Existence and Nonuniqueness for a Nonlinear Schrödinger Equation

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Cauchy problem for 1D periodic cubic NLS:

$$\begin{cases} iu_t + u_{xx} + \omega |u|^2 u = 0 \\ u(0,x) = f(x) \end{cases}$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,
$t \in \mathbb{R}$,
$\omega = \pm 1$.
(Distinction between focusing and defocusing cases plays no role in this talk.)

• Bourgain [1993]: wellposed in Sobolev space H^s $\forall\, s \ge 0$, with uniformly continuous dependence on the initial datum.
• Tsutsumi earlier showed wellposedness in $L^2(\mathbb{R})$.
• Burq-Gérard-Tzvetkov: Illposed, in sense that uniformly continuous dependence breaks down, $\forall\, s < 0$.
• Colliander-Tao-C: Unstable in a stronger sense for $s < 0$.

This lecture: Existence of solutions for wider classes of initial data. (Vargas-Vega have achieved one such extension.)

Function spaces: $\boxed{\mathcal{F}\ell^p(\mathbb{T})}$ for $p \in [1, \infty]$:

$$\|f\|_{\mathcal{F}\ell^p} = \|\widehat{f}\|_{\ell^p(\mathbb{Z})} = \Big(\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^p \Big)^{1/p}.$$

For $p > 2$ this is a space of *distributions*, larger than borderline Sobolev space H^0.

I maintain that these function spaces are rather natural in the context of NLS from the viewpoint of inverse scattering theory (more on this later).

Basic property: *conservation law*

$$\int_{\mathbb{T}} |u(t,x)|^2\,dx = \int_{\mathbb{T}} |f(x)|^2\,dx \;\forall\, t > 0.$$

In order for the Cauchy problem to make any sense in $\mathcal{F}\ell^p$ for $p > 2$ it is essential to modify NLS. Write

$$\mu(g) = (2\pi)^{-1}\int_{\mathbb{T}} g(x)\,dx = \text{ mean value of } g.$$

Consider modified NLS

$$\begin{cases} iu_t + u_{xx} + \omega\Big(\,|u|^2 - 2\mu(|u|^2)\,\Big)u = 0 \\ u(0,x) = f(x) \end{cases} \qquad (\text{NLS}^*)$$

In (NLS*), $\mu(|u|^2)$ is shorthand for $\mu(|u(t,\cdot)|^2) = (2\pi)^{-1}\|u(t,\cdot)\|_{L^2}^2$, which is *independent of* t for all sufficiently smooth solutions. We've merely introduced a trivial unimodular scalar factor $e^{2i\mu t}$, where $\mu = \mu(|f|^2)$.

Theorem. (Existence and continuity) *For any* $p \in [1, \infty)$ *and any* $R < \infty$, *there exists* $\tau > 0$ *for which the solution mapping* $f \mapsto u(t, x)$, *defined initially for* $f \in C^\infty(\mathbb{T})$, *extends to a uniformly continuous mapping from* $\mathcal{F}\ell^p$ *to* $C^0(\mathcal{F}\ell^p)$.

More precisely: From any bounded set in $\mathcal{F}\ell^p(\mathbb{T})$ to $C^0([0, \tau], \mathcal{F}\ell^p(\mathbb{T}))$, where $\tau > 0$ depends on the bounded set.

For the unmodified equation this has an obvious consequence in terms of continuous dependence for initial data with *identical* L^2 norms which are close in $\mathcal{F}\ell^p$...

The following result quantifies the relation between the nonlinear evolution (NLS*) and the corresponding linear Cauchy problem

$$\begin{cases} iv_t + v_{xx} = 0 \\ v(0,x) = f(x). \end{cases} \tag{LS}$$

Proposition. *Let $R < \infty$ and $p \in [1, \infty)$. Let $q > p/3$ also satisfy $q \geq 1$. Then there exist $\tau, \varepsilon > 0$ and $C < \infty$ such that for any initial datum f satisfying $\|f\|_{\mathcal{F}\ell^p} \leq R$, the solutions u of (NLS*) and v of* (LS) *satisfy*

$$\|u(t,\cdot) - v(t,\cdot)\|_{\mathcal{F}\ell^q} \leq Ct^\varepsilon \quad \textit{for all } t \in [0, \tau].$$

u denotes solution defined by approximating f by smooth functions, solving equation, and passing to limit. Thus for $p > 1$, the difference between (NLS*) and (LS) is smoother, in $\mathcal{F}\ell^q$ scale, than the linear evolution (LS).

The "solution" defined by approximating initial datum by smooth data and passing to limit is a solution in a more natural sense.

Definition. A *sequence of Fourier truncation operators* T_ν is any sequence

$$\widehat{T_\nu f}(n) = m_\nu(n)\widehat{f}(n)$$

such that
(i) m_ν is finitely supported $\forall\,\nu$
(ii) $\sup_{\nu,n}|m_\nu(n)| < \infty$,
(iii) $\forall\, n \in \mathbb{Z}$, $m_\nu(n) \to 1$ as $\nu \to \infty$.

(T_ν acts also on functions $v(t,x)$ in the obvious way.)

A more fundamental question than whether some function $u(t,x)$ is a solution of our partial differential equation is whether the nonlinear term $\mathcal{N}(u) = (|u|^2 - 2\mu(|u|^2))u$ has an intrinsic meaning.

Definition. $\mathcal{N}(u)$ exists in the *limiting Fourier cutoff sense* if for any sequence of Fourier truncation operators, $\lim_{\nu\to\infty} \mathcal{N}(T_\nu u)$ exists in the sense of distributions.
(This limit is then necessarily independent of the sequence m_ν.)

Definition. $u \in C^0([0,\tau], \mathcal{F}\ell^p(\mathbb{T}))$ is said to be a weak solution of (NLS*) if $\mathcal{N}(u)$ exists in the above sense, and if the differential equation then holds.

Proposition. *Let* $p \in [1, \infty)$, $s \geq 0$, *and* $f \in \mathcal{F}\ell^p$. *Define* $u(t,x)$ *by approximating* f *by smooth functions, solving (NLS*), and passing to the limit. Then for any* $R < \infty$ *there exists* $\tau > 0$ *such that whenever* $\|f\|_{\mathcal{F}\ell^p} \leq R$, u *is a weak solution of (NLS*) in the sense defined above.*

Making sense of the nonlinearity via this limiting procedure bears some superficial resemblance to general theories of multiplication of distributions, but the existence here of a limit that is *independent* of the sequence (m_ν) gives u a much stronger claim to the title of solution.

Kappeler and Topalov have used inverse scattering theory to solve the KdV equation for singular initial data, in the sense that the solution map defined for smooth initial data extends continuously.

Their work seems to leave open the question of whether their solution satisfies the equation in any stronger sense.

Takaoka and Tsutsumi have shown that this is true for a partial range of the Sobolev exponents in the Kappeler-Topalov theorem, but I don't know the precise status of this question.

Naturality of the spaces $\mathcal{F}\ell^p$

In work not yet written up for dissemination, I believe that Burak Erdogan and I have shown that for initial data with small $\mathcal{F}\ell^p$ norms, the problem is wellposed globally in time in the defocusing case.

This is (to be) proved by combining the local-in-time wellposedness with the conserved quantities furnished by inverse scattering theory.

Our calculations show that (assuming small norms) the ℓ^p norm of $\widehat{f}$ is equivalent to the ℓ^p norm of the sequence of spectral gap lengths for an associated Dirac operator with "potential" f. These Dirac gap lengths are invariant under the (NLS) flow.

I'll reformulate NLS as an (infinite) coupled system of ODEs for Fourier coefficients

$$\widehat{f}(n) = (2\pi)^{-1} \int_{\mathbb{T}} f(x) e^{-inx}\, dx \quad n \in \mathbb{Z}.$$

$$\widehat{u}(t,n) = (2\pi)^{-1} \int_{\mathbb{T}} u(t,x) e^{-inx}\, dx.$$

In terms of $u_n(t) = \widehat{u}(t,n)$, (NLS*) is

$$i\frac{du_n(t)}{dt} - n^2 u_n = -\omega \sum_{j-k+l=n}^{*} u_j \bar{u}_k u_l + \omega |u_n|^2 u_n$$

where $\sum^*$ indicates that the sum is over all terms with $j, l \neq k, n$.

Existence theorem and nonuniqueness are proved by working with this system.

The difference between (NLS) and (NLS*) is precisely a multiple of $(\sum_j |u_j|^2)u_n$ on the right-hand side.

Since L^2 norm is conserved, if initial datum has infinite L^2 norm then this term is an infinite constant times u_n.

Nonuniqueness

From the MSRI announcement of this workshop: *"If the interaction is strong enough then it may create new waves."*

Theorem. For any $p > 2$ there exists a nonzero "solution" u of (NLS*) in $C^0([0,1], \mathcal{F}\ell^p)$ satisfying $u(0,x) \equiv 0$. The same holds in $C^0([0,1], H^s)$ for all $s < 0$.

I don't know about $p \leq 2$, or $s \geq 0$. Uniqueness was established by Bourgain, in the class $C^0(H^0) \cap L^4_{x,t}$.

For the 2D incompressible periodic Euler equation, Scheffer (2nd proof by Shnirelman) has proved: There exists a nonzero solution in the class $L^2_{x,t}$ that is $\equiv 0$ for $t \leq 0$ (and $t \geq 1$). My construction does not seem to give that, but it does give nonuniqueness of generalized solutions for the 2D incompressible periodic Navier-Stokes equation in $C^0(H^s)$ for all $s < 0$.

For Navier-Stokes there are examples of nonuniqueness due to Ladyzhenskaya, but they are on bounded time-dependent domains which reduce to a single point at $t = 0$ and either involve nonstandard boundary conditions, or give solutions with infinite $L^2_{x,t}$ norm.

Here u is a solution in 3 senses:

• Consider inhomogeneous (NLS*)

$$\begin{cases} iu_t + u_{xx} + \omega\Big(|u|^2 - 2\mu(|u|^2)\big)u = F \\ u(0,x) \equiv 0 \end{cases}$$

Then $u = \lim_{\nu\to\infty} u_\nu$ where u_ν is a solution of inhomogeneous (NLS*) with driving force F_ν where

$$u_\nu \to u \text{ in } C^0(\mathcal{F}\ell^p)$$
$$e^{-it\Delta}F_\nu \to 0 \text{ in } C^{-1}(\mathcal{F}\ell^\infty).$$

• u is a weak solution of (NLS*), in the limiting Fourier cutoff sense.

• Each Fourier coefficient $\widehat{u(t,n)}$ is a C^∞ function of t. The infinite series in the ODE for the n-th Fourier coefficient of $\mathcal{N}(u)$ involves only finitely many nonzero terms for each n, and each ODE holds in the ordinary sense.

Nonuniqueness is based on an infinite cascade of energy from higher to lower Fourier modes. (Shnirelman's construction for the Euler equation shares this feature.)

High modes can interact with one another to drive low modes through the terms $u_j \bar{u}_k u_l$ in the equation for du_n/dt, where $j - k + l = n$.

These modes in turn can be driven by still (much) higher modes, and so on.

I see no reason to think that the details of the (very simple) construction have been optimally arranged.

For 1D periodic NLS with nonlinearity u^2 or $\bar{u}^2$, Kenig-Ponce-Vega have established wellposedness in H^s for a range of s extending below 0, with existence and uniqueness in a smaller space than $C^0(H^s)$.

Our construction applies here also and gives nonuniqueness in $C^0(H^s)$. (No modification of the PDE is needed in this case.)

Essence of the nonuniqueness construction is an approximation result for smooth solutions of the inhomogeneous problem.

Proposition. *Let* $p > 2$ *and* $\omega \neq 0$. *Let* $u \in C^\infty([0,1] \times \mathbb{T})$. *Suppose that for all* n, $\widehat{u}(t,n)$ *vanishes to infinite order as* $t \to 0^+$.

For any $\varepsilon > 0$ *and* $N < \infty$ *there exist* $v, F \in C^\infty([0,1] \times \mathbb{T})$, *each of whose Fourier coefficients vanishes to infinite order as* $t \to 0^+$, *such that* v *satisfies inhomogeneous (NLS*) with driving force* F, *and*

$$\|v - u\|_{C^0([0,1],\mathcal{F}\ell^p)} \le \varepsilon$$
$$\|e^{-it\Delta} F\|_{C^{-1}([0,1],\mathcal{F}\ell^\infty)} \le \varepsilon.$$
$$(\widehat{v} - \widehat{u})(t,n) \equiv \widehat{F}(t,n) \equiv 0 \ \forall\, |n| < N.$$

The same holds with $\mathcal{F}\ell^p$ replaced by H^s for any $s < 0$.

A simple limiting argument using this result produces nonunique solutions and shows that they are solutions in the various senses I discussed earlier.

The procedure is roughly this:

Regard u as being an exact solution of inhomogeneous (NLS*) with driving force G.

Set $v = u + h$ and choose h to have small norm and to produce an offsetting driving force $-G$. That is,

$$\mathcal{N}(h) = -G + \text{ other terms.}$$

If u satisfies inhomogeneous (NLS*) with driving force G, then $v = u + h$ satisfies inhomogeneous (NLS*) with driving force

$$\Big[G + \mathcal{N}(h)\Big] + \Big[\mathcal{N}(u+h) - \mathcal{N}(u) - \mathcal{N}(h)\Big] + \Big[ih_t + h_{xx}\Big].$$

$\mathcal{N}(h)$ will inevitably include many "other terms" that we don't want, and cross terms from $\mathcal{N}(u+h)-\mathcal{N}(u)-\mathcal{N}(h)$ also occur, driving unwanted Fourier modes.

Key. It's easy to arrange that all *unwanted* Fourier modes are arbitrarily high frequency modes.

If for instance

$$h(t,x) = A(t)e^{iax} + B(t)e^{ibx}$$

then (for $\omega = 1$)

$$\mathcal{N}(h) = A^2\bar{B}e^{i(2a-b)x} + B^2\bar{A}e^{i(2b-a)x} + |A|^2Ae^{iax} + |B|^2Be^{ibx}.$$

Given n, we can choose a, b arbitrarily large so that $2a-b=n$, and $2b-a$ is arbitrarily large ($b \approx 2a$).

I turn now to the **existence proof**.

The plan is to show that the mapping

$$[\text{initial datum } u(0) \mapsto \text{solution } u(t)]$$

is analytic by expanding it in power series, calculating the resulting series explicitly, and proving absolute convergence. This actually works.

The terms in the power series are now multilinear operators of all degrees, mapping

$$\mathcal{F}\ell^p \otimes \mathcal{F}\ell^p \otimes \mathcal{F}\ell^p \otimes \cdots \to \mathcal{F}\ell^q,$$

rather than numerical coefficients times monomials.

Much of the work goes into systematically describing these operators. The estimates themselves are quite elementary.

Substitute $a_n(t) = e^{in^2 t} u_n(t)$ to get system

$$\frac{da_n(t)}{dt} = i\omega \sum_{j-k+l=n}^{*} a_j \bar{a}_k a_l e^{i\sigma(j,k,l,n)t} - i\omega |a_n|^2 a_n,$$

where

$$\begin{aligned} \sigma(j,k,l,n) &= n^2 - j^2 + k^2 - l^2 \\ &= 2(n-j)(n-l) \end{aligned}$$

when $j - k + l = n$.

Convention: $\sum^*$ denotes summation over all j, k, l with $j, k \neq l, n$.

I'll simplify and discuss the slightly simpler equations

$$\frac{da_n(t)}{dt} = i\omega \sum_{j-k+l=n}^{*} a_j(t) \bar{a}_k(t) a_l(t) \, e^{i\sigma(j,k,l,n)t}.$$

Rewrite as system of integral equations

$$a_n(t) = b_n + i\omega \sum_{j-k+l=n}^{*} \int_0^t a_j(s)\bar{a}_k(s)a_l(s)e^{i\sigma(j,k,l,n)s}\,ds.$$

By repeatedly substituting system into itself, derive a formal Taylor series expansion of the form

$$a_n(t) = b_n + i\omega \sum_{j-k+l=n}^{*} b_j\bar{b}_k b_l \,\frac{e^{i\sigma(j,k,l,n)t} - 1}{i\sigma(j,k,l,n)} + \textbf{much more}$$

where "more" denotes an infinite sum of higher-order terms.

The $\boxed{\text{denominators } \sigma(j,k,l,n)}$ are an essential feature of the second term.

The higher-order terms can in principle be calculated, although in practice this is tedious because they become increasingly complicated as their degrees increase.

A sample:

$$\sum_{j-k+l=n}^{*} \sum_{j_1-j_2+j_3=j}^{*} \sum_{k_1-k_2+k_3=k}^{*} \sum_{l_1-l_2+l_3=l}^{*} \boxed{I}\; b_{j_1}\overline{b}_{j_2}b_{j_3}\overline{b}_{k_1}b_{k_2}\overline{b}_{k_3}b_{l_1}\overline{b}_{l_2}b_{l_3}$$

where

$$I = I(t, j, k, l, j_1, j_2, j_3, \cdots, l_3) = \int_{0 \le r_1, r_2, r_3 \le s \le t} e^{i\phi}\, dr_1\, dr_2\, dr_3\, ds$$

with

$$\phi = \sigma(j, k, l, n)s + \sigma(j_1, j_2, j_3, j)r_1 \\ - \sigma(k_1, k_2, k_3, k)r_2 + \sigma(l_1, l_2, l_3, l)r_3.$$

This is a multilinear operator of degree 9, applied to $(b, \overline{b}, b, \overline{b}, \cdots)$.

There are infinitely many terms in the expansion. As the degree increases, they become increasingly complicated.

A simple recursive analysis of the construction of the Taylor expansion for $a(t)$ establishes

Proposition. A solution $a(t)$ of the infinite coupled system of integral equations is given *formally* by

$$a_n(t) = \sum_{k=0}^{\infty} \sum_{T} c_T S_T(t)(b^*, \cdots, b^*)(n)$$

where for each k, the inner sum is taken over $O(C^k)$ ornamented $\boxed{\text{trees}}$ T, each satisfying $|T| = 1 + 3k$. Moreover $|c_T| = O(c^k)$.

Each b^* denotes either b or $\bar{b}$.

$S_T(t)$ are $\boxed{\text{multilinear operators}}$ of degrees $1 + 2k$, described below.

Trees and associated multilinear operators

Terms in the Taylor expansion (about $b = 0$) of the nonlinear operator $b \mapsto a(t)$ are naturally indexed by a certain class of trees.

• These trees T are finite and rooted with root $\mathbf{r}$.
• Every vertex has either 0 children, or 3 children.

Notation:
• T^∞ denotes the set of all terminal vertices;
• T^0 denotes the set of all nonterminal vertices.
• For any $v \in T^0$, the three children of v are denoted by (v, i), for $i = 1, 2, 3$.

An **ornamented tree** T is a tree together with:
(i) For each $v \in T$ there is given an element of $\{-1, 1\}$, denote by $\pm_v$.
(ii) An auxiliary space $\mathbb{Z}^T$. Elements of this space are denoted by $\mathbf{j} = (j_v)_{v \in T}$. (These are the indices we'll sum over.)

Define
(iii) For each $v \in T^0$ and each $\mathbf{j} \in \mathbb{Z}^T$,

$$\sigma_v(\mathbf{j}) = j_v^2 - j_{(v,1)}^2 + j_{(v,2)}^2 - j_{(v,3)}^2.$$

(These appear in exponents and reflect dispersive character of equation.)

(iv) Let coefficients $\varepsilon_u \in \{0, 1, -1\}$ be given for each $u \in T$.

$\boxed{\rho_v(\mathbf{j})}$ is defined recursively for all $v \in T^0$ by

$$\rho_v(\mathbf{j}) = \begin{cases} \sigma_v(\mathbf{j}) \text{ if every child of } v \text{ is terminal} \\ \sigma_v(\mathbf{j}) \boxed{+ \sum_u \varepsilon_u \rho_u(\mathbf{j})} \text{ otherwise} \end{cases}$$

where sum is over the 3 children u of v. (These sums of exponents arise in describing higher-order interactions.)

One more definition:

$\boxed{J(T)} \subset \mathbb{Z}^T$ is the set of all $\mathbf{j} \in \mathbb{Z}^T$ satisfying

$$j_v = j_{(v,1)} - j_{(v,2)} + j_{(v,3)} \; \forall\, v \in T^0$$

(reflecting the interaction relation $j - k + l = n$ in system of ODEs)

$$\{j_v, j_{(v,2)}\} \cap \{j_{(j,1)}, j_{(j,3)}\} = \emptyset \; \forall\, v \in T^0.$$

(reflecting the restriction $j, l \neq k, n$)

Tree operators.

To any ornamented tree and any $t \in \mathbb{R}$ are associated **multilinear operators**

$$S_T(t)(x_v)_{v \in T^\infty}(n) = \sum_{\mathbf{j} \in J(T): j_{\mathbf{r}} = n} \boxed{I_T(t, \mathbf{j})} \prod_{v \in T^\infty} x_v(j_v).$$

Coefficients

$$\boxed{I_T(t, \mathbf{j})} = \int_{\mathcal{R}(T,t)} \prod_{u \in T^0} \left(e^{\pm_u i \omega \sigma_u(\mathbf{j}) t_u} \, dt_u \right)$$

for $t \in \mathbb{R}$, $\mathbf{j} \in \mathbb{Z}^T$ where

$$\mathcal{R}(T,t) = \{(t_u)_{u \in T^0} : 0 \le t_u \le t_v \le t \text{ whenever } u \le v \text{ are elements of } T^0\}.$$

This is a subset of $[0,t]^{T^0}$.

$S_T(t)$ is a multilinear operator of degree $|T^\infty|$. It takes as input a $|T^\infty|$-tuple of complex-valued sequences x_v, and outputs a single complex-valued sequence, whose n-th term is given by above formula.

There is a trivial bound

$$|I_T(t,\mathbf{j})| \le t^{|T^0|}$$

for all $T, t, \mathbf{j}$.

Lemma. For all T, $\mathbf{j} \in J(T)$, and $t \in [0,1]$,

$$|I_T(t,\mathbf{j})| \le 2^{|T|} \sum_{(\varepsilon_{u,i})} \prod_{v \in T^0} \langle \rho_v(\mathbf{j}) \rangle^{-1}.$$

The sum is taken over all possible vectors $(\varepsilon_{u,i} : u \in T^0, i \in \{1,2,3\})$, with each $\varepsilon_{u,i} \in \{-1,0,1\}$. ($\rho_v(\mathbf{j})$ depends on these $\varepsilon_{u,i}$.)

This is a rather complicated bound. Recall that $\rho_v(\mathbf{j})$ are defined recursively by

$$\rho_v(\mathbf{j}) = \sigma_v(\mathbf{j}) + \sum_{u \text{ child of } v} \varepsilon_u \rho_u(\mathbf{j})$$

for nonterminal v, and $= \sigma_v(\mathbf{j}) = 2(j_v - j_{v,1})(j_v - j_{v,3})$ for terminal v. Nothing prohibits cancellation in these sums.

Simplest bound available:

$$\Big\| \sum_{\mathbf{j}\in J(T): j_{\mathbf{r}}=n} \prod_{v\in T^\infty} |x_v(j_v)| \Big\|_{\ell^1(n)} \le \prod_{v\in T^\infty} \|x_v\|_{\ell^1}.$$

(Requires ℓ^1 norms, but doesn't require any gain from factors $I_T(t,\mathbf{j})$.)

Second simplest bound:

$$\Big\| \sum_{\mathbf{j}\in J(T): j_{\mathbf{r}}=n} \prod_{v\in T^\infty} \langle \sigma_v(\mathbf{j})\rangle^{-1+\delta} |x_v(j_v)| \Big\|_{\ell^q(n)} \le C^{|T|} \prod_{v\in T^0} \|x_v\|_{\ell^p}$$

provided $p \in (1,\infty)$, $q > p/|T^0|$, and $q \ge 1$.

There exists some small strictly positive $\delta = \delta(p,q)$ for which this holds.

Proof of ℓ^1 bound is just Fubini. Second uses triangle inequality and Hölder.

There has to be a catch.

The second simplest bound required coefficients $\langle\sigma_v(\mathbf{j})\rangle^{-1+\delta}$. Instead we have $\langle\rho_v(\mathbf{j})\rangle^{-1}$.

$\rho_v(\mathbf{j})$ is a quadratic polynomial in all variables $j_u : u \le v$; there can be arbitrarily many such variables. There's no hope of any simple factorization as for $\sigma_v(\mathbf{j})$. Worse, nothing prevents cancellation in the sum. All we know is $|\rho_v(\mathbf{j})| \le (Ct)^{|T^0|}$.

This is not a defect in the analysis, simply a reflection of the nature of the evolution. Frequencies $n_1, n_2, \cdots$ with $\sum_j n_j = n$ interact strongly when $|n^2 - (\sum_j \pm n_j^2)|$ is not large. Everyone familiar with the analysis of Bourgain will recognize this issue.

What saves us is that there are few $\mathbf{j}$ for which $\rho_v(\mathbf{j})$ is not reasonably large.

Classify pairs $(v, \mathbf{j})$ into two cases:
<u>Good</u>: $|\rho_v(\mathbf{j})| \geq c|\sigma_v(\mathbf{j})|^{1-\eta}$.
<u>Bad</u>: $|\rho_v(\mathbf{j})|$ is not suitably large.

Bad case: Suppose for exposition that $\rho_v(\mathbf{j}) = 0$. Then $2(j_v - j_{v,1})(j_v - j_{v,3}) = \sum_{u \text{ child of } v} \varepsilon_u \rho_u(\mathbf{j})$.

Thus $(j_v - j_{v,1})$ is a factor of $\sum_u \pm\rho_u(\mathbf{j})$. The latter depends only on those $j_w : w < v$.

An integer $\boxed{N \text{ has } O(N^\varepsilon) \text{ factors}}$ for arbitrarily small $\varepsilon > 0$.

Thus $(j_v - j_{v,1})$ can take on at most $C_\varepsilon N^\varepsilon$ values, where $N = \max_u |\rho_u(\mathbf{j})|$ where u ranges over the 3 children of v.

Thus we can break our sum up into $O(N^{\varepsilon})$ subsums. In each, $(j_v - j_{v,i})$ are uniquely determined, for $i = 1, 2, 3$, as functions of other indices j_w. The factor $|\sigma_v(\mathbf{j})|^{-1}$ is then no longer needed.

The lost factor of $O(N^{\varepsilon})$ is harmless if each child u of v is good, for then we have factors of $\langle\sigma_u(\mathbf{j})\rangle^{-1}$ which can be used to absorb the small loss since $N^{\varepsilon}\langle\sigma_u(\mathbf{j})\rangle^{-1+\eta} \lesssim \langle\sigma_u(\mathbf{j})\rangle^{-1+\delta}$.

A small additional argument is needed to handle chains of bad vertices (for given $\mathbf{j}$), but we won't discuss that today.