POWER SERIES SOLUTION OF A NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. A slightly modified variant of the cubic periodic one-dimensional nonlinear Schrödinger equation is shown to be well-posed, in a relatively weak sense, in certain function spaces wider than L^2. Solutions are constructed as sums of infinite series of multilinear operators applied to initial data, and these multilinear operators are analyzed directly.

1. INTRODUCTION

1.1. **The NLS Cauchy problem.** The Cauchy problem for the one-dimensional periodic cubic nonlinear Schrödinger equation is

$$\begin{cases} iu_t + u_{xx} + \omega|u|^2u = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{NLS}$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $t \in \mathbb{R}$, and the parameter ω equals ± 1. Bourgain [2] has shown this problem to be wellposed in the Sobolev space H^s for all $s \geq 0$, in the sense of uniformly continuous dependence on the initial datum. In H^0 it is wellposed globally in time, and as is typical in this subject, the uniqueness aspect of wellposedness is formulated in a certain auxiliary space more restricted than $C^0([0,T], H^s(\mathbb{T}))$, in which existence is also established. For $s < 0$ it is illposed in the sense of uniformly continuous dependence [3], and is illposed in stronger senses [5] as well. The objectives of this paper are twofold: to establish the existence of solutions for wider classes of initial data than H^0, and to develop an alternative method of solution.

The spaces of initial data considered here are the spaces $\mathcal{F}L^{s,p}$ for $s \geq 0$ and $p \in [1, \infty]$, defined as follows:

Definition 1.1. $\mathcal{F}L^{s,p}(\mathbb{T}) = \{f \in \mathcal{D}'(\mathbb{T}) : \langle \cdot \rangle^s \widehat{f}(\cdot) \in \ell^p\}$.

Here $\mathcal{D}'(\mathbb{T})$ is the usual space of distributions, and $\mathcal{F}L^{s,p}$ is equipped with the norm $\|f\|_{\mathcal{F}L^{s,p}} = \|\widehat{f}\|_{\ell^{s,p}(\mathbb{Z})} = \big(\sum_{n\in\mathbb{Z}} \langle n \rangle^{ps} |\widehat{f}(n)|^p\big)^{1/p}$. We write $\mathcal{F}L^p = \mathcal{F}L^{0,p}$, and are mainly interested in these spaces since, for $p > 2$, they are larger function spaces than the borderline Sobolev space H^0 in which (NLS) is already known to be wellposed.

1.2. **Motivations.** At least four considerations motivate analysis of the Cauchy problem in these particular function spaces. The first is the desire for existence theorems for initial data in function spaces which scale like the Sobolev spaces H^s,

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for negative s. $\mathcal{F}L^p$ scales like $H^{s(p)}$ where $s(p) = -\frac{1}{2} + \frac{1}{p} \downarrow -\frac{1}{2}$ as $p \uparrow \infty$, thus spanning the gap between the optimal exponent $s = 0$ for Sobolev space wellposedness, and the scaling exponent $-\frac{1}{2}$. Moreover, $\mathcal{F}L^p$ is invariant under the Galilean symmetries of the equation.

Some existence results are already known in spaces scaling like H^s for certain negative exponents, for the nonperiodic one-dimensional setting. Vargas and Vega [12] proved existence of solutions for arbitrary initial data in certain such spaces, for a certain range of strictly negative exponents. In particular, for the local in time existence theory, their spaces contain $\mathcal{F}L^p$ for all $p < 3$, and scale like $\mathcal{F}L^p$ for a still larger though bounded range of p. Grünrock [7] has proved wellposedness for the cubic nonlinear Schrödinger equation in the real line analogues of $\mathcal{F}L^{s,p}$, and for other PDE in these function spaces, as well.

A second motivation is the work of Kappeler and Topalov [9],[10], who showed via an inverse scattering analysis that the periodic KdV and mKdV equations are wellposed for wider ranges of Sobolev spaces H^s than had previously been known. It is reasonable to seek a corresponding improvement for (NLS). We obtain here such an improvement, but with $\mathcal{F}L^p$ with $p > 2$ substituted for H^s with $s < 0$.

Thirdly, Christ and Erdoğan, in unpublished work, have investigated the conserved quantities in the inverse scattering theory relevant to (NLS), and have found that for any distribution in $\mathcal{F}L^p(\mathbb{T})$ with small norm, the sequence of gap lengths for the associated Dirac operator belongs to ℓ^p and has comparable norm.[1] Thus $\mathcal{F}L^p$ for $2 < p < \infty$ may be a natural setting for the Dirac operator inverse scattering theory relevant to the periodic cubic nonlinear Schrödinger equation.

For $p = 2$, the existing proof [2] of wellposedness via a contraction mapping argument implies that the mapping from initial datum to solution has a convergent power series expansion; that is, certain multilinear operators are well-defined and satisfy appropriate inequalities. Our fourth motivation is the hope of understanding more about the structure of these operators.

1.3. **Modified equation.** In order for the Cauchy problem to make any sense in $\mathcal{F}L^p$ for $p > 2$ it seems to be essential to modify the differential equation. We consider

$$\text{(NLS}^*\text{)} \qquad \begin{cases} iu_t + u_{xx} + \omega\big(|u|^2 - 2\mu(|u|^2)\big)u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where

$$\mu(|f|^2) = (2\pi)^{-1} \int_{\mathbb{T}} |f(x)|^2 \, dx \tag{1.1}$$

equals the mean value of the absolute value squared of f. In (NLS*), $\mu(|u|^2)$ is shorthand for $\mu(|u(t,\cdot)|^2) = \|u(t,\cdot)\|_{L^2}^2$, which is independent of t for all sufficiently smooth solutions; modifying the equation in this way merely introduces a unimodular

[1] Having slightly better than bounded Fourier coefficients seems to be a minimal condition for the applicability of this machinery, since the eigenvalues for the free periodic Dirac system are equally spaced, and gap lengths for perturbations are to leading order proportional to absolute values of Fourier coefficients of the perturbing potential.

scalar factor $e^{2i\mu t}$, where $\mu = \mu(|u_0|^2)$. For parameters p, s such that $\mathcal{F}L^{s,p}$ is not embedded in H^0, $\mu(|u_0|^2)$ is not defined for typical $u_0 \in \mathcal{F}L^{s,p}$, but of course the same goes for the function $|u_0(x)|^2$, and we will nonetheless prove that the equation makes reasonable sense for such initial data.

The coefficient 2 in front of $\mu(|u|^2)$ is the unique one for which solutions depend continuously on initial data in $\mathcal{F}L^p$ for $p > 2$.

1.4. **Conclusions.** Our main result is as follows. Recall that there exists a unique mapping $u_0 \mapsto Su_0(t,x)$, defined for $u_0 \in C^\infty$, which for all sufficiently large s extends to a uniformly continuous mapping from $H^s(\mathbb{T})$ to $C^0([0,\infty), H^s(\mathbb{T})) \cap C^1([0,\infty), H^{s-2}(\mathbb{T}))$, such that Su_0 is a solution of the modified Cauchy problem (NLS*). $C^\infty(\mathbb{T})$ is of course a dense subset of $\mathcal{F}L^{s,p}$ for any $p \in [1,\infty)$.

Theorem 1.1. *For any $p \in [1,\infty)$, any $s \geq 0$, and any $R < \infty$, there exists $\tau > 0$ for which the solution mapping S extends by continuity to a uniformly continuous mapping from the ball centered at 0 of radius R in $\mathcal{F}L^{s,p}(\mathbb{T})$ to $C^0([0,\tau], \mathcal{F}L^{s,p}(\mathbb{T}))$.*

For the unmodified equation this has the following obvious consequence. Denote by $H^0_c = H^0_c(\mathbb{T})$ the set of all $f \in H^0$ such that $\|f\|_{L^2} = c$. Denote by $S'u_0$ the usual solution [2] of the unmodified Cauchy problem (NLS) with initial datum u_0, for $u_0 \in H^0$.

Corollary 1.2. *Let $p \in [1,\infty)$ and $s \geq 0$. For any $R < \infty$ there exists $\tau > 0$ such that for any finite constant $c > 0$, the mapping $H^0_c \ni u_0 \mapsto S'u_0$ is uniformly continuous as a mapping from H^0_c intersected with the ball centered at 0 of radius R in $\mathcal{F}L^{s,p}$, equipped with the $\mathcal{F}L^{s,p}$ norm, to $C^0([0,\tau], \mathcal{F}L^{s,p}(\mathbb{T}))$.*

The unpublished result of the author and Erdoğan says that for smooth initial data, if $\|u_0\|_{\mathcal{F}L^p}$ is sufficiently small then $\|u(t)\|_{\mathcal{F}L^p}$ remains bounded uniformly for all $t \in [0,\infty)$. This result in combination with Theorem 1.1 would yield global wellposedness for sufficiently small initial data.

The following result concerns the discrepancy between the nonlinear evolution (NLS*) and the corresponding linear Cauchy problem

$$\begin{cases} iv_t + v_{xx} = 0 \\ v(0,x) = u_0(x). \end{cases} \tag{1.2}$$

Proposition 1.3. *Let $R < \infty$ and $p \in [1,\infty)$. Let $q > p/3$ also satisfy $q \geq 1$. Then there exist $\tau, \varepsilon > 0$ and $C < \infty$ such that for any initial datum u_0 satisfying $\|u_0\|_{\mathcal{F}L^p} \leq R$, the solutions $u = Su_0$ of* (NLS*) *and v of* (1.2) *satisfy*

$$\|u(t,\cdot) - v(t,\cdot)\|_{\mathcal{F}L^q} \leq Ct^\varepsilon \quad \textit{for all } t \in [0,\tau]. \tag{1.3}$$

Here u the solution defined by approximating u_0 by elements of C^∞ and passing to the limit. Thus for $p > 1$ the linear evolution approximates the nonlinear evolution, modulo correction terms which are smoother in the $\mathcal{F}L^q$ scale.

Our next result indicates that the function $u(t,x)$ defined by the limiting procedure of Theorem 1.1 is a solution of the differential equation in a more natural sense than merely being a limit of smooth solutions. Define Fourier truncation operators T_N, acting on $\mathcal{F}L^{s,p}(\mathbb{T})$, by $\widehat{T_N f}(n) = 0$ for all $|n| > N$, and $= \widehat{f}(n)$ whenever $|n| \le N$. T_N acts also on functions $v(t,x)$ by acting on $v(t,\cdot)$ for each time t separately. We denote by $S(u_0)$ the limiting function whose existence, for nonsmooth u_0, is established by Theorem 1.1.

Proposition 1.4. *Let $p \in [1,\infty)$, $s \ge 0$, and $u_0 \in \mathcal{F}L^{s,p}$. Write $u = S(u_0)$. Then for any $R < \infty$ there exists $\tau > 0$ such that whenever $\|u_0\|_{\mathcal{F}L^{s,p}} \le R$, $\mathcal{N}u(t,x) = (|u|^2 - 2\mu(|u|^2))u$ exists in the sense that*

$$\lim_{N\to\infty} \mathcal{N}(T_N u)(t,x) \text{ exists in the sense of distributions in } C^0([0,\tau],\mathcal{D}'(\mathbb{T})). \tag{1.4}$$

Moreover if $\mathcal{N}(u)$ is interpreted as this limit, then $u = S(u_0)$ satisfies (NLS*) *in the sense of distributions in* $(0,\tau)\times\mathbb{T}$.

More generally, the same holds for any sequence of Fourier multipliers of the form $\widehat{T_\nu f}(n) = m_\nu(n)\widehat{f}(n)$ where each sequence m_ν is finitely supported, $\sup_\nu \|m_\nu\|_{\ell^\infty} < \infty$, and $m_\nu(n) \to 1$ as $\nu \to \infty$ for each $n \in \mathbb{Z}$; the limit is of course independent of the sequence (m_ν). Making sense of the nonlinearity via this limiting procedure is connected with general theories of multiplication of distributions [1],[6], but the existence here of the limit over all sequences (m_ν) gives u stronger claim to the title of solution than in the general theory.

Unlike the fixed point method, our proof yields no uniqueness statement corresponding to these existence results. For any $p > 2$, solutions of the Cauchy problem in the class $C^0([0,\tau],\mathcal{F}L^p)$, in the sense of Proposition 1.4, are in fact not unique [4].

1.5. **Method.** Define the partial Fourier transform

$$\widehat{u}(t,n) = (2\pi)^{-1}\int_{\mathbb{T}} e^{-inx}u(t,x)\,dx. \tag{1.5}$$

Our approach is to regard the partial differential equation as an infinite coupled nonlinear system of ordinary differential equations for these Fourier coefficients, to express the solution as a power series in the initial datum

$$\widehat{u}(t,n) = \sum_{k=0}^{\infty} \hat{A}_k(t)(\widehat{u_0},\cdots,\widehat{u_0}) \tag{1.6}$$

where each $\hat{A}_k(t)$ is a bounded multilinear operator[2] from a product of k copies of $\mathcal{F}L^{s,p}$ to $\mathcal{F}L^{s,p}$, to show that the individual terms $\hat{A}_k(t)(\widehat{u_0},\cdots,\widehat{u_0})$ are well-defined, and to show that the formal series converges absolutely in $C^0(\mathbb{R},\mathcal{F}L^{s,p})$ to a solution in the sense of (1.4). The case $s \ge 0$ follows from a very small modification of the analysis for $s = 0$, so we discuss primarily $s = 0$, indicating the necessary modifications for $s > 0$ at the end of the paper.

The analysis is rather elementary, much of the paper being devoted to setting up the definitions and notation required to describe the operators $\hat{A}_k(t)$. A single number theoretic fact enters the discussion: the number of factorizations of an integer n as

[2]Throughout the discussion we allow multilinear operators to be either conjugate linear or linear in each of their arguments, independently.

a product of two integer factors is $O(n^\delta)$, for all $\delta > 0$; this same fact was used by Bourgain [2].

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2. A system of coupled ordinary differential equations

2.1. General discussion. Define

$$\sigma(j,k,l,n) = n^2 - j^2 + k^2 - l^2. \tag{2.1}$$

It factors as

$$\sigma(j,k,l,n) = 2(n-j)(n-l) = 2(k-l)(k-j) \text{ provided that } j-k+l=n. \tag{2.2}$$

Written in terms of Fourier coefficients $\widehat{u}_n(t) = \widehat{u}(t,n)$, the equation $iu_t + u_{xx} + \omega\big(|u|^2 - 2\mu(|u|^2)\big)u = 0$ becomes

$$i\frac{d\widehat{u}_n}{dt} - n^2\widehat{u}_n + \omega \sum_{j-k+l=n} \widehat{u}_j\overline{\widehat{u}_k}\widehat{u}_l - 2\omega\sum_m |\widehat{u}_m|^2\widehat{u}_n = 0. \tag{2.3}$$

Here the first summation is taken over all $(j,k,l) \in \mathbb{Z}^3$ satisfying the indicated identity, and the second over all $m \in \mathbb{Z}$. Substituting

$$a_n(t) = e^{in^2t}\widehat{u}(t,n), \tag{2.4}$$

(2.3) becomes

$$\frac{da_n}{dt} = i\omega \sum_{j-k+l=n}^{*} a_j\bar{a}_k a_l e^{i\sigma(j,k,l,n)t} - i\omega|a_n|^2 a_n \tag{2.5}$$

where the notation $\sum_{j-k+l=n}^*$ means that the sum is taken over all $(j,k,l) \in \mathbb{Z}^3$ for which neither $j=n$ nor $l=n$. This notational convention will be used throughout the discussion. The effect of the term $-2\omega\mu(|u|^2)u$ in the modified differential equation (NLS*) is to cancel out a term $2i\omega(\sum_m |a_m|^2)a_n$, which would otherwise appear on the right-hand side of (2.5).

Reformulated as an integral equation, (2.5) becomes

(2.6)
$$a_n(t) = a_n(0) + i\omega \sum_{j-k+l=n}^{*} \int_0^t a_j(s)\bar{a}_k(s)a_l(s)e^{i\sigma(j,k,l,n)s}\,ds - i\omega\int_0^t |a_n(s)|^2 a_n(s)\,ds.$$

However, in deriving (2.6) from (2.5), we have interchanged the integral over $[0,t]$ with the summation over j,k,l without any justification.

In terms of Fourier coefficients, (2.6) is restated as

$$
\begin{aligned}
(2.7)\quad \widehat{u}(t,n) = \widehat{u_0}(n) - in^2 \int_0^t \widehat{u}(s,n)\,ds \\
+ i\omega \sum_{j-k+l=n}^{*} \int_0^t \widehat{u}(s,j)\overline{\widehat{u}(s,k)}\widehat{u}(s,l)\,ds - i\omega \int_0^t |\widehat{u}(s,n)|^2 \widehat{u}(s,n)\,ds.
\end{aligned}
$$

Substituting for $a_j(s), a_k(s), a_l(s)$ in the right-hand side of (2.6) by means of (2.6) itself yields

(2.8)

$$
\begin{aligned}
a_n(t) &= a_n(0) + i\omega \sum_{j-k+l=n}^{*} a_j(0)\bar{a}_k(0)\bar{a}_l(0) \int_0^t e^{i\sigma(j,k,l,n)s}\,ds - i\omega |a_n(0)|^2 a_n(0) \int_0^t 1\,ds \\
&\qquad + \text{ additional terms} \\
&= a_n(0)\big(1 - i\omega t |a_n(0)|^2\big) + \tfrac{1}{2}\omega \sum_{j-k+l=n}^{*} \frac{a_j(0)\bar{a}_k(0)a_l(0)}{(n-j)(n-l)} \big(e^{i(n^2-j^2+k^2-l^2)t} - 1\big) \\
&\qquad + \text{ additional terms.}
\end{aligned}
$$

These additional terms involve the functions a_m, not only the initial data $a_m(0)$. The right-hand side of the integral equation (2.6) can then be substituted for each function a_n, replacing it by $a_n(0)$ but producing still more complex additional terms. Repeating this process indefinitely produces an infinite series, whose convergence certainly requires justification. Each substitution by means of (2.6) results in multilinear expressions of increased complexity in terms of functions $a_n(t)$ and initial data $a_n(0)$.

We recognize $1 - i\omega t |a_n(0)|^2$ as a Taylor polynomial for $\exp(-i|a_n(0)|^2 t)$, but for our purposes it will not be necessary to exploit this by recombining terms. In particular, we will not exploit the coefficient i which makes this exponential unimodular.

2.2. **A sample term.** One of the very simplest additional terms arises when (2.6) is substituted into itself twice:

$$
\begin{aligned}
(2.9)\quad (i\omega)^4 \sum_{j_1-j_2+j_3=n}^{*} \ \sum_{m_1^1-m_2^1+m_3^1=j_1}^{*} \ \sum_{m_1^2-m_2^2+m_3^2=j_2}^{*} \ \sum_{m_1^3-m_2^3+m_3^3=j_3}^{*} \\
\int_{0\le r_1,r_2,r_3\le s\le t} a_{m_1^1}(r_1)\bar{a}_{m_2^1}(r_1)a_{m_3^1}(r_1)\bar{a}_{m_1^2}(r_2)a_{m_2^2}(r_2)\bar{a}_{m_3^2}(r_2)a_{m_1^3}(r_3)\bar{a}_{m_2^3}(r_3)a_{m_3^3}(r_3) \\
e^{i\sigma(j_1,j_2,j_3,n)s} e^{i\sigma(m_1^1,m_2^1,m_3^1,j_1)r_1} e^{-i\sigma(m_1^2,m_2^2,m_3^2,j_2)r_2} e^{i\sigma(m_1^3,m_2^3,m_3^3,j_3)r_3}\,dr_1\,dr_2\,dr_2\,ds.
\end{aligned}
$$

Substituting once more via (2.6) for each function $a_n(r_j)$ in (2.9) yields a main term

$$
(2.10)\qquad (i\omega)^4 \sum_{(m_k^i)_{1\le i,k\le 3}}^{*} \mathcal{I}(t,(m_k^i)_{1\le i,k\le 3}) \prod_{i,j=1}^{3} a^*_{m_j^i}(0),
$$

which arises when $a_n(r_j)$ is replaced by $a_n(0)$, plus higher-degree terms. Here the superscript $*$ indicates that the sum is taken over only certain $(m_k^i)_{1\le i,k\le 3} \in \mathbb{Z}^9$, where $a^*_{m^i_j}(0) = a_{m^i_j}(0)$ if $i+j$ is even and $= \overline{a_{m^i_j}(0)}$ if $i+j$ is odd, and where

$$\mathcal{I}(t, (m^i_k)_{1\le i,k\le 3}) = \int_{0\le r_1,r_2,r_3\le s\le t} e^{i\phi(s,r_1,r_2,r_3,\{m^i_j:1\le i,j\le 3\})}\, dr_1\, dr_2\, dr_2\, ds, \tag{2.11}$$

with

$$\phi(s, r_1, r_2, r_3, (m^i_j)_{1\le i,j\le 3}) = \sigma(j_1, j_2, j_3, n)s + \sum_{i=1}^{3} (-1)^{i+1} \sigma(m^i_1, m^i_2, m^i_3, j_i) r_i; \tag{2.12}$$

and j_1, j_2, j_3, n are defined as functions of (m^i_j) by the equations governing the sums in (2.9). Continuing in this way yields formally an infinite expansion for the sequence $(a_n(t))_{n\in\mathbb{Z}}$ in terms of multilinear expressions in the initial datum $(a_n(0))$. This expansion is doubly infinite; the single (and relatively simple) term (2.10) is for instance an infinite sum over most elements of an eight-dimensional free $\mathbb{Z}$-module for each n.

The discussion up to this point has been purely formal, with no justification of convergence. In the next section we will describe the terms in this expansion systematically. The main work will be to show that each multilinear operator is well-defined on ℓ^p initial data, and then that the resulting fully nonlinear infinite series is convergent.

3. Trees and operators indexed by trees

3.1. **Trees.** On a formal level $a(t) = (a_n(t))_{n\in\mathbb{Z}}$ equals an infinite sum

$$\sum_{k=1}^{\infty} A_k(t)(a(0), a(0), a(0), \cdots) \tag{3.1}$$

where each $A_k(t)$ is a sum of finitely many multilinear operators, each of degree k. Throughout the paper, by a multilinear operator we mean one which with respect to each argument is either linear or conjugate linear; for instance, $(f, g) \mapsto f\bar{g}$ is considered to be multilinear. We now describe a class of trees which will be used both to name, and to analyze, these multilinear operators.

In a partially ordered set with partial order $\le$, w is said to be a child of v if $w \le v$, $w \ne v$, and if $w \le u \le v$ implies that either $u = w$, or $u = v$.

The word "tree" in this paper will always refer to a special subclass of what are usually called trees, equipped with additional structure.

Definition 3.1. A tree T is a finite partially ordered set with the following properties:

1. Whenever $v_1, v_2, v_3, v_4 \in T$ and $v_4 \le v_2 \le v_1$ and $v_4 \le v_3 \le v_1$, then either $v_2 \le v_3$ or $v_3 \le v_2$.
2. There exists a unique element $\mathbf{r} \in T$ satisfying $v \le \mathbf{r}$ for all $v \in T$.
3. T equals the disjoint union of two subsets T^0, T^∞, where each element of T^∞ has zero children, and each element of T^0 has three children.
4. For each $v \in T$ there is given a number in $\{\pm 1\}$, denoted $\pm_v$.

(5) There is given a partition of the set of all nonterminal nodes of T into two disjoint classes, called simple nodes and ordinary nodes.

Terminal nodes are neither simple nor ordinary. The distinction between ordinary and simple nodes will encode the distinction between the two types of nonlinear terms on the right-hand side of (2.6).

Definition 3.2. Elements of T are called nodes. A terminal node is one with zero children. The maximal element of T is called its root node, and will usually be denoted by $\mathbf{r}$. T^∞ denotes the set of all terminal nodes of T, while $T^0 = T \setminus T^\infty$ denotes the set of all nonterminal nodes. The three children of any $v \in T^0$ are denoted by $(v,1),(v,2),(v,3)$.

For any $u \in T$, $T_u = \{v \in T : v \le u\}$ is a tree, with root node u. The number $|T|$ of nodes of a tree is of the form $1+3k$ for some nonnegative integer k.

$$|T^\infty| = 1+2k \text{ and } |T^0| = k \tag{3.2}$$

so that T, T^∞, T^0 have uniformly comparable cardinalities, except in the trivial case $k=0$ where $T=\{\mathbf{r}\}$.

Given a tree T, we will work with the auxiliary space $\mathbb{Z}^T$; the latter symbol T denotes the set all nodes of the tree with the same name. Elements of $\mathbb{Z}^T$ will be denoted by $\mathbf{j} = (j_v)_{v\in T} \in \mathbb{Z}^T$ with each coordinate $j_v \in \mathbb{Z}$.

Definition 3.3. Let T be any tree. A function $\sigma_w : \mathbb{Z}^T \to \mathbb{Z}$ is defined by

$$\sigma_w(\mathbf{j}) = \begin{cases} 0 & \text{if } w \text{ is terminal,} \\ j_w^2 - j_{(w,1)}^2 + j_{(w,2)}^2 - j_{(w,3)}^2 & \text{if } w \text{ is nonterminal.} \end{cases} \tag{3.3}$$

$\sigma_v(\mathbf{j})$ depends only on the four coordinates $j_v, j_{(v,1)}, j_{(v,2)}, j_{(v,3)}$ of $\mathbf{j}$.

Definition 3.4. An ornamented tree is a tree T, together with a coefficient $\varepsilon_{v,i} \in \{-1,0,1\}$ for each nonterminal node $v \in T^0$, and for each $i \in \{1,2,3\}$.

Definition 3.5. Let T be an ornamented tree. The function $\rho : \mathbb{Z}^T \to \mathbb{Z}$ is defined recursively by

$$\rho_v(\mathbf{j}) = 0 \text{ if } v \in T^\infty \tag{3.4}$$

and

$$\rho_v(\mathbf{j}) = \sigma(j_{(v,1)}, j_{(v,2)}, j_{(v,3)}, j_v) + \sum_{i=1}^{3} \varepsilon_{v,i}\rho_{(v,i)}(\mathbf{j}) \text{ if } v \in T^0. \tag{3.5}$$

Whenever all children of v are terminal, $\rho_v(\mathbf{j}) = \sigma_v(\mathbf{j})$. But if T has many elements, then for typical $v \in T^0$, ρ_v will be a quadratic polynomial in many variables, which will admit no factorization like that enjoyed by σ_v. $\rho_v(\mathbf{j})$ depends only on $\{j_u, \varepsilon_{u,i} : u \le v\}$. To simplify notation and language, we will use the symbol T to denote the ornamented tree, the underlying tree, and the underlying set.

Definition 3.6. Let T be a tree. $\mathcal{J}(T) \subset \mathbb{Z}^T$ denotes the set of all $\mathbf{j} = (j_v)_{v\in T}$ satisfying the restrictions

$$j_v = j_{(v,1)} - j_{(v,2)} + j_{(v,3)} \text{ for every } v \in T^0 \tag{3.6}$$

$$\{j_v, j_{(v,2)}\} \cap \{j_{(v,1)}, j_{(v,3)}\} = \emptyset \text{ for every ordinary node } v \in T^0 \tag{3.7}$$

$$j_v = j_{(v,i)} \text{ for all } i \in \{1,2,3\} \text{ for every simple node } v \in T^0. \tag{3.8}$$

(3.6) implies that for any $v \in T^0$, j_v can be expressed as a linear combination, with coefficients in $\{\pm 1\}$, of $\{j_w : w \in T^\infty\}$.

Let $\delta, c_0 > 0$ be sufficiently small positive numbers, to be specified later. The following key definition involves these quantities.

Definition 3.7. Let T be an ornamented tree. If $\mathbf{j} \in \mathcal{J}(T)$ and $v \in T$, we say that the ordered pair $(v, \mathbf{j})$ is nearly resonant if v is nonterminal and

$$|\rho_v(\mathbf{j})| \le c_0 |\sigma_v(\mathbf{j})|^{1-\delta}. \tag{3.9}$$

$(v, \mathbf{j})$ is said to be exceptional if $v \in T^0$ and $\rho_v(\mathbf{j}) = 0$.

Whether $(v, \mathbf{j})$ is nearly resonant depends on the values of j_u for all $u \le v$.

Exceptional pairs $(v, \mathbf{j})$ are of course nearly resonant. If $v \in T^0$ is an ordinary node all three of whose children of v are terminal, then $(v, \mathbf{j})$ cannot be exceptional, for $\rho_v(\mathbf{j}) = \sigma(j_{(v,1)}, j_{(v,2)}, j_{(v,3)}, j_v) = 2(j_v - j_{(v,1)})(j_v - j_{(v,3)})$ cannot vanish, by (3.7). But if v has at least one nonterminal child, then nothing prevents $\rho_v(\mathbf{j})$ from vanishing, and if v is a simple node all of whose children are terminal, then any pair $(v, \mathbf{j})$ is certainly exceptional.

3.2. **Multilinear operators associated to trees.**

Definition 3.8. Let T be any tree, and let t be any real number. If T is not the trivial tree $\{\mathbf{r}\}$ with only element, then the associated interaction amplitudes are

$$\mathcal{I}_T(t, \mathbf{j}) = \int_{\mathcal{R}(T,t)} \prod_{u\in T^0} e^{\pm_u i\omega\sigma_u(\mathbf{j})t_u}\, dt_u \tag{3.10}$$

where $\mathcal{R}(T,t) \subset [0,t]^{T^0}$ is defined to be

$$\mathcal{R}(T,t) = \{(t_u)_{u\in T^0} : 0 \le t_u \le t_{u'} \le t \text{ whenever } u, u' \in T^0 \text{ satisfy } u \le u'\}. \tag{3.11}$$

When $T = \{\mathbf{r}\}$ has a single element, $\mathcal{J}(T) = \mathbb{Z}$, and $\mathcal{I}_T(t, \mathbf{j})$ is defined to be 1 for all $t, \mathbf{j}$.

The following upper bounds for the interaction amplitudes $\mathcal{I}_T(t, \mathbf{j})$ are the only information concerning them that will be used in the analysis.

Lemma 3.1. *Let T be any tree, and let $\mathbf{j} \in \mathcal{J}(T)$. Then for all $t \in [0,1]$,*

$$|\mathcal{I}_T(t, \mathbf{j})| \le t^{|T^0|} \tag{3.12}$$

and

$$|\mathcal{I}_T(t, \mathbf{j})| \le 2^{|T|} \sum_{(\varepsilon_{u,i})} \prod_{w\in T^0} \langle \rho_w(\mathbf{j}) \rangle^{-1}. \tag{3.13}$$

The notation $\langle x\rangle$ means $(1+|x|^2)^{1/2}$. The sum in (3.13) is taken over all of the $3^{|T^0|}$ possible choices of $\varepsilon_{u,i} \in \{0,1,-1\}$; these choices in turn determine the functions ρ_w. Lemma 3.1 will be proved in §5.

Definition 3.9. Let T be any tree, and let $t \in \mathbb{R}$. The tree operator $\mathfrak{S}_T(t)$ associated to T, t is the multilinear operator that maps the $|T^\infty|$ sequences $(x_v)_{v\in T^\infty}$ of complex numbers to the sequence of complex numbers

$$\mathfrak{S}_T(t)\big((x_v)_{v\in T^\infty}\big)(n) = \sum_{\mathbf{j}\in\mathcal{J}(T):j_{\mathbf{r}}=n} \mathcal{I}_T(t,\mathbf{j}) \prod_{w\in T^\infty} x_w(j_w) \tag{3.14}$$

indexed by $n \in \mathbb{Z}$.

$\mathfrak{S}_T(t)$ takes as input $|T^\infty|$ complex sequences, each belonging to a Banach space $\ell^p(\mathbb{Z})$, and outputs a single complex sequence, which will be shown to belong to some $\ell^q(\mathbb{Z})$.

When T is the trivial tree $\{\mathbf{r}\}$ having only one element, $\mathfrak{S}_T(t)$ is the identity operator for every time t, mapping any sequence $(x_n(0))_{n\in\mathbb{Z}}$ to itself. This corresponds to the linear Schrödinger evolution; it is independent of t because we are dealing with twisted Fourier coefficients (2.4).

4. Formalities

With all these definitions and notations in place, we can finally formulate the conclusion of the discussion in §2.

Proposition 4.1. *The recursive procedure indicated in §2 yields a formal expansion*

$$a(t) = \sum_{k=1}^{\infty} A_k(t)(a^\star_{T,1}(0), a^\star_{T,2}(0), \cdots), \tag{4.1}$$

where each $A_k(t)$ is a multilinear operator of the form

$$A_k(t) = \sum_{|T|=3k+1} c_T \mathfrak{S}_T(t), \tag{4.2}$$

each sequence $a^\star_{T,n}(0)$ equals either $a(0)$ or $\bar a(0)$, the scalars $c_T \in \mathbb{C}$ satisfy $|c_T| \le C^{1+|T|}$, and for each index k, the sum in (4.2) is taken over a finite collection of $O(C^k)$ ornamented trees T of the indicated cardinalities.

This asserts that the outcome of the repeated substitution of (2.6) into itself, as described in §2, is accurately encoded in the definitions in §3. This proposition and the following result will be proved later in the paper.

Proposition 4.2. *There exists a finite positive constant c_0 such that whenever $a(0) \in \ell^1$, the multiply infinite series $\sum_k A_k(t)(a^\star(0),\cdots)$ converges absolutely to a function in $C^0([0,\tau],\ell^1)$ provided that $\tau\|a(0)\|_{\ell^1} \le c_0$.*

Conversely, if $u \in C^0([0,\tau],\ell^1)$ then for such τ, the sequence $a_n(t) = e^{in^2t}\widehat{u}(t,n)$ equals the sum of this series, for $t \in [0,\tau]$.

By the first statement we mean that $\sum_{\mathbf{j}\in\mathcal{J}(T)} |\mathcal{I}_T(t,\mathbf{j})| \prod_{w\in T^\infty} |a(0)(j_w)|$ converges absolutely for each ornamented tree T, and that if its sum is denoted by $\mathfrak{S}_T^*(a(0),a(0),\cdots)(t)$ then the resulting series $\sum_{k=1}^\infty \sum_{|T|=3k+1} c_T \mathfrak{S}_T^*(a(0),a(0),\cdots)(t)$ likewise converges.

The operators $\mathfrak{S}_T$ and coefficients c_T were defined so that the following holds automatically.

Lemma 4.3. *There exists $c>0$ with the following property. Let $\widehat{u_0}$ be any numerical sequence and define $a(0)(n)=\widehat{u_0}(n)$. Suppose that the infinite series defining $\mathfrak{S}_T^*(a^\star(0),a^\star(0),\cdots)(t)$ converges absolutely and uniformly for all $t\in[0,\tau]$ and that its sum is $O(c^{|T|})$, uniformly for every ornamented tree T. Define $a(t)$ to be the sequence $\sum_{k=1}^\infty A_k(t)(a^\star(0),a^\star(0),\cdots)$. Then a satisfies the integral equation* (2.6) *for $t\in[0,\tau]$. Moreover, the function $u(t,x)$ defined by $\widehat{u}(t,n)=e^{-in^2t}a(t,n)$ is a solution of the modified Cauchy problem* (NLS*) *in the corresponding sense* (2.7).

The main estimate in our analysis is as follows.

Proposition 4.4. *Let $p\in(1,\infty)$. Then for any exponent $q>\frac{p}{|T^\infty|}$ satisfying also $q\ge 1$, there exist $\varepsilon>0$ and $C<\infty$ such that for all trees T and all sequences $x_v\in\ell^1$,*

$$\|\mathfrak{S}_T(t)\big((x_v)_{v\in T^\infty}\big)\|_{\ell^q} \le (Ct^\varepsilon)^{|T^\infty|} \prod_{v\in T^\infty} \|x_v\|_{\ell^p}. \tag{4.3}$$

Proposition 4.4 and Lemma 4.3 will be proved in subsequent sections. Together, they give:

Corollary 4.5. *Let $p\in[1,\infty)$. For any $R<\infty$ there exists $\tau>0$ such that the solution mapping $u_0\mapsto u(t,\cdot)$ for the modified Cauchy problem* (NLS*)*, initially defined for all sufficiently smooth u_0, extends by uniform continuity to a real analytic mapping from $\{u_0\in\mathcal{F}L^p : \|u_0\|_{\mathcal{F}L^p}\le R\}$ to $C^0([0,\tau],\mathcal{F}L^p(\mathbb{T}))$.*

We emphasize that analytic dependence on t is not asserted; solutions are Hölder continuous with respect to time.

5. Bound for the interaction amplitudes $\mathcal{I}_T(t,\mathbf{j})$

Proof of Lemma 3.1. Let $\mathbf{j}\in\mathbb{Z}^T$ be given; it will remain constant throughout the proof. The first bound of the lemma holds simply because $|\mathcal{I}_T(t,\mathbf{j})|\le|\mathcal{R}(T,t)|$. The proof of the second bound (3.13) proceeds recursively in steps. In each step we integrate with respect to t_v for certain nodes v in the integral defining $\mathcal{R}(T,t)$, holding certain other coordinates t_w fixed. Once integration has been performed with respect to some coordinate, that coordinate is of course removed from later steps.

In step 1, we hold t_v fixed whenever at least one child of v is not terminal. We also fix t_v for every simple node v having only terminal children. The former coordinates t_v, and underlying nodes v, are said to be temporarily fixed; the latter coordinates and nodes are said to be permanently fixed. We integrate with respect to all nonfixed coordinates t_w.

When $|T|=1$ there is nothing to prove. Otherwise there must always exist at least one node, all of whose children are terminal. If there exists such a node which is also ordinary, then at least one coordinate t_v is not fixed. The subset, or slice, of $\mathcal{R}(T,t)$ defined by setting each of the fixed coordinates equal to some constant is either empty, or takes the product form $\times_{u \text{ not fixed}}[0,t_{u^*}]$, where u^* denotes the parent of u. The integrand is likewise a product, of simple exponentials. Integrating over this slice with respect to all of the nonfixed coordinates thus yields

$$\prod_w e^{\pm_w i\sigma_w t_w} \prod_u \int_0^{t_{u^*}} e^{\pm_u i\sigma_u t_u}\,dt_u,$$

where the first product is taken over all fixed $w \in T^0$, and the second over all remaining nonfixed $u \in T^0$.

None of the quantities σ_u can vanish in step 1, since an ordinary node having only terminal children can never be exceptional, by (3.7). Therefore the preceding expression equals

$$\prod_w e^{\pm_w i\sigma_w t_w} \prod_u (\pm_u i\sigma_u)^{-1}\big(e^{\pm_u i\sigma_u t_{u^*}} - 1\big).$$

This may be expanded as a sum of 2^N terms, where N is the number of nonfixed nodes in T^0. Each of these terms has the form

$$\pm \prod_w e^{\pm_w i\sigma_w t_w} \prod_u (i\sigma_u)^{-1} e^{\pm_{(u^*)}\varepsilon_u i\sigma_u t_{u^*}} \tag{5.1}$$

for some numbers $\varepsilon_u \in \{0,1,-1\}$.

The other possibility in step 1 is that $|T| > 1$, but every nonterminal node that has only terminal children is simple. In that case all coordinates t_v are fixed at step 1, no integration is performed, and we move on to step 2.

Any node v that is permanently fixed at any step of the construction remains fixed through all subsequent steps; we never integrate with respect to t_v. On the other hand, once we've integrated with respect to some t_w, then the node w is also removed from further consideration.

We now carry out step 2. The set T_1 of all nodes temporarily fixed during step 1 is itself a tree. There is an associated subset $\mathcal{R}_{T_1}$ of $\{(t_w : w \in T_1)\}$, defined by the inequalities $0 \le t_w \le t_{w'} \le t$ whenever $w \le w'$, and also by $t_u \le t_w$ if $u \le w$ and u was permanently fixed in step 1. To each node $w \in T_1$ is associated a modified phase $\sigma_w^{(2)}$, defined to be $\sigma_w + \sum_i \varepsilon_{(w,i)}\sigma_{(w,i)}$, where the sum is taken over all $i \in \{1,2,3\}$ such that we integrated with respect to $t_{(w,i)}$ in the first step. Thus the product of exponentials in (5.1) can be rewritten as

$$\prod_w e^{\pm_w i\sigma_w t_w} \prod_u e^{\pm_{(u^*)}\varepsilon_u i\sigma_u t_{u^*}} = \prod_{v \in T_1} e^{\pm_v i\sigma_v^{(2)} t_v}, \tag{5.2}$$

which takes the same general form as the original integrand.

A node w is permanently fixed at step 2 if it was permanently fixed at step 1, or if w is terminal in T_1 and satisfies $\sigma_w^{(2)} = 0$. A node $w \in T_1$ is temporarily fixed at step 2 if w is not terminal in T_1. We now integrate $\prod_{w\in T_1} e^{\pm i\sigma_w^{(2)}(t_w)}$ with respect to t_u for all $u \in T_1$ that are neither temporarily nor permanently fixed at step 2. As in step 1, this integral has a product structure $\times_u[t_{u,*}, t_{u^*}]$ where the product is taken over all nodes u not fixed at this step, u^* is the parent of u, and the lower limit $t_{u,*}$ is either zero, or equals t_w for some child w of u which has been permanently fixed. Now 2^{N_2} terms are obtained after integration, where N_2 is the number of variables with respect to which we integrate.

In step 3 we consider the tree T_2 consisting of all $w \in T_1$ that were temporarily fixed in step 2. Associated to T_2 is a set $\mathcal{R}_{T_2}$, and associated to each node $v \in T_2$ is a modified phase $\sigma_w^{(3)} = \sigma_w^{(2)} + \sum_i \varepsilon_{(w,i)} \sigma_{(w,i)}^{(2)}$, the sum being taken over all $i \in \{1, 2, 3\}$ such that (w, i) was not fixed in step 2. A node $v \in T_2$ is then permanently fixed if it is terminal in T_2 and $\sigma_v^{(3)} = 0$. $v \in T_2$ is temporarily fixed if it is not terminal in T_2. We then integrate with respect to t_v for all $v \in T_2$ that are neither temporarily nor permanently fixed.

This procedure terminates after finitely many steps, when for each node $v \in T^0$, either v has become permanently fixed, or we have integrated with respect to t_v. This yields a sum of at most $2^{|T^0|}$ terms. Each term arises from some particular choice of the parameters $\varepsilon_{u,i}$, and is expressed as an integral with respect to t_v for all nodes $v \in T^0$ that were permanently fixed at some step; the vector (t_v) indexed by all such v varies over a subset of $[0, t]^M$ where M is the number of such v. At step n, each integration with respect to some t_u yields a factor of $(\sigma_u^{(n)})^{-1}$, multiplied by some unimodular factor; $\sigma_u^{(n)}$ is nonzero, since u would otherwise have been permanently fixed.

Thus for each term we obtain an upper bound of $\prod_u |\rho_u(\mathbf{j})|^{-1}$, where the product is taken over all nonexceptional nodes u; this bound must still be integrated with respect to all t_w where w ranges over all the exceptional nodes. Each such coordinate t_w is restricted to $[0, t]$. Thus we obtain a total bound

$$|\mathcal{I}(t, \mathbf{j})| \le \sum_{(\varepsilon_{u,i})} t^M \prod_{w \in T^0}^{*} |\rho_w(\mathbf{j})|^{-1} \tag{5.3}$$

where for each $(\varepsilon_{u,i})$, $M = M((\varepsilon_{u,i}))$ is the number of exceptional nodes encountered in this procedure, that is, the number of permanently fixed nodes, and where for each $(\varepsilon_{u,i})$, $\prod_{w \in T^0}^{*}$ denotes the product over all nodes $w \in T^0$ that are nonexceptional with respect to the parameters $(\varepsilon_{u,i})$ and $\mathbf{j}$. Since $t \in [0, 1]$, the stated result follows. $\square$

6. A simple ℓ^1 bound

This section is devoted to a preliminary bound for simplified multilinear operators. For any tree T and any sequences $y_v \in \ell^1$, define

$$\tilde{S}_T\big((y_v)_{v \in T^\infty}\big)(n) = \sum_{\mathbf{j}: j_{\mathbf{r}} = n}^{\star\star} \prod_{u \in T^\infty} y_u(j_u). \tag{6.1}$$

The notation $\sum_{\mathbf{j}: j_{\mathbf{r}}=n}^{\star\star}$ indicates that the sum is taken over all indices $\mathbf{j} \in \mathbb{Z}^T$ satisfying (3.6) as well as $j_{\mathbf{r}} = n$; the restrictions (3.7) and (3.8) are not imposed here. $\tilde{S}_T$ as the same general structure as $\mathfrak{S}_T$, except that the important interaction amplitudes $\mathcal{I}_T(t, \mathbf{j})$ have been omitted.

Lemma 6.1. *For any tree T and any sequences $\{(y_v) : v \in T^\infty\}$*

$$\|\tilde{S}_T\big((y_v)_{v\in T^\infty}\big)\|_{\ell^1} \le \prod_{w\in T^\infty} \|y_w\|_{\ell^1}, \tag{6.2}$$

with equality when all $y_v(j_v)$ are nonnegative.

Proof. There exists a nonnegative integer k for which $|T| = 3k+1$, $|T^\infty| = 2k+1$, and $|T^0| = k$. Consider the set $B \subset T$ whose elements are the root node $\mathbf{r}$ together with all (v,i) such that $v \in T^0$ and $i \in \{1,3\}$. Thus $|B| = 1 + 2k = |T^\infty|$. Define

$$k_{v,i} = j_v - j_{(v,i)} \text{ for } v \in T^0 \text{ and } i \in \{1,3\}. \tag{6.3}$$

Consider the $\mathbb{Z}$-linear mapping L from $\mathbb{Z}^{T^\infty}$ to $\mathbb{Z}^B$ defined so that $L(\mathbf{j})$ has coordinates $j_{\mathbf{r}}$ and all $k_{v,i}$. The definition of $k_{v,i}$ makes sense for $i = 2$, but that quantity is redundant; $k_{v,1} - k_{v,2} + k_{v,3} \equiv 0$.

j_v and $j_{(v,i)}$ are well-defined linear functionals of $\mathbf{j} \in \mathbb{Z}^{T^\infty}$, because given the quantities j_w for all $w \in T^\infty$, j_v can be recovered for all other $v \in T$ via the relations (3.6), by ascending induction on v. We claim that L is invertible. Indeed, from the quantities $j_{\mathbf{r}}$ and all $j_v - j_{(v,i)}$ with $v \in T^0$ and $i \in \{1,3\}$, j_u can be recovered for all $u \in T$ by descending induction on u, using again (3.6) at each stage. For instance, at the initial step, $j_{(\mathbf{r},i)} = j_{\mathbf{r}} + k_{\mathbf{r},i}$ for $i = 1,3$, and then $j_{(\mathbf{r},2)}$ can be recovered via (3.6). Thus L is injective, hence invertible.

By descending induction on nodes it follows in the same way from (3.6) that $\mathbf{j} = (j_w)_{w\in T^\infty}$ satisfies a certain linear relation of the form

$$j_{\mathbf{r}} = \sum_{w\in T^\infty} \pm_w j_w \tag{6.4}$$

where each coefficient $\pm_w$ equals ± 1. By the conclusion of the preceding paragraph, $(j_w)_{w\in T^\infty}$ is subject to no other relation; the sum defining $\tilde{S}_T\big((y_w)_{w\in T^\infty}\big)(j_{\mathbf{r}})$ is taken over all $\mathbf{j}$ satisfying this relation. Therefore $\sum_{j_{\mathbf{r}}} \tilde{S}_T(j_{\mathbf{r}})$ equals the summation over all $w \in T^\infty$ and all $j_w \in \mathbb{Z}$, without restriction, of $\prod_{w\in T^\infty} y_w(j_w)$. The lemma follows. $\square$

Corollary 6.2. *For any tree, the sum defining $\mathfrak{S}_T\big((y_v)_{v\in T^\infty}\big)(n)$ converges absolutely for all $n \in \mathbb{Z}$ whenever all $y_v \in \ell^1$, and the resulting sequence satisfies*

$$\|\mathfrak{S}_T\big((y_v)_{v\in T^\infty}\big)\|_{\ell^1} \le \prod_{v\in T^\infty} \|y_v\|_{\ell^1}. \tag{6.5}$$

Proof. This is a direct consequence of the preceding lemma together with the simple bound $|\mathcal{I}_T(t,\mathbf{j})| \le t^{|T^0|}$ of Lemma 3.1. $\square$

Estimates in ℓ^p for $p > 1$ are less simple; there is no bound for $\tilde{S}_T$ in terms of the quantities $\|y_w\|_{\ell^p}$ for $p > 1$. The additional factors $\langle \rho_u \rangle^{-1}$ in the second interaction amplitude bound (3.13), reflecting the dispersive character of the partial differential equation, are essential for estimates in terms of weaker ℓ^p norms.

Proof of Propositions 4.1 and 4.2. The first conclusion of Proposition 4.2 follows directly from the preceding corollary. To establish Proposition 4.1, let $y = y_n(t) = y(t,n) \in C^0([0,\tau],\ell^1)$ be any sequence-valued solution of the integral equation

$$y(t,n) = y(0,n) - i\omega \int_0^t |y(s,n)|^2 y(s,n)\,ds + i\omega \sum_{j-k+l=n}^{*} \int_0^t y(s,j)\bar{y}(s,k)y(s,l)e^{i\sigma(j,k,l,n)s}\,ds. \tag{6.6}$$

Consider any tree T, and let each node $v \in T^\infty$ be designated as either finished or unfinished. Consider the associated function

$$\int_{\mathcal{R}(T,t)} \sum_{\mathbf{j}\in\mathcal{J}(T)} \prod_{v\in T^0} e^{\pm_v i\sigma_v t_v} \prod_{u\in T^\infty} y_u(t_u, j_u)\,dt_u \tag{6.7}$$

for $0 \le t \le \tau$, with $t_{\mathbf{r}} \equiv t$, where for each $u \in T^\infty$, $y_u(t,\cdot)$ is identically equal to one of $y(t,\cdot)$, $\bar{y}(t,\cdot)$ of u is unfinished, and to one of $y(0,\cdot)$, $\bar{y}(0,\cdot)$ if u is finished. The simplest such expression, associated to the tree $T = \{\mathbf{r}\}$ having only one element, is any constant sequence $y_{\mathbf{r}}(0, j_{\mathbf{r}})$.

For each unfinished node u, substitute the right-hand side of (6.6) or its complex conjugate, as appropriate, for $y_u(t_u, j_u)$ in (6.7). The $C^0(\ell^1)$ hypothesis guarantees that an absolutely convergent integral and sum are produced. Thus we may interchange the outer integral with the sums. What results is a finite linear combination of expressions of the same character as (6.7), associated to trees $T^\sharp$. At most $3^{|T^\infty|}$ such expressions are obtained, and each is multiplied by a unimodular numerical coefficient.

Each nonterminal node of T is a nonterminal node of $T^\sharp$, and each finished node of T^∞ remains a terminal node of $T^\sharp$. When the first term on the right in (6.6) is substituted for $y_u(t_u, j_u)$ then the unfinished node u becomes a finished terminal node. When the second or third terms on the right are substituted, new unfinished terminal nodes are added to create $T^\sharp$, in which u is a nonterminal simple node or a ordinary node, respectively. Each child of u in $T^\sharp$ is a terminal node of $T^\sharp$, and is (consequently) unfinished.

When $T = \{\mathbf{r}\}$, we have simply $y(t)$. Repeatedly subsituting as above produces an infinite sum of expressions as described in Proposition 4.1. Thus the proof of that result is complete.

To prove that any solution y in $C^0([0,\tau],\ell^1)$ must agree with the sum of our power series for sufficiently small τ, regard y as being the function associated as above to $T = \{\mathbf{r}\}$ and apply the substitution procedure a large finite number of times, N. If M is given and N is chosen sufficiently large in terms of N, then what results is an expression for y as a sum of some terms of the power series, including all terms associated to trees of orders $\le M$, together with certain error terms. There are at most C^N error terms, and each is $O(\tau^{cN})$ in $C^0(\ell^1)$ norm, where the constants depend on the $C^0(\ell^1)$ norm of y. Therefore these expressions converge, as $N \to \infty$, to the sum of the power series in $C^0([0,\tau],\ell^1)$ norm provided that τ is sufficiently small relative to the $C^0(\ell^1)$ norm of y. $\square$

7. Tree sum majorants

In this section we introduce majorizing operators which are the essence of the problem, and decompose them into sub-operators.

7.1. Majorant operators associated to ornamented trees.

Definition 7.1. Let T be an ornamented tree. The tree sum majorant associated to T is the multilinear operator

$$S_T\big((y_w)_{w\in T^\infty}\big)(n) = \sum_{\mathbf{j}\in\mathcal{J}(T):j_{\mathbf{r}}=n} \prod_{u\in T^0} \langle \rho_u(\mathbf{j})\rangle^{-1} \prod_{w\in T^\infty} y_w(j_w). \tag{7.1}$$

S_T is initially defined when all $y_w \in \ell^1$, in order to ensure absolute convergence of the sum.

Lemma 7.1. *Let $p \in [1,\infty)$ and suppose that $q > |T^\infty|^{-1}p$ and $q \ge 1$. Then there exists $C < \infty$ such that for all ornamented trees,*

$$\|S_T\big((x_v)_{v\in T^\infty}\big)\|_{\ell^q} \le C^{|T|} \prod_{v\in T^\infty} \|x_v\|_{\ell^p} \tag{7.2}$$

for all sequences $x_v \in \ell^1$.

Assuming this for the present, we show how it implies Proposition 4.4.

Proof of Proposition 4.4. Let T be any tree. We already have

$$\|\mathfrak{S}_T(t)\big((x_v)_{v\in T^\infty}\big)\|_{\ell^1} \le t^{|T^0|} \prod_{v\in T^\infty} \|x_v\|_{\ell^1} \tag{7.3}$$

for all sequences $x_v \in \ell^1$ by Lemma 6.1 together with the first bound for the interaction amplitudes $\mathcal{I}_T(t,\mathbf{j})$ provided by Lemma 3.1.

On the other hand, to T are associated at most $3^{|T|}$ ornamented trees $\tilde T$, defined by specifying coefficients $\varepsilon_{v,i}$. According to the second conclusion (3.13) of Lemma 3.1, $\|\mathfrak{S}_T(t)\big((x_v)_{v\in T^\infty}\big)\|_{\ell^q}$ is majorized by $C^{|T|}$ times the sum over these $\tilde T$ of $\|S_{\tilde T}\big((x_v)_{v\in T^\infty}\big)\|_{\ell^q}$. This bound holds uniformly in t, provided that t is restricted to a bounded interval. Thus (7.2) implies that

$$\|\mathfrak{S}_T(t)\big((x_v)_{v\in T^\infty}\big)\|_{\ell^q} \le C^{|T|} \prod_{v\in T^\infty} \|x_v\|_{\ell^p} \tag{7.4}$$

under the indicated assumptions on p,q. Interpolating this with the bound for $p = q = 1$ yields

$$\|\mathfrak{S}_T(t)\big((x_v)_{v\in T^\infty}\big)\|_{\ell^q} \le (Ct^{\varepsilon})^{|T|} \prod_{v\in T^\infty} \|x_v\|_{\ell^p} \tag{7.5}$$

for some $\varepsilon > 0$. □

7.2. **Marked ornamented trees, and associated operators.** The analysis of S_T will rely on several further decompositions.

Definition 7.2. A marked ornamented tree (T, T') is an ornamented tree T together with a subset $T' \subset T^0$, the set of marked nodes, and the collection

$$\mathcal{J}(T,T') = \{\mathbf{j} \in \mathcal{J}(T) : \{v \in T : (v,\mathbf{j}) \text{ is nearly resonant}\} = T'\}. \tag{7.6}$$

Definition 7.3. Let (T, T') be a marked ornamented tree. The associated tree sum majorant is the multilinear operator

$$S_{(T,T')}\big((y_w)_{w\in T^\infty}\big)(n) = \sum_{\mathbf{j}\in\mathcal{J}(T,T'):j_{\mathbf{r}}=n} \prod_{u\in T^0} \langle\rho_u(\mathbf{j})\rangle^{-1} \prod_{w\in T^\infty} y_w(j_w). \tag{7.7}$$

Now for any ornamented tree T,

$$S_T = \sum_{T'\subset T^0} S_{(T,T')}, \tag{7.8}$$

the sum being taken over all subsets $T' \subset T^0$. The total number of such subsets is $2^{|T^0|} \le 2^{|T|} \le 2^{3|T^\infty|/2} = C^{|T^\infty|}$. Therefore in order to establish the bound stated in Lemma 7.1 for the operator S_T associated to an ornamented tree T, it suffices to prove that same bound for $S_{(T,T')}$, for all subsets $T' \subset T^0$.

7.3. **A further decomposition.** Let (T, T') be any marked ornamented tree, which will remain fixed for the remainder of the analysis. To avoid having to write absolute value signs, we assume that y_v are all sequences of nonnegative real numbers.

We seek an upper bound for the associated tree sum operator $S_{(T,T')}$. The factors $\langle\rho_v\rangle^{-1}$ in the definition of $S_{(T,T')}$ are favorable when $|\rho_v|$ is large; nearly resonant pairs are those for which $|\rho_v(\mathbf{j})|$ is relatively small, and hence these require special attention.

Denote by $\Gamma = (\gamma_u)_{u\in T'}$ any element of $\mathbb{Z}^{T'}$. Let

$$\mathcal{J}(T,T',\Gamma) = \{\mathbf{j} \in \mathcal{J}(T,T') : \rho_u(\mathbf{j}) = \gamma_u \text{ for all } u \in T'\}. \tag{7.9}$$

T' is the set of all nearly resonant nodes, so by its definition we have

$$|\gamma_u| = |\rho_u(\mathbf{j})| \le c_0|\sigma_u(\mathbf{j})|^{1-\delta} \ \forall u \in T'. \tag{7.10}$$

This leads to a further decomposition and majorization

$$\begin{aligned} S_{(T,T')}\big((y_v)_{v\in T^\infty}\big)(n) &= \sum_{\Gamma\in\mathbb{Z}^{T'}} \sum_{\mathbf{j}\in\mathcal{J}(T,T',\Gamma):j_{\mathbf{r}}=n} \prod_{u\in T^0} \langle\rho_u(\mathbf{j})\rangle^{-1} \prod_{w\in T^\infty} y_w(j_w) \\ &\le C^{|T|} \sum_{\mathbf{N}} \prod_{v\in T'} 2^{-N_v} \sum_{\mathbf{M}} \prod_{u\in T^0\setminus T'} 2^{-(1-\delta)M_u} \sum_{\Gamma} \sum_{\mathbf{j}\in\mathcal{J}(T,T',\Gamma):j_{\mathbf{r}}=n} \prod_{w\in T^\infty} y_w(j_w) \end{aligned} \tag{7.11}$$

where $\mathbf{N} = (N_v)_{v\in T'}$ and $\mathbf{M} = (M_u)_{u\in T^0\setminus T'}$. The notation in the last line means that the first two sums are taken over all nonnegative integers N_v, M_u as v ranges over T' and u over $T^0 \setminus T'$; the third sum is taken over all $\Gamma = (\gamma_u)_{u\in T'}$ such that

$$\langle\gamma_v\rangle \in [2^{N_v}, 2^{1+N_v}) \ \text{ for all } v \in T'; \tag{7.12}$$

and the sum with respect to $\mathbf{j}$ is taken over all $\mathbf{j} \in \mathcal{J}(T, T', \Gamma)$ satisfying $j_{\mathbf{r}} = n$ together with the additional restrictions

$$|\sigma_u(j_{(u,1)}, j_{(u,2)}, j_{(u,3)}, j_u)| \sim 2^{M_u} \quad \text{for all } u \in T^0 \setminus T' \tag{7.13}$$

$$\rho_v(\mathbf{j}) = \gamma_v \text{ for all } v \in T'. \tag{7.14}$$

Thus there is an upper bound $2^{N_v} \le Cc_0 |\sigma_v(\mathbf{j})|^{1-\delta}$ for all $v \in T'$.

7.4. **Rarity of near resonances.** Let δ_1 be a small constant, to be chosen later. Recall that for any positive integer n, there are at most $C_{\delta_1} n^{\delta_1}$ pairs (n', n'') of integers for which n can be factored as $n = n'n''$. This fact was exploited by Bourgain [2] in his proof of H^0 wellposedness.

The key to the control of near resonances is a strong limitation on the number of $\mathbf{j}$ satisfying (7.14), for any fixed Γ. Given $v \in T'$ any parameter γ_v, and any $\mathbf{j} \in \mathcal{J}(T, T', \Gamma)$, the equation (7.14) can be written as

$$\sigma_v(\mathbf{j}) = \gamma_v - \sum_{i=1}^{3} \varepsilon_{v,i} \rho_{(v,i)}(\mathbf{j}),$$

and $\rho_{(v,i)}(\mathbf{j})$ depends only on $\{j_w - j_{(w,i)} : w < v, i \in \{1,2,3\}\}$. Since the quantity σ_v on the left-hand side of this rewritten equation can be factored as $2(j_v - j_{(v,1)})(j_v - j_{(v,3)})$, we conclude that for any $\{j_w - j_{(w,l)} : w < v, l \in \{1,2,3\}\}$ and any γ_v there are at most $C_{\delta_1} |\gamma_v - \sum_{i=1}^{3} \varepsilon_{v,i} \rho_{(v,i)}(\mathbf{j})|^{\delta_1}$ ordered pairs $\big(j_v - j_{(v,1)}, j_v - j_{(v,3)}\big)$ satisfying (7.14).

For any nearly resonant node $v \in T'$, $|\gamma_v|$ is small relative to $\sum_{i=1}^{3} |\rho_{(v,i)}(\mathbf{j})|^{1-\delta}$, provided that the constant c_0 is chosen to be sufficiently small in the definition of a nearly resonant node. Therefore we can choose for each $\mathbf{N}, \mathbf{M}$ a family $\mathcal{F} = \mathcal{F}_{\mathbf{N},\mathbf{M}}$ of vector-valued functions $F = (f_{v,i} : v \in T', i \in \{1,3\})$ such that for any Γ satisfying (7.12) and any $\mathbf{j} \in \mathcal{J}(T, T', \Gamma)$, there exists $F \in \mathcal{F}_{\mathbf{N},\mathbf{M}}$ such that for each $v \in T'$ and each $i \in \{1, 3\}$,

$$k_{v,i} = f_{v,i}(\gamma_v, (k_{w,i} : w < v)) \tag{7.15}$$

where $k_{u,i} = j_u - j_{(u,i)}$.

The number of such functions is strongly restricted:

$$|\mathcal{F}_{\mathbf{N},\mathbf{M}}| \le C_{\delta_1}^{|T'|} \prod_{v \in T'} 2^{\delta_1 \max_i K_{(v,i)}} \tag{7.16}$$

where $K_u = N_u$ for $u \in T'$ and $K_u = M_u$ for $u \in T^0 \setminus T'$, and the maximum is taken over $i \in \{1, 3\}$. Powers of $2^{\delta_1 N_{(v,i)}}$ are undesirable; we will show in Lemma 8.2 below that the product on the right-hand side of (7.16) satisfies a better bound in which $\mathbf{N}$ does not appear.

7.5. **A final decomposition.** For $\mathbf{M}, \mathbf{N}$ as above, we set $|\mathbf{M}| = \sum_{u \in T^0 \setminus T'} M_u$ and $|\mathbf{N}| = \sum_{v \in T'} N_v$.

Definition 7.4. To any $\mathbf{M}, \mathbf{N}, \Gamma$ and any function $F \in \mathcal{F}_{\mathbf{N},\mathbf{M}}$ is associated the multilinear operator

$$S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}\big((y_w)_{w\in T^\infty}\big)(n) = \sum_{\mathbf{j}\in\mathcal{J}(T,T',\Gamma):j_{\mathbf{r}}=n} \prod_{w\in T^\infty} y_w(j_w) \tag{7.17}$$

where the sum in (7.17) is taken over all $\mathbf{j} \in \mathcal{J}(T, T', \Gamma)$ satisfying $j_{\mathbf{r}} = n$, (7.13), (7.14), and the additional restriction (7.15).

The multilinear operators $S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}$ are our basic building blocks. We have shown so far that for all nonnegative sequences y_w and all $n \in \mathbb{Z}$,

$$\begin{aligned} &|S_{(T,T')}\big((y_w)_{w\in T^\infty}\big)(n)| \\ &\qquad \le C^{|T|} \sum_{\mathbf{N},\mathbf{M}} 2^{-|\mathbf{N}|} 2^{-(1-\delta)|\mathbf{M}|} \sum_{\Gamma} \sum_{F\in\mathcal{F}_{\mathbf{N},\mathbf{M}}} |S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}\big((y_w)_{w\in T^\infty}\big)(n)| \end{aligned} \tag{7.18}$$

where the second summation in (7.17) is taken over all $\Gamma = (\gamma_u)_{u\in T'}$ satisfying both (7.12) and (7.10). The factor of $2^{-(1-\delta)|\mathbf{M}|}$ arises because for each $u \in T^0 \setminus T'$, we have by virtue of Lemma 3.1 a factor of $\langle \rho_u(\mathbf{j})\rangle^{-1}$, and this factor is $\le C 2^{-(1-\delta)M_u}$ because u is not nearly resonant.

8. Bounds for the most basic multilinear operators

Lemma 8.1. *Let $p \in [1,\infty)$ and $\delta_1 > 0$. Then for every exponent $q \ge \max(1, p/|T^\infty|)$, there exists $C < \infty$ such that for every $T, T', \mathbf{N}, \mathbf{M}, \Gamma, F$ and for every sequence y_v,*

$$\|S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}\big((y_v)_{v\in T^\infty}\big)\|_{\ell^q} \le C^{|T|} 2^{(1+\delta_1)|\mathbf{M}|} \prod_{v\in T^\infty} \|y_v\|_{\ell^p}. \tag{8.1}$$

This involves no positive power of $2^{|\mathbf{N}|}$, and thus improves on (7.16).

Proof. As was shown in the proof of Lemma 6.1, each quantity j_v in the summation defining $S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}\big((y_w)_{w\in T^\infty}\big)(j_{\mathbf{r}})$ can be expressed as a function, depending on Γ and on F, of $j_{\mathbf{r}}$ together with all $k_{w,i} = j_w - j_{(w,i)}$, where w varies over the set T^0 and i varies over $\{1,3\}$. The equation (7.15) can then be used by descending induction on T to eliminate $k_{w,i}$ for all $w \in T'$ so long as F, Γ are given. More precisely, j_v equals $j_{\mathbf{r}} + g_v$, where g_v is some function of all $k_{w,i}$ with $w \in T^0 \setminus T'$ and $i \in \{1,3\}$.

$\prod_{v\in T^\infty} y_v(j_v)$ can thus be rewritten as $\prod_{v\in T^\infty} y_v(j_{\mathbf{r}} + g_v)$. If every $k_{w,i}$ is held fixed, then as a function of $j_{\mathbf{r}}$, this product belongs to ℓ^q for $q = p/|T^\infty|$ with bound $\prod_{v\in T^\infty} \|y_v\|_{\ell^p}$, by Hölder's inequality.

The total number of terms in the sum defining $S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}$ is the total possible number of vectors $(k_{w,i})$ where w ranges over $T^0 \setminus T'$ and i over $\{1,3\}$. The number of such pairs for a given w is $\le C_{\delta_1} 2^{(1+\delta_1)M_w}$, since $|2k_{w,1}k_{w,3}| = |\sigma_w(\mathbf{j})| \le 2^{M_w+1}$. Thus in all there are at most $C_{\delta_1}^{|T|} 2^{(1+\delta_1)|\mathbf{M}|}$ terms. Minkowski's inequality thus gives the stated bound. □

A difficulty now appears. For each $v \in T'$ we have a compensating factor of $\langle \rho_v(\mathbf{j})\rangle^{-1} = \langle \gamma_v(\mathbf{j})\rangle^{-1} \sim 2^{-N_v}$, but no upper bound whatsoever is available for the ratio of $\max_i |\rho_{(v,i)}(\mathbf{j})|^{\delta_1}$ to $\langle \gamma_v(\mathbf{j})\rangle$. Thus for any particular $v \in T'$, the factor lost through the nonuniqueness of F need not be counterbalanced by the favorable factor ρ_v^{-1}. Nonetheless, the product of all these favorable factors does compensate for the product of all those factors lost, as will now be shown.

Lemma 8.2. *For any $\varepsilon > 0$ there exists $C_\varepsilon < \infty$ such that uniformly for all $T, T', \mathbf{N}, \mathbf{M}$,*

$$|\mathcal{F}_{\mathbf{N},\mathbf{M}}| \le C_\varepsilon^{|T|} 2^{\varepsilon|\mathbf{M}|}. \tag{8.2}$$

Proof. Let $\mathbf{j} \in \mathcal{J}(T, T', \Gamma)$ satisfy $\rho_v(\mathbf{j}) = \gamma_v$ for all $v \in T'$ but be otherwise arbitrary. Throughout this argument, $\mathbf{j}$ will remain fixed, and ρ_v will be written as shorthand for $\rho_v(\mathbf{j})$.

If the constant c_0 in the definition (3.9) of a nearly resonant node is chosen to be sufficiently small, then any nearly resonant node u has a child (u, i) such that $|\rho_u| \le \frac{1}{2}|\rho_{(u,i)}|^{1-\delta}$. Consider any chain $v = u_h \ge u_{h-1} \ge \cdots \ge u_1$ of nodes such that u_{k+1} is the parent of u_k for each $1 \le k < h$ (u_k is called the $(k-1)$-th generation ancestor of u_1), u_k is nearly resonant for all $k > 1$, u_1 is either not nearly resonant or is terminal, and $|\rho_{u_k}| \le \frac{1}{2}|\rho_{u_{k-1}}|^{1-\delta}$. Then

$$|\rho_{u_k}| \le |\rho_{u_1}|^{(1-\delta)^{k-1}}; \tag{8.3}$$

hence

$$2^{K_{u_k}} = 2^{N_{u_k}} \le C 2^{(1-\delta)^{k-1} M_{u_1}}. \tag{8.4}$$

If u_1 is terminal then $\rho_{u_1} = 0$ by definition, whence the inequality $|\rho_{u_k}| \le \rho_{u_1}|^{(1-\delta)^{k-1}}$ forces $\rho_{u_k} = 0$ for all u_k, as well. This means that $2^{\max_i K_{(u_k,i)}} \sim 1$. In particular, this holds for $u_k = v$, so the factor $2^{\max_i K_{(v,i)}}$ will be harmless in our estimates. We say that a node v is negligible if there exists such a chain, with $v = u_h$ for some $h \ge 1$.

Recall that $|\mathcal{F}_{\mathbf{N},\mathbf{M}}| \le C_{\delta_1}^{|T|} \prod_{v \in T'} 2^{\max_i K_{(v,i)}\delta_1}$. For each nonnegligible nearly resonant node v, choose one such chain with $u_h = v$, thus uniquely specifying h and u_1 as functions of v; we then write $(u_1, h) = A(v)$. Any node has at most one h-th generation ancestor; therefore given both u_1 and h, there can be at most one v such that $(u_1, h) = A(v)$. Consequently

$$\prod_{v \in T' \text{ nonnegligible}} 2^{\max_i K_{(v,i)}\delta_1} \le \prod_{w \in T^0 \setminus T'} \prod_{h=1}^{\infty} 2^{(1-\delta)^{h-1}\delta_1 M_w} = \prod_{w \in T^0 \setminus T'} 2^{M_w \delta_1/\delta}, \tag{8.5}$$

since each factor $2^{\max_i K_{(v,i)}\delta_1}$ in the first product is majorized by $2^{(1-\delta)^{h-1}\delta_1 M_w}$ in the second product, where $(w, h) = A(v)$. Forming the product with respect to h for each fixed v yields the desired inequality, since the series $\sum_{h=0}^{\infty}(1-\delta)^{h-1}\delta_1$ is convergent. The exponent $1 - \delta < 1$ in the definition (3.9) of a nearly resonant node was introduced solely for this purpose. If negligible nodes are also allowed in the product on the left-hand side of (8.5), then they contribute a factor bounded by $C^{|T|}$, so the conclusion remains valid for the full product.

The desired bound now follows by choosing δ_1 so that $\delta_1/\delta = \varepsilon$. $\square$

Conclusion of proof of Lemma 7.1. As already noted, it suffices to establish (7.2) with S_T replaced by $S_{(T,T')}$. Combining the preceding two lemmas gives

$$\sum_{F\in\mathcal{F}_{\mathbf{N},\mathbf{M}}} \|S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}\big((y_v)_{v\in T^\infty}\big)\|_{\ell^q} \le C_\varepsilon^{|T|} 2^{(1+\varepsilon)|\mathbf{M}|} \prod_{v\in T^\infty} \|y_v\|_{\ell^p} \tag{8.6}$$

for arbitrarily small $\varepsilon > 0$, provided $q \ge \max(1, \frac{p}{|T^\infty|})$. Since $|\Gamma| \le C^{|T|}2^{|\mathbf{N}|}$,

$$\sum_{\Gamma}\sum_{F\in\mathcal{F}_{\mathbf{N},\mathbf{M}}} \|S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}\big((y_v)_{v\in T^\infty}\big)\|_{\ell^q} \le C_\varepsilon^{|T|} 2^{|\mathbf{N}|}2^{(1+\varepsilon)|\mathbf{M}|} \prod_{v\in T^\infty} \|y_v\|_{\ell^p}. \tag{8.7}$$

On the other hand, Lemma 6.1 gives

$$\sum_{\Gamma}\sum_{F\in\mathcal{F}_{\mathbf{N},\mathbf{M}}} \|S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}\big((y_v)_{v\in T^\infty}\big)\|_{\ell^1} \le C^{|T|} \prod_{v\in T^\infty} \|y_v\|_{\ell^1}. \tag{8.8}$$

Thus if $q > \frac{p}{|T^\infty|}$ and $q \ge 1$, we may interpolate to find that there exists $\eta > 0$ depending on $q - \frac{p}{|T^\infty|}$ but not on δ such that

$$\sum_{\Gamma}\sum_{F\in\mathcal{F}_{\mathbf{N},\mathbf{M}}} \|S_{T,T',\mathbf{N},\mathbf{M},\Gamma,F}\big((y_v)_{v\in T^\infty}\big)\|_{\ell^q} \le C_\eta^{|T|} 2^{(1-\eta)|\mathbf{N}|+(1-\eta)|\mathbf{M}|} \prod_{v\in T^\infty} \|y_v\|_{\ell^p}. \tag{8.9}$$

Taking into account the factors $2^{-|\mathbf{N}|}2^{-(1-\delta)|\mathbf{M}|}$ in (7.18), and summing over $\mathbf{N}, \mathbf{M}$ as well as over all subsets $T' \subset T^0$, completes the proof of Lemma 7.1. □

9. Loose ends

We may reinterpret the sum of our power series (4.1),(4.2) as a function via the relation $\widehat{u}(t,n) = e^{in^2t}a_n(t)$ with $a(0)$ defined by $\widehat{u_0}(n) = a_n(0)$, and will do so consistently without further comment, abusing notation mildly by writing $u(t,x) = S(t)u_0(x)$.

Lemma 9.1. *Let $p \in [1,\infty)$. For any $R > 0$ there exists $\tau > 0$ such that for any $u_0 \in \mathcal{F}L^p$ with norm $\le R$, the element $u(t,x) \in C^0([0,\tau],\mathcal{F}L^p)$ defined by* (4.1),(4.2) *is a limit, in $C^0([0,\tau],\mathcal{F}L^p)$ norm, of smooth solutions of* (NLS*).

Proof. All of our estimates apply also in the spaces $\mathcal{F}L^{s,p}$ defined by the condition that $(\langle n\rangle^s\widehat{f}(n))_{n\in\mathbb{Z}} \in \ell^p$, provided that $1 \le p < \infty$ and $s > 0$. This follows from the proof given for $s = 0$ above, for the effect of working in $\mathcal{F}L^{s,p}$ is to introduce a factor of $\prod_{v\in T^0}\frac{\langle j_v\rangle^s}{\prod_{i=1}^3\langle j_{(v,i)}\rangle^s}$ in the definition of the tree operator. The relation $j_v = j_{(v,1)} - j_{(v,2)} + j_{(v,3)}$ ensures that $\max_i |j_{(v,i)}| \ge \frac{1}{3}|j_v|$, whence $\frac{\langle j_v\rangle^s}{\prod_{i=1}^3\langle j_{(v,i)}\rangle^s} \lesssim 1$, so the estimates for $s = 0$ apply directly to all $s > 0$.

More generally, if $\mathcal{F}L^{s,p}$ is equipped with the norm

$$\|f\|_{\mathcal{F}L^{s,p}_\varepsilon} = \|(1+|\varepsilon\cdot|^{2s})^{1/2}\widehat{f}(\cdot)\|_{\ell^p}$$

then all estimates hold uniformly in $\varepsilon \in [0,1]$ and $s \ge 0$. This follows from the same reasoning.

Fix a sufficiently large positive exponent s. Given any initial datum u_0 satisfying $\|u_0\|_{\mathcal{F}L^p} \le R$ with the additional property that $\widehat{u_0}(n) = 0$ whenever $|n|$ exceeds some large quantity N, we may choose $\varepsilon > 0$ so that $\|u_0\|_{\mathcal{F}L^{s,p}_\varepsilon} \le 2R$. This ε depends on N, but not on R. Thus the infinite series converges absolutely and uniformly in $C^0([0,\tau], H^{s-\frac{1}{2}+\frac{1}{p}})$ if $p \ge 2$ and in $C^0([0,\tau], H^s)$ if $p \le 2$, where τ depends only on R, not on s. By Lemma 4.3, the series sums to a solution of (NLS*) in the sense (2.7); but since the sum is very smooth as a function of x (that is, its Fourier coefficients decay rapidly) this implies that it is a solution in the classical sense. Given an arbitrary u_0 satisfying $\|u_0\|_{\mathcal{F}L^p} \le R$, we can thus approximate it by such special initial data to conclude that $S(t)u_0$ is indeed a limit, in $C^0([0,\tau], \mathcal{F}L^p)$, of smooth solutions. □

Proof of Proposition 1.4. Let $u_0 \in \mathcal{F}L^p$ be given, let $u(t,x) = S(t)(u_0) \in C^0([0,\tau], \mathcal{F}L^p)$. We aim to prove that the nonlinear expression $|u|^2u$ has an intrinsic meaning as the limit as $N \to \infty$ of $|T_N u|^2 T_N u$ in the sense of distributions in $(0,\tau)\times\mathbb{T}$. Forming $T_N S(t)(u_0)$ is of course not the same thing as forming $S(t)(T_N u_0)$.

Define $a_n(t) = e^{in^2t}\widehat{u}(t,n)$. Denote also by T_N the operator that maps a sequence-valued function $(b_n(t))$ to $(T_N b_n(t))$ where $T_N b_n = b_n$ if $|n| \le N$, and $= 0$ otherwise. It suffices to prove that

$$(9.1)\qquad \int_0^t \sum_{j-k+l=n}^{*} T_N a_j(s)\overline{T_N a_k(s)} T_N a_l(s) e^{i\sigma(j,k,l,n)s}\, ds - \int_0^t |T_N a_n(s)|^2 T_N a_n(s)\, ds$$

converges in ℓ^p norm as $N \to \infty$, uniformly for all $t \in [0,\tau]$, to

$$\sum_{j-k+l=n}^{*} \int_0^t a_j(s)\overline{a_k(s)} a_l(s) e^{i\sigma(j,k,l,n)s}\, ds - \int_0^t |a_n(s)|^2 a_n(s)\, ds.$$

Convergence in the distribution sense follows easily from this by expressing any sufficiently smooth function of the time t as a superposition of characteristic functions of intervals $[0,t]$.

Now in the term $\int_0^t \sum_{j-k+l=n}^{*} T_N a_j(s)\overline{T_N a_k(s)} T_N a_l(s) e^{i\sigma(j,k,l,n)s}\, ds$, the integral may be interchanged with the sum since the truncation operators restrict the summation to finitely many terms. Expanding a_j, a_k, a_l out as infinite series of tree operators applied to $a(0)$, we obtain finally an infinite series of the general form $\sum_{k=1}^\infty B_k(t)(a(0),\cdots,a(0))$ where $B_k(t)$ is a finite linear combination of $O(C^k)$ tree sum operators, with coefficients $O(C^k)$, applied to $a(0)$ just as before, with the sole change that the extra restriction $|j_{(\mathbf{r},i)}| \le N$ for $i \in \{1,2,3\}$ is placed on $\mathbf{j}$ in the summation defining $\mathfrak{S}_T$ for each tree T.

Since we have shown that all bounds hold for the sums of the absolute values of the terms in the tree sum, it follows immediately that this trilinear term converges as $N \to \infty$. Convergence for the other nonlinear term is of course trivial. Likewise it is trivial that $(T_N u)_t \to u_t$ and $(T_N u)_{xx} \to u_{xx}$, by linearity. □

This reasoning shows that the limit of each term equals the sum of a convergent power series, taking values in $C^0([0,\tau], \mathcal{F}L^p)$, in u_0.

Given $R > 0$, there exists $\tau > 0$ for which we have shown that for any $a(0) \in \ell^p$ satisfying $\|a(0)\|_{\mathcal{F}L^p} \le R$, our power series expansion defines $a(t) \in C^0([0,\tau], \ell^p)$, as an ℓ^p-valued analytic function of $a(0)$. Moreover for any $t \in [0,\tau]$, both cubic terms in the integral equation (2.6) are well-defined as limits obtained by replacing $a(s)$ by $T_N a(s)$, evaluating the resulting cubic expressions, and passing to the limit $N \to \infty$.

Lemma 9.2. *Whenever* $\|a(0)\|_{\ell^p} \le R$*, the function* $a(t) \in C^0([0,\tau], \ell^p)$ *defined as the sum of the power series expansion* (4.1) *satisfies the integral equation* (2.7) *when the nonlinear terms in* (2.6) *are defined by the limiting procedure described in the preceding paragraph.*

Proof. This follows by combining Lemma 4.3 with the result just proved. □

Proof of Proposition 1.3. Let $u_0 \in \mathcal{F}L^p$. If $u = Su_0$, and if v is the solution of the Cauchy problem (NLS*) for the modified linear Schrödinger equation with initial datum u_0, then $u_0 - v$ is expressed as $\sum_{k=1}^\infty B_k(t)(u_0, \cdots, u_0)$ where the n-th Fourier coefficient of the function $B_k(t)(u_0, \cdots)(t)$ equals $e^{-in^2 t} A_k(t)(a^\star(0), \cdots)$ with $a_n(0) = \widehat{u_0}(n)$. According to Proposition 4.4,

$$\|A_k(t)(a^\star(0), \cdots)\|_{\ell^q} = O(t^{k\varepsilon} \|a(0)\|_{\ell^p}^k)$$

whenever $q > \frac{p}{3}$ and $q \ge 1$. Summation with respect to k yields the conclusion. □

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