An a priori Bound and Weak Solutions for the Nonlinear Schrödinger Equation

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Cubic one-dimensional (focusing or defocusing) **NLS**:

$$i\partial_t u + \Delta_x u + \omega|u|^2 u = 0 \qquad (x \in \mathbb{R}^1)$$

with $u(0,x) \equiv f(x)$.
$\omega \in \mathbb{R}$ is nonzero; sign of ω plays no role in our work.

• The H^0 norm is conserved.

• Wellposed in $H^s(\mathbb{R}^1)$ for all $s \geq 0$ (Tsutsumi) and likewise in $H^s(\mathbb{T}^1)$ (Bourgain). This means (i) Existence; (ii) $f \mapsto u$ is continuous from H^s to $C^0(H^s)$; (iii) Solutions exist in a more restrictive class than $C^0_t(H^s_x)$, and are unique in this class.

• For $s < 0$, $f \mapsto u$ fails to be *uniformly* continuous as a map from H^s to $C^0(H^s)$. (Burq-Gérard-Tzvetkov; C-C-T)
• For $s < -\frac{1}{2}$: For any $\varepsilon > 0$ there exist a smooth solution u satisfying
$\|u(0,\cdot)\|_{H^s} < \varepsilon$ and $\|u(\varepsilon,\cdot)\|_{H^s} > \varepsilon^{-1}$.

Main Theorem. For any $s > -\frac{1}{12}$ and any $R < \infty$ there exist $T > 0$ and $R' < \infty$ such that for all $f \in H^0$, the standard solution of NLS with $u(0, \cdot) = f$ satisfies

$$\|f\|_{H^s} \leq R \Rightarrow \|u\|_{C^0([0,T],H^s)} \leq R'. \qquad \square$$

Application:

Theorem 2. For any $s > -\frac{1}{12}$ and any $R < \infty$ there exists $T > 0$ such that for all $f \in H^s$ satisfying $\|f\|_{H^s} \leq R$, there exists a $\boxed{\text{weak}}$ solution $u \in C^0([0, T], H^s)$ of NLS with initial datum f.

What is a weak solution?

As stated, Theorem 2 makes no sense, because $|u|^2 u$ has no natural meaning for arbitrary $u \in C^0(H^s)$.

We construct a space $Y = Y^{s,b}$ such that (for some but by no means all parameters s, b)
• Smooth solutions with an *a priori* bound on $\|u(0,\cdot)\|_{H^s}$ enjoy an *a priori* upper bound in $Y^{s,b}$,
• The mapping $(u, v, w) \mapsto u\bar{v}w$ is well-defined and continuous from $(Y^{s,b})^3$ to $\mathcal{S}'$.

One then has a natural notion of a weak solution in $Y^{s,b}$, and our weak solutions are of this type.

Open questions.

• There is no reason to believe that the threshold $-\frac{1}{12}$ is optimal. Do our theorems extend to all $s > -\frac{1}{2}$? To get this by tinkering with our method seems hopeless. If true, this should be connected with the complete integrability of NLS.

• Do our weak solutions enjoy any uniqueness?
Continuous (though nonuniform) dependence on initial data?
(Our analysis yields no bound for $u_1 - u_2$.)
(We do get uniqueness for $u(0) = 0$ and continuity at $u(0) = 0$.)

Comments.

• Structure of our arguments is rather general, although details are highly specific. Similar results are likely obtainable for some other equations.

• Complete integrability plays no role in our proof.

• The analysis is somewhat related to the so-called I-method.

• Continuous dependence of u on $u(0)$ in H^s norm does fail in the periodic case.

$X^{s,b}$ spaces (Bourgain): $u \in X^{s,b}$ if

$$\iint_{\mathbb{R}^2} |\widehat{u}(\tau,\xi)|^2 \langle \tau - \xi^2 \rangle^{2b} \langle \xi \rangle^{2s} \, d\xi.$$

Our principal ingredients are the following inequalities for all sufficiently smooth solutions of NLS:

$$\Big| \|u(t)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 \Big| \leq C\|u\|_{X^{r,b}}^4 \qquad \text{if } r > -\tfrac{1}{4} \text{ and } s \leq 0$$

$$\|u\|_{Y^{s,b}} \leq C\|u\|_{C^0(H^s)} + C\|u\|_{Y^{s,b}}^3 \qquad \text{if } s > -\tfrac{2}{15}$$

and two inequalities for all sufficiently smooth functions:

$$\|u\|_{X^{r,b}} \leq C\|u\|_{Y^{s,b}} \qquad \text{if } r < (1+4b)s$$

$$\|u_1 \overline{u_2} u_3\|_{Y^{s,b-1}} \leq C \prod_j \|u_j\|_{Y^{s,b}} \qquad \text{if } s > -\tfrac{2}{15}$$

for $b > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$.

For the experts: In the classical inequality

$$\|u_1\overline{u_2}u_3\|_{X^{0,b-1}} \le C\prod_j \|u_j\|_{X^{0,b}}$$

there is a certain amount of slack. For instance, one has (as is well known)

$$\|u_1\overline{u_2}u_3\|_{X^{0,0}} \le C\prod_j \|u_j\|_{X^{0,b}}$$

provided that $b > \frac{1}{2}$.

The main contribution to $u_1\overline{u_2}u_3$ in $X^{0,b-1}$ comes, in Fourier space, from the near-diagonal region where (ξ_1, ξ_2, ξ_3) is close to some (ξ_0, ξ_0, ξ_0).

Much of our analysis is a painstaking exploitation of scraps of slack.

Notation

$\vec{\xi} = (\xi_1, \cdots, \xi_4) \in \mathbb{R}^4$,
$\Xi = \{\vec{\xi} : \xi_1 - \xi_2 + \xi_3 - \xi_4 = 0\}$,
$\lambda =$ Lebesgue measure on Ξ,
$\psi(\vec{\xi}) = \langle \xi_1 \rangle^{2s} - \langle \xi_2 \rangle^{2s} + \langle \xi_3 \rangle^{2s} - \langle \xi_4 \rangle^{2s}$.

$$\begin{aligned} \sigma(\xi_1, \cdots, \xi_4) &= \xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 = (\text{for } \vec{\xi} \in \Xi) \\ &= 2(\xi_1 - \xi_2)(\xi_3 - \xi_2) = 2(\xi_1 - \xi_4)(\xi_3 - \xi_4) \end{aligned}$$

We seek an *a priori* bound on $\|u(t, \cdot)\|_{H^s}^2$.

$\dfrac{d}{dt}\|u(t, \cdot)\|_{H^s}^2 = \mathrm{Re}(c\omega\mathcal{I}(t))$ so $\Big| \|u(t, \cdot)\|_{H^s}^2 - \|u(0, \cdot)\|_{H^s}^2 \Big| \lesssim \Big| \displaystyle\int_0^t \mathcal{I}(r)\, dr \Big|$
where

$$\mathcal{I}(t) = \int_{\Xi} \widehat{u}(t, \xi_1) \overline{\widehat{u}(t, \xi_2)} \widehat{u}(t, \xi_3) \overline{\widehat{u}(t, \xi_4)}\, \psi(\vec{\xi})\, d\lambda(\vec{\xi}).$$

In the near-diagonal region where all $|\xi_j|$ have the same large order of magnitude, $|\psi(\vec{\xi})| \le C|\vec{\xi}|^{2s-2}|\sigma(\vec{\xi})|$. The factor $\sigma(\vec{\xi}) = 2(\xi_1 - \xi_2)(\xi_1 - \xi_4)$ expresses *cancellation* near the diagonal.

Space-time Fourier transform representation

Write $\vec{\tau} = (\tau_1, \cdots, \tau_4)$.

$$|\int_0^t \mathcal{I}(u,r)\,dr|$$
$$\leq C \iint_{\Xi\times\mathbb{R}^4} \prod_{j=1}^4 |\widehat{u}(\xi_j,\tau_j)| \langle \tau_1 - \tau_2 + \tau_3 - \tau_4\rangle^{-1} |\psi(\vec{\xi})|\,d\vec{\tau}\,d\lambda(\vec{\xi})$$

Substitute

$$|\widehat{u}(\xi_j,\tau_j)| = \langle \xi_j\rangle^{-r}\langle \tau_j - \xi_j^2\rangle^{-b} g_j(\xi_j,\tau_j), \text{ so } \|g_j\|_{L^2(\mathbb{R}^2)} = \|u\|_{X^{r,b}}.$$

Thus

$$|\int_0^t \mathcal{I}(u,r)\,dr| \leq C \iint_{\Xi\times\mathbb{R}^4} \prod_{n=1}^4 \left(g_n(\xi_n,\tau_n)\langle \xi_n\rangle^{-r}\langle \tau_n - \xi_n^2\rangle^{-b}\right)$$
$$|\psi(\vec{\xi})| \langle \tau_1 - \tau_2 + \tau_3 - \tau_4\rangle^{-1}\,d\vec{\tau}\,d\lambda(\vec{\xi}).$$

We need to show that this is majorized by $C\prod_{n=1}^4 \|g_n\|_{L^2(\mathbb{R}^2)}$ provided that $r > -\frac{1}{4}$, $s < 0$ and $b = \frac{1}{2} + \varepsilon$.

Review of two Strichartz-type inequalities

For $h \in L^2(\mathbb{R})$, the solution u of the linear Schrödinger equation with initial datum h belongs to $L^6(\mathbb{R}^2)$ (Stein-Tomas). Thus $u_1\overline{u_2}u_3 \in L^2$.

Rewritten on the Fourier side, this **trilinear** inequality is:

$$\Big|\int f(\xi_1-\xi_2+\xi_3, \xi_1^2-\xi_2^2+\xi_3^2)\prod_{n=1}^{3} g_n(\xi_n)\,d\xi_n\Big| \le C\prod_n \|g_n\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R}^2)}.$$

One version of the **bilinear** Strichartz inequality (Carleson-Sjölin): for any subset $E \subset \mathbb{R}^2$,

$$\Big|\int_{\mathbb{R}^2} f(\xi_1 \pm \xi_2, \xi_1^2 \pm \xi_2^2)h_1(\xi_1)h_2(\xi_2)\chi_E(\xi_1,\xi_2)\,d\xi_1\,d\xi_2\Big|$$
$$\le C\Big(\min_{(\xi_1,\xi_2)\in E}|\xi_1-\xi_2|\Big)^{-1/2}\|f\|_{L^2(\mathbb{R}^2)}\|h_1\|_{L^2(\mathbb{R}^1)}\|h_2\|_{L^2(\mathbb{R}^1)},$$

where the two $\pm$ signs agree. This represents the pairing of f with a bilinear operator applied to h_1, h_2.

Proposition Strichartz 1. Consider

$$\int_{\vec{\xi}\in S\subset\Xi}\int_{\tau_1-\tau_2+\tau_3-\tau_4=0}\prod_{n=1}^{4} g_n(\xi_n,\tau_n)$$
$$\langle\tau_n-\xi_n^2\rangle^{-\beta_n}\chi_E(L(\vec{\xi}))\,d\lambda(\vec{\tau})\,d\lambda(\vec{\xi}) \quad (1)$$

where $\phi, g_n \geq 0$, $i,j\in\{1,2,3,4\}$ are distinct, $E\subset\mathbb{R}^1$ is any measurable set, and $L:\mathbb{R}^4\to\mathbb{R}$ is a linear transformation. Suppose that

- $\beta_n > \frac{1}{2}$ for all but at most one index n, and $\beta_n > 0$ for all n.
- i,j have opposite parity.
- $L(\vec{\xi}) = \xi_\mu - \xi_\nu$ where exactly one of μ,ν belongs to $\{i,j\}$.

Then there exists $C<\infty$ such that (1) is majorized by

$$C|E|^{1/2}\max_{\vec{\xi}\in S}\left(|\xi_i-\xi_j|^{-1/2}\langle\sigma(\vec{\xi})\rangle^{-\beta}\right)\prod_{n=1}^{4}\|g_n\|_{L^2(\mathbb{R}^2)}$$

where $\beta = \min_n \beta_n$.

Recall $\sigma(\vec{\xi}) = 2(\xi_1-\xi_2)(\xi_1-\xi_4)$.

Proposition Strichartz 2. Consider (1) with $L(\vec{\xi}) = \xi_k$ for some $k \notin \{i, j\}$. Suppose again that $\beta_n > \frac{1}{2}$ for all but at most one index n, and $\beta_n > 0$ for all n. Suppose that $|E| \lesssim \min_{\vec{\xi} \in S} |\xi_i - \xi_j|$. Let $\beta = \min_n \beta_n$. Then there exists $C < \infty$ such that (1) is majorized by

$$C|E|^{1/4} \max_{\vec{\xi} \in S} \left(|\xi_i - \xi_j|^{-1/4} \langle \sigma(\vec{\xi}) \rangle^{-\beta} \right) \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)}.$$

The exponent $\frac{1}{4}$ here is optimal; the counterexamples which show this involve 3 distinct scales.

Proposition Strichartz 3. Consider

$$\int_{\Xi\times\mathbb{R}^4}\langle\tau_1-\tau_2+\tau_3-\tau_4\rangle^{-1}\prod_{n=1}^{4}g_n(\xi_n,\tau_n)\langle\tau_n-\xi_n^2\rangle^{-\beta_n}$$
$$\phi(\vec{\xi})\chi_S(\vec{\xi})\chi_E(L(\vec{\xi}))\,d\lambda(\vec{\xi})\,d\vec{\tau}$$

where $g_n \geq 0$, $\phi \geq 0$, and $\beta_n > \frac{1}{2}$ for all n. Let $i \neq j \in \{1,2,3,4\}$ and let $L : \mathbb{R}^4 \to \mathbb{R}$ be a linear functional satisfying the hypotheses of Proposition Strichartz 1. Then above expression is

$$\leq C|E|^{1/2}\prod_{n=1}^{4}\|g_n\|_{L^2(\mathbb{R}^2)}$$
$$\max_{\vec{\xi}\in S}\left[\phi(\vec{\xi})|\vec{\xi}|^{1/2}\langle\sigma(\vec{\xi})\rangle^{-1}+\phi(\vec{\xi})|\xi_i-\xi_j|^{-1/2}\langle\sigma(\vec{\xi})\rangle^{-\beta}\right]$$

where $\beta = \min_n \beta_n$.

None of these three results require ideas dramatically different from standard arguments for the bilinear Strichartz inequality.

Y space theory.

$\|u\|_{C^0(H^s)}$ has been controlled by $\|u\|_{X^{r,b}}$ for certain $r < s$; a bound in $X^{r,b}$ is needed to close the loop. Direct attempts to control the $X^{r,b}$ norm by itself together with the $C^0(H^s)$ norm fail for $r < 0$.

Proposition.

There exist spaces $Y^{s,b}$ with the following properties:

(1) $\|u\|_{X^{r,b}} \leq C\|u\|_{Y^{s,b}}$ if $r < (1+4b)s$

(2) $(i\partial_t + \Delta_x)^{-1}(Y^{s,b-1}) \subset Y^{s,b}$

(3) $Y^{s,b} \subset C^0(H^s)$ provided $b > \frac{1}{2}$

(4) $\|u_1\overline{u_2}u_3\|_{Y^{s,b-1}} \leq C \prod_{n=1}^{3} \|u_n\|_{Y^{s,b}} \qquad \forall$ distributions u_n,

(5) $\|u\|_{Y^{s,b}} \leq C\|u\|^3_{Y^{s,b}} + C\|u\|_{C^0(H^s)} \qquad \forall$ solutions u.

The last two inequalities hold for any $s > -\frac{2}{15}$, for $b = \frac{1}{2} + \varepsilon(s)$.

One can reduce by scaling to the case where the initial datum has small H^s norm, and then these inequalities suffice to close the loop.

Motivation for definition of $Y^{s,b}$

• Define $T_\lambda u(t,x) = \lambda u(\lambda^2 t, \lambda x)$; u solution $\Rightarrow T_\lambda u$ solution.

• If $\widehat{u(t,\xi)}$ is supported where $|\xi| \lesssim N$ and u is $O(1)$ in $C^0(H^s)$ then $T_\lambda u$ is $O(1)$ in $C^0(H^0)$ if $\lambda = N^{2s}$. This scaling converts H^s to H^0 at the expense of lengthening time T to $N^{-4s}T$.

• NLS is globally wellposed in H^0, but one doesn't prove this directly via a fixed-point argument in $X^{0,b}$. Instead, a contraction mapping argument in $X^{0,b}$ is used for unit time steps, and this is coupled with the H^0 conservation law.

• The resulting function has good $C^0(H^0)$ bound, but poor $X^{0,b}$ bound.

To make this work, we're forced to use a different scaling factor λ for each N. This leads to various bookkeeping issues.

Introduce Fourier side localizing operators

$$\widehat{P_{<N}u}(\xi,\tau) = \begin{cases} \widehat{u}(\xi,\tau) & \text{if } |\xi| \le N \\ 0 & \text{if } |\xi| > N. \end{cases}$$

Fix $\eta \in C_0^\infty(\mathbb{R}^1)$ satisfying $\eta(0) \neq 0$.

Definition. Let $s, b \in \mathbb{R}$ with $s \in [-\frac{1}{2}, 0]$.

$$\|u\|_{Y^{s,b}} = \sup_{t_0 \in \mathbb{R}} \sup_{N \ge 1} \|\eta(t-t_0)\, T_{N^{2s}}(P_{<N}u)\|_{X^{0,b}}.$$

Compare with work of
Bahouri-Chemin,
Koch-Tzvetkov,
Kenig,
Smith-Sogge,
Burq-Gérard-Tzvetkov

$\vdots$

$\vdots$

$\vdots$

Reversing the scaling leads to bound

$$\sup_N \int_{\langle\xi\rangle\sim N} \int_{\tau\in\mathbb{R}} |\widehat{u}(\xi,\tau)|^2 \langle\xi\rangle^{2s} \langle N^{4s}(\tau-\xi^2)\rangle^{2b} \, d\xi \, d\tau \leq C\|u\|^2_{Y^{s,b}}.$$

(The reverse inequality is not valid.)

Since

$$\langle N^{4s}(\tau-\xi^2)\rangle^b \geq N^{4sb}\langle\tau-\xi^2\rangle^b$$

for $s<0$, this implies the still weaker bound

$$\|u\|^2_{X^{r,b}} \leq C\|u\|^2_{Y^{s,b}} \text{ provided } r<(1+4b)s$$

if $s\leq 0$.

Practicum. Define

$$\|u\|^2_{X^{r,b}_M} = \iint_{\mathbb{R}^2} |\widehat{f}(\xi,\tau)|^2 \langle \xi/M\rangle^{2r} \langle \tau - \xi^2\rangle^{2b}\, d\tau\, d\xi.$$

Lemma. Let $s < 0$ and $b \in (\frac{1}{2}, 1)$. Let $r < (1+4b)s$. Then for any $u \in Y^{s,b}$, $N \geq 1$, and $t_0 \in \mathbb{R}$, $v(t,x) = \eta(t-t_0)T_{N^{2s}}f(t,x)$ satisfies

$$\|v\|_{X^{r,b}_{N^{1+2s}}} \leq C\|u\|_{Y^{s,b}}.$$

In proving fundamental bound $\|u_1\overline{u_2}u_3\|_{Y^{s,b-1}} \leq C\prod \|u_n\|_{Y^{s,b}}$
(i) One needs to make an $X^{0,b-1}$ estimate, but essentially only in the region where the "output" frequency satisfies $|\xi| \lesssim 1$.
(ii) Compared to the usual $X^{0,b}$ situation, there is a factor $\langle \xi_j\rangle^{-r}$ accompanying each $\widehat{u_j}$. These are harmful when the "input" frequencies ξ_j are large.

Weak solutions

Given this machinery, existence of weak solutions in $L^\infty(H^s) \cap C^0(H^{s-\varepsilon}) \cap Y^{s,b}$ is straightforward:

• Given initial datum $f \in H^s$, let $H^0 \ni f_n \to f$ in H^s.
Let u_n be corresponding solutions.
• Some subsequence converges to some u
weak-$*$ in $L^\infty(H^s) \cap Y^{s,b}$.
• $|u|^2 u \in Y^{s,b-1} \subset \mathcal{S}'$ is well-defined.
• Claim: For a sub-subsequence,
$|u_n|^2 u_n \to |u|^2 u$ weakly in $Y^{s-\varepsilon,b-1}$.
Proof: Split each u_n into high-and low-frequency parts.
All three factors low-frequency: Arzela-Ascoli.
High-frequency: Passage from $Y^{s,b}$ to $Y^{s-\varepsilon,b}$ gains $|\xi|^{-\varepsilon}$.
• Claim: $u(0,\cdot) \equiv f$.
Proof: For any $N < \infty$, $P_{<N} : Y^{s,b} \to \Lambda_\alpha(H^s)$ is bounded (norm depends on N) for all $\alpha < b - \frac{1}{2}$.
Therefore $u_n \in C^0(H^{s-\varepsilon})$ are *uniformly* continuous.

A further argument seems needed to put u in $C^0(H^s)$, rather than $C^0(H^{s-\varepsilon})$.

• For any initial datum $f \in H^s$, there exists a weight function φ so that $\varphi(\xi)/\langle\xi\rangle^{2s} \to \infty$ as $|\xi| \to \infty$ and $\int |\widehat{f}|^2 \varphi(\xi)\, d\xi < \infty$.

• Almost-conservation analysis applies to modified norms

$$\|f\|_{H_\varphi}^2 = \int |\widehat{f}(\xi)|^2 \varphi(\xi)\, d\xi < \infty.$$

Thus u_n is uniformly bounded in $C^0(H_\varphi)$.

• Thus no need to pass from s to $s-\varepsilon$ in the preceding slide.

Corollary. Let $s > -\frac{1}{12}$.
$\boxed{\text{If}}$ the mapping datum $\mapsto$ solution is continuous from H^s to $C^0(H^r)$ for some $r > -\infty$

then it is continuous from H^s to $C^0(H^s)$.