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Slow off-diagonal decay for Szegö kernels associated to smooth Hermitian line bundles

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Abstract. The off-diagonal rate of decay of Szegö kernels associated to k-th powers of positive Hermitian holomorphic line bundles over complex manifolds is analyzed, as $k \to +\infty$. Examples with C^∞ Hermitian metrics are constructed for which the rate of decay is subexponential, in contrast with the case of C^ω metrics. The construction is based on a nonlinear eigenvalue analysis for certain ordinary differential operators.

1. Introduction

The purpose of this note is to show that Szegö kernels associated to powers of positive Hermitian holomorphic line bundles over complex manifolds may decay more slowly away from the diagonal for C^∞ Hermitian metrics than they do for C^ω metrics. This answers a question posed by B. Shiffman and S. Zelditch.

Let $L \mapsto M$ be a holomorphic line bundle over a complex manifold of dimension $n \geq 1$, equipped with a Hermitian metric h which is C^∞, but not necessarily C^ω. We assume that (L, h) is positive, in the sense that the curvature form associated to h is everywhere strictly positive. Let M likewise be equipped with some C^∞ Riemannian metric g, and with the associated volume form. Let L^* be the line bundle dual to L.

We consider the CR manifold $X = \{(m, v) \in L^* : h(v, v) = 1\}$, which is strictly pseudoconvex if and only if (L, h) is positive — this could be taken as a definition of positivity, for our purposes. The circle group $\mathbb{T}$ acts on X by complex multiplication in the fibers, thus by isometric CR automorphisms. We denote this action by $X \times \mathbb{T} \ni x = (m, v) \mapsto e^{i\theta}x = (m, e^{i\theta}v)$.

g, h determine a Hilbert space structure $L^2(X)$. In $L^2(X)$ sits its closed subspace $H^2(X)$, consisting of all square integrable CR functions. The Szegö projection S_X is the orthogonal projection of $L^2(X)$ onto $H^2(X)$. It commutes with the $\mathbb{T}$ action, hence is associated to a distribution-kernel which, when written in local coordinates as $S_X((z, \theta); (z', \theta'))$, depends only on $\theta - \theta'$.

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On M one likewise has the Hilbert space $L^2(M, L^{\otimes\lambda})$ of all square integrable sections of $L^{\otimes\lambda}$, and its closed subspace $H^2(M, L^{\otimes\lambda})$, consisting of all holomorphic sections. Holomorphic sections over M of the powers $L^{\otimes\lambda}$ of L, where $\lambda \in \mathbb{Z}^+$, may be canonically identified with CR functions on X satisfying $f(e^{i\theta}x) \equiv e^{i\lambda\theta}f(x)$; likewise, general L^2 sections may be identified with functions in $L^2(X)$ satisfying the same identity. Under these identifications, in local holomorphic coordinates z on M, the local holomorphic sections of $L^{\otimes\lambda}$ satisfy not $\partial f/\partial\bar{z} = 0$, but rather a twisted first-order equation $\partial f/\partial\bar{z} + \lambda b(z)f = 0$, for a certain C^∞ coefficient function b. As compensation, the Hilbert space pairing for $L^2(M, L^{\otimes\lambda})$, in local coordinates under these identifications, is simply $\int f(z)\bar{g}(z)dv(z)$ where v is a volume measure independent of λ.

There is also a natural orthogonal projection operator $\Pi_\lambda : L^2(M, L^{\otimes\lambda}) \mapsto H^2(M, L^{\otimes\lambda})$, whose distribution-kernel takes values in sections of an appropriate endomorphism bundle depending on λ. Under these same identifications, in local coordinates, this distribution-kernel may be regarded as a function $\Pi_\lambda(z, w)$ of $(z, w) \in M \times M$, which takes values in $\mathbb{C}$ and is related to S_X by

$$\Pi_\lambda(z, w) = (2\pi)^{-1} \int_0^{2\pi} S_\lambda(z, \theta; w, 0)\, e^{-i\lambda\theta}\, d\theta.$$

The regularity of the kernel S_X is closely related to the size of Π_λ as $\lambda \to \infty$. Under our hypotheses of smoothness and positivity for the metric, $S_X \in C^\infty$ off of the diagonal, while Π_λ is globally C^∞. By simple Fourier analysis, the off-diagonal smoothness of the Szegö kernel S_X implies that $\Pi_\lambda(z, z') = O(\lambda^{-N})$ where $z \neq z'$ as $\lambda \to +\infty$, for all N. All that is required is that $S_X((z, \theta), (z', 0))$ be a C^∞ function of θ, where $z \neq z'$. When the metrics h, g are real analytic, then $S_X((z, \theta), (z', \theta'))$ is known to be real analytic off of the diagonal; the same reasoning implies that real analyticity of S_X with respect to θ where $z \neq z'$ is equivalent to exponential decay of Π_λ, in the sense $\Pi_\lambda = O(\exp(-c\lambda))$ for some $c > 0$ as $\lambda \to +\infty$, outside any neighborhood of the diagonal in $M \times M$.

When h, g are merely C^∞, all the data determining S_X respect the $\mathbb{T}$ action, so depend analytically on θ. Moreover, Shiffman and Zelditch have pointed out that all terms of a formal expansion for S_X given by Boutet de Monvel and Sjöstrand [**2**] are indeed C^ω where $z \neq z'$. This expansion converges in the usual asymptotic sense of full symbols of pseudodifferential operators, but gives no direct information about analyticity of S_X. Nonetheless, it is consequently reasonable to ask whether S_X is real analytic with respect to θ, when z, z' (and θ') are fixed and $z \neq z'$.

THEOREM 1.1. *There exist a one-dimensional complex manifold M, a holomorphic line bundle L over M, a C^∞ Hermitian metric h on L such that (L, h) is positive, and a C^∞ Riemannian metric g on M, with the following property: There exists no $c > 0$ such that the Szegö kernels Π_λ are $O(\exp(-c\lambda))$ away from the diagonal, as $\lambda \to +\infty$. Equivalently, the Szegö kernel $S_X(z, \theta; z', \theta')$ fails to be real analytic in θ at certain points where $z \neq z'$.*

More precisely, for any nondecreasing function k satisfying $k(\lambda)/\sqrt{\lambda\log(\lambda)} \to \infty$ as $\lambda \to +\infty$, there exist M, L, h, g as above, such that Π_λ is not $O(e^{-k(\lambda)})$ away from the diagonal.

In the opposite direction, one has

$$\Pi_\lambda(z, w) = O(e^{-c\lambda^{1/2}}) \qquad \text{away from the diagonal} \tag{1.1}$$

for some $c > 0$, under the regularity hypothesis $h \in C^{1,1}$, as a direct consequence of the arguments in [**4**],[**11**]. Thus a small gap remains between upper and lower bounds. We intend to show in a future paper that our lower bound is sharp: For C^∞ metrics, $\Pi_\lambda = O(e^{-A\sqrt{\lambda \log \lambda}})$ away from the diagonal, for all $A < \infty$.

This investigation was motivated by work of Bleher, Shiffman, and Zelditch [**1, 13**] on the distribution and correlation of zeros of random sections of such bundles.[1] Those authors showed that the correlation functions can be expressed solely in terms of associated Szegö kernels (and their covariant derivatives), and of the Hermitian metric. Thus a rough interpretation of our theorem is that zeros located at a positive distance from one another may[2] be more strongly correlated for general C^∞ metrics than for C^ω metrics as $\lambda \to \infty$ — although the correlations are still quite weak.

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2. Preliminaries

Consider a holomorphic line bundle L over a one-dimensional complex manifold M. Let $z = x + iy$ be a holomorphic coordinate for a small neighborhood U_0 of a base point $0 \in M$, so that L is trivial over U_0. We may coordinatize L^* over U_0 by $(z, \zeta) \in U_0 \times \mathbb{C}$.

Let X be the unit sphere bundle of L^*. In the coordinates (z, ζ), $X\big|)U_0$ may be expressed as $\{(z,\zeta) : h(z)|\zeta|^2 = 1\}$. We will use alternative coordinates $x = (z, \theta)$ for $X\big|_{U_0}$, with (z, θ) in this second coordinate system corresponding to $(z, h(z)^{-1}e^{i\theta}) \in U_0 \times \mathbb{C}$ in the first system. $\mathbb{T}$ acts on X by multiplication in the fibers, $((z, e^{i\theta}), e^{i\varphi}) \mapsto (z, e^{i(\theta+\varphi)})$.

The Cauchy-Riemann operator $\bar\partial_b$ maps $C^\infty(X)$ to sections of a bundle $T^{0,1*}$, as follows: The complexified tangent bundle $TX \otimes \mathbb{C}$ may be decomposed as $T^{1,0} \oplus T^{0,1} \oplus \mathbb{C}\frac{\partial}{\partial\theta}$, where $T^{1,0}X$ and $T^{0,1}X$ denote respectively the holomorphic and antiholomorphic tangent vectors in L^* that are tangent to X. Then

$$\bar\partial_b f = df\big|_{T^{0,1}}. \tag{2.1}$$

In this one-dimensional case, $\bar\partial_b$ may be identified locally with the complex vector field $\frac{\partial}{\partial \bar z} + \zeta^{-1}\frac{\partial h^{-1}}{\partial \bar z}\frac{\partial}{\partial \zeta}$, since this is tangent to X. In the local coordinates (z, θ) for X, $\bar\partial_b = \partial_{\bar z} - ib(z)\partial_\theta$ for some C^∞ function b. Since $[\partial_{\bar z} - ib(z)\partial_\theta, -\partial_z - i\bar b(z)\partial_\theta] = -2i\,\mathrm{Re}\,(\partial_z b)\cdot\partial_\theta$, strict pseudoconvexity of X means, in local coordinates, that $\mathrm{Re}\,(\partial_z b)$ is a strictly positive function.

The Hardy space $H^2(X)$, which consists of the L^2 nullspace of $\bar\partial_b$, may be identified with $\oplus_{\lambda=0}^\infty H^2_\lambda(X)$ where $H^2_\lambda = H^2 \cap L^2_\lambda$ and $L^2_\lambda = \{f \in L^2(X) : f(e^{i\theta x}) \equiv e^{i\lambda\theta}f(x)\}$ almost everywhere on X. H^2_λ may in turn be naturally identified with the Hilbert space of all square integrable holomorphic sections of $L^{\otimes\lambda}$. Likewise, L^2_λ is identified with the square integrable sections of $L^{\otimes\lambda}$. In local coordinates, a section of $L^{\otimes\lambda}$ is identified with a function $f : U_0 \mapsto \mathbb{C}$, which in turn is identified

[1]For each λ, one puts a Gaussian density on the vector space of all holomorphic sections of $L^{\otimes\lambda}$, thus establishing a notion of randomness for sections. Zero varieties are "counted" according to the positive currents associated to them.

[2]I have not examined the formulas carefully enough to verify that our lower bounds for Szegö kernels actually translate to lower bounds on correlations; in principle, some cancellation could occur although this seems unlikely.

with the $\mathbb{C}$-valued function $(z,\theta)\mapsto f(z)e^{i\lambda\theta}$. Then $\bar\partial_b, \bar\partial_b^*$ correspond to operators $\overline{D}_\lambda, \overline{D}_\lambda^*$; the latter maps $C^\infty(M)$ to sections of a certain bundle $\mathcal{T}$ over M; the latter maps sections of $\mathcal{T}^*$ to $C^\infty(M)$. In local coordinates in U_0, these operators may be written as $\overline{D}_\lambda = \partial_{\bar z} + \lambda b(z)$ and $\overline{D}_\lambda^* = -\partial_z + \lambda\bar b(z)$.

We now specialize, by considering structures for which $b(x+iy)$ depends only on x in U_0, and we write $b(x+iy) = a'(x) = \partial a/\partial x$. Then $\overline{D}_\lambda, \overline{D}_\lambda^*$ may be expressed in local coordinates as $\overline{\mathcal{L}}_\lambda, \mathcal{L}_\lambda$ where

$$\overline{\mathcal{L}}_\lambda = \partial_x + i(\partial_y - i\lambda a'(x)) \tag{2.2}$$

where

$$a \in C^\infty \text{ is real-valued,} \tag{2.3}$$

with a'' strictly positive near the origin. We choose the Riemannian metric to be the Euclidean metric in the coordinates $x+iy \leftrightarrow (x,y)\in\mathbb{R}^2$. Then the adjoint operators $\mathcal{L}_\lambda = \overline{\mathcal{L}}_\lambda^*$ take the form

$$\mathcal{L}_\lambda = -\partial_x + i(\partial_y - i\lambda a'(x))\ ; \tag{2.4}$$

in general, there would also be a lower-order term. Then

$$[\overline{\mathcal{L}}_\lambda, \mathcal{L}_\lambda] = 2\lambda a''(x). \tag{2.5}$$

We will consistently use the different notations $\overline{D}_\lambda, \overline{\mathcal{L}}_\lambda$ to distinguish between the globally defined operators $\overline{D}_\lambda$, and their realizations $\overline{\mathcal{L}}_\lambda$ in this particular local coordinate patch and coordinate system.

Suppose further that

$$a(x) = x^2 + A(x), \tag{2.6}$$

where the real-valued perturbation A is small in C^2 norm, and is supported in a small neighborhood of the origin. Such structures are easily constructed, with A arbitrary so long as it has sufficiently small C^2 norm. Indeed, starting with any positive Hermitian holomorphic line bundle over a one-dimensional base, coordinates can be chosen so that $2\,\mathrm{Re}\,(\partial_z b)(0) = 4$, then the Hermitian metric can be modified by a partition of unity so that $2\,\mathrm{Re}\,(\partial_z b)\equiv 4$ in a smaller neighborhood of 0. A change of variables of the form $(z,\theta)\mapsto(z,\theta+\varphi(z))$ keeps $2\,\mathrm{Re}\,(\partial_z b)\equiv 4$, but puts $\bar\partial_b$ in the form $\partial_x + i(\partial_y + 2x\partial_\theta)$. A small perturbation of the Hermitian metric near 0 then achieves the form desired. Finally, the Riemannian metric can be adjusted so that the adjoint operator also has the desired form.

3. The main construction

Fix a nondecreasing function k such that $k(\lambda)/\sqrt{\lambda\log\lambda}\to\infty$ as $\lambda\to+\infty$. The proof of the following lemma is the core of the analysis, and the subject of this section.

Lemma 3.1. *There exist M, L, h such that there exist neighborhoods U_0, U_1 of $0\in M$ with U_0 relatively compact in U_1, a sequence $\lambda_\nu\to+\infty$, and functions u_ν*

defined in U_1 *satisfying*

$$\bar{\mathcal{L}}_{\lambda_\nu}\mathcal{L}_{\lambda_\nu}u_\nu = 0 \qquad \textit{in } U_1 \tag{3.1}$$

$$u_\nu = O(e^{\varepsilon k(\lambda_\nu)}) \quad \textit{in } U_1 \tag{3.2}$$

$$\mathcal{L}_{\lambda_\nu}u_\nu = O(e^{\varepsilon k(\lambda_\nu)}) \quad \textit{in } U_1 \tag{3.3}$$

$$\sup_{U_0}|\mathcal{L}_{\lambda_\nu}u_\nu| \ge c > 0 \qquad \textit{as } \nu\to\infty \tag{3.4}$$

for all $\varepsilon > 0$.

In subsequent sections we will show how this implies that Π_λ cannot be $O(e^{-k(\lambda)})$ away from the diagonal. Note that proving the lemma with the weaker bounds $O(\exp(Ck(\lambda_\nu)))$ for a fixed constant C, for arbitrary functions k with the specified growth, is equivalent to proving it as stated.

Let $\{\lambda_j\}\to+\infty$ be a very rapidly increasing sequence, to be chosen later in the proof, and choose a sequence of positive numbers $\{\beta_j\}\to 0$ so that

$$\beta_j = O(\lambda_j^{-N}) \qquad \text{for every finite } N, \tag{3.5}$$

$$\beta_j\cdot e^{ck^2(\lambda_j)/\lambda_j}\to\infty \qquad \text{for every } c>0. \tag{3.6}$$

as $j\to\infty$. The hypothesis on k is equivalent to the existence of such a sequence. Write $k_j = k(\lambda_j)$.

Fix a smooth, compactly supported real-valued function $\mathfrak{a}\in C_0^\infty(\mathbb{R})$, not identically vanishing, whose Fourier-Laplace transform $\hat{\mathfrak{a}}(\zeta) = \int \mathfrak{a}(s)e^{-i\zeta s}\,ds$ has a *simple* zero at some $0\ne\zeta_0\in\mathbb{R}$.

The perturbation $A(x)$ will be expressed as

$$A(x) = \sum_{j=1}^{\infty} a_j(x), \tag{3.7}$$

where

$$a_j(\tfrac{1}{2}\lambda_j^{-1/2} + k_j^{-1}t) = \frac{\beta_j}{2\lambda_j}\mathfrak{a}(t). \tag{3.8}$$

Although the perturbation A is an infinite sum, the construction will be set up so that for each ν only the single summand a_ν significantly influences the properties of the solution u_ν. We will always assume, without loss of generality, that

$$\frac{k(\lambda)}{\lambda}\to 0 \text{ as } \lambda\to\infty. \tag{3.9}$$

Consequently if $\lambda_j\to\infty$ sufficiently rapidly, then the coefficient $\mathfrak{a}$ will be C^∞, by (3.5). We also require $\lambda_j\to\infty$ sufficiently rapidly that the functions a_j have disjoint supports, which of course accumulate at the origin.

For any λ and any $\xi\in\mathbb{C}$, a one-parameter family of solutions of $\overline{\mathcal{L}}_\lambda\mathcal{L}_\lambda u = 0$ can be constructed by separation of variables. For any parameter $\xi\in\mathbb{C}$ let

$$u(x+iy) = e^{i\xi y}f(x), \tag{3.10}$$

and require

$$0 = [\partial_x + i(i\xi - i\lambda a'(x)]\,[\partial_x - i(i\xi - i\lambda a'(x)]f = (\partial_x+\psi')(\partial_x-\psi')f, \tag{3.11}$$

where

$$\psi(x) = -\xi x + \lambda a(x), \tag{3.12}$$

and ψ' denotes $d\psi/dx$. A solution is

$$f(x) = e^{-\xi x+\lambda a(x)} \int_{-\infty}^{x} e^{2[\xi t-\lambda a(t)]}\,dt = e^{\psi(x)} \int_{-\infty}^{x} e^{-2\psi(t)}\,dt. \tag{3.13}$$

Although we ultimately aim to construct functions on the manifold M, we will work for the present on $\mathbb{C}^1$. The coefficient a may be regarded as being globally defined on $\mathbb{R}$, hence the operators $\overline{\mathcal{L}}_\lambda, \mathcal{L}_\lambda$ may be regarded as being globally defined on $\mathbb{C}^1$. A global solution on $\mathbb{C}^1$ may then be restricted to a neighborhood of 0, which is identified with a coordinate patch on the manifold M. This procedure does not directly produce global solutions on M, but we will show in §4 how they may be corrected so as to become true solutions.

The crucial condition in our analysis will be the vanishing of the Fourier-Laplace transform

$$\mathcal{N}(\xi,\lambda) = \int_{-\infty}^{\infty} e^{2[\xi t-\lambda a(t)]}\,dt. \tag{3.14}$$

To achieve this we are forced to use complex ξ, since otherwise the integrand is everywhere positive. We will show that for large ν, there exists ξ_ν satisfying

$$\mathcal{N}(\xi_\nu,\lambda_\nu) = 0 \quad\text{and}\quad |\xi_\nu| \le Ck_\nu. \tag{3.15}$$

We digress briefly to indicate why $\mathcal{N}$ should be relevant. The first indication is an explicit formula for the Szegö kernels Π_λ, for the unbounded models on $\mathbb{C}^1$ with $\overline{\mathcal{L}}_\lambda \equiv \partial_x + i(\partial_y - i\lambda a'(x))$, due to Nagel [**12**]:

$$\Pi_\lambda(x+iy, x'+iy') = c_0 \int_{\mathbb{R}} e^{i\xi(y-y')} e^{\psi_{\xi,\lambda}(x)+\psi_{\xi,\lambda}(x')} \mathcal{N}(\xi,\lambda)^{-1}\,d\xi \tag{3.16}$$

where $\psi_{\xi,\lambda}(x) = -\xi x + \lambda a(x)$. It was shown in [**10**] how this formula, together with the existence of zeros of $\mathcal{N}$, could be used to prove that the Szegö kernel for certain real-analytic Cauchy-Riemann manifolds of finite type fails to be real analytic off of the diagonal. However, it seems difficult to work directly with the formula (3.16) in our context; see the remarks in §5. Instead, we will use zeros of $\mathcal{N}$ to build slowly decaying solutions of $\overline{\mathcal{L}}_\lambda\mathcal{L}_\lambda$, then will relate those solutions to the Szegö kernels by an auxiliary argument.

Set

$$\xi_\nu = \lambda_\nu^{1/2} + ik_\nu z_\nu, \tag{3.17}$$

where $z_\nu \in \mathbb{C}$ is to be determined. We will arrange for z_ν to converge to a finite limiting value, $-\zeta_0/2 \in \mathbb{R}$, as $\nu \to \infty$.

Fix some large index ν. To analyze $\mathcal{N}(\xi_\nu,\lambda_\nu)$, substitute $x = \lambda_\nu^{-1/2}(\frac{1}{2}+s)$. In order to simplify notation in the computations to follow, we will frequently drop the subscript ν. Thus

$$\xi x - \lambda x^2 = \tfrac{1}{4} - s^2 + \frac{i}{2}\lambda^{-1/2}kz + i\lambda^{-1/2}kzs, \tag{3.18}$$

so

$$\mathcal{N}(\xi_\nu,\lambda_\nu) = e^{1/2}e^{i\lambda^{-1/2}kz}\lambda^{-1/2} \int_{\mathbb{R}} e^{-2s^2+2i\lambda^{-1/2}kzs} e^{-2\lambda\sum_j a_j(\frac{1}{2}\lambda^{-1/2}+\lambda^{-1/2}ks)}\,ds. \tag{3.19}$$

The main term of the sum in the exponent will be $j=\nu$; all others will turn out to be negligible. Recalling that for $\lambda=\lambda_\nu$, $2\lambda a_\nu(\frac{1}{2}\lambda^{-1/2}+\lambda^{-1/2}s) = \beta_\nu \mathfrak{a}(\lambda^{-1/2}s)$, we decompose

$$\mathcal{N}(\xi_\nu, \lambda_\nu) = e^{1/2} e^{i\lambda^{-1/2}kz} \lambda^{-1/2} \cdot (N_0 + N_1 + N_2 + N_3) \tag{3.20}$$

where

$$N_0 = \int_{\mathbb{R}} e^{-2s^2 + 2i\lambda^{-1/2}kzs}\, ds \tag{3.21}$$

$$N_1 = -\int_{\mathbb{R}} e^{-2s^2 + 2i\lambda^{-1/2}kzs} \beta_\nu \mathfrak{a}(\lambda^{-1/2}ks)\, ds \tag{3.22}$$

$$N_2 = \int_{\mathbb{R}} e^{-2s^2 + 2i\lambda^{-1/2}kzs} \Big[e^{-\beta_\nu \mathfrak{a}(\lambda^{-1/2}ks)} - 1 + \beta_\nu \mathfrak{a}(\lambda^{-1/2}ks) \Big]\, ds \tag{3.23}$$

$$N_3 = \int_{\mathbb{R}} e^{-2s^2 + 2i\lambda^{-1/2}kzs} e^{-\beta_\nu \mathfrak{a}(\lambda^{-1/2}ks)} \Big[e^{-2\lambda \sum_{j\neq\nu} a_j(\frac{1}{2}\lambda^{-1/2} + \lambda^{-1/2}s)} - 1 \Big], \tag{3.24}$$

and $z = z_\nu$, $\lambda \equiv \lambda_\nu$ throughout.

Assume that z_ν belongs to a small, fixed ball centered at $-\frac{1}{2}\zeta_0$. This ensures that

$$N_0 = O(e^{-ck^2\lambda^{-1}}). \tag{3.25}$$

The main term will be N_1: substituting $s = \lambda^{1/2}k^{-1}t$, we obtain

$$\begin{aligned} N_1 &= -\beta_\nu \lambda^{1/2} k^{-1} \int_{\mathbb{R}} e^{-\lambda k^{-2} t^2} e^{2izt} \mathfrak{a}(t)\, dt \\ &= -\beta_\nu \lambda^{1/2} k^{-1} [\hat{\mathfrak{a}}(-2z) + O(\lambda k^{-2})]. \end{aligned} \tag{3.26}$$

Likewise

$$N_2 = O(\beta_\nu^2 \lambda^{1/2} k^{-1}). \tag{3.27}$$

The term N_3 is also rather small. Indeed, we may make

$$e^{-2\lambda_\nu \sum_{j>\nu} a_j(\frac{1}{2}\lambda_\nu^{-1/2} + \lambda_\nu^{-1/2}s)} - 1 \tag{3.28}$$

as small as we like, by choosing $\{\lambda_j\}$ to tend sufficiently rapidly to infinity, because $a_j = O(\lambda_j^{-1})$. Thus we may reduce to the situation where the summation in the exponent appearing in brackets in the definition of N_3 may be taken over all $j < \nu$, rather than over all $j \neq \nu$. For $j < \nu$, we compute

$$2\lambda_\nu a_j(\tfrac{1}{2}\lambda_\nu^{-1/2} + \lambda_\nu^{-1/2}s) = \lambda_\nu \lambda_j^{-1} \beta_j \mathfrak{a}(k_j[\tfrac{1}{2}\lambda_\nu^{-1/2} + \lambda_\nu^{-1/2}s - \tfrac{1}{2}\lambda_j^{-1/2}]). \tag{3.29}$$

Assuming that $\lambda_{i+1} \geq 10\lambda_i$ for all i, this function is supported where $|s| \geq c(\lambda_\nu/\lambda_j)^{1/2}$, for some constant $c > 0$. On the support of $\exp(-2\lambda_\nu a_j(\frac{1}{2}\lambda_\nu^{-1/2} + \lambda_\nu^{-1/2}s)) - 1$, the factor $\exp(-2s^2)$ in the integrand is therefore $\leq \exp(-c\lambda_\nu/\lambda_j)$. On the other hand, the exponent $-2\lambda_\nu a_j(\frac{1}{2}\lambda_\nu^{-1/2} + \lambda_\nu^{-1/2}s)$ has absolute value $\leq C\lambda_\nu\lambda_j^{-1}\beta_j \ll \lambda_\nu/\lambda_j$. Thus the contribution of all $j < \nu$ in (3.24) may be made as small as desired, relative to the size of the main term N_1, by choosing λ_ν to be sufficiently large, relative to all λ_j with $j < \nu$.

In all,

$$\begin{aligned} N_0 + N_1 + N_2 + N_3 = -\beta_\nu \lambda^{1/2} k^{-1} &\cdot [\hat{\mathfrak{a}}(-2z) + O(\lambda k^{-2}) + O(\beta_\nu)] \\ &+ O(e^{-c\lambda_\nu/\lambda_{\nu-1}}) + O(e^{-ck^2\lambda^{-1}}), \end{aligned} \tag{3.30}$$

uniformly in ν, z for all z in a fixed small ball centered at $-\frac{1}{2}\zeta_0$, provided that $\{\lambda_\nu\}$ increases sufficiently rapidly. The last two terms on the right-hand side of (3.30) satisfy

$$e^{-c\lambda_\nu/\lambda_{\nu-1}} + e^{-ck_\nu^2\lambda_\nu^{-1}} \ll \beta_\nu(\lambda_\nu^{1/2}k_\nu^{-1})^3. \tag{3.31}$$

Indeed, the second term on the left is majorized because $\beta \gg \exp(-ck^2/2\lambda) \gg \exp(-ck^2/\lambda)\cdot(k^2/\lambda)^{3/2}$, since $k_\nu^2 \gg \lambda_\nu$. The first term on the left is likewise majorized by virtue of the same lower bound for β_ν, provided λ_ν is chosen sufficiently large relative to $\lambda_{\nu-1}$, since $k_\nu = o(\lambda_n)$. Moreover, λ_ν/k_ν^2 and β_ν are both $o(\lambda_\nu^{1/2}/k_\nu)$ since $\lambda_\nu^{1/2} \ll k_\nu \ll \lambda_\nu$ and $\beta_\nu = O(\lambda_\nu^{-2})$. Thus

$$N_0 + N_1 + N_2 + N_3 = -\beta_\nu\lambda^{1/2}k^{-1}\cdot[\hat{\mathfrak{a}}(-2z) + O(\lambda k^{-2})]. \tag{3.32}$$

Therefore since $\hat{\mathfrak{a}}$ has a simple zero at ζ_0, we may apply Rouché's theorem on a disk of radius $c\lambda_\nu/k_\nu^2$ centered at $-\zeta_0/2$ to conclude that for all sufficiently large ν there exists z_ν, tending to $-\zeta_0/2$ as $\nu\to\infty$, such that $\mathcal{N}(\xi_\nu,\lambda_\nu) = 0$, where $\xi_\nu = \lambda_\nu^{1/2} + ik_\nu z_\nu$, and moreover, $|z_\nu + \frac{1}{2}\zeta_0| \le C\lambda_\nu k_\nu^{-2} \ll \lambda_\nu^{1/2}k_\nu^{-1}$. Since $\zeta_0\in\mathbb{R}$, this implies that

$$k_\nu|\operatorname{Im}(z_\nu)| \le C\lambda_\nu^{1/2}. \tag{3.33}$$

z_ν, and hence ξ_ν, are fixed for the remainder of the discussion.

We next establish the conclusions of the lemma, beginning with an upper bound for $u(x+iy) = e^{i\xi_\nu y}f_\nu(x)$. The factor $e^{i\xi_\nu y}$ is

$$O(e^{C|\operatorname{Im}(\xi_\nu)|}) = O(e^{Ck_\nu}) \tag{3.34}$$

for y in any fixed compact set. At 0, we have in contrast the lower bound $|e^{i\xi_\nu y}| = 1$.

Set $\psi_\nu(x) = -\xi_\nu x + \lambda_\nu a(x)$. We have

$$\sup_x |e^{-\psi_\nu(x)}| = O(1). \tag{3.35}$$

Indeed, the real part of $-\psi_\nu(x)$ is $\lambda_\nu^{1/2}x - \lambda_\nu a(x) + k_\nu\operatorname{Im}(z_\nu)x$. Since $a(x)\ge cx^2$ for some $c>0$, it suffices to know that $k_\nu|\operatorname{Im}(z_\nu)| = O(\lambda_\nu^{1/2})$, which is (3.33).

Writing $f_\nu(x) = e^{\psi(x)}\int_{-\infty}^x e^{-2\psi(y)}\,dy$, we have

$$|f_\nu(x)| \le \int_{-\infty}^x e^{\operatorname{Re}(\psi(x)-2\psi(y))}\,dy. \tag{3.36}$$

The real part of the exponent is convex downwards, precisely because $a''>0$. Let x_ν be the unique critical point of Re ψ_ν. Then for $x\le x_\nu$,

$$\int_{-\infty}^x e^{\operatorname{Re}(\psi(x)-2\psi(y))}\,dy \le \int_{-\infty}^x e^{-\operatorname{Re}\psi(y)}\,dy \le \int_{-\infty}^{x_\nu} e^{-\operatorname{Re}\psi(y)}\,dy. \tag{3.37}$$

For $x\ge x_\nu$, we exploit the condition $\mathcal{N}(\xi_\nu,\lambda_\nu)=0$ to write

$$f_\nu(x) = -e^{\psi(x)}\int_x^\infty e^{-2\psi(y)}\,dy, \tag{3.38}$$

then follow the same reasoning to deduce that

$$|f_\nu(x)| \le \int_{x_\nu}^\infty e^{-\operatorname{Re}\psi(y)}\,dy. \tag{3.39}$$

Because Re $\psi(y)''$ has a strictly negative upper bound, uniformly in y, ν, we conclude that

$$\max_x |f_\nu(x)| \le Ce^{-\operatorname{Re}\psi(x_\nu)}. \tag{3.40}$$

It is now immediate from (3.35) that as $\nu \to \infty$,

$$||f_\nu||_{C^0} = O(1). \tag{3.41}$$

Similar reasoning shows that any derivative $d^k f_\nu/dx^k$ is $O(\lambda_\nu^N)$ for some exponent N depending on k. The same bounds hold for $e^{-i\xi_\nu y}\mathcal{L}_{\lambda_\nu}e^{i\xi_\nu y}f_\nu \equiv e^{-\psi_\nu}$, and are easier to verify for it.

The conclusions of the lemma all follow from this information, in the weaker form with arbitrary $\varepsilon > 0$ replaced by some fixed constant; as explained above, this version of the lemma is actually equivalent to the formally stronger version stated. Condition (3.1), the vanishing of $\bar{\mathcal{L}}_{\lambda_\nu}\mathcal{L}_{\lambda_\nu}u_\nu$, follows directly from the ordinary differential equation (3.11) and the definition $u_\nu = e^{i\xi_\nu y}f_\nu$. (3.2) follows from (3.41), since $|e^{i\xi_\nu y}| \le e^{C|\operatorname{Im}(\xi_\nu)|} = e^{C|k_\nu z_\nu|}$ and $|z_\nu|$ is bounded uniformly in ν. The corresponding conclusion (3.3) for $\mathcal{L}_{\lambda_\nu}u_\nu = e^{-\psi_\nu}e^{i\xi_\nu y}$ holds since $e^{-\psi_\nu}$ is uniformly bounded, as noted above, and $e^{i\xi_\nu y} = O(e^{Ck_\nu})$. Conclusion (3.4) holds because $\mathcal{L}_{\lambda_\nu}u_\nu(0) = e^{-\psi_\nu(0)} \cdot e^{i\xi_\nu 0} \equiv 1$.

4. A right parametrix for $\overline{\mathcal{L}}_\lambda$

In the preceding section we have constructed solutions u of $\overline{\mathcal{L}}_\lambda\mathcal{L}_\lambda u = 0$ in a fixed coordinate patch in M. In the present section we establish an auxiliary lemma, which will be used to globalize the construction, and to link these solutions with the Szegö kernels Π_λ.

Just as $\bar\partial_b$ maps $C^\infty(X)$ to sections of a certain line bundle $(T'')^*$, there is for each λ an operator $\overline{D}_\lambda$ corresponding to $\bar\partial_b$, mapping $C^\infty(M)$ to sections of a corresponding line bundle $\mathcal{T}$ over M; $\mathcal{T}$ is independent of λ. In local coordinates in our special coordinate patch U_0, $\overline{D}_\lambda$ is identified with $\overline{\mathcal{L}}_\lambda$. The adjoint operator $\overline{D}_\lambda^*$ maps sections of $\mathcal{T}^*$ to functions on M.

For all sufficiently large $\lambda > 0$, for all C^1 sections of $\mathcal{T}^*$,

$$||\overline{D}_\lambda^* f||^2 \ge c\lambda||f||^2, \tag{4.1}$$

where the norms are respectively those of $L^2(M, \mathbb{C})$ and $L^2(M, \mathcal{T})$. Indeed, in any small coordinate patch over which $\mathcal{T}$ is trivial, we may identify $\mathcal{T}$ with $\mathbb{C}$ and thus let $\overline{D}_\lambda, \overline{D}_\lambda^*$ act on the same class of functions. Then the positivity of (L, h) translates via a commutator identity into an operator inequality

$$\overline{D}_\lambda\overline{D}_\lambda^* \ge \overline{D}_\lambda^*\overline{D}_\lambda + c'\lambda I, \tag{4.2}$$

where I denotes the identity operator and $c'' > 0$. By means of a partition of unity, we obtain (4.1) for all sufficiently large λ.

A corollary of (4.1) is that $\overline{D}_\lambda$ is surjective from its domain onto $L^2(M, \mathcal{T})$, for all sufficiently large $\lambda \in \mathbb{R}^+$. More precisely, for any $f \in L^2(M, \mathcal{T})$ there exists a solution $u \in L^2(M, \mathbb{C})$ of $\overline{D}_\lambda u = f$, satisfying $||u||_{L^2} \le C\lambda^{-1/2}||f||_{L^2}$.

Lemma 4.1. *For any M, L, h, g of the type under consideration, for any $r > 0$ there exists $\delta > 0$ such that for any two open sets V, W with* distance$(V, W) \ge r$*,* *uniformly for all large* $\lambda \gg 1$*, for any* $f \in L^2(M)$ *supported in* W*, there exists* $u \in L^2(M)$ *satisfying* $\overline{D}_\lambda u \equiv f$ *on all of* M*, with*

$$\|u\|_{L^2(V)} \le e^{-\delta\lambda}\|f\|_{L^2}. \tag{4.3}$$

The crucial point here is that the first power of λ appears in the exponent, rather than $\lambda^{1/2}$. The price we pay is that the solution u will not in general be the canonical solution, that is, will not be orthogonal to the nullspace of $\overline{D}_\lambda$.

PROOF. Denote by μ the volume form associated to the Riemannian metric g, and by $\|\cdot\|_{\text{op}}$ the operator norm on $L^2(M, \mathcal{T})$. We will construct a kernel $K_\lambda(z,w)$ such that the operator $Tf(z) = \int K_\lambda(z,w)f(w)\,d\mu(w)$ is a right parametrix for $\overline{D}_\lambda$, in the strong sense that for all sufficiently large λ,

$$\|\overline{D}_\lambda \circ T - I\|_{\text{op}} \le e^{-c\lambda} \tag{4.4}$$

for some constant $c > 0$, and moreover such that for any $r > 0$, there exists $\delta > 0$ such that

$$|K_\lambda(z,w)| \le Ce^{-\delta\lambda} \tag{4.5}$$

for all $\lambda \ge 1$, for all (z,w) satisfying distance $(z,w) \ge r$.

Then the proof may be concluded, as follows. Given f supported in W with $\|f\|_{L^2} = 1$, write $f = \overline{D}_\lambda Tf + \tilde{f}$ where $\|\tilde{f}\|_{L^2(M)} = O(e^{-c\lambda})$. By the surjectivity of $\overline{D}_\lambda$, there exists $\tilde{u}$ satisfying $\overline{D}_\lambda\tilde{u} = \tilde{f}$ exactly, globally on M, with $\|\tilde{u}\|_{L^2(M)} = O(e^{-c\lambda})$. Then $\overline{D}_\lambda(Tf + \tilde{u}) = f$, and both Tf and $\tilde{u}$ are $O(e^{-c\lambda})$ in $L^2(V)$.

It suffices to construct $K(z,w)$ in a small neighborhood of an arbitrary point $(w_0, w_0) \in M^2$, for using a partition of unity to patch such local parametrices together loses only $O(e^{-c\lambda})$ if each local parametrix satisfies the desired off-diagonal decay bound. Near w_0, we may choose coordinates so that $\overline{D}_\lambda = e^{-\lambda\phi}\circ\bar{\partial}\circ e^{\lambda\phi}$, where $e^{\pm\lambda\phi}$ denotes both a function, and the operator on L^2 defined by multiplication by that function. Here $\bar{\partial}$ denotes the usual Cauchy-Riemann operator for $\mathbb{C}^1$. A right parametrix for $\overline{D}_\lambda$ has distribution-kernel

$$K(z,w) = c_0 e^{-\lambda\phi(z)}(z-w)^{-1}e^{\lambda\phi(w)} \tag{4.6}$$

for a certain constant c_0.

Now ϕ is uniquely determined only modulo addition of an arbitrary analytic function; moreover, $\Delta\phi(z) = \rho(z)$ is strictly positive because of the positivity of (L,h). Thus we may use as a right parametrix any kernel of the form

$$K(z,w) = c_0 e^{-\lambda[\phi(z)+h(z,w)]}(z-w)^{-1}e^{\lambda\phi(w)} \tag{4.7}$$

where $h(z,w)$ is holomorphic in z for each fixed w, depends smoothly (or merely measurably) on w, and vanishes whenever $z = w$. It suffices to prove that there exists h such that $\operatorname{Re}[\phi(z) + h(z,w) - \phi(w)]$ is strictly positive whenever $z \ne w$, assuming always that (z,w) is close to (w_0,w_0). Expand

$$\operatorname{Re}(\phi(z)-\phi(w)) = \operatorname{Re}[a(w)(z-w))+c(w)(z-w)^2]+b(w)|z-w|^2+O(|z-w|^3). \tag{4.8}$$

Choose $h(z,w) = -a(w)(z-w) - c(w)(z-w)^2$. Since $\Delta\phi > 0$, $b(w)$ is strictly positive for all w, and hence $\operatorname{Re}(\phi(z) + h(z,w))$ has a strict minimum at w as a function of z, for each w, as desired. □

5. Conclusion of proof

The Szegö projection operator Π_λ is related to $\overline{D}_\lambda$ by the formula

$$(I - \Pi_\lambda)F = \overline{D}_\lambda^* G \qquad \text{where} \qquad \overline{D}_\lambda \overline{D}_\lambda^* G = \overline{D}_\lambda F, \tag{5.1}$$

where I denotes the identity operator. This holds because the orthocomplement of the nullspace of $\overline{D}_\lambda$ is equal to the range of $\overline{D}_\lambda^*$.

Let $\{\lambda_\nu\}$ be as constructed above. In order to prove our main theorem, it suffices to exhibit sequences of functions F_{λ_ν} and U_{λ_ν}, and neighborhoods V_0, V_1, V_2 of 0 such that each V_i is relatively compact in V_{i+1}, such that for each $\lambda = \lambda_\nu$, $\overline{D}_\lambda \overline{D}_\lambda^* U_\lambda \equiv 0$ in V_2, $U_\lambda = O(e^{\varepsilon k(\lambda)})$ for all $\varepsilon > 0$ in C^2 norm on $M \backslash V_1$, and $\overline{D}_\lambda^* U_\lambda$ is not $O(e^{-k(\lambda)})$ on V_0. Indeed, fix $\eta \in C^\infty(M)$ identically equal to 1 in $\bar{V}_1$, whose support is relatively compact in V_2. Consider $\eta \cdot U_\lambda$ as a globally defined function on M. Then $\overline{D}_\lambda \overline{D}_\lambda^*(\eta U_\lambda)$ is supported outside a neighborhood of $\bar{V}_1$, and is $O(e^{\varepsilon k(\lambda)})$ for all $\varepsilon > 0$. By Lemma 4.1, there exists a solution h_λ of $\overline{D}_\lambda h_\lambda = \overline{D}_\lambda \overline{D}_\lambda^*(\eta U_\lambda)$, such that

$$h_\lambda = O(e^{-c\lambda}) \text{ in } V_1. \tag{5.2}$$

We have previously stated this bound only in L^2 norm, but it follows in any C^N norm from the L^2 case, because h_λ is the solution of an elliptic equation, whose coefficients are $O(\lambda)$. We have

$$(I - \Pi_\lambda)(h_\lambda) = \overline{D}_\lambda^*(\eta U_\lambda) \tag{5.3}$$

Suppose now that the Szegö kernel associated to Π_λ were $O(e^{-k(\lambda)})$ on $V_0 \times (M \backslash V_1)$ for some $\gamma > \sigma$. From (5.2), we would then conclude that

$$\Pi_\lambda h_\lambda = O(e^{-k(\lambda)}) \qquad \text{on } V_0. \tag{5.4}$$

Hence the same holds for $\overline{D}_\lambda^*(\eta U_\lambda)$, which on V_0 is equal to $\overline{D}_\lambda^* U_\lambda = e^{\xi(x+iy) - \lambda a(x)}$. At the origin, this equals 1, contradicting the upper bound (5.4).

6. Concluding remarks

Remark 6.1. By choosing $\{\beta_\nu\}$ to tend sufficiently rapidly to zero, the coefficient a, hence the Hermitian metric h, can be made to belong to the Gevrey class G^s for any specified $s > 1$. Repeating the same reasoning, one finds that for any $r > (s+1)/2s$, it is not the case that $\Pi_\lambda(z, w) = O(\exp(-\lambda^r))$ away from the diagonal for all metrics in G^s.

Remark 6.2. A related question, on the CR level, is whether $S_X(z, \theta; z', \theta')$ is analytic in θ wherever $\theta \neq \theta'$. We strongly believe that it is not, in general. Elements of the construction in **[8]**, together with the above arguments, should lead to a counterexample.

Remark 6.3. The analysis works for any Riemannian metric g on M for which the lower-order term in $\mathcal{L}_\lambda$ depends only on x, near the origin, in our local coordinates $x + iy$. One obtains the same lower bound for the Szegö kernels, although some formulas become slightly more complicated. In particular, g may be chosen to be the Kähler metric corresponding to the Kähler form defined by the Ricci curvature of h.

Remark 6.4. Closely related arguments have previously been used to construct and/or analyze various classes of partial differential operators that fail to be hypoelliptic, in various senses. See **[10, 3, 5, 7, 8, 9]**. The novel element here is that the degeneracy consists of a lack of appropriate smoothness of the structure, rather than geometric degeneracy fitting the framework discussed in [**6**].

REMARK 6.5. An alternative approach would be to analyze the explicit formula (3.16) for the Szegö kernels associated to the unbounded models on $\mathbb{C}^1$ with $\overline{\mathcal{L}}_\lambda \equiv \partial_x + i(\partial_y - i\lambda a'(x))$. The function $\mathcal{N}$ appears as a denominator in that formula; by shifting the contour $\mathbb{R} \ni \xi$ of integration into the complex plane, one picks up contributions from poles of $\mathcal{N}^{-1}$. Indeed, the main point of our construction is to build the coefficient a so that $\mathcal{N}^{-1}$ has poles, with appropriate control over their locations, for a sequence of values of λ. Nonetheless, we have elected not to work directly with (3.16), because poles other than those we have built in could potentially arise, and seem difficult to control. Instead we have argued by contradiction, showing that if $\{\Pi_\lambda\}$ were to decay too rapidly, then there could be no poles at all in a certain region.

REMARK 6.6. Our construction also implies lower bounds for Π_λ, much closer to the diagonal, that are larger than hold in the C^ω case. Define $\rho_\lambda(z,w) = \lambda^{1/2}\,\text{distance}\,(z,w)$, the distance on the right-hand side being computed with respect to any fixed Riemannian metric on M. Then one conclusion that can be drawn is that for any $\delta \in (0,1/2]$, there exist sequences of points $(z_\nu, w_\nu) \in M \times M$ satisfying $\text{distance}\,(z_\nu, w_\nu) \sim \lambda_\nu^{\delta - \frac{1}{2}}$, for which it is not the case that $\Pi_\lambda(z_\nu, w_\nu) = O(\exp(-k(\rho_\nu)))$, where $\rho_\nu \equiv \rho_{\lambda_\nu}(z_\nu, w_\nu)$. This should be contrasted with the upper bounds

$$|\Pi_\lambda(z,w)| \le C\lambda^d e^{-c\rho_\lambda(z,w)}, \tag{6.1}$$

where d is the complex dimension of the base space M, which can be proved as in [**4**],[**11**]. In the C^ω case, one has the improved bound

$$|\Pi_\lambda(z,w)| \le C\lambda^d e^{-c\rho_\lambda(z,w)^2}. \tag{6.2}$$

REMARK 6.7. There do exist nonanalytic metrics for which $\Pi_\lambda = O(\exp(-c\lambda))$ off of the diagonal. In particular, this is the case whenever the degenerate locus of all points in M at which the metric is nonanalytic is totally disconnected.

To justify this, consider for simplicity the case where M has dimension $n=1$. In order to analyze the fundamental solution $G_\lambda(z,w)$ for $\overline{\mathcal{L}}_\lambda\mathcal{L}_\lambda$, fix a cutoff function $\eta \in C^\infty(M)$ such that $\eta \equiv 1$ near w, $\eta \equiv 0$ near z, and $\nabla\eta$ is supported in an open set in which $h \in C^\omega$. It is elementary that $G_\lambda = O(1)$ outside any neighborhood of the diagonal, as $\lambda \to +\infty$; this can be proved by a simpler version of the arguments in [**4**], for instance. Then it is a consequence of basic results on analytic hypoellipticity that $G_\lambda(\cdot, w) = O(\exp(-c\lambda))$ on the support of $\nabla\eta$, and likewise for its derivatives. Therefore $\overline{\mathcal{L}}_\lambda\mathcal{L}_\lambda\big((1-\eta)(\cdot)G_\lambda\big)(\cdot,w) = O(\exp(-c\lambda))$ globally. Thus

$$\begin{aligned}
G_\lambda(z,w) &= \eta(z)G_\lambda(z,w) + \int G_\lambda(z,z')\,\overline{\mathcal{L}}_\lambda\mathcal{L}_\lambda\big((1-\eta)G_\lambda\big)(z',w)\,dv(z') \\
&= 0 + O\big(e^{-c\lambda}\int |G_\lambda(z',w)|\,dv(z')\big) \\
&= O\big(e^{-c\lambda}\big)
\end{aligned}$$

So $G_\lambda(\cdot, w)$ is exponentially small in a neighborhood of z. Simple elliptic regularity estimates then yield the same conclusions for all its derivatives, in particular for the associated Szegö kernel.

REMARK 6.8. The proof of the main theorem actually shows that $\Pi_\lambda(x+iy, x'+iy')$ is not $O(\exp(-k(\lambda))$ in any neighborhood of $x = x' = 0$, $|y - y'| \geq c > 0$. Our heuristic interpretation is that correlations propagate along very thin tubular neighborhoods (with width tending to 0 as $\lambda \to \infty$) of curves $x = x(\lambda)$ parametrized by the y coordinate.

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