A PRIORI BOUNDS AND WEAK SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION IN SOBOLEV SPACES OF NEGATIVE ORDER

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Abstract. Solutions to the Cauchy problem for the one-dimensional cubic nonlinear Schrödinger equation on the real line are studied in Sobolev spaces H^s, for s negative but close to 0. For smooth solutions there is an *a priori* upper bound for the H^s norm of the solution, in terms of the H^s norm of the datum, for arbitrarily large data, for sufficiently short time. Weak solutions are constructed for arbitrary initial data in H^s.

1. Introduction

The Cauchy problem for the one-dimensional cubic nonlinear Schrödinger equation is

$$\text{(NLS)} \qquad \begin{cases} iu_t + u_{xx} + \omega|u|^2 u = 0 \\ u(0,x) = u_0(x). \end{cases}$$

Here $u = u(t,x)$ with $(t,x) \in [0,T] \times \mathbb{R}^1$, and $\omega = \pm 1$. As is well known, this Cauchy problem is globally wellposed in H^0 [13]. For all negative s it is illposed in H^s, in the sense that solutions (for smooth initial data) fail to depend *uniformly* continuously on initial data in the H^s norm [9],[5]. Moreover, for $s < -\frac{1}{2}$, there is a stronger form of illposedness: the solution operator fails even to be continuous at 0; there exist smooth solutions with arbitrarily small H^s norms at time 0, yet arbitrarily large H^s norms at time ε, for arbitrarily small $\varepsilon > 0$.

Our first result, concerning smooth (or more precisely, H^0) solutions, implies continuity of the solution map at $u_0 = 0$ in the $C^0(H^s)$ norm for negative s sufficiently close to 0, in contrast with the strong illposedness for $s < -\frac{1}{2}$. It asserts an *a priori* upper bound for the H^s norm of an arbitrary smooth solution, in terms of the H^s norm of its datum.

Theorem 1.1 (A priori bound)**.** *Let $s > -\frac{1}{12}$. Then for all $R < \infty$, there exist $R' < \infty$ and $T > 0$ such that for all $u_0 \in H^0$ satisfying $\|u_0\|_{H^s} < R$, the standard solution u of* (NLS) *with initial datum u_0 satisfies $\max_{t\in[0,T]} \|u(t,\cdot)\|_{H^s} \le R'$.*

For large R, T scales like a certain negative power of R.

By the standard solution we mean the unique solution of (NLS) belonging to the function space $X^{0,b}$ for some $b > \frac{1}{2}$, or equivalently to $C^0(H^0) \cap L^4([0,T] \times \mathbb{R})$. Koch and Tataru [10] have obtained the same result in the larger range $s \ge -\frac{1}{6}$. It remains an open question whether this type of result is valid over a yet larger range.

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Wellposedness of (NLS) has been established by earlier authors in various function spaces which are wider than H^0 [14],[7],[4] and scale like negative order Sobolev spaces, but do not contain H^s for any $s < 0$. We emphasize that those results have a different character than ours; *uniformly* continuous dependence on the initial datum in the norm in question is established in those works, whereas it certainly fails to hold [9],[5] in H^s for $s < 0$.

Our second main result asserts the solvability of the Cauchy problem, in a weak sense, for all initial data in H^s for a range of negative exponents s. The precise statement involves certain function spaces $Y^{s,b}$, which will be specified in Definition 6.1. These are variants of the spaces $X^{s,b}$ commonly employed in connection with this equation. For any $u \in Y^{s,b}$, $|u|^2u$ has a natural interpretation as a distribution for the range of parameters s, b covered by our results, in the sense that when the space-time Fourier transform of $|u|^2u$ is written as an integral expression directly in terms of the space-time Fourier transforms of the factors $u, \bar{u}, u$, the resulting integral is absolutely convergent almost everywhere and defines a tempered locally integrable function; see (7.2). Thus there is a natural notion of a weak solution in $Y^{s,b}$: We say that $u \in Y^{s,b}$ is a weak solution of (NLS) if the equation holds in the sense of distributions, when $|u|^2u$ is interpreted as the inverse Fourier transform of the function defined by this absolutely convergent integral.

Theorem 1.2 (Existence of weak solutions)**.** *Let $s > -\frac{1}{12}$. Then there exists $b > \frac{1}{2}$ such that for each $R < \infty$ there exist $R' < \infty$ and $T > 0$ such that for all $u_0 \in H^s$ satisfying $\|u_0\|_{H^s} < R$, there exists a weak solution $u \in C^0([0,T], H^s) \cap Y^{s,b}$ of* (NLS) *with initial datum u_0 which satisfies $\max_{t\in[0,T]} \|u(t)\|_{H^s} \le R'$.*

$Y^{s,b}$ embeds continuously in $C^0(H^s)$ for $b > \frac{1}{2}$, so the $C^0(H^s)$ part of the conclusion is redundant, and is included only for emphasis.

The solutions guaranteed by this theorem are weak limits of smooth solutions with smooth initial data approximating given H^s data. We do not know whether these solutions are unique, that is, independent of the choice of approximating sequence, let alone whether there exists any $s < 0$ for which the mapping from datum to solution is continuous.

Our analysis does not rely on the complete integrability [1] of (NLS). Our arguments would apply, with essentially no changes, to nonintegrable vector-valued generalizations of the one-dimensional cubic nonlinear Schrödinger equation, provided that those systems obey H^0 norm conservation.

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2. Strategy of the analysis

The strategy is as follows. We begin by using the differential equation to (formally, at least) rewrite the increment $\|u(t)\|_{H^s}^2 - \|u_0\|_{H^s}^2$ as a multilinear expression in terms of the space-time Fourier transform of u. Certain cancellations arise, which have no analogues in the corresponding expression for $u(t,x) - u_0(x)$. This leads to an *a priori* inequality of the form $\big| \|u(t)\|_{H^s}^2 - \|u_0\|_{H^s}^2 \big| \le C\|u\|_{X^{r,b}}^4$, for certain r, s, b with $s < 0$ and $r < s$. It is this initial step which breaks down if u is replaced by the difference of two solutions, preventing us from establishing any continuity of the map $u_0 \mapsto u$.

Thus a bound is required for the $X^{r,b}$ norm, but a loss relative to the $C^0(H^s)$ norm is permitted in the sense that r can be less than s. In §6 we introduce certain function spaces $Y^{s,b}$. Their main relevant properties are:

(1) For $s < 0$, $Y^{s,b}$ embeds in $X^{r,b}$, provided that $r < (1 + 4b)s$.

(2) $Y^{s,b}$ embeds in $C^0_t(H^{s-\varepsilon}_x)$ for all[1] $\varepsilon > 0$, provided that $b > \frac{1}{2}$.
(3) If $u, v, w \in Y^{s,b}$ then $u\bar{v}w \in Y^{s,b-1}$, under certain restrictions on s, b.
(4) If $u \in C^0(H^s)$ and $(i\partial_t - \Delta_x)u \in Y^{s,b-1}$ then $u \in Y^{s,b}$.
(5) For solutions of (NLS), there is an *a priori* bound for the $Y^{s,b}$ norm in terms of the $C^0(H^s)$ norm, of the form $\|u\|_{Y^{s,b}} \le C\|u\|_{C^0(H^s)} + C\|u\|^3_{Y^{s,b}}$, valid under certain restrictions on s, b.

Thus one obtains a coupled system of two inequalities relating $\|u\|_{C^0(H^s)}$ and $\|u\|_{Y^{s,b}}$ to $\|u_0\|_{H^s}$. By restricting attention to a short time T and rescaling, one can reduce matters (for $s > -\frac{1}{2}$) to the case where u_0 has small H^s norm. Via a continuity argument, the coupled system then yields a bound for $\|u\|_{C^0(H^s)} + \|u\|_{Y^{s,b}}$ in terms of $\|u_0\|_{H^s}$.

Weak solutions are obtained as limits of smooth solutions. An *a priori* bound in H^s yields compactness in $H^{s-\varepsilon}$ on bounded spatial regions. It follows readily from the machinery developed below that if smooth solutions u_j with uniformly bounded $Y^{s,b}$ norms converge weakly to u, then $|u_j|^2u_j$ converges weakly to $|u|^2u$ for some subsequence.

An additional argument is needed to place these weak solutions in $C^0(H^s)$, rather than $C^0(H^{s-\varepsilon}) \cap L^\infty(H^s)$. We refine the machinery by replacing the squared H^s norm $\int |\widehat{u}(t,\xi)|^2(1+|\xi|^2)^s\,d\xi$ by $\int |\widehat{u}(t,\xi)|^2\varphi(\xi)\,d\xi$ for weight functions φ adapted to individual initial data, so that $\varphi(\xi) \gg (1+|\xi|^2)^s$ for very large $|\xi|$, and show that control of $\int |\widehat{u_0}(\xi)|^2\varphi\,d\xi$ extends to control of $\int |\widehat{v}(t,\xi)|^2\varphi\,d\xi$ for all solutions v of (NLS) with smooth initial data sufficiently close in H^s norm to u_0. This extra control at high frequencies leads to compactness in $C^0(H^s)$.

3. Bounding the norm

In this section we begin to establish an *a priori* bound for the $C^0(H^s)$ norm of any sufficiently smooth solution of (NLS), in terms of certain other norms. For technical reasons we work with the modified Cauchy problem

$$\text{(NLS*)} \qquad \begin{cases} iu_t + u_{xx} + \zeta_0(t)\omega|u|^2u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where ζ_0 is a smooth real-valued function which is $\equiv 1$ on $[0,T]$, and is supported in $(-2T, 2T)$. Standard proofs of wellposedness in H^0 (or in H^t for $t \ge 0$) apply to this modified equation. One advantage is that u can be extended to a solution defined for all $t \in \mathbb{R}$.

We will study $\zeta_1(t)u(t,x)$, where ζ_1 is another real-valued smooth cutoff function supported in $(-2T, 2T)$ which satisfies $\zeta_1\zeta_0 \equiv \zeta_0$. Because the equation is simply the linear Schrödinger equation outside the support of ζ_0, a $C^0(H^s)$ bound holds for $\zeta(t)u(t,x)$ for one real-valued cutoff function in $\zeta \in C^\infty_0(-2T, 2T)$ satisfying $\zeta\zeta_0 \equiv \zeta_0$ if and only if such a bound holds for every such function.

Recall [2] the function space $X^{s,b}$, which is defined to be the set of all space-time distributions u whose spacetime Fourier transform $\widehat{u}$ is such that

$$\|u\|^2_{X^{s,b}} := \iint_{\mathbb{R}^2} |\widehat{u}(\xi,\tau)|^2 \langle \tau - \xi^2 \rangle^{2b} \langle \xi \rangle^{2s}\,d\xi\,d\tau < \infty$$

where $\langle x \rangle := (1+|x|^2)^{1/2}$.

[1] We find it convenient to work with Besov-like spaces $Y^{s,b}$ rather than Sobolev-like versions. Their Besov character accounts for the infinitesimal loss of derivatives in the embedding into $C^0(Y^s)$.

One of the two principal inequalities underlying our theorems is as follows. The second is formulated in Proposition 8.1.

Proposition 3.1. *Let $T_0 < \infty$, $T \in [0, T_0]$, $s \in (-\frac{1}{2}, 0)$, $b \in (\frac{1}{2}, 1)$. There exists $C < \infty$ such that for any sufficiently smooth solution*[2] *u of* (NLS*) *with initial datum u_0,*

$$\left| \|u\|^2_{C^0([-2T,2T],H^s)} - \|u_0\|^2_{H^s} \right| \leq C \, \|\zeta_1 u\|^4_{X^{r,b}} \tag{3.1}$$

provided that

$$r > -\tfrac{1}{4} \text{ and } b > \tfrac{1}{2}. \tag{3.2}$$

For $s > -\frac{1}{4}$, the right-hand side involves a norm which is weaker, in terms of the number of spatial derivatives involved, than the $C^0(H^s)$ norm. The proof of this result is begun below and completed in §5, using some of the inequalities established in §4.

We will work with both spatial Fourier coefficients

$$\widehat{u}(t,\xi) := \int_{\mathbb{R}} e^{-ix\xi} u(t,x)\, dx \tag{3.3}$$

and space-time Fourier coefficients[3]

$$\widehat{u}(\xi,\tau) := \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} u(t,x)\, dx\, dt; \tag{3.4}$$

it will be clear from context and from the names of the variables which of these two is meant in any particular instance. The differential equation (NLS*) is expressed in terms of spatial Fourier coefficients as

$$\frac{d}{dt}\widehat{u}(t,\xi) = -i\xi^2 \widehat{u}(t,\xi) + i\omega\zeta_0(t) \int_{\xi_1-\xi_2+\xi_3=\xi} \widehat{u}(t,\xi_1)\overline{\widehat{u}(t,\xi_2)}\widehat{u}(t,\xi_3)\, d\lambda_\xi \tag{3.5}$$

where λ_ξ is appropriately normalized Lebesgue measure on $\{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : \xi_1 - \xi_2 + \xi_3 = \xi\}$.

Consider any sufficiently regular solution u of (NLS*). Let $\varphi : \mathbb{R} \to [0,\infty)$ and define the modified mass

$$\Phi_\varphi(t) = \Phi_\varphi(t,u) := \int_{\mathbb{R}} |\widehat{u}(t,\xi)|^2 \varphi(\xi)\, d\xi. \tag{3.6}$$

We will be primarily interested in $\varphi(\xi) = \langle \xi \rangle^{2s}$, but more general weights will be needed to establish the full conclusion of Theorem 1.2.

A short calculation establishes the "almost conservation law"

$$\frac{d\Phi}{dt} = \mathrm{Re}\,(c\omega\mathcal{I})$$

for Φ, where c is an absolute constant, $\mathcal{I}$ is the multilinear integral

$$\mathcal{I}(t) = \mathcal{I}_\varphi(u,t) := \zeta_0(t) \int_\Xi \widehat{u}(t,\xi_1)\overline{\widehat{u}(t,\xi_2)}\widehat{u}(t,\xi_3)\overline{\widehat{u}(t,\xi_4)}\, \psi(\vec{\xi}\,)\, d\lambda(\vec{\xi}\,), \tag{3.7}$$

$\vec{\xi} = (\xi_1, \cdots, \xi_4) \in \mathbb{R}^4$ is a multi-frequency, $\Xi \subset \mathbb{R}^4$ is the hyperplane

$$\Xi := \{\vec{\xi} : \xi_1 - \xi_2 + \xi_3 - \xi_4 = 0\}, \tag{3.8}$$

[2]For instance, $u_0 \in H^{10}$ would suffice.

[3]The order of the variables is reversed in our space-time transform; $u(t,x)$ is transformed to $\widehat{u}(\xi,\tau)$ where ξ, τ are dual to x, t respectively.

λ is appropriately normalized Lebesgue measure on Ξ, and

$$\psi(\vec{\xi}\,) := \varphi(\xi_1) - \varphi(\xi_2) + \varphi(\xi_3) - \varphi(\xi_4). \tag{3.9}$$

Thus[4] $|\Phi(t) - \Phi(0)| \lesssim |\int_0^t \mathcal{I}(r)\,dr|$.

Introduce also

$$\sigma(\xi_1, \cdots, \xi_4) := \xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2. \tag{3.10}$$

σ has the useful alternative expressions

$$\sigma(\vec{\xi}\,) = 2(\xi_1 - \xi_2)(\xi_1 - \xi_4) = -2(\xi_1 - \xi_2)(\xi_3 - \xi_2) = -2(\xi_1 - \xi_4)(\xi_3 - \xi_4) \quad \forall \vec{\xi} \in \Xi. \tag{3.11}$$

We have the following basic cancellation bound (cf. [6]):

Lemma 3.2 (Double mean value theorem)**.** *Let $\vec{\xi} = (\xi_1, \cdots, \xi_4) \in \Xi \subset \mathbb{R}^4$. If $\varphi \in C^2$ and all ξ_j belong to a common interval I then $|\psi(\vec{\xi}\,)| \le |\sigma(\vec{\xi}\,)| \max_{y \in I} |\varphi''(y)|$.*

Proof. $\varphi(\xi_2) - \varphi(\xi_1) = (\xi_2 - \xi_1)\int_0^1 \varphi'(\xi_1 + t(\xi_2 - \xi_1))\,dt$. Writing the corresponding expression for $\varphi(\xi_4) - \varphi(\xi_3)$, and noting that $(\xi_2 - \xi_1) = -(\xi_4 - \xi_3)$ since $\vec{\xi} \in \Xi$, gives

$$\begin{aligned}\psi(\xi) &= (\xi_2 - \xi_1)\int_0^1 [\varphi'(\xi_1 + t(\xi_2 - \xi_1)) - \varphi'(\xi_4 + t(\xi_3 - \xi_4))]\,dt \\ &= (\xi_2 - \xi_1)(\xi_1 - \xi_4)\iint_{[0,1]^2} \varphi''(\xi_1 + t(\xi_2 - \xi_1) + s(\xi_4 - \xi_1)\,ds\,dt.\end{aligned}$$

□

In order to control the contribution made by the region not close to the diagonal, express each factor $\widehat{u}(t, \xi)$ in the integral as the inverse Fourier transform of its Fourier transform with respect to t, to obtain for all $t \in [-2T, 2T]$

$$\Big|\int_0^t \mathcal{I}_\varphi(u, r)\,dr\Big| \le C\int_\Xi \int_{\mathbb{R}^4} \prod_{j=1}^4 |\widehat{u}(\xi_j, \tau_j)| \langle \tau_1 - \tau_2 + \tau_3 - \tau_4 \rangle^{-1} |\psi(\vec{\xi}\,)|\,d\vec{\tau}\,d\lambda(\vec{\xi}\,) \tag{3.12}$$

where C depends on T and $\vec{\tau} = (\tau_1, \cdots, \tau_4)$. The notation $\widehat{u}$ denotes here the Fourier transform with respect to both spatial and temporal variables.

Write

$$|\widehat{u}(\xi_j, \tau_j)| =: \langle \xi_j \rangle^{-r} \langle \tau_j - \xi_j^2 \rangle^{-b} g_j(\xi_j, \tau_j). \tag{3.13}$$

Then $\|g_j\|_{L^2(\mathbb{R})} = \|u\|_{X^{r,b}}$. The right-hand side of (3.12) becomes

$$\int_{\mathbb{R}^4}\int_\Xi \prod_{n=1}^4 \Big(g_n(\xi_n, \tau_n)\langle \xi_n \rangle^{-r} \langle \tau_n - \xi_n^2 \rangle^{-b}\Big) |\psi(\vec{\xi}\,)|\,d\lambda(\vec{\xi}\,) \langle \tau_1 - \tau_2 + \tau_3 - \tau_4 \rangle^{-1}\,d\vec{\tau}. \tag{3.14}$$

In §5 we will complete the proof of Proposition 3.1 by showing that for $\varphi(\xi) = \langle \xi \rangle^{2s}$, the integral (3.14) is majorized by $C\prod_{n=1}^4 \|g_n\|_{L^2(\mathbb{R}^2)}$ provided that s, r, b satisfy the hypotheses of the proposition.

[4]As usual, we use $X \lesssim Y$ to denote an estimate of the form $X \le CY$ for some constant C, depending only on the exponents r, s and b which will appear later in this paper.

4. Trilinear inequalities of Strichartz type

A prototypical inequality of Strichartz type says that for $h \in L^2(\mathbb{R})$, the solution u of the linear Schrödinger equation with initial datum h belongs to $L^6(\mathbb{R}^2)$. Therefore any three such solutions satisfy $u_1\bar{u}_2u_3 \in L^2$. Rewritten on the Fourier side by means of the Plancherel identity, this becomes

$$|\int f(\xi_1 - \xi_2 + \xi_3, \xi_1^2 - \xi_2^2 + \xi_3^2) \prod_{n=1}^{3} g_n(\xi_n)\, d\xi_n| \lesssim \prod_{n=1}^{3} \|g_n\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R}^2)}. \tag{4.1}$$

One version of the bilinear Strichartz inequality, expressed directly in terms of Fourier variables, states that for any subset $E \subset \mathbb{R}^2$,

$$\Big|\int_{\mathbb{R}^2} f(\xi_1 \pm \xi_2, \xi_1^2 \pm \xi_2^2) h_1(\xi_1) h_2(\xi_2) \chi_E(\xi_1, \xi_2)\, d\xi_1\, d\xi_2\Big| \tag{4.2}$$
$$\lesssim \big(\min_{(\xi_1,\xi_2)\in E} |\xi_1 - \xi_2|\big)^{-1/2} \|f\|_{L^2(\mathbb{R}^2)} \|h_1\|_{L^2(\mathbb{R}^1)} \|h_2\|_{L^2(\mathbb{R}^1)},$$

where the two $\pm$ signs are either both $+$, or both $-$; this represents the pairing of f with a bilinear operator applied to h_1, h_2. This is implicit in Carleson and Sjölin [3], and is a direct consequence of Cauchy-Schwarz via the substitution $(\xi_1, \xi_2) \mapsto (\xi_1 \pm \xi_2, \xi_1^2 \pm \xi_2^2)$. Its advantage, in practice, is that it provides a superior bound when $|\xi_1 - \xi_2|$ is large.

In this section we establish certain versions of the trilinear inequality (4.1) which incorporate improvements similar to the factor $|\xi_1 - \xi_2|^{-1/2}$ in (4.2). These arise naturally in the analysis of the Fourier transform of a threefold product $u\bar{v}w$ of functions in spaces $X^{r,b}$ or $Y^{s,b}$.

4.1. Statements of inequalities.

Proposition 4.1. *Consider*

$$\int_{\vec{\xi} \in S \subset \Xi} \int_{\tau_1 - \tau_2 + \tau_3 - \tau_4 = 0} \prod_{n=1}^{4} g_n(\xi_n, \tau_n) \langle \tau_n - \xi_n^2 \rangle^{-\beta_n} \chi_E(L(\vec{\xi}))\, d\lambda(\vec{\tau})\, d\lambda(\vec{\xi}) \tag{4.3}$$

where each $g_n \geq 0$, $i, j \in \{1,2,3,4\}$ are distinct, $E \subset \mathbb{R}^1$ is any measurable set, and $L : \mathbb{R}^4 \to \mathbb{R}$ is a linear transformation. Suppose that
- *$\beta_n > \frac{1}{2}$ for all but at most one index n, and $\beta_n > 0$ for all n.*
- *i, j have opposite parity.*
- *L belongs neither to the linear span of $\{\xi_i, \xi_j, \xi_1 - \xi_2 + \xi_3 - \xi_4\}$, nor to the linear span of $\{\xi_k, \xi_l, \xi_1 - \xi_2 + \xi_3 - \xi_4\}$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.*

Then there exists $C < \infty$ depending on L such that (4.3) *is majorized by*

$$C|E|^{1/2} \max_{\vec{\xi} \in S} \big(|\xi_i - \xi_j|^{-1/2}\big) \cdot \max_{\vec{\xi} \in S} \big(\langle \sigma(\vec{\xi}) \rangle^{-\beta}\big) \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)} \tag{4.4}$$

where $\beta = \min_n \beta_n$.

In our application, L will take the form $L(\vec{\xi}) = \xi_\mu - \xi_\nu$ for some $\mu \neq \nu$. If $\{\mu, \nu\}$ equals neither $\{i, j\}$ nor $\{1, 2, 3, 4\} \setminus \{i, j\}$ then L satisfies the hypothesis.

A variant of this inequality applies to other linear transformations L:

Proposition 4.2. *Consider* (4.3) *with $L(\vec{\xi}) = \xi_k$ for some $k \notin \{i, j\}$. Suppose again that $\beta_n > \frac{1}{2}$ for all but at most one index n, and $\beta_n > 0$ for all n. Suppose that $|E| \lesssim$*

$\min_{\vec{\xi}\in S} |\xi_i - \xi_j|$. *Let* $\beta := \min_n \beta_n$. *Then there exists* $C < \infty$ *such that* (4.3) *is majorized by*

$$|E|^{1/4} \max_{\vec{\xi}\in S} \left(|\xi_i - \xi_j|^{-1/4}\right) \cdot \max_{\vec{\xi}\in S} \left(\langle\sigma(\vec{\xi})\rangle^{-\beta}\right) \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)}. \tag{4.5}$$

Remark 4.1 (Trilinear Knapp example)**.** (4.5) is (in practice) weaker than (4.4), because $|E|/\min_S |\xi_i - \xi_j|$ is raised only to the power $\frac{1}{4}$, rather than $\frac{1}{2}$ as in Proposition 4.1. We discuss here the simplified expression

$$\int_{\mathbb{R}^3} G(\xi_1 + \xi_3 - \xi_4, \xi_1^2 + \xi_3^2 - \xi_4^2) \prod_{n\neq 2} h_n(\xi_n)\ \chi_{|\xi_4|\leq 1}\chi_S(\vec{\xi})\, d\xi_1\, d\xi_3\, d\xi_4, \tag{4.6}$$

which arises in the proof of Proposition 4.2 (see the case $\nu = 2$). The example can be adapted to the situation of the Proposition. The analogue of (4.5) for (4.6) is the bound $\min_S |\xi_1 - \xi_2|^{-1/4}\|G\|_{L^2} \prod_{n\neq 2} \|h_n\|_{L^2}$, which is established in the proof of (4.5) below. We show now that the exponent $\frac{1}{4}$ cannot be improved in this bound for (4.6).

Define h_4 to be the characteristic function of the interval $[0, 1]$, h_1 to be the characteristic function of $[N, N + N^{1/2}]$, and h_3 to be the characteristic function of $[N + N^{1/2}, N + 2N^{1/2}]$. Define S to be the set of all $\vec{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)$ for which $h_1(\xi_1)h_3(\xi_3)h_4(\xi_4) \neq 0$; ξ_2 is always regarded as a function of (ξ_1, ξ_3, ξ_4) via the relation $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0$. Define $G(x, y)$ to be the characteristic function of the set of all $(x, y) \in \mathbb{R}^2$ satisfying $|x - 2N| \leq 3N^{1/2}$ and $|y+2N^2-N-2Nx| \leq 4N$. Then a short calculation shows that $G(\xi_1+\xi_3-\xi_4, \xi_1^2+\xi_3^2-\xi_4^2) \equiv 1$ for all $\vec{\xi} \in S$, and consequently the integral (4.6) is simply $\prod_{n\neq 2} \int_{\mathbb{R}^1} h_n = N^{1/2}\cdot N^{1/2}\cdot 1 = N$.

On the other hand, $\|G\|_{L^2} = CN^{3/4}$, while $\|h_n\|_{L^2} = N^{1/4}$ for $n = 1, 3$ and $= 1$ for $n = 4$. Thus the product of the four L^2 norms has order of magnitude $N^{5/4}$, and consequently the ratio of (4.6) to the product of norms has order of magnitude $N/N^{5/4} = N^{-1/4}$. Since $\xi_2 - \xi_1 = \xi_3 - \xi_4$ has order of magnitude N for all $\vec{\xi} \in S$, this is the ratio claimed. □

The analysis of (3.14) is a bit more complicated because the relation $\tau_1 - \tau_2 + \tau_3 - \tau_4 = 0$ is replaced by the slowly decaying factor $\langle\tau_1 - \tau_2 + \tau_3 - \tau_4\rangle^{-1}$. It requires a third variant:

Proposition 4.3. *Consider*

$$\int_{\vec{\xi}\in S\subset\Xi} \int_{\vec{\tau}\in\mathbb{R}^4} \langle\tau_1 - \tau_2 + \tau_3 - \tau_4\rangle^{-1} \prod_{n=1}^{4} g_n(\xi_n, \tau_n)\langle\tau_n - \xi_n^2\rangle^{-\beta_n} \phi(\vec{\xi})\chi_E(L(\vec{\xi}))\, d\lambda(\vec{\xi})\, d\vec{\tau} \tag{4.7}$$

where $g_n \geq 0$, $\phi \geq 0$, *and* $\beta_n > \frac{1}{2}$ *for all* n. *Let* $i \neq j \in \{1, 2, 3, 4\}$ *and let* $L : \mathbb{R}^4 \to \mathbb{R}$ *be a linear functional satisfying the hypotheses of Proposition 4.1. Then* (4.7) *is majorized by*

$$\begin{aligned} C_\beta \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)}|E|^{1/2} \cdot \Big[&\max_{\vec{\xi}\in S} \left(\phi(\vec{\xi})\langle\sigma(\vec{\xi})\rangle^{-1}\right) \cdot \max_{\vec{\xi}\in S} |\vec{\xi}|^{1/2} \\ &+ \max_{\vec{\xi}\in S} \phi(\vec{\xi}) \cdot \max_{\vec{\xi}\in S} \left(|\xi_i - \xi_j|^{-1/2}\right) \cdot \max_{\vec{\xi}\in S} \left(\langle\sigma(\vec{\xi})\rangle^{-\beta}\right)\Big] \end{aligned} \tag{4.8}$$

for any $\beta < \min_n \beta_n$.

While (4.7) is formally similar to (4.3), a significant contribution to (4.7) can arise from a region in Fourier space which has no analogue in (4.3). This region contributes an additional term in (4.8). In our application, ϕ will be $|\psi|$.

The factor $|\vec{\xi}|^{1/2}$ in (4.8) can be replaced by $\max_{\vec{\xi} \in S} |\tilde{L}(\vec{\xi})|^{1/2}$ for any linear functional $\tilde{L}$ such that $\{\tilde{L}, L, \xi_1 - \xi_2 + \xi_3 - \xi_4\}$ is linearly independent.

4.2. **Proofs of inequalities.** The essence of Propositions 4.1, 4.2, and 4.3 lies in the following two simpler inequalities.

Lemma 4.4. *Let i, j, k be the three elements of $\{1, 2, 3\}$, written in any order. Let $\ell : \mathbb{R}^3 \mapsto \mathbb{R}^1$ satisfy $\partial\ell/\partial\xi_k \neq 0$. Then for any nonnegative measurable functions G, g_n of two and one real variables respectively, and for any measurable sets $E \subset \mathbb{R}^1$ and $S \subset \Xi$, the quantity*

$$\int_{\mathbb{R}^3} \prod_{n=1}^{3} g_n(\xi_n) G(\xi_1 - \xi_2 + \xi_3, \xi_1^2 - \xi_2^2 + \xi_3^2) \chi_S(\vec{\xi}) \chi_E(\ell(\vec{\xi}))\, d\xi_1\, d\xi_2\, d\xi_3 \tag{4.9}$$

is majorized by

$$\lesssim \|G\|_{L^2} \prod_{n=1}^{3} \|g_n\|_{L^2} |E|^{1/2} (\min_{\vec{\xi} \in S} |\xi_i - \xi_j|)^{-1/2} \tag{4.10}$$

where the implied constant depends on ℓ.

Proof. Consider the case where $\{i, j\} = \{1, 2\}$. Apply Cauchy-Schwarz to majorize by

$$\Big(\int_{\vec{\xi} \in S} G^2(\xi_1 - \xi_2 + \xi_3, \xi_1^2 - \xi_2^2 + \xi_3^2) g_3^2(\xi_3)\, d(\xi_1, \xi_2, \xi_3) \Big)^{1/2} \Big(\int_{\mathbb{R}^3} g_1^2(\xi_1) g_2^2(\xi_2) \chi_E(\ell(\vec{\xi}))\, d(\xi_1, \xi_2, \xi_3) \Big)^{1/2}. \tag{4.11}$$

The left-hand factor is majorized by $\lesssim \|G\|_{L^2} \|g_3\|_{L^2} (\min_S |\xi_1 - \xi_2|)^{-1/2}$; this is seen by first fixing ξ_3 and integrating with respect to (ξ_1, ξ_2), making the change of variables $(\xi_1, \xi_2) \mapsto (\xi_1 - \xi_2, \xi_1^2 - \xi_2^2)$.

To analyze the right-hand factor, first integrate with respect to ξ_3, obtaining a bound of

$$\lesssim |E|^{1/2} (\int g_1^2(\xi_1) g_2^2(\xi_2)\, d\xi_1\, d\xi_2)^{1/2} \tag{4.12}$$

since $\partial\ell/\partial\xi_3 \neq 0$. Then integrate with respect to (ξ_1, ξ_2). Multiplying these bounds for the two factors yields $\lesssim \|G\|_{L^2} \prod_{n=1}^{3} \|g_n\|_{L^2} |E|^{1/2} (\min_S |\xi_1 - \xi_2|)^{-1/2}$.

The same reasoning applies for other $\{i, j\}$; in all cases $|\xi_i - \xi_j|$ arises, rather than $|\xi_i + \xi_j|$. □

Lemma 4.5. *For $m = 1, 2$ let $L_m : \mathbb{R}^4 \to \mathbb{R}$ be linear functionals such that $\{\xi_1 - \xi_2 + \xi_3 - \xi_4, L_1, L_2\}$ is linearly independent. Then all nonnegative measurable functions $g_n \in L^2(\mathbb{R}^1)$ and all measurable sets $E_m \subset \mathbb{R}^1$,*

$$\int_\Xi \prod_{n=1}^{4} g_n(\xi_n) \prod_{m=1}^{2} \chi_{E_m}(L_m(\vec{\xi}))\, d\lambda(\vec{\xi}) \lesssim \prod_{m=1}^{2} |E_m|^{1/2} \prod_{n=1}^{4} \|g_n\|_{L^2}. \tag{4.13}$$

Proof. Consider the multlinear form $T(g_1, \cdots, g_6) := \int_\Xi \prod_{n=1}^{4} g_n(\xi_n) g_5(L_1(\vec{\xi})) g_6(L_2(\vec{\xi}))\, d\lambda(\vec{\xi})$. By Cauchy-Schwarz,

$$|T(g_1, \cdots, g_6)| \leq \Big(\int_\Xi \prod_{n=1}^{3} |g_n(\xi_n)|^2\, d\lambda(\vec{\xi}) \Big)^{1/2} \Big(\int_\Xi |g_4(\xi_4)|^2 |g_5(L_1(\vec{\xi}))|^2 |g_6(L_2(\vec{\xi}))|^2\, d\lambda(\vec{\xi}) \Big)^{1/2}.$$

The first factor is a constant multiple of $\prod_{j=1}^{3}\|g_j\|_2$. The assumption that $\{\xi_1-\xi_2+\xi_3-\xi_4, L_1, L_2\}$ is linearly independent implies that the second factor is likewise proportional to $\prod_{j=4}^{6}\|g_j\|_2$. □

Proof of Proposition 4.1. Consider the quantity (4.3) given in the statement of the proposition. Define $A=\min_{\vec{\xi}\in S}\langle\sigma(\vec{\xi})\rangle\ge 1$. Introduce $\rho_n=\tau_n-\xi_n^2$. Since $\rho_1-\rho_2+\rho_3-\rho_4=\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2=\sigma(\vec{\xi})$, we have $\langle\rho_n\rangle\gtrsim\langle\sigma(\vec{\xi})\rangle$ for some n.

Partition the region of integration into four subregions, according to the index n for which $|\rho_n|$ is largest. By symmetry, it suffices to prove the stated bound for one of these subregions. Let $\nu\in\{1,2,3,4\}$ be arbitrary, and consider the subregion consisting of all $\vec{\xi}$ satisfying $|\rho_\nu(\vec{\xi})|=\max_n|\rho_n(\vec{\xi})|$. Partition further into subregions, in each of which $\langle\rho_\nu\rangle\sim 2^\kappa A$ for some nonnegative integer κ, and consider the contribution of any one of these subregions.

Suppose first that $\nu\notin\{i,j\}$. Denote by μ the remaining index, so that $\{1,2,3,4\}=\{i,j,\mu,\nu\}$. The contribution of the subregion under examination is

$$\lesssim\int_{(\rho_i,\rho_j,\rho_\mu)\in\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}g_\nu(\pm\xi_\mu\pm\xi_i\pm\xi_j,\pm\xi_\mu^2\pm\xi_i^2\pm\xi_j^2\pm\rho_\mu\pm\rho_i\pm\rho_j)\prod_{n\neq\nu}h_n(\xi_n,\rho_n)\\ \langle\rho_\nu\rangle^{-\beta_\nu}\chi_S(\vec{\xi})\chi_E(L(\vec{\xi}))\,d\xi_i\,d\xi_j\,d\xi_\mu\Big)\prod_{n\neq\nu}\langle\rho_n\rangle^{-\beta_n}\,d(\rho_i,\rho_j,\rho_\mu) \tag{4.14}$$

where the $\pm$ sign preceding ξ_n^2 agrees with the sign preceding ξ_n for each $n\in\{\mu,i,j\}$, and where the outer integral extends only over those (ρ_i,ρ_j,ρ_μ) satisfying $|\rho_n|\le|\rho_\nu|$ for all n and $\langle\rho_\nu\rangle\sim 2^\kappa A$. Here $h_n(\xi_n,\rho_n)=g_n(\xi_n,\tau_n)=g_n(\xi_n,\rho_n+\xi_n^2)$, and consequently $\|h_n\|_{L^2}=\|g_n\|_{L^2}$.

Fix (ρ_i,ρ_j,ρ_μ). The linear transformation $\Xi\in\vec{\xi}\mapsto(\xi_i,\xi_j,\xi_\mu)\in\mathbb{R}^3$ is invertible, so there is a unique linear functional $\tilde{L}:\mathbb{R}^3\mapsto\mathbb{R}$ satisfying $\tilde{L}(\xi_i,\xi_j,\xi_\mu)=L(\vec{\xi})$. The hypothesis on L ensures that $\partial\tilde{L}/\partial\xi_\mu\neq 0$. The inner integral thus takes the form discussed in Lemma 4.4, and is consequently majorized by

$$\|g_\nu\|_{L^2(\mathbb{R}^2)}\prod_{n\neq\nu}\|g_n(\cdot,\rho_n)\|_{L^2(\mathbb{R}^1)}|E|^{1/2}\sup_S(|\xi_i-\xi_j|^{-1/2})(2^\kappa A)^{-\beta_\nu} \tag{4.15}$$

since $\langle\rho_\nu\rangle\gtrsim 2^\kappa A$.

It remains to bound $\prod_{n\neq\nu}\int_{\langle\rho_n\rangle\lesssim 2^\kappa A}\|g_n(\cdot,\rho_n)\|_{L^2(\mathbb{R}^1)}\langle\rho_n\rangle^{-\beta_n}\,d\rho_n$. If $\beta_n>\frac12$ then

$$\int_{\mathbb{R}}\|g_n(\cdot,\rho_n)\|_{L^2(\mathbb{R}^1)}\langle\rho_n\rangle^{-\beta_n}\,d\rho_n\lesssim\|g_n\|_{L^2(\mathbb{R}^2)} \tag{4.16}$$

by Cauchy-Schwarz. If $\beta_n>\frac12$ for all $n\neq\nu$ then, since A was defined to be $\min_S\langle\sigma\rangle$, the desired bound is obtained from (4.15) by summation over all integers $\kappa\ge 0$.

Otherwise there remains exactly one index $m\neq\nu$ such that $\beta_m\le\frac12$. Then $\beta=\beta_m$, and $\beta_\nu\ge\beta_m$. Since $2^\kappa A\sim\langle\rho_\nu\rangle\gtrsim\langle\rho_m\rangle$ throughout the region of integration, one has

$$(2^\kappa A)^{-\beta_\nu}\lesssim(2^\kappa A)^{-\beta_m}\langle\rho_m\rangle^{\beta_m-\beta_\nu}.$$

This factor of $\langle\rho_m\rangle^{\beta_m-\beta_\nu}$, multiplied by the factor of $\langle\rho_m\rangle^{-\beta_m}$ already present in the integral, becomes $\langle\rho_m\rangle^{-\beta_\nu}$. Since $\beta_\nu>\frac12$, the analysis can be completed as above, yielding a bound of

$$\prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)} |E|^{1/2} \sup_S (|\xi_i - \xi_j|^{-1/2})(2^\kappa A)^{-\beta_m}.$$

The desired bound again follows by summation over κ, in the case where $\nu \notin \{i, j\}$.

Suppose finally that $\nu \in \{i, j\}$; by symmetry, we may suppose that $\nu = i$. If we write $\{1, 2, 3, 4\} = \{i, j, k, l\}$, then the equation for Ξ, together with the hypothesis that i, j have opposite parity, imply that $|\xi_i - \xi_j| \equiv |\xi_k - \xi_l|$ for all $\vec{\xi} \in \Sigma$. Thus $\min_S |\xi_i - \xi_j| = \min_S |\xi_k - \xi_l|$, so $\{i, j\}$ can be interchanged with $\{k, l\}$. Denote by μ the remaining index, so that $\{1, 2, 3, 4\} = \{\mu, \nu, k, l\}$. The hypothesis on L is formulated so as to be unaffected when $\{i, j\}$ is interchanged with $\{k, l\}$. Therefore the above reasoning again applies. □

Proof of Proposition 4.2. (3.14) is invariant under the permutations $(1, 2, 3, 4) \mapsto (2, 1, 4, 3)$, $(1, 2, 3, 4) \mapsto (3, 2, 1, 4)$, $(1, 2, 3, 4) \mapsto (1, 4, 3, 2)$, and consequently also $(1, 2, 3, 4) \mapsto (3, 4, 1, 2)$ of the indices. Therefore it is no loss of generality to assume that $i = 1$, $j = 2$, and $k = 4$.

We follow the proof of Proposition 4.1. In the case when $\nu \notin \{1, 2\}$, because $L(\vec{\xi}) = \xi_4$ does not belong to the span of the three linear transformations ξ_1, ξ_2, and $\xi_1 - \xi_2 + \xi_3 - \xi_4$, that proof applies without alteration and yields the upper bound (4.4). Since $|E|/\min_S |\xi_1 - \xi_2| \lesssim 1$ by hypothesis, (4.4) is majorized by a constant multiple of the desired bound (4.5).

Consider next the case where $\nu = 2$. Then because $\xi_1 - \xi_2 \equiv -(\xi_3 - \xi_4)$, Lemma 4.4 can be applied with the roles of the indices $2, 4$ interchanged to obtain a bound

$$\lesssim |E|^{1/2} \max_{\vec{\xi} \in S} \left(|\xi_1 - \xi_3|^{-1/2} \cdot \max_{\vec{\xi} \in S} \langle \sigma(\vec{\xi}) \rangle^{-\beta}\right) \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)}. \tag{4.17}$$

Another bound is also available. Apply Proposition 4.1 with L replaced by $\tilde{L}(\vec{\xi}) = \xi_1 - \xi_3$; $\tilde{L}$ belongs to neither the span of $\{\xi_1, \xi_2, \xi_1 - \xi_2 + \xi_3 - \xi_4\}$ nor the span of $\{\xi_3, \xi_4, \xi_1 - \xi_2 + \xi_3 - \xi_4\}$, so the hypotheses are satisfied. This yields an alternative bound

$$\max_{\vec{\xi} \in S} |\xi_1 - \xi_3|^{1/2} \cdot \max_{\vec{\xi} \in S} \left(|\xi_1 - \xi_2|)^{-1/2}\right) \cdot \max_{\vec{\xi} \in S} \left(\langle \sigma(\vec{\xi}) \rangle^{-\beta}\right) \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)}. \tag{4.18}$$

If $\max_S |\xi_1 - \xi_3|$ is comparable to $\min_S |\xi_1 - \xi_3|$, then taking the geometric mean of these two upper bounds yields the desired bound (4.5). Decomposing S into subsets S_κ in which $|\xi_1 - \xi_3|$ is comparable to 2^κ for arbitrary $\kappa \in \mathbb{Z}$, invoking whichever of (4.17), (4.18) is more favorable for each κ, and summing over κ yields the same bound in the general case.

Finally, when $\nu = 1$, apply Lemma 4.4 with the roles of the indices $1, 4$ interchanged, and repeat the above discussion for the case $\nu = 2$, replacing $\xi_1 - \xi_3$ by $\xi_2 - \xi_3$ throughout. The reasoning is otherwise unchanged. □

Proof of Proposition 4.3. Substitute $\tau_n = \rho_n + \xi_n^2$ and $g_n(\xi_n, \tau_n) = \tilde{g}_n(\xi_n, \rho_n)$ for all $n \in \{1, 2, 3, 4\}$ to transform the integral into

$$\int_{\mathbb{R}^4} \int_{S \subset \Xi} \langle \rho_1 - \rho_2 + \rho_3 - \rho_4 + \sigma(\vec{\xi}) \rangle^{-1} \prod_{n=1}^{4} \tilde{g}_n(\xi_n, \rho_n) \langle \rho_n \rangle^{-\beta_n} \phi(\vec{\xi}) \chi_E(L(\vec{\xi}))\, d\lambda(\vec{\xi})\, d\vec{\rho}, \tag{4.19}$$

where $\tilde{g}_n$ has the same L^2 norm as g_n.

Begin with the region where $|\rho_n| \le \frac{1}{8}\langle\sigma(\vec{\xi})\rangle$ for all $n \in \{1,2,3,4\}$, which has no counterpart in Proposition 4.1. Its contribution is comparable to

$$\int_{S\subset\Xi}\int_{\mathbb{R}^4}\prod_{n=1}^{4}\tilde{g}_n(\xi_n,\rho_n)\langle\rho_n\rangle^{-\beta_n}\langle\sigma(\vec{\xi})\rangle^{-1}\phi(\vec{\xi})\chi_E(L(\vec{\xi}))\,d\vec{\rho}\,d\lambda(\vec{\xi}). \tag{4.20}$$

Since all β_n are assumed to be strictly $> \frac{1}{2}$, applying Cauchy-Schwarz to the integral with respect to each variable ρ_n gives an upper bound

$$\lesssim \int_{S\subset\Xi}\prod_{n=1}^{4}h_n(\xi_n)\langle\sigma(\vec{\xi})\rangle^{-1}\phi(\vec{\xi})\chi_E(L(\vec{\xi}))\,d\lambda(\vec{\xi}) \tag{4.21}$$

where $h_n(\xi_n) = \|\tilde{g}_n(\xi_n,\cdot)\|_{L^2(\mathbb{R}^1)} = \|g_n(\xi_n,\cdot)\|_{L^2(\mathbb{R}^1)}$.

According to Lemma 4.5, (4.21) is

$$\lesssim |E|^{1/2}\prod_n \|g_n\|_{L^2(\mathbb{R}^2)}\max_S\Big(\phi(\vec{\xi})\langle\sigma(\vec{\xi})\rangle^{-1}|\vec{\xi}|^{1/2}\Big), \tag{4.22}$$

since the linear functional L does not vanish identically on Ξ.

Consider next the region where $\max_n\langle\rho_n\rangle \ge \frac{1}{8}\langle\sigma(\vec{\xi})\rangle$. By symmetry, it is no loss of generality to restrict attention to the region where $|\rho_4| = \max_n|\rho_n|$. An upper bound for the integral over this region is

$$\int_{\mathbb{R}^3}\int_{S\subset\Xi}\Big(\int_{|\rho_4|\ge\max_{j\le3}|\rho_j|}\langle\rho_1-\rho_2+\rho_3-\rho_4+\sigma(\vec{\xi})\rangle^{-1}g_4(\xi_4,\rho_4)\,\langle\rho_4\rangle^{-\beta_4}d\rho_4\Big)$$
$$\phi(\vec{\xi})\chi_E(L(\vec{\xi}))\,\prod_{n=1}^{3}\big(g_n(\xi_n,\rho_n)\langle\rho_n\rangle^{-\beta_n}\big)\,d\lambda(\vec{\xi})\,\prod_{n=1}^{3}d\rho_n. \tag{4.23}$$

Consider the contribution made to (4.23) by the subregion in which $\langle\rho_4\rangle$ is comparable to an arbitrary constant $\Lambda \ge 2$. Since $\langle\rho_4\rangle = \max_n\langle\rho_n\rangle \gtrsim \langle\sigma(\vec{\xi})\rangle$, necessarily $\Lambda \gtrsim \langle\sigma(\vec{\xi})\rangle$, and thus $\langle\rho_1-\rho_2+\rho_3-\rho_4+\sigma(\vec{\xi})\rangle \lesssim \Lambda$. Therefore the innermost integral in (4.23) is majorized by the convolution of $\Lambda^{-\beta_4}g(\xi_4,\cdot)$ with $\langle\rho_4\rangle^{-1}\cdot\chi_{[-C\Lambda,C\Lambda]}(\rho_4)$, evaluated at $\rho_1-\rho_2+\rho_3+\sigma(\vec{\xi})$.

Since $\|\langle\rho_4\rangle^{-1}\cdot\chi_{[-C\Lambda,C\Lambda]}(\rho_4)\|_{L^1(\mathbb{R})} \lesssim \log\Lambda$, the contribution of the region $\langle\rho_4\rangle \sim \Lambda$ to (4.23) is majorized by

$$C\Lambda^{-\beta_4}\log\Lambda\int_{\mathbb{R}^3}\int_{\langle\sigma(\vec{\xi})\rangle\lesssim\Lambda}\chi_S(\vec{\xi})G_4(\xi_4,\rho_1-\rho_2+\rho_3+\sigma(\vec{\xi}))$$
$$\prod_{n=1}^{3}g_n(\xi_n,\rho_n)\phi(\vec{\xi})\chi_E(L(\vec{\xi}))\,d\lambda(\vec{\xi})\,\prod_{n=1}^{3}\langle\rho_n\rangle^{-\beta_n}d\rho_n \tag{4.24}$$

where $\|G_4\|_{L^2} \le C\|g_4\|_{L^2}$. $G_4(\xi_4,\rho_1-\rho_2+\rho_3+\sigma(\vec{\xi}))$ can be reexpressed as $\tilde{G}_4(\xi_4,\rho_1-\rho_2+\rho_3+\xi_1^2-\xi_2^2+\xi_3^2)$ where $\|\tilde{G}_4\|_{L^2} = \|G_4\|_{L^2}$. Summing over dyadic values of Λ yields for (4.23) the upper bound

$$C_\beta\int_{\mathbb{R}^3}\int_{S\subset\Xi}G_4(\xi_4,\rho_1-\rho_2+\rho_3+\sigma(\vec{\xi}))$$
$$\prod_{n=1}^{3}g_n(\xi_n,\rho_n)\phi(\vec{\xi})\langle\sigma(\vec{\xi})\rangle^{-\beta}\chi_E(L(\vec{\xi}))\,d\lambda(\vec{\xi})\,\prod_{n=1}^{3}\langle\rho_n\rangle^{-\beta_n}d\rho_n \tag{4.25}$$

for any $\beta < \beta_4$.

This is nearly identical to the expression (4.14) reached in the proof of Proposition 4.1, with the factor $\langle\rho_\nu\rangle^{-\beta_\nu}$ in (4.14) now replaced by $C_\beta\langle\sigma(\vec{\xi}\,)\rangle^{-\beta}$. It suffices to repeat the analysis above of (4.14), with the simplification that here all β_m are $> \frac{1}{2}$. □

5. Conclusion of the proof of Proposition 3.1

Lemma 5.1. *Suppose that $s < 0$, $r > -\frac{1}{4}$, and $b > \frac{1}{2}$. Let $\psi(\vec{\xi}\,) := \sum_{n=1}^{4}(-1)^n\langle\xi_n\rangle^{2s}$. Then*

$$\int_{\mathbb{R}^4}\int_{\Xi}\prod_{n=1}^{4}\Big(g_n(\xi_n,\tau_n)\langle\xi_n\rangle^{-r}\langle\tau_n-\xi_n^2\rangle^{-b}\Big)|\psi(\vec{\xi}\,)|\,d\lambda(\vec{\xi}\,)\,d\vec{\tau} \lesssim \prod_{n=1}^{4}\|g_n\|_{L^2(\mathbb{R}^2)}. \tag{5.1}$$

Proof. It is no loss of generality to assume throughout the proof that $\|g_n\|_{L^2} = 1$ for all n. We analyze the integral (3.14) using Proposition 4.3, with $\phi \equiv \psi$. Recall the symmetries discussed in the proof of Proposition 4.2. These will be used to reduce the number of cases that must be discussed in the proof.

Let $N \geq 1$, and consider the contribution to the integral made by the subregion S_N of integration in which all $\langle\xi_j\rangle$ are comparable to N. Because of the symmetries listed above, we may restrict attention to the region where $|\xi_1-\xi_2| \leq |\xi_1-\xi_4|$. Let $S_{N,A,B}$ be the subregion where $|\xi_1-\xi_2| \sim AN$ and $|\xi_1-\xi_4| \sim B$, for arbitrary $0 < A \leq B \lesssim 1$. We majorize the contribution of $S_{N,A,B} = S$ via the bound given by Proposition 4.3, with $L(\vec{\xi}\,) = \xi_1-\xi_2$ and $E = [-CAN, CAN]$. Since $|\psi(\vec{\xi}\,)| \lesssim N^{2s-2}|\sigma(\vec{\xi}\,)| \lesssim N^{2s}AB$, this yields the sum of the following two quantities:

$$C|E|^{1/2}\max_S|\psi(\vec{\xi}\,)|\max_S(\langle\sigma(\vec{\xi}\,)\rangle^{-1})\max_S|\vec{\xi}\,|^{1/2}$$
$$\lesssim (AN)^{1/2}N^{2s}AB\langle ABN^2\rangle^{-1}N^{1/2} \leq A^{1/2}N^{2s-1}$$

and

$$C|E|^{1/2}\max_S(|\xi_1-\xi_4|^{-1/2})\max_S|\psi(\vec{\xi}\,)|\max_S(\langle\sigma(\vec{\xi}\,)\rangle^{-\beta})$$
$$\lesssim (AN)^{1/2}(BN)^{-1/2}(N^{2s}AB)\langle ABN^2\rangle^{-\beta} \leq AN^{2s-1}$$

since $\beta \geq \frac{1}{2}$.

Summing over dyadic values of $A \leq B \lesssim 1$ gives a total bound of $\lesssim N^{2s-1}$ for the contribution of S_N. Taking the factors $\langle\xi_n\rangle^{-r}$ into account yields a net bound of $\lesssim N^{2s-4r-1}$ for the contribution of S_N to (3.14). Provided that $-r < \frac{1}{4} - \frac{1}{2}s$, this is $\lesssim N^{-\delta}$ for some $\delta > 0$ and hence we can sum over dyadic values of $N \geq 1$ to majorize the contribution of the entire region on which all four quantities $\langle\xi_n\rangle$ are mutually comparable. Since $s < 0$, this is a less stringent condition on r than the hypothesis $-r < \frac{1}{4}$. □

The relation $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0$ defining Ξ implies that the largest two of the four quantities $|\xi_n|$ must remain uniformly comparable. Consider next the contribution of a region of integration in which some two variables ξ_n with indices n of opposite parity are large, and at least one of the other two variables is comparatively small. Because of symmetries, it is then no loss of generality to restrict attention to the region where $|\xi_1|, |\xi_2| \sim N_2$, $\langle\xi_3\rangle \sim N_1$, and $\langle\xi_4\rangle \sim N_0$, where the parameters $N_0, N_1, N_2 \geq 1$ satisfy $N_0 \leq N_1 \leq N_2$.

In the subcase in which $N_0 \sim N_1$, consider the subregion S_Δ where $|\xi_4-\xi_3| \equiv |\xi_1-\xi_2|$ has some fixed order of magnitude Δ; necessarily $\Delta \lesssim N_1$. There $|\sigma(\vec{\xi}\,)| = |\xi_1-\xi_4|\cdot|\xi_1-\xi_2| \sim$

ΔN_2, and

$$|\psi(\vec{\xi})| \le |\varphi(\xi_3) - \varphi(\xi_4)| + |\varphi(\xi_1) - \varphi(\xi_2)| \lesssim N_0^{2s-1}\Delta + N_2^{2s-1}\Delta \lesssim N_0^{2s-1}\Delta \tag{5.2}$$

since $|\xi_1 - \xi_2| = |\xi_3 - \xi_4|$.

Apply Proposition 4.3 with $L(\vec{\xi}) = \xi_4 - \xi_3$ and $\phi(\vec{\xi}) = |\psi(\vec{\xi})|$ to the contribution made by the region of integration S_Δ to (3.14).

$$\begin{aligned} &\max_{S_\Delta} |L(\vec{\xi})|^{1/2} \max_{S_\Delta} |\psi(\vec{\xi})| \max_{S_\Delta}(\langle \sigma(\vec{\xi})\rangle^{-1}) \max_{S_\Delta} |\vec{\xi}|^{1/2} \\ &\qquad \lesssim \Delta^{1/2} \cdot N_0^{2s-1}\Delta \cdot \langle \Delta N_2\rangle^{-1} N_2^{1/2} \\ &\qquad \lesssim \Delta^{1/2} N_0^{2s-1} N_2^{-1/2} = (\Delta/N_0)^{1/2} N_0^{2s-\frac{1}{2}} N_2^{-1/2}, \end{aligned} \tag{5.3}$$

while

$$\begin{aligned} &\max_{S_\Delta} |L(\vec{\xi})|^{1/2} \max_{S_\Delta} |\xi_1 - \xi_4|^{-1/2} \max_{S_\Delta} |\psi(\vec{\xi})| \max_{S_\Delta}(\langle \sigma(\vec{\xi})\rangle^{-\beta}) \\ &\qquad \lesssim \Delta^{1/2} \cdot N_2^{-1/2} \cdot N_0^{2s-1}\Delta \cdot \langle \Delta N_2\rangle^{-1/2} \lesssim (\Delta/N_0) N_0^{2s} N_2^{-1}. \end{aligned} \tag{5.4}$$

Since $\Delta \lesssim N_0 \lesssim N_2$, the maximum of these two maxima is $\lesssim (\Delta/N_0)^{1/2} N_0^{2s-\frac{1}{2}} N_2^{-1/2}$. Incorporating the factors $\langle \xi_n\rangle^{-r}$ from (3.14) introduces an additional factor of $N_0^{-2r} N_2^{-2r}$, leaving a net bound

$$\lesssim (\Delta/N_0)^{1/2} N_0^{2s-\frac{1}{2}-2r} N_2^{-\frac{1}{2}-2r}.$$

Summing over dyadic values of $\Delta \lesssim N_0$ yields a bound

$$\lesssim N_0^{2s-\frac{1}{2}-2r} N_2^{-\frac{1}{2}-2r}$$

for the original region. This quantity is $\lesssim N_2^{-\delta}$ for some $\delta > 0$ if (and only if) $-r < \frac{1}{4}$. We may then sum over dyadic $N_0 \lesssim N_2$, then over all dyadic N_2. □

If on the other hand $N_0 \le \frac{1}{10} N_1$ then $\Delta \sim N_1$ and $|\psi(\vec{\xi})| \lesssim N_0^{-2s}$, so

$$\max |L(\vec{\xi})|^{1/2} \max |\psi(\vec{\xi})| \max(\langle \sigma(\vec{\xi})\rangle^{-1}) \max |\vec{\xi}|^{1/2} \lesssim N_1^{1/2} \cdot N_0^{2s} \cdot \langle N_1 N_2\rangle^{-1} N_2^{1/2},$$

giving a net bound of $N_0^{2s-r} N_1^{-\frac{1}{2}-r} N_2^{-\frac{1}{2}-2r}$, which again is $\lesssim N_2^{-\delta}$ for some $\delta > 0$ if and only if $-r < \frac{1}{4}$. Likewise

$$\begin{aligned} &\max |L(\vec{\xi})|^{1/2} \max |\xi_1 - \xi_4|^{-1/2} \max |\psi(\vec{\xi})| \max(\langle \sigma(\vec{\xi})\rangle^{-\beta}) \\ &\qquad \lesssim N_1^{1/2} \cdot N_2^{-1/2} \cdot N_0^{2s} \cdot (N_1 N_2)^{-1/2} \lesssim N_0^{2s} N_1^{0} N_2^{-1}, \end{aligned} \tag{5.5}$$

leading once again to the less stringent requirement $-r < \frac{1}{4} - \frac{1}{2}s$. □

Because the roles of the four variables ξ_n are not completely symmetric, it is necessary to analyze separately the subcase in which again $N_0 \le \frac{1}{10} N_1 \le \frac{1}{10} N_2$, but $\langle \xi_4\rangle \sim N_0$, $\langle \xi_2\rangle \sim N_1$, and $|\xi_1|, |\xi_3| \sim N_2$. Thus $|\sigma(\vec{\xi})|$ is at least as large as in the above analysis. Since it was raised to negative powers above, this new situation is more favorable. Therefore the hypothesis $-r < \frac{1}{4}$ again suffices. □

When the various symmetries between the indices $\{1, 2, 3, 4\}$ are taken into account, the above discussion exhausts all possible cases, and the proof is complete. □

Proof of Proposition 3.1. It suffices to bound $\|u(t,\cdot)\|_{H^s}$ for t in the support of ζ_0, since $\mathcal{I}(t) \equiv 0$ for other t. For such t, $u(t,x) \equiv \zeta_1(t)u(t,x)$ and hence $\widehat{u}$ can be replaced by $\widehat{\zeta_1(t)u}$ throughout the above discussion. Thus $\|u\|_{X^{r,b}}$ can be replaced by $\|\zeta_1 u\|_{X^{r,b}}$ on the right-hand side of the inequality. □

6. $Y^{s,b}$ NORMS

The purpose of this section is to introduce certain function spaces $Y^{s,b}$, variants of the spaces $X^{s,b}$ employed by Bourgain [2] and then Kenig, Ponce, and Vega [8] to establish wellposedness of the nonlinear Schrödinger and Korteweg-de Vries equations. An *a priori* bound for $|u|^2u$ in these spaces, in terms of u, will be proved in the following section.

Proposition 3.1 asserts an *a priori* upper bound for a solution in $C^0(H^s)$ in terms of an $X^{r,b}$ bound. Rather than establishing an $X^{r,b}$ bound directly, we will work with $Y^{s,b}$. Whereas the usual argument establishing an *a priori* $X^{0,b}$ bound for a solution breaks down for $X^{s,b}$ for s strictly negative, it continues to apply for $Y^{s,b}$ when an upper bound in $C^0(H^s)$ is known. $Y^{s,b}$ strictly contains $X^{s,b}$, but embeds in $X^{r,b}$ for certain $r < s$; see Lemma 6.2.

Define the scaling operator

$$T_\lambda u(t,x) := \lambda u(\lambda^2 t, \lambda x); \tag{6.1}$$

T_λ acts on distributions u defined on $\mathbb{R}^2$. It maps any solution of the cubic nonlinear Schrödinger equation to another solution. We use the same notation for functions of x alone: $T_\lambda f(x) := \lambda f(\lambda x)$.

Define also the (rough) Littlewood-Paley projections

$$\widehat{P_{<N}u}(\xi,\tau) := \begin{cases} \widehat{u}(\xi,\tau) & \text{if } |\xi| \le N \\ 0 & \text{if } |\xi| > N. \end{cases} \tag{6.2}$$

We say that a function f is *M-band-limited* if $\widehat{f}(\xi,\tau) = 0$ whenever $|\xi| > M$.

Fix an infinitely differentiable, compactly supported cutoff function $\eta \in C_0^\infty(\mathbb{R}^1)$ satisfying $\eta(0) \neq 0$.

Definition 6.1 ($Y^{s,b}$ norm)**.** Let $s, b \in \mathbb{R}$ with $s \in [-\frac{1}{2}, 0]$. For any tempered distribution u defined on $\mathbb{R}^2$ whose space-time Fourier transform $\widehat{u}(\xi,\tau)$ belongs to $L^2_{\text{loc}}(\mathbb{R}^2)$,

$$\|u\|_{Y^{s,b}} := \sup_{t_0 \in \mathbb{R}} \sup_{N \ge 1} \|\eta(t-t_0) T_{N^{2s}}(P_{<N}u)\|_{X^{0,b}}\,. \tag{6.3}$$

It would be slightly more natural to form an ℓ^2 norm over a dyadic sequence of values of N, rather than a supremum, but the definition used here is a bit simpler to work with, and is sufficient for our purpose. Observe that if f is N-band-limited, then $T_{N^{2s}}P_{<N}f$ is N^{1+2s}-band-limited.

For functions f supported in any fixed bounded interval with respect to time t,

$$\sup_N \int_{\langle\xi\rangle \sim N} \int_{\tau \in \mathbb{R}} |\widehat{f}(\xi,\tau)|^2 \langle\xi\rangle^{2s} \langle N^{4s}(\tau - \xi^2)\rangle^{2b} \, d\xi \, d\tau \lesssim \|f\|^2_{Y^{s,b}}, \tag{6.4}$$

although the reverse inequality does not hold;[5] this inequality can be derived as in the proof of Lemma 6.2 below. Because s is negative and b positive, the factor $\langle N^{4s}(\tau-\xi^2)\rangle^{2b}$ is

[5] For $r = (1+4b)s$ and $|\xi|$ of some fixed order of magnitude $N \ge 1$, the left-hand side of (6.4) is equivalent to the $X^{r,b}$ norm squared in the region where $|\tau - \xi^2| \gtrsim N^{-4s}$; it becomes larger as $|\tau-\xi^2|$ becomes smaller than this threshold.

weaker than the corresponding factor $\langle\tau-\xi^2\rangle^{2b}$ that appears in the $X^{s,b}$ norm. Thus $X^{s,b}$ embeds continuously in $Y^{s,b}$.

Our first lemma is a simple consequence of the definition; the proof is omitted.

Lemma 6.1 (Insensitivity to smooth cutoffs)**.** *(i) If $h:\mathbb{R}\to\mathbb{C}$ is compactly supported and infinitely differentiable then $\|hu\|_{Y^{s,b}}\lesssim\|u\|_{Y^{s,b}}$ for all $u\in Y^{s,b}$.*

(ii) Changing the cutoff function η in the definition of $Y^{s,b}$ leads to an equivalent norm, provided that $\eta\in C^\infty$ is compactly supported, and not identically zero.

Remark 6.1. For $s<0$, the spaces $Y^{s,b}$ are natural from the point of view of the extant H^0 theory. If an initial datum u_0 for (NLS) is N-band-limited in the sense that $\widehat{u_0}(\xi)$ is supported where $|\xi|\sim N$, and if $\|u_0\|_{H^s}\sim 1$, then $u_0\in H^0$, but with large norm $\|u_0\|_{H^0}\sim N^{-s}$. Hence the Cauchy problem with initial datum u_0 has a solution belonging to $X^{0,b}$. This does not follow from the usual fixed point argument, since u_0 may be quite large in H^0. Instead one can partition the interval $[0,t]$ into sufficiently short subintervals that a fixed point argument applies on each, and invoke H^0 norm conservation.

An equivalent way to do the first time step is to solve the Cauchy problem for unit time with rescaled initial datum $T_{\lambda_N}u_0$, where $\lambda_N=N^{2s}$, then to reverse the scaling. The exponent is chosen so that $\|T_{\lambda_N}u_0\|_{H^0}\lesssim 1$ uniformly in $N\ge 1$. Successive time steps are done in the same way.

The next simple lemma makes possible the conversion of bounds in $Y^{s,b}$ to the more standard spaces $X^{r,b}$.

Lemma 6.2 (Y controls X)**.** *Let $s<0$ and $b\ge 0$. For any $A<\infty$ and any $r<(1+4b)s$ and all Schwartz class functions $f(t,x)$ supported where $|t|\le A$, we have*

$$\|f\|_{X^{r,b}}\lesssim\|f\|_{Y^{s,b}}\,. \tag{6.5}$$

The converse inequality is not true; in the region where $|\tau-\xi^2|\ll\langle\xi\rangle^{-4s}$, the $Y^{s,b}$ norm is stronger than the $X^{r,b}$ norm even for $r=(1+4b)s$. We make this conversion both for the sake of conceptual simplicity, and because it simplifies certain calculations later on.

While Lemma 6.2 is needed to control $d\Phi/dt$, a variant will be used in establishing the $Y^{s,b}$ norm bound. For any real number $M\ge 1$ define the $X^{r,b}_M$ and $\hat X^{r,b}_M$ norms by

$$\|f\|^2_{X^{r,b}_M}:=\iint_{\mathbb{R}^2}|\widehat{f}(\xi,\tau)|^2\langle\xi/M\rangle^{2r}\langle\tau-\xi^2\rangle^{2b}\,d\tau\,d\xi.$$

$$\|g\|^2_{\hat X^{r,b}_M}:=\iint_{\mathbb{R}^2}|g(\xi,\tau)|^2\langle\xi/M\rangle^{2r}\langle\tau-\xi^2\rangle^{2b}\,d\tau\,d\xi.$$

Likewise define

$$\|g\|^2_{\hat X^{r,b}}:=\iint_{\mathbb{R}^2}|g(\xi,\tau)|^2\langle\xi\rangle^{2r}\langle\tau-\xi^2\rangle^{2b}\,d\tau\,d\xi.$$

Lemma 6.3 (Y controls X, refined)**.** *Let $s<0$, $b\in(\frac12,1)$, and suppose that $\eta\in C^\infty(\mathbb{R})$ has compact support. Let $r<(1+4b)s$. Then there exists $C<\infty$ such that for any $f\in Y^{s,b}$, any $N\ge 1$, and any $t_0\in\mathbb{R}$, the function $g(t,x)=\eta(t-t_0)T_{N^{2s}}f(t,x)$ belongs to $X^{r,b}_{N^{1+2s}}$ with bound*

$$\|g\|_{X^{r,b}_{N^{1+2s}}}\le C\,\|f\|_{Y^{s,b}}\,. \tag{6.6}$$

The constant C can be taken to depend only on s,b,r,η.

Choose any smooth, compactly supported function η such that $\sum_{j\in\mathbb{Z}} \eta(t-j) \equiv 1$ for all $t \in \mathbb{R}$.

Lemma 6.4 (Almost-orthogonality)**.** *Let $b \in \mathbb{R}$. Let g be any Schwartz function, and define $g_j = \eta(t-j)g$ so that $g = \sum_{j\in\mathbb{Z}} g_j$. Then the summands g_j are almost orthogonal in $X^{0,b}$ norm, in the sense that*

$$\|g\|_{X^{0,b}} \leq C\Big(\sum_j \|g_j\|^2_{X^{0,b}}\Big)^{1/2} \tag{6.7}$$

where $C < \infty$ depends only on b, η.

Proof. Introduce the spatial Fourier transform $\mathcal{F}g(t,\xi) = \int_{\mathbb{R}} g(t,x)e^{-ix\xi}\,dx$. Let $J(t)$ be the distribution in $\mathcal{S}'(\mathbb{R}^1)$ whose Fourier transform is $\langle\tau\rangle^b$. Then J may be decomposed as $J = J_0 + J_\infty$ where J_0 is compactly supported and J_∞ belongs to the Schwartz class.

Now

$$\|g\|_{X^{0,b}} = \Big\|\mathcal{F}g * (e^{i\xi^2 t}J(t))\Big\|_{L^2} \tag{6.8}$$

where $*$ denotes convolution, taken with respect to the t variable alone for each fixed value of ξ. Since J_∞ is a Schwartz function,

$$\Big\|\mathcal{F}g * (e^{i\xi^2 t}J_\infty(t))\Big\|_{L^2} \lesssim (\sum_j \|g_j\|^2_{L^2})^{1/2}, \tag{6.9}$$

and since $b \geq 0$, $\|g_j\|_{L^2} \lesssim \|g_j\|_{X^{0,b}}$.

There exists a finite constant C_0, depending only on η and on the support of J_0, such that no point (t,x) belongs to the support of g_j for more than C_0 integers j. Because the cutoff functions $\eta(t-j)$ are independent of x, the same bounded overlap property holds for their spatial Fourier transforms $\mathcal{F}g_j(t,\xi)$. Because J_0 has compact support, it follows that likewise no point (t,ξ) belongs to the support of $\mathcal{F}g_j * (e^{i\xi^2 t}J_0(t))$ for more than C_0 integers j.

Therefore

$$\begin{aligned}\Big\|\mathcal{F}g * (e^{i\xi^2 t}J_0(t))\Big\|^2_{L^2} &\lesssim \sum_j \Big\|\mathcal{F}g_j * (e^{i\xi^2 t}J_0(t))\Big\|^2_{L^2}\\ &\lesssim \sum_j \iint |\widehat{g_j}(\tau,\xi)|^2 |\widehat{J_0}(\tau-\xi^2)|^2\,d\tau\,d\xi\\ &\lesssim \sum_j \|g_j\|^2_{X^{0,b}}\end{aligned}$$

since $|\widehat{J_0}| = |\widehat{J} - \widehat{J_\infty}| \leq |\widehat{J}| + C \lesssim \langle\tau\rangle^b + C \leq \langle\tau\rangle^b$ since $b \geq 0$. □

Proof of Lemma 6.2. Let f be given. Let $r := (1+4b)s$. It suffices to show that for all $N \geq 1$,

$$\int_{\langle\xi\rangle\sim N}\int_{\tau\in\mathbb{R}} |\widehat{f}(\xi,\tau)|^2 \langle\tau-\xi^2\rangle^{2b}\langle\xi\rangle^{2r}\,d\xi\,d\tau \lesssim \|f\|^2_{Y^{s,b}}, \tag{6.10}$$

since summing over all $N = 1, 2, 4, 8, \ldots$ then yields the required bound for all r strictly less than $(1+4b)s$.

Define $g_j := \eta(t-j)\cdot T_{N^{2s}}P_{<N}f$, and $g := \sum_{j\in\mathbb{Z}} g_j$, as in Lemma 6.4. All but at most CN^{-4s} terms in this decomposition vanish identically, because of the hypothesis restricting

the support of f with respect to t. Moreover $\widehat{f}(\xi,\tau) = N^{4s}\widehat{g}(N^{2s}\xi, N^{4s}\tau)$. Consequently a trivial majorization of the ℓ^2 outer norm in (6.7) gives

$$\|g\|_{X^{0,b}} \lesssim N^{-2s} \max_j \|g_j\|_{X^{0,b}} \lesssim N^{-2s} \|f\|_{Y^{s,b}} . \tag{6.11}$$

Now (since $1+2s>0$)

$$\begin{aligned}
&\int_{\langle\xi\rangle\sim N}\int_{\tau\in\mathbb{R}} |\widehat{f}(\xi,\tau)|^2\langle\tau-\xi^2\rangle^{2b}\langle\xi\rangle^{2r}\,d\xi\,d\tau \\
&\qquad = N^{8s}\int_{\langle\xi\rangle\sim N}\int_{\tau\in\mathbb{R}} |\widehat{g}(N^{2s}\xi, N^{4s}\tau)|^2\langle\tau-\xi^2\rangle^{2b}\langle\xi\rangle^{2r}\,d\xi\,d\tau \\
&\qquad = N^{2s}\int_{\langle\xi\rangle\sim N^{1+2s}}\int_{\tau\in\mathbb{R}} |\widehat{g}(\xi,\tau)|^2\langle N^{-4s}(\tau-\xi^2)\rangle^{2b}\langle N^{-2s}\xi\rangle^{2r}\,d\xi\,d\tau \\
&\qquad \sim N^{2s}\int_{\langle\xi\rangle\sim N^{1+2s}}\int_{\tau\in\mathbb{R}} |\widehat{g}(\xi,\tau)|^2\langle N^{-4s}(\tau-\xi^2)\rangle^{2b}N^{2r}\,d\xi\,d\tau \\
&\qquad \lesssim N^{2s-8bs+2r}\int_{\langle\xi\rangle\sim N^{1+2s}}\int_{\tau\in\mathbb{R}} |\widehat{g}(\xi,\tau)|^2\langle\tau-\xi^2\rangle^{2b}\,d\xi\,d\tau \\
&\qquad \le N^{2s-8bs+2r}\|g\|^2_{X^{0,b}} . \\
&\qquad \lesssim N^{-2s-8bs+2r}\|f\|^2_{Y^{s,b}}
\end{aligned}$$

by (6.11). This is $\lesssim \|f\|^2_{Y^{s,b}}$ under the hypothesis that $r \le (1+4b)s$. □

Proof of Lemma 6.3. This argument is nearly identical to the proof of Lemma 6.2, except that additional parameters are involved.

Let $f \in Y^{s,b}$ be arbitrary. Let $g(t,x) = \eta(t-t_0)T_{N^{2s}}f(t,x)$ and $M = N^{1+2s}$. Consider $\int_{|\xi/M|\sim\Lambda}\int_{\tau\in\mathbb{R}} |\widehat{g}(\xi,\tau)|^2\langle\xi/M\rangle^{2r}\langle\tau-\xi^2\rangle^{2b}\,d\tau\,d\xi$ for arbitrary $\Lambda\ge 1$. The contribution of the region $|\xi|\le N^{1+2s}$ to this integral is controlled directly by $\|f\|^2_{Y^{s,b}}$, and hence requires no further discussion.

Since $\widehat{T_{N^{2s}}f}(\xi,\tau) = \widehat{f}(N^{-2s}\xi, N^{-4s}\tau)$, the above integral can be reexpressed in terms of $\widehat{f}(\tilde\xi,\tilde\tau)$ with $|\tilde\xi|\sim\Lambda M N^{-2s} = \Lambda N$, thus in terms of $F = \eta(t-t_0)T_{N^{2s}}(P_{<C\Lambda N}f)$. This function F can be naturally decomposed as $F=\sum_j F_j$ where each function $F_j(t,x)$ is supported where $t\in I_j$, each $I_j\subset\mathbb{R}$ is an interval of length Λ^{4s}, no point of $\mathbb{R}$ belongs to more than 2 intervals I_j, the sum extends over at most $C\Lambda^{-4s}$ indices j, and F_j satisfies

$$\iint |\widehat{F_j}(\xi,\tau)|^2\langle\Lambda^{4s}(\tau-\xi^2)\rangle^{2b}\Lambda^{-4s}\,d\tau\,d\xi \le C\Lambda^{-2s}\|f\|^2_{Y^{s,b}}. \tag{6.12}$$

This decomposition is obtained by a smooth partition of unity in the t variable, which decomposes the portion of $f(t,x)$ with Fourier transform (with respect to x) supported where $|\xi|\le C\Lambda N$ into summands which (as functions of t) are supported on intervals of lengths $(\Lambda N)^{4s}$. The $Y^{s,b}$ norm directly gives a bound for each summand, and substitution via the dilations $T_{N^{2s}}$ yields (6.12).

By dilating time by a factor of $|\Lambda|^{-4s}$, invoking Lemma 6.4, and reversing the dilation, we conclude that $F=\sum_j F_j$ satisfies

$$\iint |\widehat{F}(\xi,\tau)|^2\langle\Lambda^{4s}(\tau-\xi^2)\rangle^{2b}\Lambda^{-4s}\,d\tau\,d\xi \lesssim \Lambda^{-2s}\cdot\Lambda^{-4s}\|f\|^2_{Y^{s,b}}; \tag{6.13}$$

whereas an application of the triangle inequality would yield a factor of Λ^{-8s} on the right-hand side, the orthogonality expressed by Lemma 6.4 saves a factor of Λ^{4s}. Since $\langle\Lambda^{4s}(\tau-$

$\xi^2)\rangle \gtrsim \Lambda^{4s}\langle\tau-\xi^2\rangle$, it follows that

$$\iint |\widehat{F}(\xi,\tau)|^2\langle\tau-\xi^2\rangle^{2b}\,d\tau\,d\xi \lesssim \Lambda^{-2s(1+4b)}\|f\|^2_{Y^{s,b}}. \tag{6.14}$$

Since $-2s(1+4b) \le -2r$, this yields the desired bound for the contribution made by F to g, that is, the contribution of the region where $|\xi/M| \sim \Lambda$. Since $-2s(1+4b)$ is strictly less than $-2r$, summation over dyadic values of $\Lambda \ge 1$ completes the proof. □

Proposition 3.1 together with the embedding of $Y^{s,b}$ in $X^{r,b}$ established in Lemma 6.2 yield

Proposition 6.5. *Let $T_0 < \infty$, $T \in [0,T_0]$, $s \in (-\frac{1}{2},0)$, $b \in (\frac{1}{2},1)$. For any sufficiently smooth solution u of* (NLS*) *with initial datum u_0,*

$$\|u\|^2_{C^0([-2T,2T],H^s)} \le \|u_0\|^2_{H^s} + C\,\|\zeta_1 u\|^4_{Y^{s,b}} \tag{6.15}$$

provided that $s<0$, $b>\frac{1}{2}$, and $-s<\frac{1}{4}(1+4b)^{-1}$.

To use this bound we of course need to control the $Y^{s,b}$ norm of u. This will be accomplished in the next two sections.

7. Bound for $|u|^2u$

The objective of this section is to prove the following nonlinear estimate.

Proposition 7.1 (Trilinear estimate in $Y^{s,b}$)**.** *Suppose that $s > -\frac{2}{15}$ and $b \in (\frac{1}{2},1)$ satisfy*

$$-s < (1+4b)^{-1}\min\Big(\tfrac{1}{10}+\tfrac{3}{5}(1-b),\tfrac{1}{12}+\tfrac{2}{3}(1-b)\Big). \tag{7.1}$$

Then for any $u,v,w \in Y^{s,b}$,

$$\|u\bar{v}w\|_{Y^{s,b-1}} \lesssim \|u\|_{Y^{s,b}}\|v\|_{Y^{s,b}}\|w\|_{Y^{s,b}}. \tag{7.2}$$

The product $u\bar{v}w$, by virtue of having a locally integrable space-time Fourier transform, consequently has a natural interpretation as a distribution.

(7.2) is a variant of a well-known inequality in which $Y^{s,c}$ is replaced by $X^{0,c}$ throughout. Here there is a tradeoff: Once the parameter N in the definition of $Y^{s,b-1}$ is fixed, no bound is asserted for $\widehat{u\bar{v}w}(\xi,\tau)$ for $|\xi| \gg N$, but u,v,w are allowed to lie in spaces of mildly negative order.

The right-hand side of (7.1) equals $\frac{2}{15}$ when $b=\frac{1}{2}$. Thus for any $s > -\frac{2}{15}$ there does exist $b \in (\frac{1}{2},1)$ satisfying (7.1).

Proof. The definition of the $Y^{s,b}$ norm involves a supremum over $N \ge 1$; fix N. Set $M := N^{1+2s}$. Choose r very slightly less than $(1+4b)s$, and recall the $X^{r,b}_{N^{1+2s}}$ bound formulated in Lemma 6.3.

Pair the space-time Fourier transform of $u\bar{v}w$ with $\langle\tau-\xi^2\rangle^{b-1}g_4(\xi,\tau)$ where $g_4 \in L^2(\mathbb{R}^2)$. Substitute for the Fourier transforms of u,v,w as in (3.13). Matters then reduce to showing that

$$\int_{\vec{\xi}\in\Xi}\int_{\vec{\tau}\in\Xi}\prod_{n=1}^{4}\Big(g_n(\xi_n,\tau_n)\langle\xi_n/M\rangle^{-r}\langle\tau_n-\xi_n^2\rangle^{-\beta_n}\Big)\chi_{S_0}(\vec{\xi}\,)\,d\lambda(\vec{\tau})\,d\lambda(\vec{\xi}\,) \lesssim \prod_{n=1}^{4}\|g_n\|_{L^2(\mathbb{R}^n)}$$

uniformly for all $M \ge 1$, where $\beta_n := b$ for $n \le 3$ and $\beta_4 := 1-b$, and $S_0 := \{\vec{\xi} : |\xi_4| \lesssim M\}$. Assume with no loss of generality that $\|g_n\|_{L^2(\mathbb{R}^2)} = 1$ for all indices n.

An important special case arises when all $|\xi_n|$ are $\lesssim M = N^{1+2s}$. For this subregion, the desired inequality is nothing more than the well-known $X^{0,b-1}$ bound for $|u|^2u$ in terms of $\|u\|^3_{X^{0,b}}$ (see e.g. [11]).

Consider next the contribution to the integral of the region where $|\xi_n| \sim AM$ for all $n \neq 4$ for some single $A \gg 1$. For all such $\vec{\xi}$, $|\sigma(\vec{\xi})| \sim (AM)^2$, so since $\min(b, 1-b) = 1-b$, an application of Proposition 4.2 with $L(\vec{\xi}) = \xi_4$ yields an upper bound of the form

$$\frac{M^{1/4}}{(AM)^{1/4}}(AM)^{-2(1-b)}A^{-3r} = M^{-2(1-b)}A^{-\frac{1}{4}-2(1-b)-3r} \tag{7.3}$$

and we need both exponents to be negative. The exponent $-2(1-b)$ on M is certainly negative since $b < 1$. Thus we need

$$-r < \tfrac{1}{12} + \tfrac{2}{3}(1-b). \tag{7.4}$$

$Y^{s,b}$ embeds in $X^{r,b}_M$ for all $r < (1+4b)s$ uniformly in $M \geq 1$, in the sense expressed by Lemma 6.3, so this expression is appropriately controlled by the product of $Y^{s,b}$ norms provided that (7.1) is satisfied.

A more delicate case arises when $|\xi_j| \sim AM$ with $A \gg 1$ for two values of $j \in \{1,2,3\}$, but $|\xi_n| \sim BM$ where $B \leq A/10$ for the third index. If $n = 2$, then $\sigma(\vec{\xi}) \sim (AM)^2$, and the above analysis applies; the sole change is that one factor of A^{-r} is now merely $\lesssim B^{-r}$, which is a more favorable bound since $B \leq A$ and $r < 0$. Thus it remains only to discuss the case where n is odd; by virtue of the symmetries of the problem, it is then no loss of generality to suppose that $n = 3$.

In the subcase where $B \gtrsim 1$, we have $|\sigma| \gtrsim AMBM$ and Proposition 4.2, again with $L(\vec{\xi}) = \xi_4$, yields the upper bound

$$\frac{M^{1/4}}{(AM)^{1/4}}(ABM^2)^{-(1-b)}A^{-2r}B^{-r} = M^{-2(1-b)}A^{-\frac{1}{4}-2r-(1-b)}B^{-r-(1-b)}. \tag{7.5}$$

Provided that $-r < 1-b$, the exponent on B is negative, so when $B \gtrsim A^{1/2}$ this is $\lesssim M^{-2(1-b)}A^{-\frac{1}{4}-\frac{5}{2}r-\frac{3}{2}(1-b)}$. In the case $1 \lesssim B \lesssim A^{1/2}$ we invoke instead Proposition 4.1 with $L = \xi_4 - \xi_3$ to obtain an upper bound

$$\begin{aligned}\frac{(BM)^{1/2}}{(AM)^{1/2}}(ABM^2)^{-(1-b)}A^{-2r}B^{-r} &= M^{-2(1-b)}B^{\frac{1}{2}-(1-b)-r}A^{-\frac{1}{2}-(1-b)-2r}\\ &\lesssim M^{-2(1-b)}A^{-\frac{1}{4}-\frac{3}{2}(1-b)-\frac{5}{2}r}\end{aligned} \tag{7.6}$$

since the exponent $\frac{1}{2} - (1-b) - r$ is positive for $b > \frac{1}{2}$ and $r < 0$, and $B \lesssim A^{1/2}$. This is the same bound as obtained above for $B \gtrsim A^{1/2}$. The exponent on M is negative since $b < 1$, while the exponent on A is negative if

$$-r < \tfrac{1}{10} + \tfrac{3}{5}(1-b). \tag{7.7}$$

Under those conditions, this bound is summable over dyadic values of M, A, B.

$1 - b > \frac{1}{2} > \min(\frac{1}{10} + \frac{3}{5}(1-b), \frac{1}{12} + \frac{2}{3}(1-b))$ for all $b \in (\frac{1}{2}, 1)$, so the condition that $-r < 1-b$ does not appear in the hypotheses of the Proposition.

If $ABM^2 \lesssim 1$ then we use the upper bound $\lesssim 1$ for $\langle\sigma\rangle$ in place of $(ABM^2)^{-(1-b)}$, and obtain the upper bound

$$(BM)^{1/2}(AM)^{-1/2}A^{-2r} = B^{1/2}A^{-\frac{1}{2}-2r} \lesssim (A^{-1}M^{-2})^{1/2}A^{-\frac{1}{2}-2r} = A^{-1-2r}M^{-1}. \tag{7.8}$$

Both exponents are negative for all $-r < \frac{1}{2}$, so this is a less stringent requirement than (7.7).

Choosing r to be sufficiently close to $(1+4b)s$ reduces all these restrictions to the stated hypothesis on s. □

8. *A priori* bound in $Y^{s,b}$

The next result is the second main inequality underlying our theorems.

Proposition 8.1. *For any $s > -\frac{2}{15}$ and $b \in (\frac{1}{2}, 1)$ satisfying*

$$-s < (1+4b)^{-1} \min\left(\tfrac{1}{10} + \tfrac{3}{5}(1-b), \tfrac{1}{12} + \tfrac{2}{3}(1-b)\right) \tag{8.1}$$

any sufficiently smooth solution u of (NLS*) *with initial datum u_0 satisfies*

$$\|u\|_{Y^{s,b}} \lesssim \|u\|_{C^0(H^s)} + \|u\|^3_{Y^{s,b}} \tag{8.2}$$

where $\|\cdot\|_{C^0(H^s)} := \|\cdot\|_{C^0([-2T,2T],H^s)}$.

Proof. Choose $r < (1+4b)s$ sufficiently close to $(1+4b)s$. Let $N \geq 1$, let η be a smooth, compactly supported function, and let $t_0 \in \mathbb{R}$. Recall that u may be considered to be defined, and to satisfy the modified equation (NLS*), for all $t \in \mathbb{R}$.

Consider $w(t,x) := \eta(t-t_0)T_{N^{2s}}(u)$, which satisfies the equation

$$iw_t + w_{xx} = \eta'(t-t_0)T_{N^{2s}}u + \eta(t-t_0)\zeta_0(N^{4s}(t-t_0))|T_{N^{2s}}u|^2 T_{N^{2s}}u. \tag{8.3}$$

It suffices to bound $\widehat{w}(\xi,\tau)$ in the region where $|\tau-\xi^2| \geq 1$, for the contribution of the region $|\tau-\xi^2| \leq 1$ to the $X^{0,b}$ norm of w is majorized by $\lesssim \|w\|_{L^2(dt\,dx)}$, hence by $\lesssim \|w\|_{C^0(H^0)}$ because as a function of t, $w(t,x)$ is supported in an interval of uniformly bounded length; hence this contribution is majorized by $\lesssim \|u\|_{C^0(H^s)}$.

We may express $\widehat{w}(\xi,\tau)$ as a constant times $(\tau-\xi^2)^{-1}$ times the Fourier transform of the right-hand side of (8.3). The contribution of the first term on the right is then easily handled, for $\|\eta'(t-t_0)T_{N^{2s}}u\|_{L^2(dt\,dx)} \leq C\,\|T_{N^{2s}}u\|_{C^0(H^0)} \leq C\,\|u\|_{C^0(H^s)}$. After dividing by $\langle\tau-\xi^2\rangle^{-1}$ we therefore have a quantity whose norm in $X^{0,1}$ is majorized by $\lesssim \|u\|_{C^0(H^s)}$.

The function $\eta(t-t_0)\zeta_0(N^{4s}(t-t_0))$ may be expressed as $\tilde{\eta}^3(t-t_0)$ where $\tilde{\eta} \in C^\infty$ is real-valued, is supported in a bounded interval independent of N, and is bounded in any C^k norm uniformly in N. The second term on the right-hand side of (8.3) thus becomes $|\tilde{\eta}(t-t_0)T_{N^{2s}}u|^2\tilde{\eta}(t-t_0)T_{N^{2s}}u$.

By Lemma 6.3, the norm of $\tilde{\eta}(t-t_0)T_{N^{2s}}u$ in $X^{r,b}_{N^{1+2s}}$ is $\lesssim \|u\|_{Y^{s,b}}$. Proposition 7.1 says that the $X^{0,b}$ norm of the function whose Fourier transform is $(\tau-\xi^2)^{-1}$ times the characteristic function of the region $|\xi| \lesssim N^{1+2s}$ times the space-time Fourier transform of $|\tilde{\eta}(t-t_0)T_{N^{2s}}u|^2\tilde{\eta}(t-t_0)T_{N^{2s}}u$ is majorized by $\lesssim \|u\|^3_{Y^{s,b}}$, provided that $-2+2b \leq 1-2b$. □

Proof of Theorem 1.1. For any finite T and $\delta' > 0$, there exists $\delta > 0$ such that the bounds of Propositions 8.1 and 6.5 together imply an *a priori* upper bound $\|u\|_{C^0([0,T],H^s)} \leq \delta'$ provided that $\|u_0\|_{H^s} \leq \delta$ and $\|u\|_{C^0([0,T],H^s)} \leq 2\delta'$.

To prove the theorem, it suffices to show that given any $R < \infty$, there exists $\varepsilon_0 > 0$ such that for any $u_0 \in H^0$ satisfying $\|u_0\|_{H^s} \leq R$, if u denotes the solution of (NLS) with initial datum u_0, then $T_{\varepsilon_0}u$ satisfies an *a priori* $C^0([0,1],H^s)$ bound. Because $s > -\frac{1}{2}$, the equation is subcritical in H^s; there exists ε_0 so that $\|\varepsilon u_0(\varepsilon x)\|_{H^s} \leq \delta$ whenever $\|u_0\|_{H^s} \leq R$ and $0 < \varepsilon \leq \varepsilon_0$. We know that $u \in C^0(H^0)$, hence $u \in C^0(H^s)$. For very small ε, depending on $\|u_0\|_{H^0}$, we have $\|T_\varepsilon u\|_{C^0([0,1],H^s)} \leq \delta'$.

Now a continuity argument can be applied. If $\varepsilon > 0$ has the property that $\|T_\varepsilon u\|_{C^0([0,1],H^s)} \le \delta'$, then there exists $\varepsilon' > \varepsilon$ such that $\|T_{\varepsilon'} u\|_{C^0([0,1],H^s)} \le 2\delta'$, and provided that $\varepsilon' \le \varepsilon_0$ and ε_0 is chosen to be sufficiently small but depending only on R, this implies that $\|T_{\varepsilon'} u\|_{C^0([0,1],H^s)} \le \delta'$. Standard reasoning shows that this must then hold for $\varepsilon' = \varepsilon_0$. □

9. Existence of weak solutions

We now prove a weakened variant of Theorem 1.2 on the existence of weak solutions, showing merely that weak solutions exist in $L^\infty(H^s) \cap C^0(H^{s'}) \cap Y^{s,b}$ for all $s' < s$. The last detail, existence in $C^0(H^s)$, will be addressed in §10.

Lemma 9.1. *Let $s > -\frac{1}{12}$. Let $u_0 \in H^s$, $\varepsilon > 0$, and $M < \infty$ be given. There exist $T' > 0$ and $\delta > 0$ such that for any initial datum $v_0 \in H^0$ satisfying $\|v_0 - u_0\|_{H^s} < \delta$, the standard solution v of* (NLS) *with initial datum v_0 satisfies*

$$\int_{|\xi|\le M} |\widehat{v}(t_1,\xi) - \widehat{v}(t_2,\xi)|^2 \langle\xi\rangle^{2s}\, d\xi < \varepsilon \text{ for all } t_1, t_2 \in [0,T'] \text{ satisfying } |t_1 - t_2| < \delta. \tag{9.1}$$

Proof. Fix any $b > \frac{1}{2}$. For any $\varepsilon' > 0$ there exists $\delta' > 0$ such that any $w \in X^{0,b}$ satisfies $\|w(t_1,\cdot) - w(t_2,\cdot)\|_{L^2} \lesssim |t_1 - t_2|^\gamma \|w\|_{X^{0,b}}$ for all $\gamma < b - \frac{1}{2}$ whenever $|t_1 - t_2| \le 1$, as follows from a standard Cauchy-Schwarz calculation. By rescaling we conclude that

$$\|P_{<M} v(t_1,\cdot) - P_{<M} v(t_2,\cdot)\|_{H^s} \le C_M |t_1 - t_2|^\gamma \|v\|_{Y^{s,b}} \tag{9.2}$$

whenever $|t_1 - t_2| \lesssim M^{4s}$.

We have already established an *a priori* upper bound for $\|v\|_{Y^{s,b}}$ in terms of $\|v_0\|_{H^s}$, hence in terms of $\|u_0\|_{H^s}$ so long as $\delta \le 1$. Consequently

$$\int_{|\xi|\le M} |\widehat{v}(t_1,\xi) - \widehat{v}(t_2,\xi)|^2 \langle\xi\rangle^{2s}\, d\xi \le C'_M \varepsilon'^2 \tag{9.3}$$

provided that $|t_1 - t_2| < \delta' M^{4s}$. The claim follows. □

Proof of Theorem 1.2. Let $s \in (-\frac{1}{12}, 0)$, and then let $s' \in (-\frac{1}{12}, s)$ be arbitrary. Consider any initial datum $u_0 \in H^s$. Let $(v_{0,j})$ be any sequence of functions in $H^0(\mathbb{R})$ such that $v_{0,j} \to u_0$ in H^s norm as $j \to \infty$. Let $v^{(j)} \in X^{0,b}$ be the unique standard solution of the Cauchy problem (NLS) with initial datum $v_{0,j}$.

There exist $b > \frac{1}{2}$ and T such that the sequence $v^{(j)}$ is uniformly bounded in $C^0((-2T,2T),H^s) \cap Y^{s,b}$ norm. Moreover, the mappings $(-2T,2T) \ni t \mapsto v^{(j)}(t,\cdot) \in H^{s'}$ are equicontinuous, by virtue of Lemma 9.1 and the inequality

$$\int_{|\xi|\ge M} |\widehat{f}(\xi)|^2 \langle\xi\rangle^{2s'}\, d\xi \le C M^{2s'-2s} \|f\|_{H^s}^2. \tag{9.4}$$

For any large N, decompose $v^{(j)}$ as

$$v^{(j)} = v^{(j)}_{N;\,\mathrm{high}} + v^{(j)}_{N;\,\mathrm{low}}$$

where $\widehat{v^{(j)}_{N;\,\mathrm{low}}}(t,\xi) := \widehat{v^{(j)}}(t,\xi)$ when $|\xi| \le N$ and $:= 0$ otherwise. The equicontinuity of the mapping $t \mapsto v^{(n)}(t,\cdot) \in H^{s'}$ implies precompactness of $\{v^{(j)}_{N;\,\mathrm{low}}\}$ in $C^0_t(C^\infty_x)$ for x in every bounded region, for every N. A diagonal argument produces a subsequence, denoted again by $v^{(j)}$, such that for every N, $v^{(j)}_{N;\,\mathrm{low}}$ converges in the $C^0(C^\infty)$ topology in every bounded region. Since $v^{(j)}$ is uniformly bounded in $C^0(H^s)$, there exists a distribution $u \in \mathcal{D}'$ such that $v^{(j)} \to u$ in the topology of $\mathcal{D}'$.

Equicontinuity, the uniform upper bound on $v^{(n)}$ in $C^0(H^s) \cap Y^{s,b}$, and (9.4) together ensure (possibly after passage to the limit of some subsubsequence) that $u \in C^0(H^{s'}) \cap L^\infty(H^s) \cap Y^{s,b}$. It follows likewise that $u(0,\cdot) \equiv u_0(\cdot)$. The proof that the limit of some subsequence actually belongs to $C^0(H^s)$ will be completed in §10.

It remains to show that u is a weak solution of the equation. To simplify notation, denote the nonlinearity by $\mathcal{N}(v) := |v|^2 v$. It follows directly from the above convergence that $\mathcal{N}(v^{(j)}_{N;\,\mathrm{low}})$ converges to $\mathcal{N}(u_{N;\,\mathrm{low}})$ in $C^0(C^\infty_{\mathrm{loc}})$ for every N.

For any $\varepsilon > 0$ there exists N such that

$$\|\mathcal{N}(v^{(j)}) - \mathcal{N}(v^{(j)}_{N;\,\mathrm{low}})\|_{Y^{s',b-1}} \le \varepsilon \quad \text{for all } j. \tag{9.5}$$

This follows from the basic trilinear estimate, Proposition 7.1, since $v^{(j)}_{N;\,\mathrm{high}}$ is arbitrarily small in $Y^{s',b}$ provided N is sufficiently large, while the low part is bounded uniformly in N. Likewise $\mathcal{N}(u) - \mathcal{N}(u_{N;\,\mathrm{low}})$ is $\le \varepsilon$ in $Y^{s',b-1}$ for all j.

These conclusions together imply that $\mathcal{N}(v^{(j)}) \to \mathcal{N}(u)$ in the topology of $\mathcal{D}'$. Since $v^{(j)}$ is a solution of (NLS), it follows that u is likewise a solution. □

10. Continuity in time

Since weak limits cannot be taken directly in spaces $C^0(H^s)$, some additional argument is needed to ensure that the weak limits constructed above do belong to these spaces. In this section we bridge that gap by establishing a certain limited equicontinuity with respect to time.

Recall the expressions $\Phi_\varphi(t,u) = \int_{\mathbb{R}} |\widehat{u}(t,\xi)|^2 \varphi(\xi)\, d\xi$. Additional control on the solution u can obtained by analyzing $\Phi_\varphi(t,u)$ for weights φ which are more general than $\langle\xi\rangle^{2s}$, and are specifically adapted to the initial datum u_0 (cf. the "frequency envelopes" used for instance in [12]). We have actually proved the following statement more general than that announced earlier.

Lemma 10.1. *Let $s > -\frac{1}{12}$ and $s' \in (s,0)$. For any nonnegative C^2 weight function φ satisfying*

$$\varphi(\xi) \le \langle\xi\rangle^{2s'}, \qquad \varphi'(\xi) \le \langle\xi\rangle^{2s'-1}, \qquad \varphi''(\xi) \le \langle\xi\rangle^{2s'-2}, \tag{10.1}$$

for any initial datum $u_0 \in H^0$, the standard solution $u(t,x)$ of (NLS) *satisfies*

$$\Big|\Phi_\varphi(t,u) - \Phi_\varphi(0,u)\Big| \le C\,\|u\|^4_{Y^{s,b}}. \tag{10.2}$$

From this can be extracted a high-frequency continuity result.

Lemma 10.2. *Let $s > -\frac{1}{12}$. Let $u_0 \in H^s$ and $\varepsilon > 0$ be given. There exist $\delta > 0$ and $N < \infty$ such that for all $v_0 \in H^0$ satisfying $\|v_0 - u_0\|_{H^s} < \delta$, the standard solution v of* (NLS) *with initial datum v_0 satisfies*

$$\int_{|\xi|\ge N} |\widehat{v}(t,\xi)|^2 \langle\xi\rangle^{2s}\, d\xi < \varepsilon \tag{10.3}$$

for all $t \in [0,T]$.

Here the timespan $T \in (0,\infty)$ is fixed, and it is assumed that $\|u_0\|_{H^s}$ is sufficiently small that the proof of Theorem 1.1 applies to all smooth solutions with initial data satisfying $\|v_0 - u_0\|_{H^s} \le \delta_0$, where δ_0 depends on T.

Proof. Fix any exponent $s' \in (s,0)$. Let $\varepsilon > 0$ be given. Choose $M < \infty$ so that $\int_{|\xi|\ge M} |\widehat{u_0}(\xi)|^2 \langle\xi\rangle^{2s}\, d\xi < \varepsilon^2$. Then there exist a large parameter $M' \ge M$ and a weight function φ satisfying the three inequalities hypothesized in Lemma 10.1, with exponent s', such that

$$\varepsilon^{-1}\langle\xi\rangle^{2s} \ge \varphi(\xi) \ge \langle\xi\rangle^{2s} \qquad \text{for all } \xi, \tag{10.4}$$

$$\varphi(\xi) = \varepsilon^{-1}\langle\xi\rangle^{2s} \qquad \text{for all } |\xi| \ge M', \tag{10.5}$$

$$\varphi(\xi) = \langle\xi\rangle^{2s} \qquad \text{for all } |\xi| \le M. \tag{10.6}$$

M', φ depend on ε and on s'. The conclusion (10.2) of Lemma 10.1 holds with a constant C independent of M, ε.

Thus by (10.2),

$$\begin{aligned}\int_{|\xi|\ge M'} |\widehat{v_0}(\xi)|^2\varphi(\xi)\,d\xi &\le 2\int_{|\xi|\ge M'} |\widehat{u_0}(\xi)|^2\varphi(\xi)\,d\xi + 2\int_{|\xi|\ge M'} |\widehat{v_0}(\xi)-\widehat{u_0}(\xi)|^2\varphi(\xi)\,d\xi \\ &\le 2\varepsilon + C_\varphi\|v_0-u_0\|_{H^s}^2\end{aligned} \tag{10.7}$$

where C_φ depends on φ, hence ultimately on ε. Therefore there exists $\delta > 0$ such that

$$\int_{|\xi|\ge M'} |\widehat{v_0}(\xi)|^2\varphi(\xi)\,d\xi \le 3\varepsilon \tag{10.8}$$

for every $v_0 \in H^0$ satisfying $\|v_0 - u_0\|_{H^s} < \delta$.

For such initial data v_0, the associated solutions v have uniformly bounded $Y^{s,b}$ norms, with a bound independent of ε, provided that δ is sufficiently small. Therefore by Lemma 10.1, $\Phi_\varphi(t,v) = \int_{\mathbb{R}} |\widehat{v}(t,\xi)|^2\varphi(\xi)\,d\xi$ is bounded by a finite constant independent of ε, M, M' uniformly for all $t \in [0,T]$. Therefore since $\langle\xi\rangle^{2s} \le \varepsilon\varphi(\xi)$ for all $|\xi| \ge M'$,

$$\int_{|\xi|\ge M'} |\widehat{v}(t,\xi)|^2\langle\xi\rangle^{2s}\,d\xi \le \varepsilon\int_{|\xi|\ge M'} |\widehat{v}(t,\xi)|^2\varphi(\xi)\,d\xi \lesssim \varepsilon, \tag{10.9}$$

provided that $t \in [0,T]$ and $\|v_0 - u_0\|_{H^s} < \delta$. □

Thus if $u_0, v_0^{(j)}$ are initial data with $u_0 \in H^s$ and $v_0^{(j)} \in H^0$, and if $v^{(j)} \to u_0$ in the H^s norm, then the corresponding standard solutions $v^{(j)}$ form an equicontinuous family in $C^0(H^s)$. Therefore passage to the limit through an appropriate subsequence produces a solution in $C^0(H^s)$, satisfying the other conclusions of Theorem 1.2.

Remark 10.1. Lemma 10.2 has the following direct consequence. Let $s > -\frac{1}{12}$. If there exists $r > -\infty$ for which the solution mapping from datum to solution of (NLS) is continuous from H^s to $C^0([0,T], H^r)$, then the solution mapping is continuous from H^s to $C^0([0,T], H^s)$.

References

[1] M. Ablowitz, D. Kaup, A. Newell, and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Appl. Math. 53 (1974), no. 4, 249-315.

[2] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I. Schrödinger equations*, Geom. Funct. Anal. 3 (1993) 107-156.

[3] L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disk*, Studia Math. 44 (1972), 287–299.

[4] M. Christ, *Power series solution of a nonlinear Schrödinger equation*, Annals of Mathematics Studies, to appear, math.AP/0503368.

[5] M. Christ, J. Colliander and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math. 125 (2003), no. 6, 1235–1293.
[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *A refined global well-posedness for the Schrodinger equations with derivative*, Siam J. Math. **34** (2002), 64–86.
[7] A. Grünrock, *Bi- and trilinear Schrdinger estimates in one space dimension with applications to cubic NLS and DNLS*, Int. Math. Res. Not. 2005, no. 41, 2525–2558.
[8] C. Kenig, G. Ponce, L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. 71 (1993), 1-21.
[9] C. Kenig, G. Ponce, L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. 106 (2001), 617-633.
[10] H. Koch and D. Tataru, *A-priori bounds for the 1-D cubic NLS in negative Sobolev spaces*, math/0612717, to appear.
[11] T. Tao, *Multilinear weighted convolution of L^2 functions, and applications to non-linear dispersive equations*, Amer. J. Math. 123 (2001), 839-908.
[12] T. Tao, *Global well-posedness of the Benjamin-Ono equation in $H^1(R)$*, J. Hyperbolic Diff. Eq. **1** (2004), 27–49.
[13] Y. Tsutsumi, *L^2 solutions for nonlinear Schrodinger equations and nonlinear groups*, Funk. Ekva. 30 (1987), 115-125.
[14] A. Vargas, L. Vega, *Global well-posedness for 1D nonlinear Schrödinger equation for data with an infinite L^2 norm*, J. Math. Pures Appl. 80, (2001), 1029-1044.

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