ILLPOSEDNESS OF A SCHRÖDINGER EQUATION WITH DERIVATIVE NONLINEARITY

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ABSTRACT. The nonlinear Schrödinger equation with nonlinearity uu_x is shown to be illposed in any Sobolev space $H^s(\mathbb{R}^1)$, for any exponent s. Solutions are constructed which have arbitrarily small norms at time zero, and attain arbitrarily large norms after an arbitrarily short time. Results of the same type are established in the periodic case, for nonlinearities $u^{p-1}u_x$ where p is any integer ≥ 2.

1. INTRODUCTION

This is the fourth in a series of papers [2, 3, 4] devoted to the illposedness of the Cauchy problem for various nonlinear evolution equations. Here we study a nonlinear Schrödinger equation

$$u_t + iu_{xx} + uu_x = 0 \tag{1.1}$$

whose nonlinear term incorporates a first derivative. Whereas earlier papers in the series were largely concerned with establishing sharp thresholds for illposedness in the scale of Sobolev spaces, here we treat an equation which is illposed in *all* Sobolev spaces, in any reasonable sense.

Molinet, Saut, and Tzvetkov [9] proved that the mapping from initial datum to solution, considered as a mapping from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$, fails to be C^2 at $t = 0$, no matter how large the index s may be. This shows that the equation is not amenable to analysis by the fixed-point method, but leaves open the possibility of weaker forms of wellposedness, in which one might for instance have C^0 dependence of solution on initial datum. We establish here (by a fixed-point method) a more extreme form of illposedness: There exist initial data in H^s with arbitrarily small norm, for which the solution attains an arbitrarily large norm, after an arbitrarily short time.

Burgers' equation $u_t + uu_x = 0$ and the derivative nonlinear Schrödinger equation (1.1) share a symmetry: If $u(t, x)$ is a solution, so is $u(t, x - vt) + v$ for any constant v. When v is not real, $x - vt$ immediately leaves the real domain and hence $u(t, x - vt)$ makes no sense unless u is analytic or has other special properties. Thus for complex-valued initial data, in the periodic case $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ where constant functions belong to Sobolev spaces, it is apparent that there can be no wellposedness in any Sobolev space.

We begin with a discussion of Burgers' equation on $\mathbb{T}$, constructing solutions with *complex*-valued initial data which are arbitrarily small in H^s, yet achieve genuine blowup in finite time. This sets the stage for a modification incorporating the dispersive term iu_{xx}. Both equations are analyzed in the simpler periodic case, where the conclusion should be considered obvious in light of the symmetry discussed in the preceding paragraph.

Lastly we carry out a second modification to convert the construction from $\mathbb{T}$ to $\mathbb{R}$. In this final step a significant further issue arises. Whereas the more general equation $u_t + iu_{xx} + u^{p-1}u_x$ is illposed in all Sobolev spaces $H^s(\mathbb{T})$ for all integers $p \geq 2$, it is wellposed in $H^s(\mathbb{R})$ for $p \geq 3$ and s sufficiently large. The distinction between the two emerges in the details of the conversion from periodic to nonperiodic.

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2. Complex-valued Burgers' Equation

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and consider Burgers' equation with complex-valued periodic initial data:

$$\begin{cases} u_t = -uu_x & \text{for } (t,x) \in \mathbb{R} \times \mathbb{T} \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{T}. \end{cases} \tag{2.1}$$

Definition 1. By a very weak solution on $[0,T]\times\mathbb{T}$ we mean a generalized function $u(t,\cdot)$ whose Fourier coefficients $\widehat{u}(t,n)$ vary continuously with time, vanish identically for all $n < 0$, and are $O(e^{C|n|})$ for some $C < \infty$, and satisfies the equation in the weak sense with respect to test functions $\varphi(t,x)$ which for each $t \in [0,T]$ are trigonometric polynomials.

A trigonometric polynomial is a function whose Fourier transform has bounded support. Here integrals such as $\int \varphi(t,x)u(t,x)\partial_x u(t,x)\,dx$ are interpreted in terms of Fourier coefficients; the assumption that $\widehat{u}(t,n) \equiv 0$ for $n < 0$ guarantees convergence of the resulting infinite series.

Theorem 1. *Let $s < \infty$ and $\varepsilon > 0$. There exist a complex-valued initial datum $u_0 \in H^s$ satisfying $\|u_0\|_{H^s} < \varepsilon$, a time $0 < T < \varepsilon$, and a solution $u \in C^1([0,T], C^\infty(\mathbb{T}))$ of (2.1) such that $\|u(T,\cdot)\|_{L^2(\mathbb{T})} > \varepsilon^{-1}$.*

There exist a very weak solution defined for all $t \in [0,\infty)$, and times $0 < T'' < T' < \varepsilon$, such that $\|u_0\|_{H^s} < \varepsilon$, $u \in C^1([0,T''], H^s))$ every Sobolev norm $\|u(t,\cdot)\|_{H^\sigma}$ is a nondecreasing function of t, and $\|u(T',\cdot)\|_{H^\sigma(\mathbb{T})} = \infty$ for all $\sigma > -\infty$.

This should be contrasted with the same equation for real-valued initial data, which is wellposed in H^s for all $s > 3/2$.

The second conclusion of the theorem is a bit awkward. Another way to put it is that there exists $\tau < \varepsilon$ such that $u \in C^\infty$ in $[0,\tau) \times \mathbb{T}$, and all Sobolev norms are infinite for all $t > \tau$. But we have not determined whether any norm is infinite at $t = \tau$.

Hounie and dos Santos Filho [5] have shown that there is a large (in the sense of Baire category) class of complex-valued initial data for which there exists no solution in C^1. Our initial data, in contrast, are real analytic, so solutions do exist for some time interval.

Proof. Let $N \in \mathbb{N}$ be a large parameter, and set

$$u_0(x) = iA + iBe^{iNx} \tag{2.2}$$

where $A, B \in \mathbb{R}^+$. We will eventually choose

$$A = \varepsilon,\ B = \varepsilon N^{-s}, \tag{2.3}$$

and show that the resulting solution has the properties stated in the theorem, but will first carry out the computation on a formal level for arbitrary A, B, N.

We seek a solution in the form

$$u(t,x) = \sum_{n=0}^{\infty} a_n(t)e^{inNx} \tag{2.4}$$

with the initial conditions $a_0(0) = iA$, $a_1(0) = iB$ and $a_n(0) = 0$ for all $n \geq 2$. This is formally equivalent to the infinite system of ordinary differential equations

$$\frac{da_n}{dt} = -\sum_{k=0}^{n} ikNa_k a_{n-k}. \tag{2.5}$$

In particular,

$$a_0(t) \equiv a_0(0) \equiv iA \tag{2.6}$$

is constant, and so for $n \geq 1$

$$\frac{da_n}{dt} = Nna_n - \sum_{k=1}^{n-1} ikNa_k a_{n-k}. \tag{2.7}$$

Thus

$$a_1(t) \equiv Be^{ANt}. \tag{2.8}$$

Substitute

$$a_n(t) = i\alpha_n(t)t^{n-1}e^{NAnt} \qquad \text{for } n \geq 1 \tag{2.9}$$

and set $a_0(t) \equiv i\alpha_0$ where $\alpha_0 = A$, $a_1(t) \equiv i\alpha_1 e^{ANt}$ where $\alpha_1 = B$. The differential equation for the remaining coefficients becomes

$$\frac{d}{dt}(\alpha_n t^{n-1}) + NAn\alpha_n t^{n-1} = NAn\alpha_n t^{n-1} + Nt^{n-2}\sum_{k=1}^{n-1} k\alpha_k\alpha_{n-k} \qquad n \geq 2. \tag{2.10}$$

Since $\alpha_n(0) = 0$ for all $n \geq 2$,

$$\alpha_n(t) = Nt^{-(n-1)}\sum_{k=1}^{n-1} k\int_0^t \alpha_k(s)\alpha_{n-k}(s)s^{n-2}\,ds. \tag{2.11}$$

There is a solution with *constant* α_n satisfying the recurrence

$$\alpha_n = N\sum_{k=1}^{n-1}\frac{k}{n-1}\alpha_k\alpha_{n-k}. \tag{2.12}$$

The denominator $n-1$ partially mitigates the derivative in the nonlinearity and will be useful below. All the coefficients α_n are real and strictly positive.

To analyze the growth of these coefficients as $n \to \infty$ it is convenient to substitute $\alpha_n = \delta N^{-1}\lambda^n\beta_n$ for $n \geq 1$, where $\delta, \lambda > 0$ are to be chosen. (2.12)(2.12) becomes

$$\beta_n = \delta\sum_{k=1}^{n-1}\frac{k}{n-1}\beta_k\beta_{n-k} \leq \delta\sum_{k=1}^{n-1}\beta_k\beta_{n-k} \qquad \text{for } n \geq 2, \tag{2.13}$$

so we choose δ to be a certain small constant. Moreover $\beta_1 = \delta^{-1}N\lambda^{-1}B$, so we choose $\lambda = \delta^{-1}NB$ in order to normalize $\beta_1 = 1$.

This normalized recursion defines coefficients β_n satisfying

$$0 < \beta_n \leq 2n^{-2} \tag{2.14}$$

provided that δ is chosen to be sufficiently small. Indeed, this bound follows directly by induction since

$$\delta\sum_{k=1}^{n-1} 4k^{-2}(n-k)^{-2} \leq n^{-2}$$

for small δ. In particular $\beta_n \leq 2$ for all n. There is also a lower bound

$$\beta_n \geq \delta^n,$$

as follows by induction since

$$\beta_n = \delta\sum_{k=1}^{n-1}\frac{k}{n-1}\beta_k\beta_{n-k} > \delta\beta_1\beta_{n-1}.$$

Thus for $n \geq 2$

$$0 < -ia_n(t) \leq C^{n+1}B^n N^{n-1}t^{n-1}e^{ANnt} \tag{2.15}$$

for some finite constant C independent of A, B, N. We have also shown that conversely

$$|a_n(t)| \geq C^{-n-1} B^n N^{n-1} t^{n-1} e^{ANnt}, \tag{2.16}$$

and this will be of significance.

Now specialize to $A = \varepsilon$, $B = \varepsilon N^{-s}$. The factor $B^n Nn - 1$ decays exponentially as $n \to \infty$. The exponent $ANt = \varepsilon Nt$ becomes large for $t \gg \varepsilon^{-1}N^{-1}$, but the product te^{ANt} remains small for a slightly larger span of time. Thus there exist a timespan $0 < T_1$ such that $NT_1 \to \infty$ as $N \to \infty$, such that $|a_n(t)| \leq C_N^{1+n}$ for all n, for all $t \in [0, T_1]$. From this it follows easily that the series converges in H^σ for every $\sigma < \infty$ for all $t \in [0, T_1]$, and defines a function $u(t, x) \in C^\infty([0, T_1], H^\sigma)$.

The coefficients $|a_n(t)|$ are increasing functions of t. According to (2.16), the series $\sum_n |a_n|^2 = c\|u(t, \cdot)\|_{L^2}^2$ diverges as soon as $te^{\varepsilon Nt}N^{1-s}$ exceeds a certain finite constant, independent of all parameters. This certainly happens for $t = N^{-\rho}$ for any fixed $0 < \rho < 1$, for all sufficiently large N. Thus there exists $T \in (\varepsilon^{-1}N^{-1}, N^{-\rho})$ for which $u(t, \cdot) \in L^2$ for all $t \in [0, T)$, but $\|u(t, \cdot)\|_{L^2} \to \infty$ as $t \to T^-$. Plainly the series converges in $C^\infty([0, T) \times \mathbb{T})$ and defines a solution of (2.1) there. $\square$

Remark. Let $\sigma > 0$, and fix an arbitrary $\varepsilon > 0$. Rapid growth of $\|u(t, \cdot)\|_{H^\sigma}$ comes about in two distinct phases. $\|u(t, \cdot)\|_{H^\sigma}^2 \geq cN^{2\sigma}|a_1|^2 = c\varepsilon^2 N^{2\sigma-2s}e^{2\varepsilon Nt}$. A short algebraic calculation shows that there exists a time $0 < T' = T'_N \ll T$ such that $N^{2\sigma}|a_1(T'_N)|^2$ tends to infinity as $N \to \infty$, yet $\|\sum_{n>1} a_n(t)e^{iNnx}\|_{H^\sigma}$ remains uniformly bounded. Thus there is rapid norm growth in this first phase, due exclusively to growth of $|a_1|$. However, $|a_1(t)|$ never actually blows up. In the second phase, $T' < t < T$, growth of the higher-frequency modes dominates and u attains infinite Sobolev norms although it continues to exist as a very weak solution.

3. A Schrödinger equation in the periodic case

Consider next a nonlinear Schrödinger equation

$$\begin{cases} u_t = -iu_{xx} - uu_x & \text{for } (t, x) \in \mathbb{R} \times \mathbb{T} \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{T}. \end{cases} \tag{3.1}$$

We discuss its illposedness in $H^s(\mathbb{T})$, as a prelude to the discussion of $H^s(\mathbb{R})$.

Theorem 2. *For any $s < \infty$ and any $\varepsilon, \sigma > 0$ there exist an initial datum satisfying $\|u_0\|_{H^s} < \varepsilon$, a time $0 < T < \varepsilon$, and a solution $u \in C^\infty([0, T], C^\infty)$ of* (3.1) *such that*

$$\|u(T, \cdot)\|_{H^\sigma} > \varepsilon^{-1}. \tag{3.2}$$

As in the preceding discussion set $u_0(x) = iA + iBe^{iNx}$ and seek a solution $u(t, x) = \sum_{n=0}^{\infty} a_n(t)e^{iNnx}$. The coefficients are to satisfy

$$\frac{da_n}{dt} = iN^2n^2a_n - iN\sum_{k=0}^{n} ka_k a_{n-k}. \tag{3.3}$$

Once again $a_0(t) \equiv iA$, but now

$$a_1(t) \equiv iBe^{ANt}e^{iN^2t}. \tag{3.4}$$

Substitute

$$a_n(t) = \alpha_n(t)t^{n-1}e^{ANnt}e^{in^2N^2t} \tag{3.5}$$

to find the infinite system of integral equations[1]

$$\alpha_n(t) = N\sum_{k=1}^{n-1} t^{-(n-1)}k\int_0^t s^{n-2}e^{i2k(n-k)N^2s}\alpha_k(s)\alpha_{n-k}(s)\,ds \qquad \text{for all } n \geq 2. \tag{3.6}$$

[1]The factor of e^{ANt} is simply a manifestation of the symmetry $u(x, t) \mapsto A + u(t, x - At)$.

Since the equation for α_n involves α_k only for $k < n$, this system defines a sequence of complex-valued functions $\alpha_n \in C^\infty([0,\infty))$.

The only change here is the insertion of the unimodular factor $e^{i2k(n-k)N^2 s}$. In particular,[2]

$$|\alpha_n(t)| \le N \sum_{k=1}^{n-1} t^{-(n-1)} k \int_0^t s^{n-2} |\alpha_k(s)\alpha_{n-k}(s)|\, ds,$$

from which it follows that

$$|\alpha_n(t)| \le \alpha_n^{\text{Burger}} \tag{3.7}$$

for all n, t, where $\{\alpha_n^{\text{Burger}}\}$ is the sequence of constants defined by the recursion formula (2.12). Therefore we may conclude that the coefficients $a_n(t)$ arising in our solution of (3.1) satisfy

$$|a_n(t)| \le C^{n+1} B^n N^{n-1} t^{n-1} e^{ANnt} \qquad \text{for all } n \ge 1,\ t \in [0,\infty). \tag{3.8}$$

Consequently $u \in C^\infty([0,T], C^\infty(\mathbb{T}))$. Now $|a_1(t)| = Be^{ANt} = \varepsilon N^{-s} e^{\varepsilon N t}$, and the conclusions of Theorem 2 follow from this together with the discussion in the final remark of the preceding section. □

Remark. In this periodic case, the same reasoning yields the same conclusion for the equation $u_t + iu_{xx} + u^{p-1}u_x = 0$ for any integer $p \ge 2$. The only significant change is that the factors t^{n-1} must be replaced by $t^{(n-1)/(p-1)}$ in (3.5).

4. The nonperiodic case

Consider next the nonperiodic Cauchy problem

$$\begin{cases} u_t = -iu_{xx} - uu_x & \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}^1 \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^1. \end{cases} \tag{4.1}$$

Theorem 3. *For any $s < \infty$, any $\varepsilon, \sigma > 0$, and any $\sigma > s - \frac{1}{2}$ there exist an initial datum satisfying $\|u_0\|_{H^s} < \varepsilon$, a time $0 < T < \varepsilon$, and a solution $u \in C^\infty([0,T], C^\infty(\mathbb{R}) \cap H^s(\mathbb{R}))$ of* (4.1) *such that*

$$\|u(T,\cdot)\|_{H^\sigma} > \varepsilon^{-1}. \tag{4.2}$$

Remark. The restriction $\sigma > s - \frac{1}{2}$ is not essential; the conclusion holds for any $\sigma > -\infty$, and can be proved via an elaboration of the same argument.

Define $a(t,\xi) = \widehat{u}(t,\xi)$, the Fourier transform of $u(t,x)$ with respect to the spatial variable x. As in the periodic case, we seek solutions a supported where $\xi > 0$. The partial differential equation thus becomes the integrodifferential system

$$\frac{da}{dt} = \tfrac{i}{2}\xi^2 a - i\xi \int_0^\xi a(\eta)a(\xi-\eta)\, d\eta. \tag{4.3}$$

We consider initial data of the form

$$a(0,\xi) = i\rho\nu^{-1/2}\big(h(\nu^{-1}\xi) + Bh(\nu^{-1}(\xi - N))\big) \tag{4.4}$$

where ρ, ν, B are positive real parameters, $N \gg 1$, $\nu \ll 1$, $B \le N^{-s}$, and $0 \le h \in C^1$ is supported in $[\frac{3}{4}, 1]$, and does not vanish identically. Thus $\|u(0,\cdot)\|_{H^s} \sim \rho$, uniformly in ρ, ν, N, B.

Via the unimodular substitution

$$a(t,\xi) = ie^{i\xi^2 t}\alpha(t,\xi) \tag{4.5}$$

[2]This dispersive effect makes the coefficients α_n appreciably smaller, for $n \ge 2$, than in our discussion of Burgers' equation, for the relevant times $t \gtrsim N^{-1}$. However we will not exploit this effect at all.

we obtain the equivalent integral equation

$$\alpha(t,\xi) = -ia(0,\xi) + \tfrac{1}{2}\xi \int_0^t \int_0^\xi e^{2i\eta(\xi-\eta)t}\alpha(s,\eta)\alpha(s,\xi-\eta)\,d\eta\,ds. \tag{4.6}$$

Any solution of (4.6) gives rise to a solution of (4.3) by reversing the substitution. A contraction mapping argument establishes the existence of a C^1 solution $\alpha(t,\xi)$, defined for all $(t,\xi) \in [0,\infty) \times \mathbb{R}$ and supported where $\xi \geq \frac{3}{4}\nu$; moreover, this solution is unique within that class (including the support restriction). However, this does not yield satisfactory bounds as $\xi \to \infty$.

There is a new difficulty, not present in the periodic case: The oscillatory factor $e^{2i\eta(\xi-\eta)t}$ potentially decreases the integral unless $\eta = \xi$ or $\eta = 0$. In the discrete case we had a sum, to which $\eta = 0$ made a definite contribution, but that is no longer the case. We have compensated for this by taking $h(\nu^{-1}\xi)$ to be supported in a short interval $0 \leq \xi \leq \nu$; the smallness of ν will be an essential ingredient in forcing our solution to become large.

The next step is a digression which has to do not with largeness of the solution, but rather with its remaining reasonably small for some positive timespan. We seek an upper bound on $|a(t,\xi)|$ which is sufficiently strong to ensure that the inverse Fourier transform of $a(t,\cdot)$ has finite H^σ norm, for any $\sigma < \infty$, for all $t \in [0,T']$ for some strictly positive T', depending on ρ,ν,N. Thus we will have constructed a genuine solution of (4.1) for some timespan; for longer times our estimates do not preclude exponential growth of $a(t,\xi)$ as a function of ξ, so we can only assert existence of a solution in a very weak sense like that of Definition 1. The following lemma is the main step in the proof of this bound. Write $\langle y\rangle = 1 + |y|$.

Lemma 4. *Let $\nu \in (0,1]$ and $N \geq 1$. Let $\mu, B, C_0 \in \mathbb{R}^+$. Let $f(t,\xi)$ be a nonnegative C^1 function supported where $t,\xi \geq 0$ and satisfying a bound*

$$f(t,\xi) \leq \sum_{m+n\geq 1} \chi_{[mN,(m+n)\nu]}(\xi) C_0 (Nt)^{m+n-1} B^m \nu^{-1} \mu^{m+n} \langle m+n\rangle^{-4}, \tag{4.7}$$

where the double summation extends over all nonnegative integers m,n which are not both 0. *Then the function*

$$g(t,\xi) = \xi \int_{s=0}^t \int_{\eta=0}^\xi f(s,\eta) f(s,\xi-\eta)\,d\eta\,ds$$

is majorized for all $t,\xi \geq 0$ by CC_0 times the right-hand side of (4.7)*, for a certain constant $C < \infty$ independent of μ,ν,N,B,C_0.*

Proof. We have

$$g(t,\xi) \leq \sum_{i+j\geq 1}\sum_{k+l\geq 1} \xi \int_0^t \int_0^\xi \chi_{[iN,iN+(i+j)\nu]}(\eta)\chi_{[kN,kN+(k+l)\nu]}(\xi-\eta)$$
$$C_0 B^i (Ns)^{i+j-1}\nu^{-1}\mu^{i+j}\langle i+j\rangle^{-4}\, C_0 B^k (Ns)^{k+l-1}\nu^{-1}\mu^{k+l}\langle k+l\rangle^{-4}\,d\eta\,ds$$

where the sums are over nonnegative integers i,j,k,l.

Set $m = i+k$ and $n = j+l$, and temporarily fix a four-tuple (i,j,k,l). The integrand vanishes identically unless $iN \leq \eta \leq iN + (i+j)\nu$ and $kN \leq \xi - \eta \leq kN + (k+l)\nu$. Thus the contribution of the four-tuple (i,j,k,l) to $g(t,\xi)$ is supported where $mN \leq \xi \leq mN + (m+n)$. It suffices to prove that the sum of the contributions of all four-tuples with $(i+k, j+l) = (m,n)$ is bounded by CC_0^2 times the corresponding factor on the right-hand side of (4.7).

The integral with respect to s may be performed explicitly, yielding a majorization of this contribution by

$$\xi C_0^2\langle i+j\rangle^{-4}\langle k+l\rangle^{-4}N^{m+n-2}B^m\nu^{-2}\mu^{m+n}$$
$$(m+n-1)^{-1}t^{m+n-1}\int_0^\xi \chi_{[iN,iN+(i+j)\nu]}(\eta)\chi_{[kN,kN+(k+l)\nu]}(\xi-\eta)\,d\eta.$$

This last integral is $\le \nu\min(i+j,k+l)$, so we have an upper bound of

$$C\xi\langle i+j\rangle^{-4}\langle k+l\rangle^{-4}(m+n-1)^{-1}\min(i+j,k+l)N^{-1}\cdot\Big(C_0^2(Nt)^{m+n-1}B^m\nu^{-2+1}\mu^{m+n}\Big).$$

Since $\xi\le(m+n)(N+\nu)\le 2(m+n)N$, this is

$$\le C\sum_{i+k=m}\sum_{j+l=n}\langle i+j\rangle^{-4}\langle k+l\rangle^{-4}\min(i+j,k+l)\Big(C_0^2(Nt)^{m+n-1}B^m\nu^{-1}\mu^{m+n}\Big).$$

To conclude the proof it suffices to observe that

$$\begin{aligned}\sum_{i+k=m}\sum_{j+l=n}&\langle i+j\rangle^{-4}\langle k+l\rangle^{-4}\min(i+j,k+l)\\ &\le C\langle m+n\rangle^{-4}\sum_{(i+k,j+l)=(m,n);i+j\le k+l}\langle i+j\rangle^{-4}(i+j)\\ &\le C\langle m+n\rangle^{-4}\sum_{0\le i,j\le m+n}\langle i+j\rangle^{-3}\\ &\le C'\langle m+n\rangle^{-4},\end{aligned}$$

as desired. □

Remark. The same reasoning shows that if $\widehat{u}(0,\cdot)$ is supported in a compact subinterval of $[0,\infty)$ then $\widehat{u}(t,\xi)$ decays exponentially as a function of ξ as $\xi\to+\infty$, for all sufficiently small $t>0$.

Lemma 5. *The solution $\alpha(t,\xi)$ of (4.6) satisfies*

$$|\alpha(t,\xi)|\le\sum_{m+n\ge1}\chi_{[mN,(m+n)\nu]}(\xi)C_0(Nt)^{m+n-1}B^m\nu^{-1}\mu^{m+n}\langle m+n\rangle^{-4}, \tag{4.8}$$

where B,N are as in (4.4), and $\mu=C_1\rho\nu^{1/2}$, where C_1 is a large constant and C_0 is a (small) constant.

Proof. When $t=0$ α satisfies (4.8) with these parameters, with the sum taken only over $(m,n)=(0,1)$ and $=(1,0)$ on the right; by choosing C_1 large we can make C_0 as small as desired. A contraction mapping argument, based on the preceding lemma, then plainly establishes the claim. □

In particular, for any ρ,N,B,ν there exist $T',\varepsilon>0$ such that $|a(t,\xi)|\le\varepsilon^{-1}e^{-\varepsilon\xi}$ for all $(t,\xi)\in[0,T']\times[0,\infty)$. Thus the spatial inverse Fourier transform $u(t,\cdot)$ of $a(t,\cdot)$ is a C^∞ solution of (4.1) on $[0,T']$, and $u\in C^\infty([0,T'],H^r(\mathbb{R}))$ for all $r<\infty$.

Define $\tilde\alpha(t,\xi)$ to be the solution of (4.6), supported where $\xi\ge0$, with the simpler initial condition $\tilde\alpha(0,\xi)=\rho\nu^{-1/2}h(\nu^{-1}\xi)$. The same reasoning as for α gives the simpler majorization

$$|\tilde\alpha(t,\xi)|\le\sum_{n\ge\xi\nu^{-1}}C_0t^{n-1}\rho\nu^{-1/2}(\rho\nu^{1/2})^{n-1}. \tag{4.9}$$

We now begin showing that $a(t,\cdot)$ becomes large by isolating a key term.

Lemma 6. *For all $\xi\in[N,N+2\nu]$,*

$$\alpha(t,\xi)=\tilde\alpha(t,\xi)+\xi\int_0^t\int_0^\xi\tilde\alpha(0,\eta)(\alpha-\tilde\alpha)(0,\xi-\eta)\,d\eta\,ds. \tag{4.10}$$

Proof. Since α is obtained via the contraction mapping principle, it may be expressed as the sum of a convergent series

$$\alpha(t,\xi) = \alpha(t,0) + \tfrac{1}{2}\xi t \int_0^\xi \alpha(0,\eta)\alpha(0,\xi-\eta)\,d\eta \\ + \tfrac{1}{4}\xi t^2 \iint_{0\le\zeta\le\eta\le\xi} \eta\alpha(0,\zeta)\alpha(0,\eta-\zeta)\alpha(0,\xi-\eta)\,d\zeta\,d\eta \;+\;\cdots$$

where the remaining terms involve integrands with factors $\alpha(0,\eta_1)\alpha(0,\eta_2)\cdots\alpha(0,\eta_k)$ where $\sum_{j=1}^k \eta_j = \xi$, and $k \ge 4$.

Decompose $\alpha(0,\xi) = \tilde\alpha(0,\xi) + \beta(0,\xi)$ where $\beta(0,\xi)$ is supported where $\xi \in [N+\frac{3}{4}\nu, N+nu]$ and $\tilde\alpha(0,\xi)$ is supported where $\xi \in [\frac{3}{4}\nu,\nu]$. Use this decomposition to expand $\alpha(0,\cdot)$ wherever it appears in the power series expansion. Any resulting integral which involves two or more factors $\beta(0,\eta_j)$ will be supported where $\xi \ge 2(N+\frac{3}{4}\nu) \ge 2N$, and this is $\gg N+2\nu$ since $N \gg 1$ and $\nu \le 1$. Thus all such terms contribute nothing to $\alpha(t,\xi)$ for $\xi \in [N, N+2\nu]$.

Likewise any term which has one factor of $\beta(0,\cdot)$, multiplied by at least two factors of $\tilde\alpha(0,\cdot)$, is supported where $\xi \ge (N+\frac{3}{4}\nu) + \frac{3}{4}\nu + \frac{3}{4}\nu > N+2\nu$. Those terms involving only $\tilde\alpha(0,\cdot)$ sum of course to $\tilde\alpha(t,\xi)$. The only remaining term is $\frac{1}{2}\xi t\int_0^\xi \tilde\alpha(0,\eta)\beta(0,\xi-\eta)\,d\eta$, as claimed. ☐

Inserting the definition of $\alpha(0,\xi)$ into (4.10) gives

$$\alpha(t,\xi) = \tilde\alpha(t,\xi) + \xi\int_0^t\int_0^\xi e^{2i\eta(\xi-\eta)s}\rho\nu^{-1/2}h(\nu^{-1}\eta)B\rho\nu^{-1/2}h(\nu^{-1}(\xi-\eta-N))\,d\eta\,ds$$

for $\xi \in [N, N+2\nu]$. Now $e^{2i\eta(\xi-\eta)s} = 1 + O(\nu N t)$ since $\eta \le \nu$ and $\xi - \eta \le \xi \le N+2\nu \le 3N$. Therefore for all $\xi \in [N, N+2\nu]$,

$$\alpha(t,\xi) = \tilde\alpha(t,\xi) + t\rho^2\nu^{-1}B\xi\int_0^\xi h(\nu^{-1}\eta)h(\nu^{-1}(\xi-\eta-N))\,d\eta \\ + O(t\rho^2\nu^{-1}BN\nu\cdot\nu N t);$$

Since h is nonnegative and not $\equiv 0$, there exists a subinterval I_N of $[N, N+2\nu]$ of length $c\nu$ on which $\int_0^\xi h(\nu^{-1}\eta)h(\nu^{-1}(\xi-\eta-N))\,d\eta \ge c\nu$, for some constant $c > 0$ depending only on h. For $\xi \in I_N$ we thus have

$$\begin{aligned}|a(t,\xi)| &= |\alpha(t,\xi)| \\ &\ge ct\rho^2 N\cdot B(1 - C\nu N t) - |a(0,\xi)| - |\tilde\alpha(t,\xi)| \\ &\ge ct\rho^2 N\cdot B(1-C\nu N t) - C\rho\nu^{-1/2}B - Ct^{\nu^{-1}\xi-1}\rho\nu^{-1/2}(\rho\nu^{1/2})^{\nu^{-1}\xi-1}.\end{aligned}$$

Let $\rho > 0$ be a sufficiently small but otherwise arbitrary constant, whose only role will be to make the initial datum $O(\rho)$ in H^s after the other parameters have been chosen to make it $O(1)$. Define τ,ν,B in terms of N by setting

$$\begin{cases} \nu = N^{-1+\varepsilon} \\ \tau = \nu^{-1}N^{-1-\varepsilon} = N^{-2\varepsilon} \\ B = N^{-s} \end{cases} \tag{4.11}$$

where $\varepsilon > 0$ is a small parameter which will be held constant as $N \to \infty$.

In particular, $\tau \to 0$ as $N \to \infty$, and $\nu N \tau = N^{-\varepsilon} \to 0$. Therefore for any $\xi \in I_N$ we have

$$\begin{aligned}
|a(\tau,\xi)| &\geq c'\tau\rho^2 NB - C\rho\nu^{-1/2}B - Ce^{-N} \\
&= c'\tau\rho\nu^{1/2}N \cdot B\rho\nu^{-1/2} - C\rho\nu^{-1/2}B - Ce^{-N} \\
&\gtrsim \rho N^{\frac{1}{2}-2\varepsilon} \cdot \rho\nu^{-1/2}B - C\rho\nu^{-1/2}B - Ce^{-N} \\
&\gtrsim \rho N^{\frac{1}{2}-2\varepsilon} \cdot \rho\nu^{-1/2}N^{-s}.
\end{aligned}$$

Define the weighted norms

$$\|f\|^2_{L^2_\sigma} = \int_0^\infty |f(\xi)|^2(1+\xi^2)^\sigma \, d\xi.$$

We have

$$\|a(\tau,\cdot)\|^2_{L^2_\sigma} \gtrsim N^{2\sigma} \int_{I_N} |a(\tau,\xi)|^2 \, d\xi \gtrsim N^{2\sigma+1-4\varepsilon-2s}\rho^4 \gtrsim N^{2\sigma+1-2s-4\varepsilon}\rho^2 \|a(0,\cdot)\|^2_{L^2_s}.$$

If $\sigma > s - \frac{1}{2}$ then $\varepsilon > 0$ may be chosen so that $2\sigma + 1 - 4\varepsilon - 2s > 0$. The theorem follows by letting $N \to \infty$. □

Remark. The quadratic form of the nonlinearity is essential here, since the Cauchy problem for $u_t + iu_{xx} + u^2 u_x = 0$ is wellposed in $H^s(\mathbb{R})$ for all sufficiently large s, in the sense of continuous dependence; see Theorem 3.1 of [6]. The above construction breaks down because the analogue of $\xi \int_0^\xi \alpha(s,\eta)\alpha(s,\xi-\eta)\,d\eta$ is a $p-1$-fold integral, which ultimately brings in an additional factor of $\nu^{(p-2)/2}$. The large factor $\rho\nu^{1/2}N$ becomes $\rho\nu^{(p-1)/2}N$, which for $p \geq 3$ is $O(\nu N)$; but the construction requires $\nu N \to 0$ in order that the dispersive effect should not come into play. Thus this factor is no longer large, and the argument collapses.

Remark. An elaboration of the construction produces a solution, in the very weak sense, whose H^s norm is as small as desired at time 0, and is infinite for all $t > 0$. In the periodic case one takes initial data of the form $iA + i\sum_{k=1}^\infty B_k e^{iN_k x}$ where the sequence N_k tends rapidly to infinity and is chosen by induction on k. The above reasoning shows that a solution takes the form $iA + i\sum_{k=1}^\infty B_k e^{AN_k t + iN_k^2 t} e^{iN_k x}$ plus secondary terms. If the initial datum is any trigonometric polynomial then the sum of all the secondary terms is a real analytic function of x, that is, has exponentially decaying Fourier coefficients. Choosing N_{k+1} sufficiently large, and choosing B_{k+1} appropriately, thus yields sufficient control to ensure that the $k+1$-st main term is large relative to the sum of all previously occurring secondary terms.

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