Three Remarks on Degenerate Kohn Laplacians

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Topics:
(i) Off-diagonal behavior of Szegö kernels associated to high powers of positive line bundles
(ii) Hypoellipticity for $\Box_b$ on domains of infinite type
(iii) Compactness of the Neumann problem for very weakly pseudoconvex Hartogs domains.

Part (iii) is joint work with Siqi Fu.

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Schrödinger operators with both electric and magnetic potentials arise in connection with CR manifolds with S^1 symmetry. This situation arises in two ways:

(1) If L is a holomorphic line bundle over a compact complex manifold M, equipped with a Hermitian metric h, and if X is the unit circle bundle in L^*, then X is such a CR manifold.

(2) If $\Omega \subset \mathbb{C}^{n+1}$ is a Hartogs domain

$$|z_0|^2 + \varphi(z_1, \ldots, z_n) < 1$$

where $\Delta\varphi \geq 0$ ($\Leftrightarrow$ pseudoconvexity), then $\partial\Omega$ is such a CR manifold.

In local coordinates $(z, \theta) \in \mathbb{C}^n \times (\mathbb{R}/2\pi\mathbb{Z})$, Cauchy-Riemann operators take the form

$$\overline{L}_j = (\partial_{x_j} + \phi_{y_j}\partial_\theta) + i(\partial_{y_j} - \phi_{x_j}\partial_\theta)$$

Taking $n = 1$ for simplicity of notation, and performing Fourier decomposition in θ, associated second-order operators are

$$-(\partial_x + i\lambda\phi_y)^2 - (\partial_y - i\lambda\phi_x)^2 + \lambda\Delta\phi$$

where Δ is the Laplacian in $z = x + iy$.
$\lambda \in \mathbb{Z}^+$ is Fourier variable dual to θ.

Electric potential $= \lambda\Delta\phi$.
Magnetic potential $= (\lambda\phi_y, -\lambda\phi_x)$.

Always assume: $\Delta\varphi \geq 0$.

Szego kernels associated to high powers of positive line bundles

- L line bundle over complex manifold M
- $h =$ positive C^∞ Hermitian metric on L
- M equipped with associated Kähler metric
- $X =$ unit circle bundle in L^*
- h is positive $\Leftrightarrow$ X is strictly pseudoconvex.

- $S(z, \theta; z', \theta')$ Szegö kernel on $X \times X$.
- $\mathcal{H}_\lambda =$ space of all CR functions on X satisfying $f(z, \theta + \beta) \equiv e^{i\lambda\beta} f(z, \theta)$.
- $\mathcal{H}_\lambda$ is in natural isometric correspondence with the Hilbert space of all sections of L^λ.
- Π_λ orthogonal projection of $L^2(X)$ onto $\mathcal{H}_\lambda$.

In local coordinates on M

$$\Pi_\lambda(z, \theta, z', \theta') \equiv e^{i\lambda(\theta - \theta')} \Pi_\lambda(z, z')$$
$$\Pi_\lambda(z, z') = (2\pi)^{-1} \int S(z, \theta; z', 0) e^{-i\lambda\theta} \, d\theta.$$

Question (Zelditch) How rapidly does $\Pi_\lambda(z, z')$ decay as $\lambda \to +\infty$, for distance$(z, z') \geq \delta > 0$? Is it $O(e^{-c\lambda})$?

For C^ω metrics h, YES. (Consequence of analytic hypoellipticity theory + exponential decay of Fourier coefficients of C^ω functions.)

This is question about **partial** regularity (with respect to θ only). Data is C^ω WRT θ.

Boutet de Monvel–Sjöstrand have given a complete asymptotic expansion of S (in sense of ψdo/FIO theory). Every term in that expansion is C^ω as function of θ, off of diagonal in $X \times X$.

Theory of Schrödinger operators with electric (only) potentials suggests $O(e^{-c\sqrt{\lambda}})$ and no better (Agmon metric ideas).

"Electric" effect does always imply $O(e^{-c\sqrt{\lambda}})$, even for C^2 metrics.

Motivation:
Long-range correlations of random zeros

Bleher, Shiffman, and Zelditch have developed a theory of random zeros (and zero varieties) of holomorphic sections of high powers L^λ of *positive* holomorphic line bundles over compact complex manifolds.

(Example 0: Homogeneous complex polynomials of degree λ, as $\lambda \to \infty$.)

This is a variant of the theory of random matrices (random eigenvalues).

They have shown that statistical properties of interest — e.g. **correlations** of random zeros — can be expressed entirely in terms of associated Szegö kernels Π_λ (and their first and second covariant derivatives).

Theorem For any C^∞ positive metric h,

- $\Pi_\lambda(z,z') = O(e^{-A\sqrt{\lambda}\sqrt{\log\lambda}})$ $\forall A < \infty$ away from the diagonal $z = z'$
- and no better, in general.

Theorem (universality)
Π_λ exhibits universal behavior on scale $c\sqrt{\frac{\log\lambda}{\lambda}}$, in the sense that

$$\frac{\lambda^{-n}\Pi_\lambda(z' + \lambda^{-1/2}\zeta, z',)}{\Pi_*(\zeta,0)} \to 1$$

as $\lambda \to +\infty$, uniformly for $|\zeta| \leq C\sqrt{\log\lambda}$, in appropriate coordinates around any $z' \in M$.

Here the universal limiting situation is a trivial bundle over $\mathbb{C}^n$, with metric $e^{-|z|^2}$, and

$$\Pi_*(\zeta,\zeta') = \pi^{-n} e^{\zeta\cdot\bar{\zeta}'} e^{-\frac{1}{2}|\zeta|^2 - \frac{1}{2}|\zeta'|^2}$$

Remarks

There certainly exist nonanalytic metrics for which Π_λ does decay exponentially off of the diagonal. Don't know any general statements.

One can formulate similar results in other function spaces than C^∞. Rate of decay varies with regularity assumption.

The example is based on a nonlinear eigenvalue construction which has been used to produce various examples of nonhypoellipticity, in C^ω and C^∞, in various situations where there is some sort of "geometric" degeneracy. The novelty here is its application to a different kind of degeneracy (lack of smoothness).

C^∞ **hypoellipticity of** $\Box_b$

Theorem. For any pseudoconvex C^∞ **tube type domain** in $\mathbb{C}^2$, the following are equivalent:
(i) $\overline{\partial}_b$ is C^∞ hypoelliptic, modulo its nullspace.
(ii) The CR structure is not **exponentially degenerate**.
(iii) The quadratic form $Q(u) = \|\overline{\partial}_b u\|^2 + \|\overline{\partial}_b^* u\|^2$ satisfies a **superlogarithmic** Sobolev estimate.

• Tube type: Im$(z_2) >$ Re(z_1).
Pseudoconvex $\Leftrightarrow$ **convex**.
These domains have two-dimensional family of symmetries.

• Hypoelliptic modulo nullspace:
$\overline{\partial}_b^* u \in C^\infty$ wherever $\Box_b u = \overline{\partial}_b \overline{\partial}_b^* u \in C^\infty$.

In coordinates $(x+iy,t)$ such that $\overline{\partial}_b = \partial_x + i(\partial_y + \phi'(x)\partial_t)$:

• **Exponentially degenerate** means there exists a sequence of intervals $I_\nu \subset \mathbb{R}^1$ and $\delta > 0$ such that $|I_\nu| \to 0$ and

$$\int_{I_\nu} \phi''(x)\,dx \le e^{-\delta/|I_\nu|}.$$

Example: $\phi''(x) = e^{-c/|x|}$
but the behavior of a general nonnegative smooth ϕ'', satisfying no finite type condition, can be quite wild.

• **Superlogarithmic Sobolev estimate** means

$$\|\overline{\partial}_b u\|^2 + \|\overline{\partial}_b^* u\|^2 \ge \int |\widehat{u}(\xi)|^2 w^2(\xi)\,d\xi$$

for some w satisfying

$$\frac{w(\xi)}{|\log|\xi||} \to \infty$$

as $|\xi| \to \infty$.

It is hereby emphasized that C^∞ hypoellipticity is in general **NOT** equivalent to the superlogarithmic Sobolev estimate.

For instance, I've investigated three-dimensional CR structures which are strictly pseudoconvex everywhere except on a single real curve, transverse to the complex tangent space. In this case hypoellipticity can hold in some cases where the Levi form vanishes arbitrarily rapidly. On the other hand, C^∞ hypoellipticity does not always hold for such structures.

Moreover, $\lambda(z) \geq e^{-d(z,\Gamma)^{-M}}$ implies hypoellipticity for any finite M, where Γ is the degenerate locus and $d(z, \Gamma)$ is the distance to Γ.

This is a problem about the 2-parameter family of ODEs ($z \in \mathbb{C},\ \ \tau \in \mathbb{R}^+$) as $\tau \to +\infty$

$$\mathcal{L}_{z,\tau} = (-\partial_x + (z - \tau\phi'(x))) \circ (\partial_x + (z - \tau\phi'(x)))$$

Make ϕ'' equal to a nonzero constant outside a compact set. Then superlogarithmic estimate $\Leftrightarrow \langle \mathcal{L}_{z,\tau}u,\, u\rangle \gg (\log\tau)^2\|u\|^2$ as $\tau \to +\infty$, uniformly for $z \in \mathbb{R}$.

Morimoto (1987) showed that superlogarithmic Sobolev estimate implies hypoellipticity.

So proof in positive direction boils down to showing that less than exponential degeneracy of "Levi form" ϕ'' implies this lower bound on spectrum.

This boils down to simple estimates for 1D Schrödinger operators, using Poincaré-type inequalities.

Kohn has established related inequalities of Poincaré type for $\Box_b$ in higher dimensions. This is much more subtle and I believe it is an important advance.

New part: the negative implication
[Exponentially degenerate] $\Rightarrow$ [Not hypoelliptic]

Then this amounts to

Proposition. If ϕ is exponentially degenerate then there exists a sequence $(z_\nu, \tau_\nu) \in \mathbb{C} \times \mathbb{R}^+$ such that $\tau_\nu \to +\infty$, $|\mathrm{Im}(z_\nu)| \leq C \log \tau_\nu$, and $\mathcal{L}_{z_\nu, \tau_\nu}$ has nontrivial nullspace in the Schwartz class.

This is equivalent to finding zeros (z_ν, τ_ν) of the Wronskian

$$\mathcal{N}(z, \tau) = \int_{\mathbb{R}} e^{2(zs - \tau\phi(s))}\, ds.$$

Third topic:
Compactness of $\Box_b^{-1}$ for Hartogs domains.

• Global regularity problem: Under what circumstances does Bergman projection $L^2(\Omega) \mapsto H^2(\Omega)$ preserve $C^\infty(\overline{\Omega})$?
• Not always, for C^∞ pseudoconvex domains.
• *Compactness* of the quadratic form

$$\int_\Omega |\bar{\partial}u|^2 + |\bar{\partial}^* u|^2$$

(for u satisfying certain boundary condition) implies this type of global regularity, but is a *local* property of $\partial\Omega$ and hence should be stabler/more robust.
• Matheos (1998) gave an example of a pseudoconvex Hartogs domain in $\mathbb{C}^2$, whose boundary contains no *complex disks*, for which compactness nonetheless fails.

His analysis was based on making electric field $\Delta\phi \equiv 0$ on "thick" set with no interior (but nonempty "fine interior" in language of potential theory), *and* making magnetic field sufficiently weak.

Theorem/Example (joint with Siqi Fu):
The magnetic field can be a source of compactness when the electric field is too weak.

Thus compactness is an even subtler issue than previous examples had shown.

But our example possesses little regularity. I am very skeptical about producing a C^∞ domain by our construction.

For a domain $V_0 \subset \mathbb{C}^1$, with Dirichlet boundary condition: Assume $\Delta\psi \geq 0$.

$$\mathcal{D}_\psi = (\partial_x + i\psi_y,\, \partial_y - i\psi_x)$$
$$\lambda_\psi = \inf_{\|u\|_{L^2}=1} \int_{V_0} \left[|\mathcal{D}u|^2 + \Delta\psi|u|^2\right]$$
$$\tilde{\lambda}_\psi = \inf_{\|u\|_{L^2}=1} \int_{V_0} \left[|\nabla u|^2 + \Delta\psi|u|^2\right]$$

Theorem/Example (restated):
There exists subharmonic φ such that

- $\tilde{\lambda}_{n\varphi}$ remains bounded as $n \to +\infty$
- $\lambda_{n\varphi} \to \infty$ as $n \to +\infty$

and $\varphi \in C^0$, $\nabla\varphi \in L^2$.

To make $\tilde{\lambda}_{n\varphi}$ bounded, one essentially needs a nontrivial function with finite Dirichlet norm, supported on $\{z : \Delta\varphi(z) = 0\}$. Thus this set must be "thick".

To make $\lambda_{n\varphi} \to +\infty$, one needs to show that the magnetic Dirichlet form $\int |\mathcal{D}_{n\varphi} u|^2$ is very strong as $n \to +\infty$, for functions u supported where $\Delta\varphi = 0$.

Major complication: unlike the electrical potential $n\Delta\varphi$, this expression is not at all *monotonic* in n.

Slides 19-22 were not included in the lecture given at the Gunning-Kohn lecture.

Proof that Π_λ decays "slowly" in some cases

Let dimension $n = 1$, coordinates $z = x + iy$. Let

$$\overline{D}_\lambda = \partial_x + i\partial_y + \lambda a'(x)$$

where $a'' > 0$.
Let $k(\lambda)$ grow faster than $\sqrt{\lambda \log \lambda}$.

Because
- $I - \Pi_\lambda = \overline{D}_\lambda^* \left(\overline{D}_\lambda \overline{D}_\lambda^*\right)^{-1} \overline{D}_\lambda$

and
- $\overline{D}_\lambda$ does have a right inverse which is $O(\exp(-c\lambda\,\text{distance}(z, z')))$ (easily constructed),

it suffices to construct example where

Green's kernel for $\overline{D}_\lambda \overline{D}_\lambda^*$
is not $O(e^{-k(\lambda)})$ off diagonal.

Thus want u_λ satisfying:

$\overline{D}_\lambda \overline{D}_\lambda^* u_\lambda \equiv 0$

$u_\lambda = O(e^{\varepsilon k(\lambda)})$ for all $\varepsilon > 0$
and same for $\overline{D}_\lambda^* u_\lambda$.

$|\overline{D}_\lambda^* u_\lambda(0)| \geq c_0 > 0$.

First two bounds hold in a fixed neighborhood of 0.

All this for a sequence of values $\{\lambda_j\} \to +\infty$.

Nonlinear Eigenvalue argument

- Acting on functions $u_\lambda(x) = e^{i\xi y} f(x)$,

$\overline{D}_\lambda \overline{D}_\lambda^* \leftrightarrow (\partial_x - \xi + \lambda a'(x)) \circ (-\partial_x - \xi + \lambda a'(x))$.

- Nontrivial solution:

$$f(x) = e^{-\xi x + \lambda a(x)} \int_{-\infty}^{x} e^{2[\xi t - \lambda a(t)]} \, dt.$$

- For $\xi \in \mathbb{R}$, this function is much larger for $x \sim \pm 1$ than for $x = 0$, by an unacceptable factor $\sim e^{c\lambda}$.

- Introduce

$$\mathcal{N}_\lambda(\xi) = \int_{\mathbb{R}} e^{2[\xi t - \lambda a(t)]} \, dt.$$

If $\xi \in \mathbb{C}$ satisfies the nonlinear eigenvalue equation

$$\mathcal{N}_\lambda(\xi) = 0$$

(and if $\mathrm{Re}(\xi)$ is appropriate) then f can be essentially maximal near $x = 0$.

• We lose a factor of $e^{c|\text{Im}(\xi)|}$ here from nonunimodularity of $e^{i\xi y}$.
• Thus it is essential to have a zero in the strip $|\text{Im}(\xi)| \le k(\lambda)$.

Counterexample boils down to:
Construct $a(x) \in C^\infty$, a small perturbation of x^2, such that there exists a sequence $\lambda_\nu \to +\infty$, $\xi_\nu \in \mathbb{C}$ such that

$$0 = \mathcal{N}_{\lambda_\nu}(\xi_\nu) \text{ and } |\text{Im}(\xi_\nu)| \le k(\lambda_\nu).$$

• Ordinarily one would analyze

$$\mathcal{N}_\lambda(\xi) = \int_{\mathbb{R}} e^{2[\xi t - \lambda a(t)]}\, dt$$

by stationary phase and deformation of contour of integration into complex domain — but here $a(t)$ is not analytic. We use this to our advantage, adding perturbations to a to create a sequence of zeros, optimally located.

Construction of an example for which $\lambda_{n\varphi} \to \infty$, yet $\tilde{\lambda}_{n\varphi}$ remains bounded

• (Kato:) For any real-valued ψ,

$$|\mathcal{D}_\psi u| \geq \Big| \nabla |u| \Big|$$

a.e. for $u \in C^1$.

• A model situation in which magnetic field has strong effect:
Let A be an annulus centered at $z = 0$, and D be the disk inside the inner radius of A. Suppose that $\Delta\psi \equiv 0$ on A. Then

$$\int_A |\mathcal{D}_\psi u|^2 \geq c w^2 \int_A |u(z)|^2 |z|^{-2}$$

where

$$w = \int_D \Delta\psi \quad \underline{\underline{\text{modulo } 2\pi\mathbb{Z}}}.$$

• If the outer radius of A is small, the factor $|z|^{-2}$ represents a strong magnetic effect.

• Situation is highly nonmonotonic as function of $\Delta\psi$. If $w \in 2\pi\mathbb{Z}$ then via a gauge transformation, $\mathcal{D}_\psi$ is unitarily equivalent to ∇ in A; there is no magnetic effect.

Form of example:

- Start with $\Omega_0 =$ disk.

- Construct $\Omega = \Omega_0 \setminus \cup_{k=1}^{\infty} \cup_j D_j^k$ where $D_j^k = B(z_j^k, \rho_k)$ are many small disks.

- For each k, the centers z_j^k form an ε_k-net in $\Omega_0 \setminus \cup_{m=1}^{k-1} \cup_j D_j^m$.

- $\varepsilon_k = 2^{-k}$ and $\rho_k = e^{-k^4 2^{2k}}$. (Tiny disks $\Rightarrow$ nonempty fine interior.)

- $\Delta\varphi =$ sum of pieces supported on the disks D_j^k.

• Let $N_k = e^{c2^k}$. **Key:** Arrange that for each $n \in [N_k, N_{k+1})$, there exist lots of "good" disks D_j^k such that

$$n \int_{D_j^k} \Delta\varphi \in [\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \;\; \text{modulo } 2\pi\mathbb{Z}.$$

• More specifically:
Each $\varepsilon_k^{1/2}$–ball in $\Omega_0 \setminus \cup_{m<k} \cup_j D_j^k$ contains at least one such good disk D_j^k.

Combine:

• Favorable magnetic inequality in a small "annulus" about each good disk D_j^k

• Global control of ∇

• High "density" of good disks.

Slides 27-32 were not included in the lecture given at the Gunning-Kohn conference.

Proof that $\Pi_\lambda(z,z') = O(\exp(-A\sqrt{\lambda \log \lambda}))$ off the diagonal:

• Suffices to estimate fundamental solution $G(z,z')$ for $\Box_\lambda = \overline{D}_\lambda \overline{D}_\lambda^*$, because

$$I - \Pi_\lambda = \overline{D}_\lambda^* {\Box_\lambda}^{-1} \overline{D}_\lambda.$$

• Choice of coordinates (origin at $z' = 0$) plus gauge transformation effects reduction to
$\overline{D}_\lambda = e^{-\lambda\phi}\bar{\partial}e^{\lambda\phi}$
where $\phi(z) = |z|^2 + O(|z|^3)$.

• Decompose $\phi = Q + R$ where
$Q =$ polynomial, $R = O(|z|^N)$.

• Apply analytic hypoellipticity theory to fundamental solution for $\Box_{\lambda,Q}$ to deduce that its fundamental solution (with pole at 0) is

$$G_{\lambda,Q}(z,0) = O(e^{-c\lambda|z|^2})$$

(ignoring singularity at $z = 0$).

• *Uniformity*: Bound is <u>independent of Q</u>, provided that λ is sufficiently large, depending on order of $Q =$ Taylor polynomial.

• Smoothly truncate resulting kernel $G_{\lambda,Q}(z,z')$ to region where distance$(z,z') < r$, where $r = r(\lambda)$. Get approximate inverse P_λ for $\Box_\lambda$.

• How small is $\Box_\lambda \circ P_\lambda - I$?
Two sources of error:
(1) Derivatives of cutoff function: $O(e^{-c\lambda r^2})$.
(2) Contribution of $R = \phi - Q$: $O(r^N)$.
Optimize: these match at $r = A\sqrt{\log\lambda/\lambda}$.
Resulting error: $O(\lambda^{-cA})$.

• Get better parametrix via Neumann series; iterate $\Box_\lambda \circ P_\lambda - I$ about $cr^{-1} = cA^{-1}\big(\lambda/\log\lambda\big)^{1/2}$ times. Do bookkeeping.

Corollary/ small application.

• Correlations between zeros at z, z' are $O(e^{-A\sqrt{\lambda \log \lambda}})$, provided distance$(z, z') \geq \delta$.

• Correlations appear to be much stronger than $O(\exp(-c\lambda))$.

However: Last part is not proved. Correlations are expressed as particular homogeneous expressions in terms of $\Pi_\lambda(z, z')$, $\Pi_\lambda(z, z)$, and $\Pi_\lambda(z', z')$, and their first and second covariant derivatives. I have not yet verified that there isn't some miraculous cancellation in these particular expressions (but believe such cancellation to be highly unlikely).

The universality proposition extends result of Bleher, Shiffman, and Zelditch, who showed this for the natural scale $\lambda^{-1/2}$.

I believe that $\sqrt{(\log \lambda)/\lambda}$ is best possible universality scale for C^∞ metrics. Certainly λ^δ is impossible for $\delta > -1/2$.

More details than you want to see:

- $a(x) = x^2 + \sum_j a_j(x)$

- $\{\lambda_j\}$ very rapidly increasing to $+\infty$.

- $\xi_j = \lambda_j^{1/2} + ik_j z$ ($z = z_j$ is TBA)

- $a_j(\frac{1}{2}\lambda_j^{-1/2} + k_j{}^{-1}t) = \frac{\beta_j}{2\lambda_j}\alpha(t)$

- $\alpha \in C^\infty$ with compact support is fixed.

- $\beta_j = O(\lambda_j^{-N})$ for all N.

- (Consequently) the real part of the phase $\xi_j t - \lambda_j a(t)$ has critical point in the support of a_j.

- Those a_i with $i \neq j$ contribute very little to $\mathcal{N}_{\lambda_j}(\xi_j)$, provided $z_j = O(1)$.

Modulo less significant terms,

$$\mathcal{N}_{\lambda_j}(\xi_j) \approx \int_{\mathbb{R}} e^{-2s^2+2i\lambda_j^{-1/2}k_j s}\,ds \\ - \beta_j \lambda_j^{1/2} k_j^{-1}\Big(\widehat{\alpha}(-2z) + O(\lambda_j k_j^{-2})\Big).$$

• For $z = O(1)$, the "main" (first) term is $O(e^{-ck_j^2\lambda_j^{-1}})$.

• The second term is larger, if $\beta_j \gg e^{-ck_j^2\lambda_j^{-1}}$.

• This is consistent with the requirement $\beta_j = O(\lambda_j^{-N})$ for all N ($\Leftrightarrow$ smoothness of a) provided $k(\lambda)^2 \gg \lambda \log \lambda$.

• Apply Rouché's theorem near a fixed zero of the Fourier transform $\widehat{\alpha}$ to obtain a zero $z = z_j$. $\square$

Two-point correlations:

$$\frac{\text{Prob}\,(\text{zero at } z \text{ and zero at } z')}{\text{Prob}\,(\text{zero at } z) \times \text{Prob}\,(\text{zero at } z')} - 1$$

where Prob ($\star$) denotes probability density.

• Estimates for such Szegö kernels, asymptotically as $\lambda \to \infty$, thus have implications for this theory.

• Topic of much physical interest: *universality*.