COMMENTARY ON TWO PAPERS OF A. P. CALDERÓN

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> Is she worth keeping? Why, she is a pearl Whose price hath launched above a thousand ships. (William Shakespeare, *Troilus and Cressida*)

Let u be a harmonic function in the upper half space $\mathbb{R}^{n+1}_+ = \{(y,t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}$. Fundamental aspects of such functions are measured by

$$N_\alpha u(x) = \sup_{(y,t)\in\Gamma_\alpha(x)} |u(y,t)| \text{ (nontangential maximal function)} \tag{1}$$

$$S_\alpha u(x) = \Big(\int_{\Gamma_\alpha(x)} t^{1-n} |\nabla u(y,t)|^2 \, dy\, dt \Big)^{1/2} \text{ (Lusin area integral)} \tag{2}$$

$$\|u\|_{H^p} = \sup_{t>0} \|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \text{ (Hardy space norm)} \tag{3}$$

$$\lim_{\Gamma_\alpha(x)\ni(y,t)\to x} u(y,t). \text{ (nontangential limit)} \tag{4}$$

Here $x \in \mathbb{R}^n$, and $\Gamma_\alpha(x) \subset \mathbb{R}^{n+1}_+$ is the open cone $\{(y,t) : |x-y| < \alpha t\}$. If $1 < p < \infty$ then for any $\alpha, \beta > 0$, $\|S_\alpha u\|_p$, $\|N_\beta u\|_p$, and $\|u\|_{H^p}$ are comparable uniformly for all Schwartz functions. Moreover for any harmonic function in $H^1 + H^\infty$, the limit (4) exists for almost every $x \in \mathbb{R}^n$.

Two 1950 papers of Calderón established local qualitative versions of some of these relations.

Theorem 1. [2] *Let $E \subset \mathbb{R}^n$ have positive Lebesgue measure. Suppose that for each $x \in E$ there exists $\beta > 0$ such that u is bounded in $\Gamma_\beta(x)$. Then for almost every $x \in E$, the nontangential limit*[1] $\lim_{\Gamma_\alpha(x)\ni(y,t)\to x} u(y,t)$ *exists for all $\alpha > 0$.*

Theorem 2. [3] *Under the same hypotheses, for almost every $x \in E$, $S_\alpha u(x)$ is finite for all $\alpha > 0$.*

For $n + 1 = 2$, these results had been established earlier by Plessner (1923) and by Marcinkiewicz and Zygmund (1938), who exploited the connection with holomorphic functions and mappings, including the conformal invariance of $\int |f'|^2$ for holomorphic f, and properties of conformal mappings of domains bounded by rectifiable curves. Concerning his motivation, Calderón discloses only that he aims to give a new proof that generalizes to functions of several variables, and that the topic was proposed by Antoni Zygmund.

These two theorems, and elements of their proofs, have been prototypes for subsequent refinements and extensions at the hands of other authors. They have exerted

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Date: June 2, 2005.
The author was supported in part by NSF grant DMS-040126.

[1] This was proved in [2] in the more general product setting. Although Calderón did little work in that setting thereafter, it was the subject of his student K. Merryfield's dissertation.

an enormous impact on the development of a broad swath of modern analysis. To discuss this impact requires an outline of the proofs.[2]

The first step in the proofs of both theorems is to uniformize the situation. Let $\varepsilon > 0$, and assume that E has finite measure. A simple real variable argument, using the Hardy-Littlewood maximal function, shows that there exist a compact subset $E' \subset E$ and $h > 0$, $C < \infty$ satisfying $|E \setminus E'| < \varepsilon$, such that for every $x \in E'$, $\sup_{\Gamma^h_\alpha(x)} |u| \le C$. Here $\Gamma^h_\alpha(x)$ denotes the truncated cone $\{(y,t) \in \Gamma_\alpha(x) : 0 < t < h\}$.

Introduce the open "sawtooth domain" $\mathcal{R} = \mathcal{R}_\alpha = \cup_{x \in E'} \Gamma^h_\alpha(x) \subset \mathbb{R}^{n+1}_+$. Then u is a bounded harmonic function in $\mathcal{R}$; the local problem has been converted back to a global problem on the sawtooth domain.

Such domains enjoy two principal properties. Firstly, they are Lipschitz domains; the lower part of the boundary of $\mathcal{R}$ is the graph of a Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}^+$, which vanishes identically on E'. Calderón's papers [2],[3] are in fact pioneering works on analysis in Lipschitz domains, a subject which has subsequently undergone an immense development. Secondly, a technical point on which the analysis heavily leans is that the harmonic function $u|_{\mathcal{R}}$ is potentially poorly behaved only[3] near $E' \subset \partial\mathcal{R}$, a portion of the boundary which is contained in the hyperplane $t = 0$, rather than in a general Lipschitz surface. More general stopping-time regions, which are essentially unions with respect to $x \in \mathbb{R}^n$ of truncated cones $\Gamma_\alpha(x) \cap \{(y,t) : t > \rho(x)\}$ for arbitrary nonnegative functions ρ, later played an important part in Carleson's proof of the Corona theorem and related developments in one complex variable; see [11].

The two proofs now diverge. The second step in the proof of Theorem 1 is to split[4] $u = \varphi + \psi$ in $\mathcal{R}$, where the good part φ is harmonic in all of $\mathbb{R}^{n+1}_+$ and is essentially the Poisson integral of $u^* \cdot \chi_{E'}$. Here χ_S denotes the characteristic function of a set S, and u^* the boundary limit of u; more precisely, this product is defined as the weak limit of $u(y, t + n^{-1})\chi_{E'}$ as $n \to \infty$. ψ is defined to be $u - \varphi$. Since φ is a bounded harmonic function in the entire upper half space, it has nontangential limits almost everywhere by the global theorem of Plessner. This step exploits the fact that E' lies in a hyperplane.

Step three is to prove that ψ has nontangential limit equal to *zero* at almost every point of E'. For this it formally suffices, by the maximum principle, to find a positive function v harmonic on $\mathcal{R}$ such that $-v \le \psi \le v$ on $\partial\mathcal{R}$, and such that v has nontangential limit zero at each point of E'. Again this reasoning can be made rigorous by replacing u with $u(y, t+n^{-1})$, comparing the associated function ψ_n to $\pm(v + \varepsilon_n)$ where $\varepsilon_n \to 0$ sufficiently slowly as $n \to \infty$, and letting $n \to \infty$. v is defined to be the Poisson integral of the characteristic function of $\mathbb{R}^n \setminus E'$; v is bounded below by a strictly positive constant on the part of $\partial\mathcal{R}$ where $t > 0$, and has nontangential limit zero almost everywhere on $\partial\mathcal{R} \cap \{t = 0\} = E'$. This concludes the proof of Theorem 1.

Theorem 1 has been extended in various ways. Carleson [6] obtained the same conclusion under the weaker assumption that u is bounded below; more about this momentarily. Hunt and Wheeden [13] extended Carleson's theorem to functions

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[2]Proof sketches here slur over essential details treated correctly in the original sources.

[3]An oversimplification.

[4]This splitting hints at the decomposition of a function into good and bad parts soon to be exploited by Calderón and Zygmund in their work on singular integrals.

harmonic in arbitrary Lipschitz domains, with a caveat: "almost everywhere" was interpreted with respect to harmonic measure, rather than surface measure on the boundary. Dahlberg [8] subsequently proved that harmonic measure and surface measure are mutually absolutely continuous for all Lipschitz domains. In conjunction with the theorem of Hunt and Wheeden, this gave the natural (global) conclusion that bounded, or more generally nonnegative, functions in a Lipschitz domain have nontangential limits almost everywhere with respect to surface measure.

The proof of Theorem 2 begins with the same uniformization step. Then (modifying the integral defining $S_\alpha u$ by integrating only over $t < h$)

$$\int_{E'} S_\alpha u(x)^2\,dx = \int_{\mathcal{R}} |\nabla u(y,t)|^2 m(y,t)\,dy\,dt \tag{5}$$

where $m(y,t) = |\{x \in E' : (y,t) \in \Gamma_\alpha(x)\}|$ is $\le C_\alpha t$. The next step is to majorize m by an appropriate harmonic function v. Since[5] $2v|\nabla u|^2 = v\Delta(u^2) - u^2\Delta(v)$, Green's theorem reexpresses $\int_{\mathcal{R}} |\nabla u|^2 v$ as a constant times

$$\int_{\partial\mathcal{R}} \Big(v\nu \cdot \nabla u^2 - u^2 \nu \cdot \nabla v \Big)\,d\sigma \tag{6}$$

where ν is the normal vector to the boundary. If $|v(y,t)| \le t$ then the first term of the integrand is harmless, for the boundedness of u in $\mathcal{R}_\beta$ for $\beta > \alpha$ implies boundedness of $t|\nabla u(y,t)|$ in $\mathcal{R}_\alpha$, by the mean value theorem applied on $B((y,t),ct)$. The remainder of the argument was simplified by Stein, who chose $v(y,t) = t$. Then $u^2\nabla v$ is likewise bounded. The boundary of $\mathcal{R}$ has finite surface measure, so $S_\alpha(u) \in L^2(E')$.

In the simple case where u is globally bounded and E' is a ball B of some radius r, this calculation gives the Carleson measure estimate $\int_{B\times(0,r)} t|\nabla u|^2 \le C|B|\|u\|^2_{L^\infty}$. The use of Green's theorem here foreshadows later developments concerning boundary value problems and harmonic measure on Lipschitz domains. Identities based on integration by parts played pivotal roles in Dahlberg's proof [8] of the mutual absolute continuity of harmonic and surface measures, and especially in a later alternative proof due to Jerison and Kenig [14]; as well as in Verchota's solution [22] of boundary value problems for Laplace's equation by the method of layer potentials, in which an identity of Rellich was used to prove the equivalence of $\int_{\partial\Omega} |\nabla_n u|^2\,d\sigma$ with $\int_{\partial\Omega} |\nabla_t u|^2\,d\sigma$, where ∇_n, ∇_t denote respectively the normal and tangential parts of the gradient.

The converse of Theorem 2 was proved by Stein [19] via closely related reasoning. Given that $S_\alpha(u) \in L^2(E')$, Green's identity applied to $\int_{\mathcal{R}} t|\nabla u|^2$ leads to a bound for (6) with $v = t$. (6) then amounts to one term which directly controls $\int_{E'} u^2$, plus other relatively harmless terms.

Calderón's original proof uses a different auxiliary function v, and invokes Green's theorem on the domain $\mathcal{D} = \{v > 0,\ 0 < t < h\}$, rather than on $\mathcal{R}$. He arranges for $\mathcal{D}$ to contain (much of) $\mathcal{R}$. The advantage is that the term $v\nu \cdot \nabla u^2$ is identically zero because v vanishes, except where $t = h > 0$, where all terms are harmless because u is smooth on $\mathbb{R}^{n+1}_+$. More precisely, $v(y,t)$ is defined to be Ct minus the Poisson integral of $d(x) = \text{distance}\,(x, E')$, for a large constant C. There is no

[5] More generally, $\Delta(u^p) = p(p-1)u^{p-2}|\nabla u|^2$ for positive harmonic functions; in particular, u^p is subharmonic when $p > 1$. This subharmonicity had already been exploited by Calder'on in his unpublished proof of the L^p boundedness of the conjugate function; see [17] for a refinement of that argument.

satisfactory pointwise upper bound for this Poisson integral, but via an application of the integral of Marcinkiewicz a useful bound was obtained on a set $E'' \subset E'$ of nearly full measure, leading to a bound for $S_\alpha(u)$ in $L^2(E'')$ via Green's theorem.

In extending (1) to nonnegative harmonic functions a decade later, Carleson worked with harmonic measure and Green's function for the same sawtooth domains. For general Lipschitz domains, these remained poorly understood until later work of Dahlberg [8], but for $\mathcal{R}$ Carleson observed [6], p. 395 that Calderón's conclusions about v could be interpreted as a pivotal bound for Green's function, namely $\partial G/\partial n \ge c_\varepsilon > 0$ on a subset of E having Lebesgue measure $> |E| - \varepsilon$. Inequalities for Green's function and harmonic measure, obtained in part via various comparisons relying on the maximum principle, were fundamental to later developments concerning general Lipschitz domains.

In subsequent years, both complex and real methods were further developed, with advances via complex techniques[6] repeatedly stimulating further advances in real techniques that yielded more general or precise results. Burkholder, Gundy, and Silverstein obtained a striking new global result for harmonic functions in the upper half-plane: For all $0 < p < \infty$, rather than merely for $p > 1$, $N(u) \in L^p$ if and only if $u + i\tilde{u} \in H^p$, where $\tilde{u}$ denotes the conjugate function. Fefferman and Stein [10] then obtained a more quantitative version of Theorem 2:

$$\lambda_{S(u)}(r) \le C\lambda_{N(u)}(r) + Cr^{-2}\int_0^r s\lambda_{N(u)}(s)\,ds \tag{7}$$

where $\lambda_f(r)$ is the distribution function $|\{x : |f(x)| > r\}|$. One proves this, essentially, by examining more closely $\int_{\{x:N(u)(x)\le r\}} S(u)(x)^2\,dx$. Calderón and Torchinsky made important contributions to the nascent real-variable Hardy space theory, for $0 < p \le 1$, in two subsequent papers.

A refinement due to Burkholder and Gundy [1] is the relative distributional inequality, also called a good λ inequality, of which one version [20] is

$$|\{x : S(u)(x) > \lambda \text{ and } N(u) : (x) \le \gamma\lambda\}| \le C_k\gamma^k|\{x : S(u)(x) > \gamma\lambda\}| \tag{8}$$

for $\lambda > 0$ and $0 < \gamma \le 1$. This implies for instance the comparability of the L^p norms of $S(u)$ and $N(u)$ for all $p \in (0, \infty)$. There is an analogous inequality with the roles of S, N reversed. Chang, Wilson, and Wolff [7] found a sharp limiting result as $p \to \infty$: if u is the Poisson integral of f, and if $S(u) \in L^\infty$, then there exists $c < \infty$ such that $e^{c|f|^2}$ is integrable over any bounded set.

A proof of (8) meshes naturally with the theory of Muckhenhoupt's A_p weights to lead to the global inequality

$$\int_{\mathbb{R}^n} S(u)(x)^p w(x)\,dx \le C_w \int_{\mathbb{R}^n} N(u)(x)^p w(x)\,dx \tag{9}$$

for all $w \in A_p$. Weighted norm inequalities for S were used by Calderón [5] in his analysis of Cauchy integrals on Lipschitz curves.

An alternative natural approach to analysis on Lipschitz domains is to map to a smoothly bounded domain via a change of variables. This converts the Laplacian to

[6]Calderón used complex methods in conjunction with Littlewood-Paley theory in his papers on commutators of singular integral operators [4] and on the Cauchy integral on Lipschitz curves [5]. Real-variable methods regained a measure of ascendancy through subsequent works of Coifman-Meyer, Coifman-Meyer-McIntosh, and David-Journé.

an elliptic second-order operator with variable coefficients of possessing limited regularity. Thus it becomes natural to investigate analogues of all the issues discussed above for such operators. See [16] for an introduction to this topic and references.

An underlying theme is the equivalence between a measure of size ($N(u)$) and a measure of variation ($S(u)$). This theme is at work in the theory of analytic capacity of compact subsets of the complex plane, as developed by Calderón, David, Melnikov, Tolsa, and many other authors; see the bibliography and Mathematical Reviews discussion of [21] for this story. In the proof that a set of positive analytic capacity must contain a rectifiable subset of positive length, a holomorphic function whose size satisfies a certain upper bound is constructed. This size bound is then reinterpreted, via an identity due to Melnikov, as an upper bound on Menger curvature, which measures the deviation of the support of the measure from a subset of a line, and is strongly analogous to an area integral. This theme, along with an interplay between global and local versions, is also seen in Jones' traveling salesman theorem [15], where the global version of the theorem relates the length of a curve to its deviation from linearity at all scales, and a local version then characterizes subsets of rectifiable curves.

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