the negative[2], and has gone further by establishing examples which simultaneously (i) satisfy the bracket hypothesis, (ii) not only fail to be subelliptic but genuinely lose derivatives, yet (iii) are nonetheless hypoelliptic. This note is a comment on [5], showing that even the weaker property of hypoellipticity can fail, for complex vector fields satisfying the bracket condition.

Earlier, Heller [4] had studied the hypoellipticity (and analytic hypoellipticity) of left-invariant differential operators of arbitrary order on the Heisenberg group, subject to a hypothesis of transversal ellipticity. He showed that such an operator is (C^∞ and C^ω) hypoelliptic whenever it loses at most finitely many derivatives, and he gave an example of a fourth order operator[3] which does genuinely lose derivatives, yet is hypoelliptic. This extended an analysis of Stein [8], who had proved hypoellipticity (as well as analytic hypoellipticity) for certain second order operators[4] which do not gain derivatives, but do not genuinely lose them either.

Parenti and Parmeggiani [7] have proved general theorems establishing hypoellipticity and nonhypoellipticity for classes of operators of Grushin type. In particular, their Example 6.5(iv) describes certain fourth-order differential operators that genuinely lose one derivative in our terminology (five derivatives in theirs), yet are hypoelliptic.

Hypoellipticity with genuine loss of derivatives is a delicate matter, because estimates without any gain in regularity are inevitably quite unstable. Any analysis of hypoellipticity must involve deformation of $\mathfrak{L}$, for instance via the introduction of some type of cutoff operators, potentially destroying the estimates (1.3).

From this point of view our main result is not surprising:

Proposition 1.1. *There exist finite families of complex vector fields Z_j with C^∞ coefficients which satisfy the bracket condition and lose at most finitely many derivatives in the sense* (1.3)*, but for which $\sum_j Z_j^* Z_j$ fails to be C^∞ hypoelliptic.*

To describe these consider $\mathbb{R}^3$ with coordinates (x, t, s). We consider always the Hilbert space $L^2(\mathbb{R}^3)$ associated to Lebesgue measure in these coordinates. Define

$$\bar{L} = \partial_x - ix\partial_t, \qquad L = \partial_x + ix\partial_t. \tag{1.4}$$

Fix an integer $k \geq 1$ and define $Z_1 = \bar{L}$, $Z_2 = x^k L$, and $Z_3 = \partial_s$. Here $\partial_x = \frac{d}{dx}$, with no factor of $\sqrt{-1}$, and so forth. Proposition 1.1 can now be more precisely restated.

Proposition 1.2. *Let k be any positive integer. The complex vector fields Z_1, Z_2, Z_3 satisfy the bracket condition at each point of $\mathbb{R}^3$, and in any bounded open set $V \subset \mathbb{R}^3$, the operator $\mathfrak{L} = \sum_{j=1}^3 Z_j^* Z_j$ loses at most finitely many derivatives. Nonetheless, $\mathfrak{L}$ is not C^∞ hypoelliptic in any neighborhood of the origin.*

When $k = 1$, $\mathfrak{L}$ actually satisfies (1.3) with $s = 0$, that is, it does not genuinely lose derivatives; yet it fails to be hypoelliptic.

Z_1, Z_2 can also be regarded as vector fields in $\mathbb{R}^2$ rather than in $\mathbb{R}^3$. The operator $\mathfrak{L}_{\mathbb{R}^2} = Z_1^* Z_1 + Z_2^* Z_2$ in $\mathbb{R}^2$ is then a simplified version of Kohn's examples, and can be shown to be hypoelliptic although we will not do so here. Adding the extra variable s and the extra term $-\partial_s^2$ to create $\mathfrak{L}$ destroys hypoellipticity, due to propagation of singularities along curves such as $\{(0, 0, s)\}$.

[2]But has shown that it does imply subellipticity if $\{Z_j\}$ together with their brackets with only two factors suffice to span the tangent space.

[3]Namely $\Box_b^2 + X$ on $\mathbb{H}^1$.

[4]Such as $\Box_b + 1$.

Our example is closely analogous to two well-known examples concerning C^∞ and analytic hypoellipticity [1], [6]. Firstly, the operator $-\partial_x^2 - x^2\partial_t^2$ is analytic hypoelliptic in $\mathbb{R}^2$, whereas $-\partial_x^2 - x^2\partial_t^2 - \partial_s^2$ fails to be analytic hypoelliptic in $\mathbb{R}^3$. Secondly, consider a C^∞ function $a : \mathbb{R}^1 \to \mathbb{R}$ such that $a(x) = 0$ if and only if $x = 0$. Then $-\partial_x^2 - a(x)^2\partial_t^2$ is always C^∞ hypoelliptic in $\mathbb{R}^2$, while $-\partial_x^2 - a(x)^2\partial_t^2 - \partial_s^2$ may or may not be hypoelliptic in $\mathbb{R}^3$, depending on the rate at which $a(x)$ tends to zero as $x \to 0$. For an attempt to place these examples in perspective see [2], [3].

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2. Spectral analysis of certain ODEs

For $\tau \in \mathbb{R}^+$ consider the ordinary differential operators

$$P_\tau = -(\partial_x - x\tau)(\partial_x + x\tau) - (\partial_x + x\tau)x^{2k}(\partial_x - x\tau). \tag{2.1}$$

These are obtained by separation of variables;

$$\mathfrak{L}_{\mathbb{R}^2}(e^{i\tau t}f(x)) = e^{i\tau t}P_\tau f(x).$$

P_τ is formally selfadjoint on $L^2(\mathbb{R})$ with respect to Lebesgue measure, and is nonnegative.

For any $\tau > 0$, P_τ is unitarily equivalent, via the change of variables $y = \tau^{1/2}x$ and substitution $F(y) = \tau^{-1/4}f(x)$, to τQ_τ where

$$Q_\tau = -(\partial_y - y)(\partial_y + y) - \tau^{-k}(\partial_y + y)y^{2k}(\partial_y - y). \tag{2.2}$$

Setting $g(y) = e^{-y^2/2}$ we have

$$\langle Q_\tau g,\, g\rangle = \tau^{-k}\|y^k(\partial_y - y)e^{-y^2/2}\|_{L^2}^2 = c\tau^{-k}. \tag{2.3}$$

Conversely we claim that for all $\tau \ge 1$ and all $f \in C_0^2(\mathbb{R})$,

$$\tau^{-k}\|f\|_{L^2}^2 + \tau^{-k}\|yf\|_{L^2}^2 + \tau^{-k}\|\partial_y f\|_{L^2}^2 \le C\langle Q_\tau f,\, f\rangle. \tag{2.4}$$

Indeed,

$$\begin{aligned}
\langle Q_\tau f,\, f\rangle &= \tau^{-k}\|y^k(\partial_y - y)f\|_{L^2}^2 + \|(\partial_y + y)f\|_{L^2}^2 \\
&\ge \tau^{-k}\int_{|y|\ge 1} |(\partial_y - y)f|^2\,dy + \int |(\partial_y + y)f|^2\,dy \\
&\ge \tau^{-k}\int_{|y|\ge 1} |yf(y)|^2\,dy + \int_{|y|\le 1} |((\partial_y + y)f|^2\,dy \\
&\ge c\tau^{-k}\|f\|_{L^2}^2,
\end{aligned}$$

and (2.4) follows from this together with the majorization $\langle Q_\tau f,\, f\rangle \ge \|(\partial_y + y)f\|_{L^2}^2$.

It follows readily that the L^2 closure of Q_τ is selfadjoint and has discrete spectrum, and that every eigenfunction of Q_τ belongs to the Schwartz space. Define $\lambda(\tau)$ to be the lowest eigenvalue of Q_τ. By (2.3) and (2.4), there exist $0 < c < c' < \infty$ such that

$$c'\tau^{-k} \le \lambda(\tau) \le c\tau^{-k} \qquad \forall\, \tau \in [1, \infty). \tag{2.5}$$

Let $\psi_\tau \in L^2(\mathbb{R})$ be an eigenfunction of Q_τ with eigenvalue $\lambda(\tau)$, normalized so that $\|\psi_\tau\|_{L^2(\mathbb{R})} = 1$. We claim that $\|y\psi_\tau\|_{L^2}$ and $\|\partial_y\psi_\tau\|_{L^2}$ are bounded above, uniformly in τ for all $\tau \ge 1$. To prove this, decompose $\psi_\tau = ah_0 + g$ where $h_0(y) = e^{-y^2/2}$, $a \in \mathbb{C}$, and $g \perp h_0$. Since $\partial_y + y$ annihilates h_0, since $\|(\partial_y + y)\psi_\tau\|_{L^2}^2 \le \langle Q_\tau\psi_\tau,\, \psi_\tau\rangle$, and since $\|g\|_{L^2} \lesssim \|(\partial_y + y)g\|_{L^2}$, it follows that for large τ one has $\|g\|_{L^2} \lesssim \lambda(\tau) \ll 1$, and consequently

$|a| \sim 1$. Since $\|yg\|_{L^2} + \|\partial_y g\|_{L^2} \lesssim \|(\partial_y + y)g\|_{L^2} + \|g\|_{L^2}$ for all functions g orthogonal to h_0, and since h_0 is a Schwartz function and is independent of τ, the claim follows.

From this we conclude firstly that $\|\psi_\tau\|_{L^\infty(\mathbb{R})}$ is bounded above, uniformly for all $\tau \geq 1$. Secondly there exists $B < \infty$ such that

$$\sup_{|y| \leq B} |\psi_\tau(y)| \geq B^{-1} \tag{2.6}$$

uniformly for all $\tau \geq 1$.

3. Conclusion of proof

Consider the family of functions u_τ defined for $\tau \in [1, \infty)$ by

$$u_\tau(x, t, s) = e^{i\tau t} e^{\sigma(\tau) s} \psi_\tau(\tau^{1/2} x) \tag{3.1}$$

where $\sigma(\tau) > 0$ is the positive solution of $\sigma^2 = \tau\lambda(\tau)$. Then $\mathfrak{L} u_\tau \equiv 0$ in $\mathbb{R}^3$. By (2.5), $\sigma(\tau) = O(\tau^{(1-k)/2})$; in particular, $\sigma(\tau)$ is uniformly bounded as $\tau \to +\infty$.

As is well known, hypoellipticity implies certain inequalities via the Baire category theorem. If $\mathfrak{L}$ were hypoelliptic, then for any open sets $V \Subset V'$ and any $N \in \mathbb{N}$ there would exist $C, M < \infty$ such that for all $u \in C^\infty(V')$,

$$\|u\|_{C^N(V)} \leq C \|\mathfrak{L} u\|_{C^M(V')} + C\|u\|_{C^0(V')}. \tag{3.2}$$

Fix $V \Subset V' \Subset \mathbb{R}^3$ with $0 \in V$. Consider the inequality (3.2) for u_τ, for large positive τ. By (2.6), for all sufficiently large τ we have

$$\|\partial_t u_\tau\|_{C^0(V)} \geq c\tau. \tag{3.3}$$

On the other hand $\mathfrak{L} u_\tau \equiv 0$, while the uniform boundedness of ψ_τ in L^∞ implies that

$$\|u_\tau\|_{C^0(V')} \leq C \|\psi_\tau\|_{C^0(\mathbb{R})} e^{C\sigma(\tau)} \leq C' e^{C\sigma(\tau)}; \tag{3.4}$$

the factor $e^{\sigma(\tau)s}$ is $O(e^{C\sigma(\tau)})$ because V' is a bounded set. Since $\sigma(\tau) = O(\tau^{(1-k)/2})$ remains bounded as $\tau \to \infty$, $\|u_\tau\|_{C^0(V')}$ likewise remains uniformly bounded. Thus (3.2) fails to hold for $N = 1$. □

4. On loss of derivatives

Definition 1.1 is only one possible notion of genuine loss of derivatives. A more common notion, as Kohn has pointed out, is essentially this: $\mathfrak{L}$ is said to genuinely lose at least δ derivatives in an open set U if there exist an open subset $V \subset U$, an exponent s, and a distribution $u \in \mathcal{D}'(V)$ such that $\mathfrak{L} u \in H^s_{\text{loc}}(V)$, yet $u \notin H^t_{\text{loc}}(V)$ for any $t > s - \delta$. $\mathfrak{L}$ is then said to genuinely lose derivatives if it genuinely loses at least δ derivatives for some $\delta > 0$. It is thus formally conceivable that an operator could lose at most a certain number of derivatives in the sense of Definition 1.1, yet lose more derivatives, or even infinitely many, in this alternative sense.

A global inequality of the form (1.3) expresses a very weak property of an operator. Hypoellipticity amounts to having a family of inequalities that are stronger in two ways, incorporating both (i) spatial localization and (ii) a type of localization (expressed by weighted L^2 inequalities) with respect to frequency variables in phase space. An inequality corresponding to an implication $\mathfrak{L} u \in H^s_{\text{loc}} \Rightarrow u \in H^t_{\text{loc}}$ expresses one of these two types of localization, but not the other. We regard such an inequality as expressing a type of partial hypoellipticity, whereas (1.3) is a minimal *a priori* inequality involving no localization. We note that such inequalities, with $\mathfrak{L}$ replaced by its transpose, are fundamental to the theory of local solvability. (1.3) appears at one extreme of a (partially ordered) spectrum of possible inequalities, with hypoellipticity lying at the opposite end of the spectrum and the notion of loss discussed in the preceding paragraph lying somewhere in between.

Other variants formulated in terms of the quadratic form $Q(u,u) = \sum_j \|Z_j u\|^2_{H^0}$, rather than some norm of $\mathfrak{L}u$, are also reasonable.

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