ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SLOWLY DECAYING POTENTIALS: SPECTRA AND ASYMPTOTICS

OR

BABY FOURIER ANALYSIS MEETS TOY QUANTUM MECHANICS

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These informal notes survey investigations carried out by the authors over the last few years. The text heavily emphasizes our own efforts, with limited discussion of the extensive literature.[1] Among many possible sources for an introduction to that literature are [57, 66, 26, 70, 15]. The papers cited in the bibliography contain more complete references.

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1. Pre-Introduction

Consider three simple facts concerning small perturbations of even simpler facts:

(1) If a sequence $v = \{v_n\}$ belongs to ℓ^1, then the series $\sum_n v_n$ converges. (Here we regard v as a perturbation of 0.)
(2) If $V \in L^1(\mathbb{R})$ then any solution f of the ordinary differential equation $df/dt = V$ remains bounded as $x \to +\infty$. Moreover, $f(t)$ is asymptotic to constants $c_\pm$ as $t \to \pm\infty$.
(3) Consider a one-dimensional Schrödinger operator $H_V u(x) = -u'' + V(x)u$, on $L^2(\mathbb{R}^1)$. Suppose that $V \in L^1(\mathbb{R})$. Then all eigenvalues of H_V are < 0. Moreover, when H_V is restricted to the orthocomplement of the sum of those eigenspaces, it is unitarily equivalent to the free Hamiltonian H_0 on $L^2(\mathbb{R})$. For any $\lambda \in \mathbb{R}\backslash\{0\}$, every solution of the generalized eigenfunction equation $H_V u = \lambda^2 u$ behaves asymptotically like a constant linear combination of $e^{i\lambda x}, e^{-i\lambda x}$ as $x \to +\infty$, and like another such linear combination as $x \to -\infty$.

In each of these statements, the hypothesis on V (or v) is essentially optimal for (at least part of) the stated conclusion. These lectures are concerned with a family of extensions of these simple facts, concerning families of perturbations V depending (smoothly) on an external parameter, which generally takes values in $\mathbb{R}^1$. They are characterized by a tradeoff: weaker decay hypotheses are imposed on V, but the conclusions then hold only for *generic* values of the parameter.

A simple classical example of such a statement is that if $v_n = O(|n|^{-\gamma})$ for some $\gamma > \frac{1}{2}$, then the Fourier series $\sum_n v_n e^{inx}$ converges for almost every x; here we have a one-parameter family of numerical series, parametrized by x.

The motivation for this work, and the main applications, come from the study of one-dimensional Schrödinger operators. There the spectral variable λ^2 plays the role of the "external" parameter.

2. Introduction and background

The basic object of study is a time-independent Schrödinger operator on the real line

$$H = -\frac{d^2}{dx^2} + V(x). \tag{2.1}$$

Standing hypotheses throughout these notes (except for the discussion of a few examples in §2) are that V is real-valued and that $\int_{|x-y|<1} |V(y)|\,dy \to 0$ as $|x| \to \infty$, although Theorem 12.1 can be formulated more generally. Then H is self-adjoint on $L^2(\mathbb{R})$. Considered instead as an operator on $L^2[0,\infty)$ with (say) Dirichlet or Neumann boundary conditions, it is likewise self-adjoint.

A quantum-mechanical interpretation is that the free Hamiltonian $H_0 = -d^2/dx^2$ describes the behavior of a free electron, while $H_0 + V$ describes one electron interacting with an external electrical field, described by the potential V. One can sometimes think of V

[1] All errors and omissions are the sole responsibility of the first author.

as representing some disorder. We will be interested in the case of small disorder, where $V \to 0$ at ∞ in some sense.

If V is sufficiently small, then the spectrum of $H_0 + V$ should resemble that of H_0. One of the goals of the theory surveyed in these notes is to justify this expectation for certain classes of potentials. In particular, reasonably precise and sharp conditions will be given for the persistence of absolutely continuous spectrum under perturbations.

To any self-adjoint operator H on a Hilbert space $\mathcal{H}$ and any vector $\varphi \in \mathcal{H}$ is associated a spectral measure μ_φ, satisfying

$$\langle f(H)\varphi, \varphi\rangle = \int_{\mathbb{R}} f(\lambda)\, d\mu_\varphi(\lambda) \tag{2.2}$$

for any Borel measurable, bounded function f.

Any finite measure μ decomposes as $\mu_{\mathrm{s}} + \mu_{\mathrm{ac}}$ where the summands are respectively singular with respect to, and mutually absolutely continuous with respect to, Lebesgue measure. The singular component decomposes further as $\mu_{\mathrm{pp}} + \mu_{\mathrm{sc}}$ where the last summand contains no atoms, while μ_{pp} is a countable linear combination of Dirac masses. H is said to have (some) absolutely continuous spectrum if there exists $\varphi \neq 0$ such that $(\mu_\varphi)_{\mathrm{ac}} \neq 0$, and to have purely absolutely continuous spectrum if $\mu_\varphi = (\mu_\varphi)_{\mathrm{ac}}$ for every φ. We often abbreviate "absolutely continuous" as "ac". Similarly one speaks of pure point and purely sc spectrum. $\mathcal{H}_{\mathrm{ac}}$ denotes the maximal subspace of $\mathcal{H}$ on which H has purely absolutely continuous spectrum.

The point spectrum is dictated by the eigenfunctions, that is, the L^2 solutions of $Hu = Eu$. By a *generalized eigenfunction*[2] of $H = H_0 + V$ we mean a solution u of $Hu \equiv Eu$ for some $E \in \mathbb{R}$. A generalized eigenfunction has at most exponential growth if V is uniformly in L^1_{loc}; the spectrum is related to those with at most polynomial growth. More precise links between growth rates and the ac spectrum will be discussed in §4.

Suppose that a Borel measure μ on $\mathbb{R}$ is absolutely continuous with respect to Lebesgue measure. By an essential support of μ is meant a Borel set S such that $\mu(\mathbb{R}\backslash S) = 0$, and $\mu(E) > 0$ whenever $E \subset S$ has positive Lebesgue measure.

To μ_φ is associated its Borel transform

$$\mathcal{M}_{\mu_\varphi}(z) = \langle (H-z)^{-1}\varphi, \varphi\rangle = \int_{\mathbb{R}} (\lambda - z)^{-1}\, d\mu_\varphi(\lambda). \tag{2.3}$$

Since μ_φ is a finite measure, $\mathcal{M}(z)$ is well-defined whenever $\mathrm{Re}\,(z) > 0$. Its imaginary part is

$$\mathrm{Im}\,(\mathcal{M}_{\mu_\varphi}(E + i\varepsilon)) = \int_{\mathbb{R}} \frac{\varepsilon}{(\lambda - E)^2 + \varepsilon^2}\, d\mu_\varphi(\lambda). \tag{2.4}$$

Let μ be any locally finite positive measure. Define

$$D^\alpha \mu(x) = \limsup_{\varepsilon \to 0} \frac{\mu(x - \varepsilon, x + \varepsilon)}{(2\varepsilon)^\alpha}\ . \tag{2.5}$$

By differentiation theory, in order to prove that a finite measure μ has a nonzero absolutely continuous component, it suffices to prove that $D^1\mu(x) > 0$ for all x in some set having positive Lebesgue measure. If $\limsup_{\varepsilon \to 0^+} \mathrm{Im}\,(\mathcal{M}_\mu)(E + i\varepsilon) > 0$, then $D^1\mu(E) > 0$.

A brief tour of some basic classes of potentials, and the spectral properties of the associated Schrödinger operators:

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[2]This term is often used in with a more specific meaning, but in these notes will always mean any solution of the eigenfunction equation, with no growth restriction.

(1) The free Hamiltonian $H_0 = -d^2/dx^2$ has purely ac spectrum.
(2) If $V(x) \to +\infty$ as $|x| \to \infty$ then the spectrum consists of a discrete sequence of eigenvalues tending to $+\infty$.
(3) If $V \in L^1(\mathbb{R})$ then $H_0 + V$ has only point spectrum (aka bound states) in $\mathbb{R}^-$, with 0 as its only possible accumulation point. In $\mathbb{R}^+$ the spectrum is purely absolutely continuous, and an essential support is $\mathbb{R}^+$ itself. If $xV(x) \in L^1$ then there are only finitely many bound states.
(4) (Wigner-von Neumann potential) There exists a potential with asymptotic behavior $V(x) \sim c\sin(2x)/x$ which has an eigenvalue at $E = +1$, embedded in the continuous spectrum.
(5) In $\mathbb{R}^n$ for[3] $n > 1$, if $V(x) = O(|x|^{-r})$ for some $r > 1$, then there are no positive eigenvalues [36, 69, 1]. Moreover, by the theorem of Agmon-Kato-Kuroda [57], $\mu_{\text{sc}} = \emptyset$, and an essential support of μ_{ac} is $\mathbb{R}^+$.
(6) If V is periodic in $\mathbb{R}$ then the spectrum is purely absolutely continuous, and consists of a countable sequence of intervals $[a_j, b_j]$ with $b_j \le a_{j+1}$ and $a_j, b_j \to +\infty$.
(7) The Almost Mathieu operators h act on ℓ^2 by

$$hu(n) = u(n-1) + u(n+1) + v(n)u(n), \tag{2.6}$$

$$v(n) = \lambda\cos(\pi\alpha n + \theta) \tag{2.7}$$

where λ, α, θ are parameters. They exhibit all manner of spectra, including purely absolutely continuous, dense pure point, and purely singular continuous, depending on the magnitude of λ and Diophantine properties of α.
(8) ([56], [41]) Let $h \ge 0$ be continuous and compactly supported, but not $\equiv 0$. Consider potentials with large gaps:

$$V(x) = \sum_n a_n h(x - x_n) \tag{2.8}$$

where $x_n \to +\infty$, and $a_n \to 0$. If $x_n/x_{n+1} \to 0$ sufficiently rapidly then the spectrum is purely absolutely continuous if $a \in \ell^2$, and is purely singular otherwise.
(9) Consider instead a family of potentials

$$V_\omega(x) = \sum_{n=-\infty}^{\infty} a_n(\omega)h(x - n) \tag{2.9}$$

where $a \in \ell^\infty$, and $\omega \in \Omega$, a probability space. Suppose that $\int h = 0$, and that $a_n = b_n r_n(\omega)$ where $b_n \in \mathbb{R}$ and $\{r_n\}$ are independent, identically distributed random variables, whose distributions are bounded and absolutely continuous with respect to Lebesgue measure. If $b_n \equiv b$, a nonzero constant, then for almost every ω, the spectrum of $H_0 + V_\omega$ consists entirely of (a dense in $\mathbb{R}^+$ set of) eigenvalues; there is no continuous spectrum. The same holds more generally, if $b_n \sim n^{-r}$ and $r \le 1/2$.
(10) In the preceding example, if instead $\{b_n\} \in \ell^2$, then for almost every ω, the spectrum is purely absolutely continuous.
(11) For any $r < 1$, there exist potentials satisfying $V(x) = O(|x|^{-r})$ for which the set of all eigenvalues is *dense* in $\mathbb{R}^+$ [52],[71].
(12) There exist potentials V which are $O(|x|^{-r})$ for all $r < 1$, for which $H_V = H_0 + V$ has nonempty singular continuous spectrum [40].

[3]With rare exceptions, we discuss only operators on $\mathbb{R}^1$; this is one exception.

When $V \in L^1$, there is no point spectrum embedded in the continuous spectrum. When for instance $V = O(|x|^{-r})$ and $r < 1$, it remained an open question until around 1996 whether there was necessarily any continuous spectrum. The first progress, for $r > 3/4$, was due to Kiselev [38]; a series of papers, culminating in [16] and Remling [58], established existence of ac spectrum for all $r > 1/2$. The situation is highly unstable, in the sense that for $r < 1$ there is sometimes *dense* point spectrum embedded in the continuous spectrum.

Deift and Killip obtained a definitive result, in some respects, by quite a different method [27].

Theorem 2.1 (Deift and Killip). *If $V \in L^1 + L^2(\mathbb{R})$ then $H_0 + V$ has nonempty absolutely continuous spectrum; moreover, an essential support equals $(0, \infty)$.*

These notes describe a further development of the method of [16], which requires slightly stronger hypotheses on V, but yields additional information. In particular, this additional information can be used to study the associated Schrödinger group $\exp(itH)$.

3. Three (sample) principal results

The main purpose of the work outlined in these notes is to better understand Schrödinger operators with rather slowly decaying potentials by

(1) Analyzing the behavior of the associated generalized eigenfunctions.
(2) Applying the resulting estimates to analyze the evolution group $\exp(itH)$.
(3) Extending the discussion to wider classes of potentials.

In the course of doing so, we will also

(4) Extend the range of the much-used WKB approximation, establishing a theory in which WKB asymptotics hold for parametrized families of functions, almost surely but not uniformly with respect to the parameter.
(5) Establish a kind of almost-sure long-term stability for solutions of ordinary differential equations under perturbations which are moderately small for large time.
(6) Develop general machinery concerning multilinear integral operators, and maximal versions thereof.

To indicate more concretely where we are heading, we now formulate three sample theorems.

Theorem 3.1. [17] *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. Then an essential support for the absolute spectrum of $H = -\partial_x^2 + V$ is $\mathbb{R}^+$. For almost every $\lambda \in \mathbb{R}$ there exists a generalized eigenfunction satisfying*

$$u(x, \lambda) - e^{i\phi(x,\lambda)} \to 0 \text{ as } x \to +\infty, \tag{3.1}$$

where[4] $\phi(x, \lambda) = \lambda x - (2\lambda)^{-1} \int_0^x V$.

du/dx has corresponding asymptotics $i\lambda \cdot e^{i\phi(x,\lambda)}$.

Lack of smoothness of the potential is not the issue here; assuming that $\partial^k V/\partial^k x \in L^p + L^1$ for all k would not change the conclusions, nor would it make the theorem any easier to prove.[5] Indeed, the examples (2.9) are smooth in this sense. This is as one might expect, from the uncertainty principle; spectral properties of H at energies in any fixed compact subinterval of $\mathbb{R}$ should not depend strongly on behavior of the potential on scales

[4]A perhaps more familiar form for the exponent is $i \int_0^x \sqrt{\lambda^2 - V(y)}\, dy$. If $V \in L^2$, this is equivalent to $i\phi(x, \lambda)$.

[5]Throughout these notes, the class $L^p + L^1$ can be replaced by the Birman-Solomjak space $\ell^p(L^1)$, which is the Banach space of all functions satisfying $\sum_n (\int_n^{n+1} |V|)^p < \infty$.

$\lesssim 1$. The situation is quite different if V satisfies symbol-type hypotheses, ensuring that successive derivatives of V decay successively more rapidly [34].

This theorem captures a certain tradeoff: it has weaker hypotheses than $V \in L^1$, but offers a (necessarily) weaker conclusion. The improvement from hypothesizing merely that $V \in L^p$, rather than power decay $V = O(|x|^{-r})$, has the physical interpretation that long gaps in which the potential vanishes identically do not affect the ac spectrum (so long as V is sufficiently small; gaps are the essential feature of the examples of Pearson described above).

Our second sample result concerns long-time asymptotics for the associated evolution e^{itH}. For definitions of wave and scattering operators see §11.

Theorem 3.2. [21] *Let $H = H_0 + V$ on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition at the origin. Suppose that $V \in L^p + L^1$ for some $1 < p < 2$. Suppose further that*

$$\lim_{x\to+\infty} \int_0^x V(y)\,dy \qquad \textit{exists.} \tag{3.2}$$

Then for each $f \in L^2(\mathbb{R}^+)$, the wave operators *$\Omega_\pm$ exist in L^2 norm as $t \to \mp\infty$. Moreover, $\Omega^\pm$ are bijective isometries from $\mathcal{H} = L^2(\mathbb{R}^+)$ to $\mathcal{H}_{ac}$.*

More concretely, this means that for any $g \in L^2(\mathbb{R}^+)$ there exist $f_\pm \in L^2(\mathbb{R}^+)$ such that $\varepsilon^{itH_0}g - e^{itH}f_\pm \to 0$ in $L^2(\mathbb{R}^+)$ norm, as $t \to \pm\infty$. For the physical interpretation, see §11.

Moreover, the scattering operator $(\Omega^+)^{-1} \circ \Omega^-$ can be identified as a "Fourier multiplier" operator, which can be explicitly described in terms of the asymptotics of the phase $\phi(x, \lambda)$. See Theorem 11.3.

The third sample result is perhaps of more academic interest. Consider a one-parameter family of nonlinear ordinary differential equations

$$\frac{d\theta}{dt} = \lambda + \mathcal{V}(t, \theta), \tag{3.3}$$

where $\mathcal{V}(t, \beta + 2\pi) \equiv \mathcal{V}(t, \beta)$, and $\mathcal{V}$ is a C^∞ function of β for each t. Here $\theta = \theta_\lambda$ is a function of t, for each real parameter λ in some open interval.

Theorem 3.3. *Let $\gamma > \frac{1}{2}$. Suppose that $\sup_\beta |\partial_\beta^k \mathcal{V}(t, \beta)| \le C_k(1 + |t|)^{-\gamma}$ for every integer $k \ge 0$. Suppose moreover that $\int_0^{2\pi} \mathcal{V}(t, \beta)\,d\beta = 0$ for all t. Then for almost every λ, for every $c \in \mathbb{R}$ there exists a solution θ of* (3.3) *such that*

$$\theta(t) - (\lambda t + c) \to 0 \textit{ as } t \to +\infty. \tag{3.4}$$

Conversely, for almost every λ, every solution of (3.3) *satisfies* (3.4) *for some $c \in \mathbb{R}$.*

That such a conclusion holds for every λ when $\mathcal{V} = O(t^{-1-\varepsilon})$ is elementary. The theorem asserts that *generic* solutions are still asymptotically stable under larger perturbations which are merely $O(t^{-1/2-\varepsilon})$.

4. A criterion for ac spectrum

How can one get a grip on the spectral measure for a selfadjoint operator H? A criterion of Weyl characterizes points E of the essential[6] spectrum by the existence of sequences of unit vectors φ_n tending weakly to zero, for which $\|(H - E)\varphi_n\|_{\mathcal{H}} \to 0$. For Schrödinger operators (in any dimension, satisfying mild hypotheses not stated here), an extension by Simon of a theorem of Sch'nol [63] states that the spectrum, as a set[7], coincides with

[6] The essential spectrum is the spectrum minus all isolated eigenvalues of finite multiplicity.

[7] More precisely, for almost every energy with respect to any fixed μ_φ, there exists a polynomially bounded generalized eigenfunction.

the closure of the set of energies for which there is a polynomially bounded[8] generalized eigenfunction; see also [70], page 501.

Several devices are potentially available for studying the ac spectrum:

(1) Analyze the associated group e^{itH} ($e^{it\sqrt{H}}$ being problematic because H need not be positive), and recover μ_φ by Fourier inversion from the formula $\langle e^{itH}\varphi, \varphi\rangle = \int e^{it\lambda}\, d\mu_\varphi(\lambda)$.
(2) Estimate resolvents $(H-\lambda^2)^{-1}$, and apply Stone's formula via the Weyl m-function.
(3) Apply the subordinacy theory developed by Gilbert and Pearson.
(4) Apply a criterion [22] relating the spectral measure to the growth properties of approximate eigenfunctions. See Proposition 4.1 and Corollary 4.2, below.

We next discuss these criteria in slightly greater detail. While the time-dependent first strategy has been enormously successful in other aspects of spectral theory, caution is required here. One seeks to recover the absolutely continuous component of some measure, which may have also have a singular component, from the asymptotics of its Fourier transform; to succeed, one must first separate out the singular part of the spectral measure.

The Weyl m-function is $m(\lambda) = u_\lambda^{+\prime}(0)/u_\lambda^{+}(0)$; it equals $\partial^x_{x,y} G_{\lambda^2}(x,y)$ evaluated at $(0,0)$ where G_ζ is Green's function, that is, the kernel associated to $(H-\zeta)^{-1}$. A formula of Stone reads

$$\pi^{-1}\operatorname{Im}\left(m(E+i\varepsilon)\right) dE \to d\mu(E) \quad \text{as } \varepsilon \to 0^+ \tag{4.1}$$

in the sense of weak limits, where μ is a positive measure on $\mathbb{R}$ such that H is unitarily equivalent to multiplication by E on[9] $L^2(\mathbb{R},\mu)$.

m can be related to other quantities, specifically to the reciprocals a, b of transmission and reflection coefficients (for the definitions see (7.1) below), by

$$m(\lambda^2) = i\lambda \frac{a(\lambda)-b(\lambda)}{a(\lambda)+b(\lambda)}. \tag{4.2}$$

From this, it follows that

$$\operatorname{Im}\left(m(\lambda)\right) \geq 4\lambda^{-1}|a(\lambda)|^{-2}. \tag{4.3}$$

In this way, upper bounds for a lead to lower bounds for $\operatorname{Im}(m)$, thence to the presence of absolutely continuous spectrum by (4.1) plus a limiting argument. Later we will discuss in detail upper bounds for a and for related quantities.

A generalized eigenfunction $u(x)$ with energy E, for a one-dimensional Schrödinger operator, is said to be *subordinate* at $+\infty$ if

$$\lim_{y\to+\infty} \frac{\int_0^y |u(x)|^2\,dx}{\int_0^y |v(x)|^2\,dx} = 0 \tag{4.4}$$

for any linearly independent generalized eigenfunction v with the same energy. Subordinacy at $-\infty$ is defined analogously. Then [31] the singular spectrum is supported on the set of all E for which there exists a generalized eigenfunction that is subordinate at both $+\infty$ and $-\infty$, and the absolutely continuous spectrum on the set of all E for which there exists no such generalized eigenfunction.

A consequence of the subordinacy theory is [74] that if V is uniformly in L^1_{loc}, and if all generalized eigenfunctions are globally bounded on $\mathbb{R}$ for all E in some set Λ, then there is ac spectrum everywhere on Λ, and no singular spectrum. See also [68] for a proof

[8] Subexponential growth suffices.

[9] Functions in $L^2(\mathbb{R},\mu)$ taking values in an appropriate auxiliary Hilbert space.

and discussion, including further references to the general subordinacy theory. For further development see [35].

This consequence of subordinacy theory is all that is required to deduce spectral implications from properties of generalized eigenfunctions which will be established in these notes. However, before proceeding to that analysis, we discuss here an alternative approach, which may offer potential advantages for other problems, in particular, for higher dimensions. This may turn out to be of some use, because the subordinacy theory and m-function approaches are special to dimension one. (The space of all generalized eigenfunctions with given energy is two-dimensional for $\mathbb{R}^1$, but infinite-dimensional in higher dimensions.)

This criterion relies on a notion of approximate eigenfunctions.

Proposition 4.1. [22] *For any spectral measure μ_φ associated to a self-adjoint operator H on a Hilbert space $\mathcal{H}$, any $E \in \mathbb{R}$, and any $\varepsilon > 0$,*

$$\operatorname{Im} \mathcal{M}_{\mu_\varphi}(E + i\varepsilon) \geq c_0 \varepsilon^{-1} \sup_\psi |\langle \varphi, \psi \rangle|^2 \tag{4.5}$$

where the supremum is taken over all $\psi \in \mathcal{H}$ satisfying

$$\|\psi\| = 1 \text{ and } \|(H - E)\psi\| \leq \varepsilon. \tag{4.6}$$

Thus in order to prove that the absolutely continuous component of μ_φ charges a set S, it suffices to show that for almost every $E \in S$ there exists a sequence $\psi_j \in \mathcal{H}$ such that $\|(H - E)\psi_j\|/\|\psi_j\| \to 0$ and

$$\frac{|\langle \varphi, \psi_j \rangle|^2}{\|\psi_j\| \cdot \|(H - E)\psi_j\|} \geq c > 0, \tag{4.7}$$

uniformly as $j \to \infty$. Indeed, normalizing to make each ψ_j a unit vector and setting $\varepsilon_j = \|(H - E)\psi_j\|$, we deduce that $\limsup_{\varepsilon \to 0} \operatorname{Im} \mathcal{M}(E + i\varepsilon)$ is strictly positive. Hence $D^1\mu_\varphi(E)$ is likewise strictly positive, by the standard majorization of the maximal Poisson integral by the Hardy-Littlewood maximal function.

Proposition 4.1 is quite easy to prove. By the spectral theorem, we may assume that H is multiplication by the coordinate λ on $L^2(\mathbb{R}^+, d\nu)$ for some positive measure ν; then $d\mu_\varphi = |\varphi|^2 d\nu$. Now

$$\begin{aligned} &|\int \varphi\psi d\nu(\lambda)|^2 \\ &\leq \Big[\int |\varphi(\lambda)|^2 \frac{\varepsilon}{(\lambda - E)^2 + \varepsilon^2} d\nu(\lambda)\Big] \Big[\int |\psi(\lambda)|^2 \frac{(\lambda - E)^2 + \varepsilon^2}{\varepsilon} d\nu(\lambda)\Big] \\ &\leq |\operatorname{Im} \mathcal{M}_{\mu_\varphi}(E + i\varepsilon)| \varepsilon^{-1} (\|(H - E)\psi\|^2 + \varepsilon^2 \|\psi\|^2), \end{aligned} \tag{4.8}$$

from which the conclusion follows by invoking the conditions $\|\psi\| = 1$ and $\|(H - E)\psi\| \leq \varepsilon$. $\square$

A more concrete criterion for the existence of absolutely continuous spectrum is as follows.

Corollary 4.2. *Suppose that $V \in L^1_{loc}(\mathbb{R})$, uniformly on unit intervals. Let $S \subset \mathbb{R}$ and suppose that for each $E \in S$, there exists a generalized eigenfunction u_E of $H_0 + V$ satisfying the growth restriction*

$$\liminf_{R \to \infty} R^{-1} \int_{|x| \leq R} |u_E(x)|^2 \, dx < \infty. \tag{4.9}$$

Then for any compact subset $S' \subset S$ of positive Lebesgue measure, there exists φ such that $\mu_\varphi(S') > 0$.

In particular, (4.9) is satisfied if $u_E \in L^\infty$.

This criterion may seem paradoxical, for it asserts in particular that the existence for each E in some interval of a genuine eigenfunction (that is, a solution $u_E \in L^2(\mathbb{R})$ of $(H-E)u_E = 0$) implies ac spectrum; the paradox is that genuine eigenfunctions correspond not to ac spectrum, but to point spectrum. However, such a situation is impossible; eigenfunctions can occur only for a countable set of values of E. A key point here is that approximate eigenfunctions satisfying the relevant growth bound are required to exist for a set of E having positive Lebesgue measure.

In order to deduce the Corollary from Proposition 4.1, one must produce a sequence of approximate eigenfunctions. This is done by multiplying generalized eigenfunctions by sequences of cutoff functions. Fix $h \in C_0^2(\mathbb{R})$, with $h \equiv 1$ in a neighborhood of the origin. Set $u_E^{(R)}(x) = h(x/R)u(x)$. Then

$$(4.10) \qquad (H_0 + V - E)(u_E^{(R)}) \equiv -R^{-2}h''(x/R)u_E(x) - 2R^{-1}h'(x/R)u_E'(x).$$

Fix any nonnegative φ supported in a sufficiently small interval near 0 but in $[0,\infty)$; then it is easy to see that $|\langle u_E, \varphi\rangle| = |\langle u_E^{(R)}, \varphi\rangle| \neq 0$ for all $E \in S'$ for which $|u_E(0)/u_E'(0)|$ is sufficiently large; other E may be handled similarly by a different choice of φ. One-dimensional elliptic regularity theory applied to the equation $Hu = Eu$, in conjunction with (4.9), reveals that u_E' likewise satisfies (4.9). From (4.10), we then deduce that $\|(H-E)u_E^{(R)}\| \cdot \|u_E^{(R)}\| = O(R^{-1/2}) \cdot O(R^{1/2})$, so the criterion (4.7) holds. □

This corollary is rather general. The criterion (4.9) applies in Euclidean space of any dimension. It likewise applies to $-L+V$ on any Lie group, where L is a self-adjoint left-invariant (subelliptic, not necessarily elliptic) sub-Laplacian and $V \in L^\infty$, with $\{|x| \le R\}$ replaced by the ball of radius R, with fixed center, with respect to the Carnot-Caratheodory metric associated to L.

5. Expansions for generalized eigenfunctions

To begin to analyze the generalized eigenfunctions, suppose that $V \in L^1$. The equation $-u'' - \lambda^2 u + Vu = 0$ may be written formally as $u = (\partial_x^2 + \lambda^2)^{-1}Vu$, modulo adding an element of the nullspace of $\partial_x^2 + \lambda^2$. One of several inverse operators is

$$(5.1) \qquad (\partial_x^2 + \lambda^2)^{-1}g(x) = (2i\lambda)^{-1}\int_{y>x}\left[e^{i\lambda(x-y)} - e^{-i\lambda(x-y)}\right]g(y)\,dy\ .$$

Seeking a solution asymptotic to $e^{i\lambda x}$ as $x \to +\infty$, we arrive at the integral equation

$$(5.2) \qquad u(x) = e^{i\lambda x} + (2i\lambda)^{-1}\int_x^\infty \left[e^{i\lambda(x-y)} - e^{-i\lambda(x-y)}\right]V(y)u(y)\,dy.$$

If $V \in L^1$ then the usual contraction mapping argument yields for every $\lambda \neq 0$ the existence of a solution satisfying $u_\lambda(x) - e^{i\lambda x} \to 0$ as $x \to +\infty$.

Alternatively, one can iterate the equation, at least formally, to arrive at

$$(5.3) \quad u(x) = e^{i\lambda x} + (2i\lambda)^{-1}\int_x^\infty \left[e^{i\lambda(x-y)} - e^{-i\lambda(x-y)}\right]V(y)e^{i\lambda y}\,dy + (2i\lambda)^{-2}\iint_{x\le y_1\le y_2}\left[e^{i\lambda(x-y_1)} - e^{-i\lambda(x-y_1)}\right]V(y_1) \cdot \left[e^{i\lambda(y_1-y_2)} - e^{-i\lambda(y_1-y_2)}\right]V(y_2)u(y_2)\,dy_1\,dy_2\ .$$

The first line has no unknown u. The terms involving V are scalar multiples of

$$e^{-i\lambda x}\int_x^\infty e^{i2\lambda y}V(y)\,dy \tag{5.4}$$

$$e^{i\lambda x}\int_x^\infty V(y)\,dy \tag{5.5}$$

(5.4) is $e^{-i\lambda x}\widehat{V_x}(-2\lambda)$, where $V_x(y) = V(y)\cdot\chi_{[x,\infty)}(y)$. At this point we recall one formulation of the theorem of Carleson on almost everywhere convergence of Fourier series and integrals: If $f \in L^2$ then $\int_{-\infty}^s e^{i\lambda\xi}\hat f(\xi)\,d\xi$ converges as $s \to +\infty$, for almost every λ; as a function of (λ, s), the indefinite integral belongs to the space $L^2_\lambda L^\infty_s$. If $V \in L^2$ then by Plancherel's theorem we may write $V = \hat f$, so (5.4) is bounded in s for almost every λ.

Even for fixed s, say $s = +\infty$, one merely has square integrability in λ, rather than locally uniform bounds. This lack of uniformity is related to the possible presence of dense point spectrum. The connection with Fourier integrals is an indication of the natural role played by L^2 in the analysis of the generalized eigenfunctions.

(5.5) behaves quite differently; if $V \notin L^1$ then it may have no reasonable interpretation. One could rearrange matters to replace the interval of integration by $[0, x]$, so that it would be finite for fixed x. But we seek $L^\infty(dx)$ estimates (or at least the L^2 analogues $\int_0^R |u|^2\,dx \le CR$ needed to apply Corollary 4.2, and no such estimates would hold uniformly in x.

If this iteration process is carried out to infinite order, one obtains a power series expansion for $u(x,\lambda)$ in terms of V. A sample quadratic term is

$$e^{i\lambda x}\iint_{x\le y_1\le y_2} e^{-i2\lambda y_1}e^{i2\lambda y_2}V(y_1)V(y_2)\,dy_1\,dy_2, \tag{5.6}$$

and then higher-order terms. Each term defines a function of (x,λ) by applying a multilinear operator to m copies of V, $m = 1, 2, 3, \dots$. One can hope that this last sample term is not much worse behaved than the expression $e^{i\lambda x}\hat V(2\lambda)\hat V(-2\lambda)$ obtained by integrating over all $y_1, y_2 \in \mathbb{R}$. On the other hand, infinitely many terms arise which share the defect of (5.5). We thus face three difficulties: (i) justifying the hope just expressed, (ii) summing bounds over $m = 1, 2, 3, \dots$, and (iii) dealing with summands that fail to satisfy the bounds sought for the sum itself.

This last difficulty is familiar; individual terms of the Maclaurin series for the bounded function $\exp(ix)$ are unbounded. We will see in §8 how the power series expansion for u_λ can be reorganized by grouping certain terms together, so that no obviously unbounded terms remain. In fact, this grouping process amounts to nothing more than summation of the Maclaurin series for the imaginary exponential function.

6. WKB approximation

Suppose temporarily that V satisfies symbol-type hypotheses:

$$|\partial_x^k V(x)| \le C_k|x|^{-\delta-k\rho} \tag{6.1}$$

for some $\delta, \rho > 0$, for all k, for large x. We seek now a formal asymptotic approximation to a generalized eigenfunction $u_\lambda(x)$ for $H = H_0 + V$, as $x \to +\infty$, for fixed $\lambda \ne 0$. Set

$$u_\lambda(x) \sim e^{i\psi(x)}, \tag{6.2}$$

expand $\psi \sim \sum_{n=0}^{\infty} \psi_n$, and set $\psi_0 = \lambda x$. We seek a solution of symbol type, with each ψ_{k+1} decaying more rapidly than ψ_k. ψ is to satisfy

$$(\psi')^2 - i\psi'' = \lambda^2 - V. \tag{6.3}$$

Thus $(\lambda + \psi_1')^2 - i\psi_1'' \approx \lambda^2 - V$. Dropping the terms ψ_1'' and $(\psi_1')^2$ because we expect them to decay more rapidly than ψ_1' itself, we find that

$$\psi_1' = -(2\lambda)^{-1}V. \tag{6.4}$$

Thus $\psi_1' = O(|x|^{-\delta})$, while $\psi_1'' = O(|x|^{-\delta-\rho})$ and $(\psi_1')^2 = O(|x|^{-2\delta})$. This procedure may be repeated to obtain ψ_n' for every n, satisfying symbol-type estimates with gain of $|x|^{-\min(\rho,\delta)}$ at each iteration.

In the case where V does not satisfy symbol-type estimates, we will seek generalized eigenfunctions of the form

$$u_\lambda(x) = e^{i\phi(x,\lambda)} + o(1) \text{ as } x \to +\infty, \text{ with} \tag{6.5}$$

$$\phi(x,\lambda) = \lambda x - (2\lambda)^{-1} \int_0^x V(y)\, dy\,. \tag{6.6}$$

The lower limit of integration may equally well be chosen differently.

Observe two things. Firstly, the WKB phase shift $-(2\lambda)^{-1} \int^x V$ is in general unbounded, if $V \notin L^1$. Secondly, we may try to measure the quality of an approximate solution $\tilde{u}$ by the remainder $-\tilde{u}'' + V\tilde{u} - \lambda^2\tilde{u}$. For $\tilde{u} = \exp(i\lambda x)$, this remainder has modulus $|V(x)|$. For $\tilde{u} = \exp(i\phi(x,\lambda))$, it has modulus $|c_1V^2(x) + c_2V'(x)|$ for certain constants c_j depending on λ. The term V^2 is on the average smaller than V. However, V' is in general only defined in the sense of distributions, and in general decays no more rapidly than V itself, even with a liberal measure of its size, for instance in a Sobolev space H^s_{loc} with $s = -1$. Nonetheless our main results show that for $V \in L^1 + L^p$, $1 < p < 2$, without any differentiability hypothesis, the approximation (6.5), (6.6) is accurate for Lebesgue-almost every λ.

See [9, 43, 80] for original papers on the WKB approximation.

7. Transmission and reflection coefficients

Suppose temporarily that V has compact support. Fix $\lambda \in \mathbb{R}$. There exists a unique generalized eigenfunction u_λ^+ that is $\equiv \exp(i\lambda x)$ for x near $+\infty$. Near $-\infty$,

$$u_\lambda^+ \equiv a(\lambda)e^{i\lambda x} + b(\lambda)e^{-i\lambda x} \tag{7.1}$$

for certain coefficients a, b. The quantitites

$$t(\lambda) = 1/a(\lambda), \qquad r(\lambda) = b(\lambda)/a(\lambda). \tag{7.2}$$

are called respectively the transmission and reflection coefficients. Their interpretation is that an incoming wave $e^{i\lambda x}$ from $-\infty$ interacts with the potential and splits into a reflected wave $r(\lambda)e^{-i\lambda x}$ plus a transmitted wave $t(\lambda)e^{i\lambda x}$.

If V has compact support, then for each $\lambda \in \mathbb{R}$ there exists a unique generalized eigenfunction $\psi_+(x,\lambda)$ satisfying

$$\psi_-(x,\lambda) = \begin{cases} t(\lambda)e^{ix\lambda} \ + o(1) & \text{as } x \to +\infty, \\ e^{ix\lambda)} + r(\lambda)e^{-ix\lambda} \ + o(1) & \text{as } x \to -\infty, \end{cases}$$

and $1 = |t(\lambda)|^2 + |r(\lambda)|^2$ for all λ. One of our principal results (see below) states that these scattering coefficients are well-defined for $V \in L^1 + L^p$, for almost every λ, provided that $1 < p < 2$. For some of our purposes, though, a, b are more fundamental than t, r.

The constancy of the Wronskian of $u^+, \overline{u^+}$ can be equivalently rephrased as[10]

$$|a(\lambda)|^2 = |b(\lambda)|^2 + 1 \quad \text{for all } \lambda \in \mathbb{R}. \tag{7.3}$$

Temporarily allowing λ to be complex, we find that $E = \lambda^2$ is an eigenvalue if $\lambda \in i\mathbb{R}$ with negative imaginary part, and $a(\lambda) = 0$. A small computation shows that $|a| + |b|$, and hence $|a|$ alone, control the magnitude of the vector $(u^+(x), du^+(x)/dx)$ for x to the left of the support of V.

For compactly supported $V \in L^2$, the following remarkable identity[11] [10, 30] holds:

$$\int_{\mathbb{R}} \log|a(\lambda)|\lambda^2\, d\lambda + \frac{2\pi}{3}\sum_k |\lambda_k|^3 = \frac{\pi}{8}\int_{\mathbb{R}} V^2\, dx \tag{7.4}$$

where $\{\lambda_k\}$ is the collection of all eigenvalues of $-\partial_x^2 + V$.[12] This set is necessarily finite, and each λ_k is negative. An outline of a simple, direct proof may be found in [27]; it involves a deformation of the contour of integration into the upper half space in the complex plane.[13] The second term on the right arises from any poles coming from zeros of a, while the first arises in the limit as the contour is pushed to infinity.

This has the following consequence. Let $\Lambda \subset \mathbb{R}\backslash\{0\}$ be a compact interval. Denote by $u_\lambda(x)$ the unique generalized eigenfunctions with $(u_\lambda(0), u_\lambda'(0))$ equal either to $(1,0)$, or to $(0,1)$. Then if $V \in L^2[0,x]$,

$$\int_\Lambda \log(1 + |u_\lambda(x)|)\, d\lambda \le C + C\int_{[0,x]} V^2, \tag{7.5}$$

where $C < \infty$ depends only on Λ. In particular, if $V \in L^2(\mathbb{R})$, then the left-hand side is bounded, uniformly in x. This bound may seem extraordinarily weak; u is only logarithmically integrable in λ. But no more can be expected. If one thinks of V as $\sum_j V_j$ with each V_j supported in $[j, j+1]$, then the map sending $(u(j), u'(j))$ to $(u(j+1), u'(j+1))$ is multiplication by a certain matrix whose entries depend on V_j; these matrices are multiplied together in sequence to yield the asymptotic behavior as $x \to +\infty$; taking a logarithm converts this back to an additive process.

Another way to see this is by writing the generalized eigenfunction equation in Prüfer variables: It is well known in Schrödinger theory that the equation $-u'' + Vu = \lambda^2 u$ may be rewritten as the system

$$\begin{cases} \rho' = (2\lambda)^{-1} V \sin(2\theta) \\ \theta' = \lambda - (2\lambda)^{-1} V + (2\lambda)^{-1} V \cos(2\theta). \end{cases} \tag{7.6}$$

[10] In the time-dependent picture, this translates to an infinitesimal form of conservation of energy; it says that the energy of the incoming wave equals the combined energies of the transmitted and reflected waves. In the time-independent framework under discussion here, it can be interpreted as a conservation of probability densities.

[11] This identity is related to the inverse scattering theory for the Korteweg-de Vries equation; the right-hand side is one of the basic quantities invariant under the KdV flow. Dropping the first term on the left, one obtains a fundamental bound of Lieb-Thirring type for the negative eigenvalues.

[12] The exponent 3 is not a typographical error.

[13] $a(\lambda)$ is an entire function. Since $a(-\lambda) \equiv \overline{a(\lambda)}$ for $\lambda \in \mathbb{R}$, $\log|a|$ may be replaced by $\log(a)$ in the integral. The integral equation (5.2) can be used to obtain an asymptotic expansion for a as $|\lambda| \to \infty$ in the upper half plane. $a(\lambda) = 0$ if and only if λ^2 is an eigenvalue of $H_0 + V$; deforming the contour, taking zeros into account, and invoking an identity which amounts to Plancherel's theorem to control the integral over the limiting contour yields the identity. (It is a bit strange that the series arising from the integral equation (5.2) works here, since many individual terms of that series fail to satisfy the conclusion desired for the sum.)

where $u = R\sin(\theta)$, $u' = \lambda R\cos(\theta)$, and $\rho = \ln(R)$. Thus if we make the first approximation $\theta(x) \approx \lambda x$, we obtain $(\ln R)'(x) \approx V(x)\sin(2\lambda x)$. Thus one merely expects some power of $\ln|u|$ to be locally integrable. From this equation one sees also that lower bounds on $|R|$ might be amenable to study.

(7.5) does not seem sufficient for a direct application of the approximate eigenfunction criterion of Proposition 4.1, but suffices for analysis of the Weyl m-function, in conjunction with a limiting argument. This is how Deift and Killip [27] proved the existence of ac spectrum for potentials in L^2. Adding an L^1 perturbation is harmless, either by functional analysis (a relative trace class perturbation does not change the ac spectrum), or because the generalized eigenfunctions for H_0+V_1 can be used to construct the associated resolvents, whence the generalized eigenfunctions for $H_0+V_1+V_2$ can be analyzed by solving an integral equation $u = -(H_0 - V_1 + \lambda^2)^{-1}V_2 u$ modulo an element of the nullspace of $H_0 - V_1 + \lambda^2$ as above; this always works if $V_2 \in L^1$ and $H_0 + V_1$ has (for the set of energies in question) bounded generalized eigenfunctions.

Here is a result on the existence of (modified) scattering coefficients for slowly decaying potentials:

Theorem 7.1. *Let $1 < p < 2$ and $V \in L^1 + L^p$. Then for almost every $\lambda \in \mathbb{R}$ there exist $t(\lambda), r(\lambda) \in \mathbb{C}$ satisfying $|t(\lambda)|^2 + |r(\lambda)|^2 = 1$ and a generalized eigenfunction $\psi_+(x,\lambda)$ such that*

$$\psi_+(x,\lambda) = \begin{cases} t(\lambda)e^{i\phi(x,\lambda)} + o(1) & \text{as } x \to +\infty, \\ e^{i\phi(x,\lambda)} + r(\lambda)e^{-i\phi(x,\lambda)} + o(1) & \text{as } x \to -\infty. \end{cases} \tag{7.7}$$

If $\lim_{x\to\pm\infty} \int_0^x V$ exists, then $\phi(x,\lambda)$ can be replaced by $x\lambda$ in this result.

8. Reduction and expansion

In this section we write the generalized eigenfunction equation $-u'' + Vu = \lambda^2 u$ as a first-order system, in a way that incorporates the WKB approximation. We then iterate the resulting equation in a fashion parallel to that of §5, to obtain a modified power series expansion for the nonlinear operator mapping V to $u_+(x,\lambda)$ where u_+ is, formally[14], the unique generalized eigenfunction asymptotic to $\exp(i\phi(x,\lambda))$ as $x \to +\infty$.

$y = \begin{pmatrix} u \\ u' \end{pmatrix}$ satisfies

$$y' = \begin{pmatrix} 0 & 1 \\ V - \lambda^2 & 0 \end{pmatrix} y\ . \tag{8.1}$$

Writing $\phi = \phi(x,\lambda) = \lambda x - (2\lambda)^{-1}\int_0^x V$, we substitute

$$y = \begin{pmatrix} e^{i\phi} & e^{-i\phi} \\ i\lambda e^{i\phi} & -i\lambda e^{-i\phi} \end{pmatrix} w\ . \tag{8.2}$$

Thus boundedness of w (as a function of x for given λ) is equivalent to boundedness of both u and u'. And if $w \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $x \to +\infty$ then $u - \exp(i\phi) \to 0$. The new unknown w satisfies

$$w' = \frac{i}{2\lambda} \begin{pmatrix} 0 & -V(x)e^{-2i\lambda x + \frac{i}{\lambda}\int_0^x V(t)\,dt} \\ V(x)e^{2i\lambda x - \frac{i}{\lambda}\int_0^x V(t)\,dt} & 0 \end{pmatrix} w. \tag{8.3}$$

[14]Since existence is not yet proved.

w is directly linked to certain reflection/transmission coefficients. Define $a(x,\lambda), b(x,\lambda)$ by

$$\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = \begin{pmatrix} a(x,\lambda)e^{i\lambda x} + b(x,\lambda)e^{-i\lambda x} \\ i\lambda a(x,\lambda)e^{i\lambda x} - i\lambda b(x,\lambda)e^{-i\lambda x} \end{pmatrix} = \begin{pmatrix} e^{i\lambda x} & e^{-i\lambda x} \\ i\lambda e^{i\lambda x} & -i\lambda e^{-i\lambda x} \end{pmatrix} \cdot \begin{pmatrix} a(x,\lambda) \\ b(x,\lambda) \end{pmatrix} . \tag{8.4}$$

Then

$$\begin{pmatrix} a(x,\lambda) \\ b(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{-i(2\lambda)^{-1}\int_0^x V} & 0 \\ 0 & e^{i(2\lambda)^{-1}\int_0^x V} \end{pmatrix} w . \tag{8.5}$$

In particular, the two components of $w(x) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ have the same magnitudes as a, b, respectively. The conservation identity $|a|^2 \equiv 1 + |b|^2$ thus is equivalent to

$$|w_1(x)|^2 \equiv 1 + |w_2(x)|^2 . \tag{8.6}$$

Introduce

$$(Tf)(\lambda) = \int_{\mathbb{R}} e^{2i\lambda x - i\lambda^{-1}\int_0^x V(t)\,dt} f(x)\,dx, \tag{8.7}$$

defined initially on integrable functions of compact support. We also introduce multilinear operators

$$T_n(f_1,\dots,f_n)(x,\lambda) = \left(\frac{i}{2\lambda}\right)^n \int_x^\infty \int_{t_1}^\infty \cdots \int_{t_{n-1}}^\infty \prod_{j=1}^n \left[e^{(-1)^{n-j}2\phi(t_j,\lambda)} f_j(t_j)\,dt_j \right] . \tag{8.8}$$

Iterating system (8.3) starting from the vector $(1,0)$ we obtain a formal "Taylor series" expansion for the putative generalized eigenfunction $u_+(x,\lambda)$ with the desired asymptotic $\exp(i\phi)$ as $x \to +\infty$:

$$\begin{aligned} u_+(x,\lambda) &= e^{i\lambda x - \frac{i}{2\lambda}\int_0^x V(t)\,dt} \sum_{n=0}^\infty (-1)^n T_{2n}(V,\dots,V)(x,\lambda) \\ &+ \; e^{-i\lambda x + \frac{i}{2\lambda}\int_0^x V(t)\,dt} \sum_{n=1}^\infty (-1)^n T_{2n-1}(V,\dots,V)(x,\lambda) \end{aligned} \tag{8.9}$$

We have set $T_0(V)(x,\lambda) \equiv 1$. This is not exactly a Taylor series, since V still appears in a nonlinear fashion in the exponents.

In a sense, our approach is a part of the program of Calderón [12] and of Coifman and Meyer [24, 25] of analyzing nonlinear operators (in the present case, mapping V to the collection of all generalized eigenfunctions) via power series expansions in terms of multilinear operators.

9. Maximal operators

We seek to analyze multilinear operators (8.8). If we simplify by discounting the WKB phase correction for the present, these are built up out of a well-understood operator, the Fourier transform, by two processes. Firstly, integration over $\mathbb{R}$ is replaced by integration over the nested family of sets $(-\infty, x]$, and a supremum estimate in x is sought. Secondly, multilinear operators are generated by iterated integrals over such sets. In this section we develop a robust, though crude, method for analyzing the first of these two processes. The theory to be developed here depends heavily on the order structure of $\mathbb{R}$, as of course does the definition of the multilinear operators (10.1).

Denote by χ_E the characteristic function of a set E, and by $\|T\|$ the operator norm of $T : L^p(\mathbb{R}) \mapsto L^q(\mathbb{R})$. To any operator T defined on $L^p(\mathbb{R})$ can be associated a maximal operator

$$T^*f(x) = \sup_s |T(f \cdot \chi_{(-\infty,s]})(x)|. \tag{9.1}$$

Theorem 9.1. [18, 17] *Let $1 \le p, q \le \infty$, and suppose that $T : L^p(\mathbb{R}) \mapsto L^q(\mathbb{R})$ is a bounded linear operator. Then T^* is likewise bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$, provided that $p < q$. Moreover, $\|T^*\|$ is bounded by an absolute constant, depending only on p, q, times $\|T\|$.*

More generally, there is an analogue for any linear operator from $L^p(Y)$ to $L^q(X)$, for arbitrary measure spaces, provided that the sets $(-\infty, s]$ are replaced by an arbitrary nested family of sets $Y_n \subset Y$ (that is, $Y_n \subset Y_{n+1}$).

Suppose that T is represented as an integral operator $Tf(x) = \int_{\mathbb{R}} K(x,y)f(y)\,dy$.

Corollary 9.2. *If $p < q$ and if T is bounded from L^p to L^q then*

$$\tilde{T}f(x) = \int_{y<x} K(x,y)f(y)\,dy \tag{9.2}$$

is likewise bounded from L^p to L^q.

The hypothesis $p < q$ is in general necessary, except in the trivial cases $p = 1$ or $q = \infty$, even in the corollary. Consider for example the Hilbert transform, for which $K(x,y) = (x-y)^{-1}$; the associated operator with kernel $(x-y)^{-1}\chi_{y<x}$ is unbounded on all L^p.

These results apply as well to functions taking values in Banach spaces. Tao [76] pointed out its applicability to Strichartz-type estimates (see Section 15 below). This generalization has been applied by Smith and Sogge [72] to the obstacle problem, and by Takaoka and Tzvetkov, by Colliander and Kenig, and by others to nonlinear evolution equations.

Definition. A martingale structure on $\mathbb{R}$ is a collection $\{E_j^m\}$ of intervals, indexed by $m \in \{0, 1, 2, \dots\}$ and $1 \le j \le 2^m$, satisfying

- For each m, $\{E_j^m : 1 \le j \le 2^m\}$ is a partition of $\mathbb{R}$ into disjoint intervals.
- E_j^m lies to the left of E_{j+1}^m for all m, j.
- Each $E_j^m = E_{2j-1}^{m+1} \cup E_{2j}^{m+1}$.

A martingale structure is said to be adapted to f in L^p if

$$\int_{E_j^m} |f|^p = 2^{-m} \int_{\mathbb{R}} |f|^p \text{ for all } m, j.$$

Outline of proof. Let $1 \le p < \infty$ and let $0 \ne f \in L^p(\mathbb{R})$ be fixed. Construct a martingale structure $\{E_j^m\}$ on $\mathbb{R}$, adapted to f in L^p. Let χ_j^m denote the characteristic function of E_j^m, and set $f_j^m = f \cdot \chi_j^m$. For any $s \in \mathbb{R}$, the interval $(-\infty, s]$ can be partitioned, modulo a set on which f vanishes almost everywhere, as $\cup_\nu E_{j_\nu}^{m_\nu}$ for some sequences such that $m_1 < m_2 < m_3 \cdots$, and each $E_{j_\mu}^{m_\nu}$ lies to the left of $E_{j_{\nu+1}}^{m_{\nu+1}}$. Thus

$$T^*f(x) \le \sum_{m=0}^{\infty} \sup_{1 \le j \le 2^m} |T(f_j^m)(x)| \le G_{T,r}f(x) \tag{9.3}$$

where the last quantity is defined by

$$G_{T,r}f(x) = \sum_m \big(\sum_j |T(f_j^m)(x)|^r \big)^{1/r} \tag{9.4}$$

for any positive exponent r. Choosing $r = q$, we have

$$\|Gf\|_q \le \sum_m \Big(\int \sum_j |T(f_j^m)|^q \Big)^{1/q} \le \|T\| \sum_m \Big(\sum_j \|f_j^m\|_p^q \Big)^{1/q}$$
$$\le \|T\| \sum_m \big(2^m 2^{-mq/p} \|f\|_p^q \big)^{1/q} = C\|T\| \cdot \|f\|_p$$

where $C < \infty$ by the hypothesis $q > p$. To justify the final inequality we have used the bound $\|f_j^m\|_p \le 2^{-m/p}\|f\|_p$ □

The following two theorems, dating roughly from the 1930's, are immediate corollaries.

Theorem 9.3. *Let $1 \le p < 2$. For any $f \in L^p(\mathbb{R}^1)$,*

$$\lim_{y\to\infty} \int_0^y e^{-i\lambda x} f(x)\, dx$$

exists for almost every λ. Moreover

$$\sup_y \Big| \int_0^y e^{-i\lambda x} f(x)\, dx \Big| \in L^q(\mathbb{R}, d\lambda), \quad \text{where } q = p/(p-1)\ .$$

Theorem 9.4. [48] *Let $1 \le p < 2$. For any*[15] *orthonormal family $\{\phi_n\}$ of functions in L^2 of any measure space, and for any sequence $c_n \in \ell^p$, the series $\sum_n c_n\phi_n(x)$ converges for almost every x.*

The first result was obtained in various versions in separate papers by Menchoff, Paley, and Zygmund. The former result continues to hold for $p = 2$ and then is essentially a restatement of Carleson's almost everywhere convergence theorem.[16]

A closely related theorem of Menchoff and of Rademacher asserts that for any arbitrary orthonormal system, if $\sum_n |c_n|^2 \log(n)^2 < \infty$, then $\sum_n c_n\phi_n$ converges almost everywhere[17]; in particular, if $\int |\hat f(\xi)|^2 \log(2 + |\xi|)\, d\xi < \infty$ then $(2\pi)^{-1} \int_{-N}^N e^{ix\xi} \hat f(\xi)\, d\xi$ converges almost everywhere to $f(x)$, as $N \to \infty$. See [81], chapter XIII.

The following variant is perhaps better known nowadays. To any orthonormal system $\{\phi_n\}$ on any measure space, and any $N \in \mathbb{N}$, we may associate a maximal function, mapping sequences $a \in \ell^2$ to functions, defined by

$$M_N(a)(x) = \sup_{1\le n\le N} \Big| \sum_{k=1}^n c_k\phi_k(x) \Big|.$$

Theorem 9.5. *There exists a universal constant $C < \infty$ such that for any orthonormal system and any $N \ge 2$, $\|M_N(a)\|_{L^2} \le C \log N \|a\|_{\ell^2}$ for all $a \in \ell^2$.*

The Menchoff-Rademacher theorem is a corollary of this one.

The following argument is by no means original, but goes back to Menchoff and/or Rademacher (see page 82 of [2]).

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[15] The version stated in [81], Theorem (10.1) of chapter XIII, is a refinement by Paley of Menchoff's original theorem. It, like the original, requires uniform boundedness of $\{\phi_n\}$. It applies to the same class of coefficients c, but uses a different scale of function spaces and thus its conclusion involves boundedness of an associated maximal operator in a different norm than we obtain.

[16] Carleson's proof does not seem to yield the strong type L^2 estimate.

[17] Our main results have similar extensions to the case where $(\log|x|)^c f(x) \in L^2$ for a certain constant c.

Proof. Without loss of generality we may assume that $N = 2^M$ for some $M \in \mathbb{N}$. Define $E^0 = \{1, 2, \ldots, 2^M\}$. Partition $E^0 = E_1^1 \cup E_2^1$ where $E_1^1 = \{1, 2, \ldots, 2^{M-1}\}$ and $E_2^1 = \{2^{M-1}+1, \ldots, 2^M\}$. Continue inductively, partitioning E^0 for each $1 \le m \le M$ as a union of 2^m sets E_j^m of 2^{M-m} consecutive integers, where $j \in \{1, 2, \ldots, 2^m\}$.

To $a \in \ell^2$ associate $A_j^m(x) = \sum_{k \in E_j^m} a_k \phi_k(x)$. Define $S^m(a)(x) = \big(\sum_{j=1}^{2^m} |A_j^m(x)|^2\big)^{1/2}$. Then $\|S^m(a)\|L^2 = \|a\|_{\ell^2}$ for every m. Of course, $|A_j^m(x)| \le S^m(a)(x)$ for all m, j, x.

For any $1 \le n \le 2^M$, $\{1, 2, \ldots, n\}$ may be expressed in a unique way as $\cup_m \tilde{E}^m_{j(m,n)}$, where each $\tilde{E}^m_{j(m,n)}$ equals either $E^m_{j(m,n)}$ or the empty set. Thus we majorize $|\sum_{k=1}^n a_k \phi_k(x)|$ by $|\sum_m' A^m_{j(m,n)}(x)| \le \sum_m' |A^m_{j(m,n)}(x)|$, where the sum extends over those m for which $\tilde{E}^m_{j(m,n)}$ is not empty.

This last sum is majorized by $\sum_{m=0}^M S^m(a)(x)$, which is $O(M\|a\|_{\ell^2})$ in L^2 norm by the triangle inequality. □

In the opposite direction, Menchoff proved that general orthonormal series $\sum_n c_n \phi_n(x)$ need not converge almost everywhere for arbitrary coefficients $c_n \in \ell^2$. A series of refinements were obtained by Menchoff and other authors, including the following theorem of Tandori [75]. For some of these results, some discussion, and further references see [2], in particular, Lemma 2.4.4.

Theorem 9.6. (Tandori [75]) *Let $\{q_n\}$ be a decreasing sequence of positive numbers satisfying $\sum_n q_n^2 \log^2(n) = \infty$. Then for any sequence of coefficients $\{c_n\}$ such that $|c_n| \ge q_n$ for all n, there exists a uniformly bounded orthogonal family of functions $\phi_n \in L^2([0,1])$ such that the series $\sum_n c_n \phi_n$ diverges almost everywhere on $[0,1]$.*

To fit Theorem 9.4 into our framework, regard $c \mapsto \sum_n \phi_n$ as a map from $L^p(\mathbb{Z})$ to L^2. This map is bounded for all $1 \le p \le 2$. Partial summation is integration over $(-\infty, N]$ for some N; these sets are nested; so Theorem 9.1 implies boundedness of the maximal partial sum operator, and hence almost everywhere convergence.

Our third application is to the variants arising in our generalized eigenfunction analysis. We need a preliminary lemma. We say that $V \to 0$ in L^1_{loc} if $\int_{|y-x|<1} |V(y)|\,dy \to 0$ as $|x| \to \infty$. As usual, $\phi(x, \lambda) = \lambda x - (2\lambda)^{-1} \int_0^x V$.

Lemma 9.7. *Suppose that $V \to 0$ in $L^1_{loc}(\mathbb{R})$. Then for any compact subset Λ of $\mathbb{R}\backslash\{0\}$, the mapping*

$$f \mapsto \int_0^\infty f(y) e^{i\phi(y,\lambda)}\,dy \tag{9.5}$$

maps $L^p(\mathbb{R})$ boundedly to $L^q(\Lambda, d\lambda)$ for all $1 \le p \le 2$, where $q = p' = p/(p-1)$.

For L^2 this is proved by dualizing, then integrating by parts. Since the $L^1 \mapsto L^\infty$ estimate is trivial, the general conclusion follows by interpolation. By combining Theorem 9.1 with the lemma, we deduce a variant of the Hausdorff-Young inequality.

Corollary 9.8. *For all $1 \le p < 2$, the sublinear operator*

$$f \mapsto \sup_{x \in \mathbb{R}} \Big| \int_0^x e^{i\phi(y,\lambda)} f(y)\,dy \Big| \tag{9.6}$$

maps $L^p(\mathbb{R})$ boundedly to $L^{p'}(\Lambda)$, for every compact subset Λ of $\mathbb{R}\backslash\{0\}$.

10. Multilinear operators and maximal variants

A multilinear variant of Theorem 9.1 is as follows. Let $T_j : L^p(\mathbb{R}, dx) \mapsto L^q(\Lambda, d\lambda)$ be bounded linear operators with locally integrable distribution kernels $K_j(\lambda, x)$. Define

$$(10.1)\quad \mathcal{M}_n^*(f_1, f_2, \dots, f_n)(\lambda) = \sup_{y \le y' \in \mathbb{R}} \left| \int \cdots \int_{y \le x_1 \le x_2 \le \dots \le x_n \le y'} \prod_{i=1}^n \big(K_i(\lambda, x_i) f_i(x_i) dx_i \big) \right| .$$

If the factors in the integrand are all nonnegative, then this is dominated by the corresponding integral over $[y, y']^n$, thus by a simple product $\prod T_j(f_j \cdot \chi_{[y,y']})$. The whole difficulty for us is that our integrals are oscillatory, and taking absolute values renders them hopelessly divergent.

Theorem 10.1. [18] *Suppose that $p < q$. Then for every $n \ge 1$, there exists $B_n < \infty$ such that $(f_1, \dots, f_n) \mapsto \mathcal{M}_n^*(f_1, \dots f_n)$ maps $\otimes^n L^p(\mathbb{R})$ boundedly to $L^{q/n}(\Lambda)$, with operator norm $\le B_n \prod_{j=1}^n \|T_j\|_{p,q}$.*

Here B_n is finite and depends only on n. The exponent q/n is natural; the product mapping $(f_1, \dots, f_n) \mapsto \prod_j T_j(f_j)$ maps into $L^{q/n}$ by Hölder's inequality, and we don't expect the iterated integrals to do better. This result is stated in [18] only for $q \ge 2$, but that assumption can be eliminated by replacing $r = 2$ by $r = q$ in the definition (9.4) of the auxiliary functional G.

Our next variant demonstrates a substantial improvement, in the special case when all the functions f_i and kernels K_i are the same, modulo complex conjugation.

Theorem 10.2. [18] *Suppose that $p < q$ and that $2 \le q$. Suppose that for some function $f \in L^p$ and some kernel K, each f_i equals either f or $\overline{f}$, and each kernel K_i equals either K or $\overline{K}$. Then for every $n \ge 1$,*

$$(10.2)\qquad \|\mathcal{M}_n^*(f_1, f_2, \dots, f_n)\|_{L^{q/n}(\Lambda)} \le \frac{B^n \|T\|_{p,q}^n \|f\|_{L^p}^n}{\sqrt{n!}} .$$

Here T denotes of course the operator associated to the given kernel K.

Our applications require a slight generalization, which follows from the same proof. Namely, a factor $1/\sqrt{n!}$ is still gained, if both the functions f_j and the operators T_j are drawn from finite sets of *bounded* cardinality $\le k$, *independent* of n. Labeling these two finite sets $\{\tilde{f}_\nu\}$ and $\{\tilde{T}_\mu\}$, the conclusion becomes

$$(10.3)\qquad \|\mathcal{M}_n^*(f_1, f_2, \dots, f_n)\|_{L^{q/n}(\Lambda)} \le \frac{B^n \big(\sum_\mu \|\tilde{T}_\nu\|_{p,q} \big)^n \big(\sum_\nu \|\tilde{f}_\nu\|_{L^p} \big)^n}{\sqrt{n!}} ,$$

where the indices μ, ν range over the given sets of cardinality $\le k$. The constant B then depends on k, p, q, but not on any other quantities.

In applications, one obtains a better conclusion by pulling out normalizing factors, so that the inequality is applied when the norm of each f_μ and the operator norm of each T_ν equals one.

This bound improves that of the preceding theorem by the factor $1/\sqrt{n!}$. No such factor arises in Theorem 10.1; B_n can't be taken to be smaller than B^n for some universal B. It is easy to see why there might be some improvement in this "diagonal" case:

$$\int_{y \le t_1 \le \dots \le t_n \le y'} \prod K(\lambda, t_i) f(t_i) dt_i \equiv \Big[\int_y^{y'} K(\lambda, t) f(t)\, dt \Big]^n / n! .$$

In our main application, however, $K_i(\lambda, t)$ will essentially be $\exp(\pm 2i\lambda t)$, with the $\pm$ signs alternating; there will still be only a single function f. Then there is no obvious majorization of the left-hand side of the preceding inequality by the right.[18]

I believe that the second theorem remains valid without the assumption $q \geq 2$, with $(n!)^{-1/2}$ replaced by an appropriate modified power depending on the exponents, but have not worked out all the details of the proof. The version stated here is easier than the general case, and is precisely what is most relevant for our applications.

The factor $1/\sqrt{n!}$ plays a twofold role in our analysis. Firstly, it is used to deduce convergence of the "Taylor series" (8.9), for almost every λ. Secondly, it leads to a bound for the generalized eigenfunctions:

Proposition 10.3. *Let $1 < p < 2$, and let $V \in L^1 + L^p(\mathbb{R})$. Denote by $u(x, \lambda)$ a generalized eigenfunction for $H_0 + V$, normalized so that $(u(0), u'(0))$ equals either $(1, 0)$ or $(0, 1)$. Then for any compact interval $\Lambda \subset \mathbb{R}\backslash\{0\}$,*

$$\int_\Lambda \log \sup_{x \in \mathbb{R}} (1 + |u(x, \lambda)|)\, d\lambda < \infty \tag{10.4}$$

What is significant here, or at least pleasing, is that this bound has the same general form as the inequality deduced from the trace identity (7.4), involving an L^1 norm on $\log^+ |u|$, except that we have a stronger conclusion in that the supremum over x is inside the integral. (On the other hand, (10.4) is slightly weaker in that V is assumed to belong to $L^1 + L^p$ for some p strictly less than 2.) The factor $1/\sqrt{n!}$ turns out to be exactly what is needed in order to obtain local integrability of the first power of the logarithm.

Nonetheless, it is perhaps worth understanding that our main conclusions, almost sure boundedness and WKB asymptotics for generalized eigenfunctions and presence of absolutely continuous spectrum everywhere in $\mathbb{R}^+$, could be deduced instead without the improved numerical factor, roughly as follows: Fix a compact interval Λ. If V has sufficiently small norm in $L^p + L^1$, then the factor of $\|V\|^n$ on the right-hand side more than compensates for B^n. Combining this with the pointwise bound of Lemma 10.4 below (in the weaker form without the factor $1/\sqrt{n!}$), one can deduce convergence and uniform boundedness for a subset of Λ having positive Lebesgue measure; moreover, the measure of the set where convergence and boundedness are not obtained approaches zero as $\|V\|$ does. Since the norm of the restriction to $[x, \infty)$ of V tends to zero as $x \to +\infty$, the exceptional set of energies has measure zero.

There is one flaw in this scheme: the bounds for $\mathcal{M}_n^*$ are in $L^{q/n}$, and $q/n \to 0$, so no triangle inequality is available to sum the infinite series. This can be dealt with in two ways. The first way is to apply Chebyshev's inequality to obtain bounds for the distribution functions of $\mathcal{M}_n^*$, then to show almost everywhere finiteness of the sum by a bare hands computation; the factor of $1/\sqrt{n!}$ is essential here. The second, and preferable, route, which could be used without these favorable numerical factors, is to exploit stronger pointwise

[18] One could try to explain the factor of $\sqrt{n!}$ by assuming n to be even, and letting the symmetric group $S_{n/2}$ act on the coordinates $(t_1, \dots, t_n)$ by permuting them in blocks of two, grouping t_{2j-1} with t_{2j} for each j. Alternatively, one could argue for a larger denominator $n!$ by considering the action of $S_{n/2} \times S_{n/2}$ defined by letting one factor permute $\{t_{2j-1}\}$, and the other permute $\{t_{2j}\}$. There are two objections, besides the fact that these two heuristics yield different conclusions. Firstly, neither argument is correct, since the region of integration is not a fundamental domain for the action of either group on the full space $\mathbb{R}^n$ of coordinates, and since cancellation plays an essential role here. Secondly, as stated above, the conclusion holds with the same factor $\sqrt{n!}$, the kernels and functions are drawn from finite sets of uniformly bounded cardinality; in that more general case the symmetry is broken.

versions of the above two theorems, which we now discuss. We state only the analogue of Theorem 10.2.

Suppose we are given a martingale structure $E_j^m \subset \mathbb{R}$ adapted to f in L^p. Define the functionals

$$\tilde{\mathfrak{g}}(f) = \sum_{m=0}^{\infty} (\sum_{j=1}^{2^m} |\int f \cdot \chi_j^m|^2)^{1/2}, \tag{10.5}$$

$$\mathfrak{g}(f) = \sum_{m=0}^{\infty} m(\sum_{j=1}^{2^m} |\int f \cdot \chi_j^m|^2)^{1/2}, \tag{10.6}$$

These operators depend on $\{E_j^m\}$. They are essentially linear operations, being norms in Banach spaces like $\ell^1(\ell^2)$ of a linear operator applied to f. But in our final application, the martingale structure will itself depend on f, so they will become rather nonlinear.

Consider integrals

$$M_n(f_1, \dots, f_n)(y, y') = \int \cdots \int_{y \le t_1 \le t_2 \le \cdots \le t_n \le n'} \prod_{k=1}^{n} f_k(t_k)\, dt_k \tag{10.7}$$

$$M_n^*(f_1, \dots, f_n) = \sup_{y, y' \in \mathbb{R}} |M_n((f_1, \dots, f_n)(y, y')| \ . \tag{10.8}$$

Lemma 10.4. *There exists a finite constant B such that for any martingale structure, any n, and any $f \in L^1$, if each $f_k \in \{f, \bar{f}\}$ then*

$$|M_n(f_1, \dots, f_n)(y, y')| \le B^n \tilde{\mathfrak{g}}(f)^n / \sqrt{n!} \tag{10.9}$$

$$|M_n^*(f_1, \dots, f_n)| \le B^n \mathfrak{g}(f)^n / \sqrt{n!}\ . \tag{10.10}$$

The bound is also independent of the choices of $f_k \in \{f, \bar{f}\}$.

From this lemma there follows a stronger form of Theorem 10.2. Let $f \in L^p(\mathbb{R})$ be given. Let $\{E_j^m\}$ be a martingale structure, constructed so as to be compatible with f in the sense that all $f_j^m = f \cdot \chi_j^m$ satisfy $\|f_j^m\|_p^p = 2^{-m}\|f\|_p^p$. Let $K(\lambda, x)$ be the kernel function associated to a linear operator T. Define

$$G(f)(\lambda) = \sum_{m=0}^{\infty} m(\sum_{j=1}^{2^m} |T(f_j^m)(\lambda)|^2)^{1/2}\ . \tag{10.11}$$

Corollary 10.5. [18] *There exists a constant $B < \infty$ such that for any n, λ,*

$$\mathcal{M}_n^*(f_1, \dots, f_n)(\lambda) \le \frac{B^n G(f)(\lambda)^n}{\sqrt{n!}}\ , \tag{10.12}$$

provided that each $f_j = f$ and each $K_j \in \{K, \bar{K}\}$.

In the general setup, where $f_j \in \{\tilde{f}_\nu : 1 \le \nu \le k\}$ and $K_j \in \{\tilde{K}_\mu : 1 \le \mu \le k\}$, this should be modified by choosing a martingale structure compatible with $f = \sum_\nu |\tilde{f}_\nu|$. In the definition of G, $|T(f_j^m)(\lambda)|^2$ should be replaced by $\sum_{\nu,\mu} |\tilde{T}_\mu(\tilde{f}_{\nu,j}^m)(\lambda)|^2$, where $\tilde{f}_{\nu,j}^m = \tilde{f}_\nu \cdot \chi_{E_j^m}$.

For our application to generalized eigenfunctions, this corollary expresses a sort of conspiracy; heuristically it says that the terms of the "Taylor" series tend to be simultaneously all good or simultaneously all bad, in the weak sense that a single functional controls them all.

This implies Theorem 10.2 by

Lemma 10.6. *Suppose that $p < q$ and $2 \le q$. Then there exists $C < \infty$ such that for any linear operator T bounded from L^p to L^q, for any $f \in L^p$,*

$$\|G(f)\|_q \le C\|T\|_{p,q} \cdot \|f\|_p \tag{10.13}$$

Here $\|T\|_{p,q}$ denotes the operator norm. It is essential that $G(f)$ be defined via a martingale structure compatible with f, in the sense described above. This lemma is a simple consequence of the triangle inequality, as in §9.

An immediate application of the corollary and the "Taylor series" representation of generalized eigenfunctions is the formal estimate

$$\sup_{x\in\mathbb{R}} |u(x,\lambda)| \le C\exp(CG(V)(\lambda)^2), \tag{10.14}$$

obtained by majorizing the sum of the series by $\sum_{n=0}^{\infty} B^n G(V)(\lambda)^n/\sqrt{n!}$. Here the relevant operator T has kernel $K(\lambda,x) = \exp(i\lambda x - i(2\lambda)^{-1}\int_0^x V)$. From (10.14) we conclude that $\log\sup_x |u(x,\lambda)|$ is locally integrable on $\mathbb{R}\backslash\{0\}$.

To conclude this section, we outline the proof of Lemma 10.4. We seek to bound $M_n(f_1,\dots,f_n)$ as in (10.9), where each $f_j \in \{f,\bar f\}$ for some single function f (the analysis of M_n^* requires a small additional step, omitted here). Simplify notation by writing $M_n(f)$ for $M_n(f_1,\dots,f_n)$. we first replace y,y' by $-\infty,+\infty$ respectively, and note the recursion

$$\begin{aligned}|M_n(f)| \le |M_n(f_1^1)| &+ \Big|\int_{E_2^1} f\Big| \cdot M_{n-1}(f_1^1) \\ &+ \sum_{j=2}^{n-2} |M_{n-j}(f_1^1)|\cdot|M_j(f_2^1)| + \Big|\int_{E_1^1} f\Big|\cdot M_{n-1}(f_2^1) + |M_n(f_2^1)| \,.\end{aligned} \tag{10.15}$$

This is obtained by partitioning the region $t_1 \le \cdots \le t_n$ of integration into $\cup_{k=0}^n \Omega_k$, where $\Omega_k = \{t : t_j \in E_1^1$ for all $j \le k$ and $t_j \in E_2^1$ for all $j > k\}$. Integration over each Ω_k gives rise to one term in (10.15).

The next step rests on a variant of the binomial identity.

Lemma 10.7. [18] *There exists $\gamma \in \mathbb{R}^+$ such that the numbers c_k defined by*

$$\beta_k = k^{-k/2}k^{-\gamma} \qquad \textit{for all } k \ge 2 \tag{10.16}$$

satisfy for every $k \ge 6$ the inequalities

$$y^k + \sum_{j=2}^{k-2} \frac{\beta_j\beta_{k-j}}{\beta_k} x^j y^{k-j} + x^k \le (x^2+y^2)^{k/2} \qquad \textit{for all } x,y \ge 0. \tag{10.17}$$

The ratios $\beta_j\beta_{k-j}/\beta_k$ behave roughly like binomial coefficients $\binom{k/2}{j/2}$. The only role of the factor $k^{-\gamma}$ and assumption $k \ge 6$ is to make the proof work. Because the lemma is to be used inside a recursive argument, it is essential that the right-hand side of the inequality be exactly $(x^2+y^2)^{k/2}$, rather than a constant multiple.

The proof of Lemma 10.7 uses Cauchy-Schwarz and term-by-term comparison with the binomial series for $(x^2+y^2)^{k/2}$ (taking into account that our series has twice as many terms) in the case where k is even, with appropriate modifications in the odd case.

To deduce the desired majorization for $M_n(f)$, we combine Lemma 10.7 with (10.15), and argue by induction on the generation number $m = 1,2,3,\dots$. Thus $M_n(f_j^m)$ can be expressed in terms of $\{M_k(f_i^{m+1}) : k < n,\ i \le 2^k\}$. The terms $|\int_{E_2^1} f|\cdot M_{n-1}(f_1^1)$ and $|\int_{E_1^1} f|\cdot M_{n-1}(f_2^1)$ cannot be handled in this way, essentially because $1+x$ cannot be dominated by $(1+Cx^2)^{1/2}$ for small x, so an extra step is required to incorporate them. See [18].

A final step is needed to handle the supremum over y, y'; it is similar to the argument for the linear case, Theorem 9.1.

Lemma 10.7 is the source of the factor $1/\sqrt{n!}$. It is a lemma about nonnegative numbers; the square root does not come about through any orthogonality.

In the above discussion we have omitted one aspect of the problem. The validity of WKB-type asymptotics is a kind of almost-everywhere convergence problem; one wants $\exp(-i\phi(x,\lambda))u(x,\lambda) \to 1$ as $x \to +\infty$, for almost every λ. The usual strategy for proving such a result is to first prove a maximal function inequality in some L^q norm, then to observe the (usually obvious) fact that the convergence holds (usually in a rather strong sense) for some appropriate dense class of functions. The almost everywhere convergence follows by combining these.

Because we have multilinear operators rather than linear ones, this last step is a bit more complicated. One must compare $T_m(V, V, \dots, V)$ with $T_m(W, W, \dots, W)$ where W has compact support, and $V - W$ has small L^p norm. This is of course done in part by expressing the difference as a telescoping sum and analyzing expressions $T_m(V, V, \dots, V, V - W, W, \dots, W)$. For details see [17].

Remark. The method applies to more general multiple integrals with variables which are partially, rather than linearly, ordered, such as

$$\int_\Omega K(\lambda,t)K(\lambda,s_1)K(\lambda,s_2)f_0(t)f_1(s_1)f_2(s_2)\,dt\,ds_1\,ds_2$$

where $\Omega = \{(t, s_1, s_2) : t \le s_1 \text{ and } t \le s_2\}$. Such expressions, with a branching factor of 2 at each level, arose in our analysis [16] of the power-decaying case $V = O(|x|^{-r})$, because we used a different expansion for the generalized eigenfunctions. They also arise, with arbitrarily large branching factors, in the proof of Theorem 3.3; see §16.

Remark. If $V = O(|x|^{-r})$ for some $r > 1/2$ then the issue of summation of an infinite series can be avoided altogether; see [16].

Remark. Our theory admits a limited generalization to higher dimensions.[19] Consider a bounded linear operator $T : L^p(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^n)$, with distribution-kernel $K(x,y)$; we assume for simplicity that $K \in L^1_{\text{loc}}$. Write $y \gtrsim z$ to mean that $y_j \ge z_j$ for all $1 \le j \le n$, where $y = (y_1, \dots, y_n)$ and likewise for z. Consider the associated maximal operators

$$T^*f(x) = \sup_{z\in\mathbb{R}^n} \left| \int_{y \gtrsim z} K(x,y)f(y)\,dy \right| . \tag{10.18}$$

Then T^* is likewise bounded from L^p to L^q, provided that $q > p$.

In spirit, this generalization bears the same relation to the one-dimensionsal case that the Marcinkiewicz multiplier theorem, and the strong maximal function of Jessen, Marcinkiewicz, and Zygmund, bear to their one-dimensional counterparts: it is controlled by an iteration of one-dimensional maximal operators. For simplicity, consider the representative case $n = 2$. The operator

$$\mathcal{T}^*(f)(x) = \sup_{z_2} \left| \int_{y_2 \ge z_2} K(x,y)f(y)\,dy \right| \tag{10.19}$$

[19] I am grateful to Alan McIntosh for posing this question.

is bounded, from L^p to L^q, by the one-dimensional theory; it suffices to view T as an operator from $L^p(\mathbb{R}^1, B_1)$ to $L^1(\mathbb{R}^1, B_2)$ where the Banach spaces B_i are respectively $L^p(\mathbb{R}^1)$, $L^q(\mathbb{R}^1)$. Then $\mathcal{T}^*$ is merely sublinear, rather than linear, but Theorem 9.1 applies equally well to sublinear operators, with the same proof as outlined above. We have

$$T^* f(x) \le \sup_{z_1} \left| \int_{y_1 \ge z_1} K(x,y) \chi_{y_2 > z_2} f(y)\, dy \right| , \tag{10.20}$$

so that the proof of Theorem 9.1 reduces matters to the boundedness of $\mathcal{T}^*$. □

11. Wave operators and time-dependent scattering

11.1. **Introduction.** If we aspire to at least a caricature of quantum mechanics, we ought to study the Schrödinger group e^{itH}, and in particular, its long term dynamics, including scattering. To the audience for whom these notes are intended, the question of Strichartz estimates may leap to mind, but caution is required. For the class of potentials under discussion, the point spectrum can be nonempty, and indeed dense in $\mathbb{R}^+$. Bound states evolve without dispersion, so no Strichartz estimates can hold for arbitrary initial data. The same difficulty arises for e^{itH}.

A second difficulty is the distinction between short and long range forces. A scattered particle cannot be expected to behave asymptotically like a free particle, if the forces acting on it are of sufficiently long range, as is the case for a slowly decaying potential V, even one of "symbol type" whose derivatives decay faster than V itself. This effect is already seen in the phase correction in our WKB asymptotics: $u_\lambda(x) \sim \exp(i\lambda x - i(2\lambda)^{-1} \int_0^x V)$. The correction term indicates heuristically that particles with energy λ^2 propagate with velocities slightly different from $\pm\lambda$. A nonintegrable discrepancy in velocities may well imply an unbounded discrepancy in positions.

In this section I describe some results in this direction, representing the culmination of our work to date on one-dimensional Schrödinger operators. Here is a special case of the main result. Denote by H_0 the free Hamiltonian $-d^2/dx^2$, and let $H = H_V = H_0 + V$, both operating on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition. Denote by $\mathcal{H}_{\mathrm{ac}}$ the maximal subspace of $\mathcal{H}$ on which H has purely absolutely continuous spectrum.

Theorem 11.1. [21] *Let $H = H_0 + V$ on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition at the origin. Suppose that $V \in L^p + L^1$ for some $1 < p < 2$. Suppose further that*

$$\lim_{x\to+\infty} \int_0^x V(y)\, dy \qquad \textit{exists.} \tag{11.1}$$

Then for each $f \in \mathcal{H}_{ac}$ there exist $g_\pm \in \mathcal{H}$ such that

$$\| e^{itH} f - e^{itH_0} g_\pm \|_{L^2} \to 0 \qquad \textit{as } t \to \pm\infty. \tag{11.2}$$

The mappings $f \mapsto g_\pm$ thus defined are isometric bijections from $\mathcal{H}_{ac}$ to $\mathcal{H} = L^2(\mathbb{R}^+)$.

V is not assumed to be nonnegative, so the supplementary hypothesis on existence of $\int_0^\infty V$ is not a restriction on the size of V. Examples of such potentials are those of the form $V(x) = \sum_{\nu\in\mathbb{Z}} a_\nu \varphi(x-\nu)$, where $\varphi \in C^0$ has compact support, $\int \varphi = 0$, and $a \in \ell^p$. Heuristically, a hypothesis with this flavor is needed for particles to have any chance of being asymptotically free. As is well known in this subject, conclusions like this often hold without such supplementary hypotheses, provided that the free asymptotics are appropriately modified; such modifications will be discussed below.

11.2. **Wave operators.**

Definition. The *wave operators* $\Omega_\pm$ associated to a perturbed Hamiltonian[20] $H = H_0 + V$ are

$$\Omega_\pm f = \lim_{t\to\mp\infty} e^{itH}e^{-itH_0}f, \tag{11.3}$$

where the limit is taken in the strong operator topology, provided it exists.

If these exist, then they are necessarily unitary operators onto closed subspaces of the underlying Hilbert space $\mathcal{H}$. $\Omega_\pm$ are simply the maps $g_\pm \mapsto f$ in (11.2).

$$\begin{aligned} e^{isH}\Omega_\pm f = e^{isH} \lim_{t\to\mp\infty} e^{itH}e^{-itH_0}f = \lim_{t\to\mp\infty} e^{i(t+s)H}e^{-itH_0}f \\ = \lim_{t\to\mp\infty} e^{i(t)H}e^{-i(t-s)H_0}f = \Omega_\pm e^{isH_0}f, \end{aligned}$$

so $\Omega_\pm$ formally intertwine H on their ranges with H_0.

Definition. Suppose that the wave operators $\Omega_\pm$ both exist, and that both have range equal to $\mathcal{H}$. Then $S = (\Omega_+)^{-1} \circ \Omega_-$ is called the *scattering operator.*

The physical interpretation of the definition and its hypothesis on the ranges of $\Omega_\pm$ are as follows: Firstly, the hypothesis of surjectivity of $\Omega_\pm$ means that the perturbed system has a full family of asymptotically free trajectories, as $t \to \pm\infty$. Secondly, any incoming particle that is asymptotically free at $t = -\infty$ will necessarily be asymptotically free at $t = +\infty$; it cannot be trapped. The scattering operator S describes the transition from pre-interaction to post-interaction asymptotics.

A well-known result, going back to the very first years of the rigorous scattering theory, says that if $V \in L^1$, then the wave operators exist and are asymptotically complete. Moreover, in this case the spectrum on the positive semi-axis is purely absolutely continuous, and there can be only discrete spectrum below zero, possibly accumulating at zero.

Theorem 11.1 may be restated in terms of wave operators (results of this type are more commonly formulated in that way in the literature), as follows.

Theorem 11.2. *Under the hypotheses of Theorem 11.1, for any $f \in L^2(\mathbb{R}^+)$, the two limits* (11.3) *exist in L^2 norm as $t \to \mp\infty$. Moreover, the wave operators $\Omega_\pm$ thus defined are bijective isometries from $\mathcal{H} = L^2(\mathbb{R}^+)$ to $\mathcal{H}_{ac}$.*

To go further, note that as a consequence of the theory developed earlier in these notes, we know that under the hypothesis (11.1), for almost every $\lambda \in \mathbb{R}$ there exists a unique pair $(u_\lambda, \omega(\lambda))$, where u_λ is a generalized eigenfunction with spectral parameter λ^2 satisfying the boundary condition $u_\lambda(0) = 0$, and $\omega(\lambda) \in \mathbb{R}$, with asymptotic behavior

$$u_\lambda(x) = \sin(\phi(x,\lambda)) + o(1) \qquad \text{as } x \to +\infty \tag{11.4}$$

where

$$\phi(x,\lambda) = \lambda x + \omega(\lambda) + (2\lambda)^{-1}\int_x^\infty V\ . \tag{11.5}$$

Theorem 11.3. [21] *Under the hypotheses of Theorem 11.1, the scattering operator $S = (\Omega_+)^{-1} \circ \Omega_-$ is the unitary "Fourier multiplier" operator on $L^2(\mathbb{R}^+)$ mapping $\sin(\lambda x)$ to $e^{2i\omega(\lambda)}\sin(\lambda x)$ for every $\lambda \in \mathbb{R}$.*

[20] Either on $\mathcal{H} = L^2(\mathbb{R})$, or on $\mathcal{H} = L^2(\mathbb{R}^+)$ with a suitable boundary condition at the origin.

11.3. **Modified wave operators.** We continue to discuss H_0+V in $L^2(\mathbb{R}^+)$, with Dirichlet boundary condition.

There has been much work extending the existence of wave operators to wider classes of potentials, with some additional conditions on derivatives, or for potentials of certain particular forms. Generally, for more slowly decaying potentials one needs to modify the free dynamics in the definition of wave operators in order for the limits to exist. The first work of this type was that of Dollard [28], who studied the Coulomb potential. Further developments used computation of asymptotic classical trajectories by means of a Hamilton-Jacobi equation to build an appropriate phase correction to the free dynamics, which was used to prove existence of modified wave operators (in any dimension). See for instance Buslaev and Matveev [11], Alsholm and Kato [3]; the strongest results are contained in Hörmander [34]. For example, existence of wave operators if $|V(x)| \le C(1+|x|)^{-1/2-\epsilon}$ and $|D^\alpha V(x)| \le C(1+|x|)^{-3/2-\epsilon}$ for any $|\alpha| = 1$; this result holds in all dimensions. More conditions on derivatives are required to compensate for slower decay.

Let

$$W(\lambda, t) = -(2\lambda)^{-1} \int_0^{2\lambda t} V(s)\, ds. \tag{11.6}$$

Define $e^{-iH_0 t \pm iW(H_0, \mp t)}$ to be the Fourier multiplier operator on $L^2(\mathbb{R}^+)$ that maps $\int_0^\infty F(\lambda) \sin(\lambda x)\, d\lambda$ to $\int_0^\infty e^{-i\lambda^2 t \pm iW(\lambda^2, \mp t)} F(\lambda) \sin(\lambda x)\, d\lambda$, for all $F \in L^2(\mathbb{R}^+, d\lambda)$. Define the modified wave operators

$$\Omega_\pm^m f = \lim_{t \to \mp\infty} e^{itH_V} e^{-itH_0 \pm iW(H_0, \mp t)} f \tag{11.7}$$

for all $f \in L^2(\mathbb{R}^+)$ for which the limit exists in L^2 norm.

Theorem 11.4. *Let V be a potential in the class $L^1 + L^p(\mathbb{R}^+)$ for some $1 < p < 2$, and let H_V be the associated Schrödinger operator on $L^2(\mathbb{R}^+)$ with any self-adjoint boundary condition at 0. Then for every $f \in L^2(\mathbb{R}^+)$, the limits in (11.7) exist in $L^2(\mathbb{R}^+)$ norm, as $t \to \mp\infty$. The modified wave operators $\Omega_\pm^m$ thus defined are both unitary bijections from $\mathcal{H} = L^2(\mathbb{R}^+)$ to $\mathcal{H}_{ac}(H_V)$.*

Our results extend to operators on the whole real line, rather than on a half line, and to Dirac-type operators as well. For statements of some of those extensions see below.

11.4. **Asymptotic completeness.**

Definition. The wave operators $\Omega_\pm$ will be called asymptotically complete[21] if the range of each of them coincides with $\mathcal{H}_{\text{continuous}} = \mathcal{H} \ominus \mathcal{H}_{\text{pp}}$, that is, with the orthogonal complement of the subspace spanned by all eigenvectors of the operator H_V.

An alternative equivalent characterization is that the range of each wave operator is equal to the absolutely continuous subspace $\mathcal{H}_{\text{ac}}(H_V)$ of H_V, and that the singular continuous spectrum $\sigma_{\text{sc}}(H_V)$ is empty.

The physical interpretation is that for an asymptotically complete system, all states are superpositions of bound states and scattering states, the latter being those states which are asymptotically free as $t \to \pm\infty$. Thus the long-time dynamics can be split into two distinct parts. One part consists of asymptotically free motion. The other is quite different, but relatively simple, consisting of bound motions.

[21] B. Simon has informed me that there is a lack of unanimity concerning the definition of asymptotic completeness; some authors require stronger properties such as discrete point spectrum which certainly do not hold, in general, for the class of potentials under discussion in these notes.

For the class of potentials which we are studying, the point spectrum of H_V can be dense. Therefore the bound part of the dynamics could potentially still be rather complicated, so asymptotic completeness in the sense of this definition would perhaps not be as satisfactory a conclusion as in more classical cases.

Recently Kiselev [40] has proved that there exist potentials which are $O(|x|^{-1+\varepsilon})$ for all $\varepsilon > 0$, for which H_V has nonempty singular continuous spectrum. Thus asymptotic completeness fails to hold, in general, for the class of potentials which we study. However, it does hold generically in various contexts. For instance:

Let V be given. We denote by H_V^β the associated self-adjoint operator on $L^2(\mathbb{R}^+)$ with boundary condition $u(0) + \beta u'(0) = 0$.

Theorem 11.5. *Assume that $V \in L^1 + L^p$ for some $1 < p < 2$. Then for almost every $\beta \in \mathbb{R}$, the modified wave operators defined in* (11.7) *are asymptotically complete. Moreover if $\int_0^\infty V(x)\,dx$ exists, then the usual wave operators, defined without the long-range phase correction, are asymptotically complete.*

Next, instead of varying the boundary condition, consider a family of random potentials

$$V_\omega(x) = \sum_{n=1}^{\infty} a_n(\omega) c_n f(x-n), \tag{11.8}$$

where the bump function $f(x)$ is C^∞ and supported in $(0,1)$, $\int f(x)\,dx = 0$, $c \in \ell^p$ for some $p < 2$, and $\{a_n(\omega)\}$ are independent, identically distributed, bounded random variables with expectation zero. For each ω, let H_{V_ω} be the associated Schrödinger operator on $L^2(\mathbb{R})$.

Theorem 11.6. *With probability one, the wave operators $\Omega_\pm$ associated to H_{V_ω} exist and are asymptotically complete.*

Without the condition $\int f = 0$ one still has a corresponding conclusion, but involving modified wave operators, since $\int_0^x V$ is almost surely unbounded.

11.5. **Strategy of the proof.** A Fourier analyst understands $e^{it\Delta}$, on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition, by analyzing the operator

$$\int_0^\infty f(\lambda)\sin(\lambda x)\,d\lambda \mapsto \int_0^\infty e^{it\lambda^2} f(\lambda)\sin(\lambda x)\,d\lambda.$$

We proceed in this way, with $\sin(\lambda x)$ replaced by $u_\lambda(x) = \sin(\lambda x + \omega(\lambda) + (2\lambda)^{-1}\int_x^\infty V) = o(1)$. For simplicity I will discuss the proofs only in the simplest case, that of $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition.

In outline, the argument goes as follows:

- *Complex spectral parameters.* The first step is to analyze solutions of $-u''+Vu = zu$, for complex z with $\operatorname{Im}(z) > 0$. There exists a single exponentially decaying solution (with no boundary condition imposed at $x = 0$), up to a scalar factor; to analyze it is quite easy so long as $\operatorname{Im}(z) \geq \delta > 0$. We analyze the solutions as $\operatorname{Im}(z) \to 0$, obtaining bounds and asymptotics which hold uniformly in that limit.
- *Concrete identification of the projection operator from $\mathcal{H}$ to $\mathcal{H}_{ac}$ and of the wave group.* For $\operatorname{Im}(z) > 0$, the resolvent operator $(H_V - z)^{-1}$ may be expressed in terms of these solutions. This leads to a concrete formula for the spectral resolution of H_V restricted to $\mathcal{H}_{\text{ac}}$, in terms of the generalized eigenfunctions u_λ, for $\lambda \in \mathbb{R}^+$. In particular, via functional calculus the wave group $\exp(itH_V)$, as well as the orthogonal projection operator from $\mathcal{H}$ to $\mathcal{H}_{\text{ac}}$, may thus be expressed in these

same terms. The operator from $L^2(\mathbb{R}, d\lambda)$ to $L^2([0,\infty))$ defined formally by

$$S(F)(x) = c_0 \int_0^\infty F(\lambda) u_\lambda(x)\, d\lambda, \tag{11.9}$$

with an appropriate constant $c_0 \in \mathbb{R}^+$, is an isometry onto $\mathcal{H}_{\mathrm{ac}}$. $S \circ S^*$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{\mathrm{ac}}$. Thus $\exp(itH)$ maps $\int F(\lambda) u_\lambda \, d\lambda$ to $\int F(\lambda) e^{it\lambda^2} u_\lambda \, d\lambda$.

- *Local decay.* For $f \in \mathcal{H}_{\mathrm{ac}}$, $e^{itH} f \to 0$ in L^2 norm on any compact subset of $[0,\infty)$, as $|t| \to \infty$. This step is routine and easy.
- *Elimination of nonlinear terms.* Decompose $e^{itH} f$ as the sum of two terms. In the main term,

$$\int_{\mathbb{R}} e^{it\lambda^2} F(\lambda) u_\lambda(x)\, d\lambda \text{ is replaced by } \int_{\mathbb{R}} e^{it\lambda^2} F(\lambda) \sin(\phi(x,\lambda))\, d\lambda. \tag{11.10}$$

The difference is shown to tend to zero in $L^2([R,\infty))$ as $R \to \infty$, uniformly in $t \in \mathbb{R}$, for a dense subspace of $\mathcal{H}_{\mathrm{ac}}$.

This is the most technically demanding step. The proof relies on multilinear expansion, but there is a new element: estimates in L^2 norm are required. This means that we need to control multilinear operators like those discussed earlier in these notes, but on $L^2 \otimes (L^p)^{n-1}$, where $1 \le p < 2$, rather than on $(L^p)^n$.
- *Digression: multilinear analysis on $L^2 \otimes (L^p)^{n-1}$.* The multilinear analysis can indeed be extended to the case where one function belongs to the endpoint space L^2, but this requires reworking all the proofs.
- *Simplification of the phase.* Another easy step replaces the phase $\lambda x + \omega(\lambda) + (2\lambda)^{-1}\int_x^\infty V$ by $\lambda x + \omega(\lambda)$ in the main term, in the special case where $\int_0^\infty V$ exists. Curiously, $p = 2$ seems also to be a natural threshold for this step (at least, for our proof).
- *Calculation of wave operators.* The final step, evaluation of $\Omega^\pm$ in terms of the function ω, is routine:

$$\Omega^\pm\Big(\int_0^\infty F(\lambda)\sin(\lambda x)\, d\lambda\Big) = \int_0^\infty F(\lambda) e^{\pm i\omega(\lambda)} u_\lambda(x)\, d\lambda. \tag{11.11}$$

11.6. **Complex spectral parameters.** Let $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$. Consider the generalized eigenfunction equation

$$-u'' + V(x)u = zu. \tag{11.12}$$

Fix a branch of $\sqrt{z}$ which has nonnegative imaginary part for all $z \in \mathbb{C}^+ \cup \mathbb{R}^+$. Define the phases

$$\xi(x,z) = \sqrt{z}x - (2\sqrt{z})^{-1}\int_0^x V. \tag{11.13}$$

Theorem 11.7. *Let $1 \le p < 2$ and assume that $V \in L^1 + L^p$. For each $z \in \mathbb{C}^+$ there exists a (unique) solution $u(x,z)$ of the generalized eigenfunction equation* (11.12) *satisfying*

$$\begin{cases} u(x,z) - e^{i\xi(x,z)} \to 0 \\ \dfrac{\partial u(x,z)}{\partial x} - i\sqrt{z}e^{i\xi(x,z)} \to 0 \end{cases} \quad \text{as } x \to +\infty. \tag{11.14}$$

$u(x,z)$ is continuous as a function on $\mathbb{C}^+ \times \mathbb{R}$, and is holomorphic with respect to z for each fixed x.

Likewise, there exists such a (unique) solution for almost every $z \in \mathbb{R}^+$. *For almost every* $E \in \mathbb{R}$, $u(x, z)$ *converges to* $u(x, E)$ *uniformly for all* x *in any interval bounded below, as* $\mathbb{C}^+ \ni \mathbb{C}^+ \ni z \to E$ *nontangentially.*

For $\operatorname{Im}(z) > 0$ this is well known and quite easy, under weaker hypotheses on V. The point here is the convergence as $z \to \mathbb{R}^+$, and in particular, global uniformity with respect to $x \in [0, \infty)$.

By rewriting (11.12) as a first-order system, performing a couple of algebraic transformations, reducing to an integral equation, and solving it by iteration, one arrives [17] at a formal series representation for solutions of (11.12):

$$\begin{pmatrix} u(x,z) \\ u'(x,z) \end{pmatrix} = \begin{pmatrix} e^{i\xi(x,z)} & e^{-i\xi(x,z)} \\ i\sqrt{z}e^{i\xi(x,z)} & -i\sqrt{z}e^{-i\xi(x,z)} \end{pmatrix} \cdot \begin{pmatrix} \sum_{n=0}^{\infty} T_{2n}(V,\dots,V)(x,z) \\ -\sum_{n=0}^{\infty} T_{2n+1}(V,\dots,V)(x,z) \end{pmatrix} \tag{11.15}$$

where

$$T_n(V_1,\dots,V_n)(x,z) = (2\sqrt{z})^{-n} \int_{x \le t_1 \le \cdots \le t_n} \prod_{j=1}^{n} e^{2i(-1)^{n-j}\xi(t_j,z)} V_j(t_j)\, dt_j, \tag{11.16}$$

with the convention $T_0(\cdot) \equiv 1$. We show that each multilinear expression T_n is well-defined for all $z \in \mathbb{C}^+$, that $T_n(\cdot)(x,z) \to 0$ as $x \to +\infty$ for all $n \ge 1$, that they have the natural limits as $z \to \mathbb{R}^+$ nontangentially, and that these expressions satisfy bounds sufficiently strong to enable us to sum the infinite series.

To establish these claims, substitute $\zeta = \sqrt{z}$ and write $\zeta = \lambda + i\varepsilon$, noting that $\varepsilon > 0$. Also write $\phi(x,\zeta) = \xi(x,z)$, $S_n(V_1,\dots)(x,\zeta) = T_n(V_1,\dots)(x,z)$. Let $\{E_j^m\}$ be a martingale structure on $\mathbb{R}^+$. Denote by $t_{m,j}^{\pm}$ respectively the right $(+)$ and left $(-)$ endpoints of the interval E_j^m.

Neglecting terms involving V, the real part of the exponent $2i\sum_{j=1}^{n}(-1)^{n-j}\xi(t_j,z)$ is

$$-2\operatorname{Im}(\sqrt{z}[(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots]) = -2\varepsilon \cdot [(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots],$$

which is nonnegative (for $x \ge 0$) for all $z \in \mathbb{C}^+$ since $t_1 \le t_2 \cdots$; the exponential factor decays rapidly as $[(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots] \to \infty$ so long as $\operatorname{Im}(z)$ is strictly positive. On the other hand, when $\operatorname{Im}(z) > 0$, the factors $\exp\big(2i(-1)^{n-j}\xi(t_j,z)\big)$ are not individually bounded functions of $t_j \in \mathbb{R}^+$. Thus when these expressions are manipulated, it is essential to keep such factors together in pairs.

Define

$$\begin{aligned} s_j^{m,-}(V,\zeta) &= \int_{E_j^m} e^{2i[\phi(t,\zeta) - \phi(t_{m,j}^-,\zeta)]} V(t)\, dt \\ s_j^{m,+}(V,\zeta) &= \int_{E_j^m} e^{2i[\phi(t_{m,j}^+,\zeta) - \phi(t,\zeta)]} V(t)\, dt. \end{aligned} \tag{11.17}$$

Because of the negativity properties of the real parts of the exponents, each of these integrals converges absolutely, for any $V \in L^1 + L^\infty$ and $\zeta \in \mathbb{C}^+$, and defines a holomorphic function of ζ. For each m, j and choice of $\pm$ sign, the map $V \mapsto S_j^{m,\pm}$ is a variant of the Fourier-Laplace transform. From these basic quantities we construct the sublinear functionals

$$(11.18)\qquad \begin{aligned} G_m(V)(\zeta) &= \Big(\sum_{j=1}^{2^m} |s_j^{m,-}(V,\zeta)|^2 + |s_j^{m,+}(V,\zeta)|^2 \Big)^{1/2} \\ G(V) &= \sum_{m=1}^{\infty} m G_m(V). \end{aligned}$$

When $j = 2^m$, and only then, the right endpoint of E_j^m is infinite. To simplify notation we make the convention that for $j = 2^m$, the second term $|s_j^{m,+}(V,\zeta)|^2$ is always to be omitted, in the definition of G and anywhere else that the quantities $\phi(t_{m,j}^+,\zeta)$ arise. In order to handle the general case where $V_1, \dots, V_n$ are not necessarily all equal to one another, we require a variant:

$$(11.19)\qquad G(\{V_i\})(\zeta) = \sum_{m=1}^{\infty} m \cdot \Big(\sum_{j=1}^{2^m} \sum_i^{*} |s_j^{m,+}(V_i,\zeta)|^2 + |s_j^{m,-}(V_i,\zeta)|^2 \Big)^{1/2},$$

where $\sum^{*}$ indicates that the sum is taken over a maximal set of indices i for which the functions V_i are all distinct. Thus if all V_i are equal to V, $G(\{V_i\}) \equiv G(V)$.

It is helpful to be able to regard G as a *linear* operator, or more precisely, as the norm of a certain vector-valued operator, in order that properties of holomorphic functions can be exploited below.

Definition. $\mathcal{B}$ denotes the Banach space consisting of all sequences

$$\{\mathbb{C}^2 \ni s_j^m : m \geq 0,\ 1 \leq j \leq 2^m\},$$

with the norm

$$\|s\|_{\mathcal{B}} = \sum_m m \Big(\sum_{j=1}^{2^m} |s_j^m|^2 \Big)^{1/2}.$$

$\mathfrak{G} : L^p \mapsto \mathcal{B}$ denotes the operator

$$(11.20)\qquad \mathfrak{G}(V)(\zeta) = \{ (s_j^{m,+}(V,\zeta), s_j^{m,-}(V,\zeta)) : 1 \leq m < \infty, 1 \leq j \leq 2^m \}.$$

Thus

$$G(V)(\zeta) = \|\mathfrak{G}(V)(\zeta)\|_{\mathcal{B}}.$$

Likewise $G_m(V) = \|\mathfrak{G}_m(V)\|_{\mathcal{B}_m}$ with the analogous definition of $\mathfrak{G}_m$, where $\mathcal{B}_m = \mathbb{C}^{2^m}$. $\mathfrak{G}(V)$ may be regarded as a $\mathcal{B}$–valued holomorphic function in any open set where it can be established that the infinite series defining $\|\mathfrak{G}(V)\|_{\mathcal{B}}$ converges uniformly.

The following three lemmas are variants of facts established earlier in these notes.

Write $p' = p/(p-1)$.

Lemma 11.8. *For any compact interval $\Lambda \Subset (0,\infty)$, there exists $C < \infty$ such that for any $1 \leq p \leq 2$, for all $t' \in \mathbb{R}$ and $f \in L^p(\mathbb{R})$, for every $\varepsilon \geq 0$,*

$$(11.21)\qquad \Big\| \int_{t \geq t'} e^{2i[\phi(t,\lambda+i\varepsilon) - \phi(t',\lambda+i\varepsilon)]} f(t)\,dt \Big\|_{L^{p'}(\Lambda,d\lambda)} \leq C\|f\|_{L^p}$$

$$(11.22)\qquad \Big\| \int_{t \leq t'} e^{2i[\phi(t',\lambda+i\varepsilon) - \phi(t,\lambda+i\varepsilon)]} f(t)\,dt \Big\|_{L^{p'}(\Lambda,d\lambda)} \leq C\|f\|_{L^p}.$$

This variant of the Parseval and Hausdorff-Young inequalities is proved in the main case $p = 2$ by the usual procedure of squaring, integrating with respect to λ, interchanging the order of integration, and integrating by parts with respect to λ. The stated relations between t, t' are essential since the exponents are not imaginary.

Lemma 11.9. *For all V, n and all $\zeta \in \mathbb{C}^+ \cup \mathbb{R}^+$,*

$$\sup_{x\in\mathbb{R}^+} |S_n(V, V, \dots, V)(x, \zeta)| \le \frac{C^{n+1}}{\sqrt{n!}} G(V)(\zeta)^n. \tag{11.23}$$

More generally, for all n and all $\{V_1, \dots, V_n\}$,

$$\sup_{x\in\mathbb{R}^+} |S_n(V_1, V_2, \dots, V_n)(x, \zeta)| \le \frac{C_k^{n+1}}{\sqrt{n!}} G(\{V_i\})(\zeta)^n, \tag{11.24}$$

provided that $\{V_i\}_{i=1}^n$ has cardinality $\le k$.

Sketch of proof. The new feature here is the introduction of the modifying factors $\exp(\pm 2i\phi(t^\pm_{m,j}, \zeta))$; without these, this is proved in [18]. The proof in [18] is an induction based on a repeated application of this decomposition; each step of that recursion involves a "cut point" $t^+_{m,j} = t^-_{m,j+1}$ playing the same role as $t^+_{1,1} = t^-_{1,2}$ in the above formula. At each step, the region of integration is decomposed into subregions as above, and corresponding to each subregion there is a splitting of the terms in the phase into two subsets. At any step which results in an odd number of terms appearing in one (hence both) subsets, we modify the resulting phases by introducing a factor $1 = \exp(\pm 2i[\phi(t^+_{m,j}, \zeta) - \phi(t^-_{m,j+1}, \zeta)])$, factoring it as a product of two exponentials, and splitting those exponential factors as above. This, together with the argument in [18], yields the assertion of the lemma for even n. A small modification is needed for odd n. □

Let an exponent $p < \infty$ be specified. Recall that a martingale structure is said to be adapted to f in L^p if $\int_{E_j^m} |f|^p = 2^{-m} \int |f|^p$ for all m, j. Denote by $\Gamma_{\alpha,\delta}(x)$, for $x \in \mathbb{R}$ and $\alpha, \delta > 0$, the cones $\{\zeta \in \mathbb{C}^+ : |\operatorname{Re}(\zeta) - x| < \alpha \operatorname{Im}(\zeta) \text{ and } \operatorname{Im}(\zeta) < \delta\}$. Denote by $N_{\alpha,\delta}$ the corresponding nontangential maximal functions, which map functions defined on $\mathbb{C}^+$ to functions defined on $\mathbb{R}$.

Corollary 11.10. *Let $\alpha < \infty$, let $1 \le p \le 2$, and let $\Lambda \Subset (0, \infty)$ be any compact subinterval. Then there exist $C < \infty, \delta > 0$ such that for any $f \in L^p(\mathbb{R})$, any martingale structure $\{E_j^m\}$ on $\mathbb{R}^+$, and for any $n \ge 1$,*

$$\|N_{\alpha,\delta} G_m(f)(\lambda)\|_{L^{p'}(\Lambda, d\lambda)} \le C\|f\|_{L^p}. \tag{11.25}$$

Moreover for each $1 \le p < 2$ there exists $\rho > 0$ such that for any $f \in L^p$ and for any martingale structure adapted to f in L^p,

$$\|N_{\alpha,\delta} G_n(f)(\lambda)\|_{L^{p'}(\Lambda, d\lambda)} \le C 2^{-\rho n}\|f\|_{L^p}. \tag{11.26}$$

Consequently under these additional hypotheses,

$$\|N_{\alpha,\delta} G(f)(\lambda)\|_{L^{p'}(\Lambda, d\lambda)} \le C\|f\|_{L^p}. \tag{11.27}$$

Moreover, for almost every $\lambda \in \Lambda$,

$$\|\mathfrak{G}(f)(\zeta) - \mathfrak{G}(f)(\lambda)\|_{\mathcal{B}} \to 0 \quad \textit{as } \zeta \to \lambda \textit{ nontangentially.} \tag{11.28}$$

This is proved by combining the preceding two lemmas with the manipulations of exponents and the triangle inequality introduced in §10, with a local version of the basic fact of H^p theory, the majorization of the $L^p(\mathbb{R})$ norm of the boundary values of a holomorphic function f by $\sup_{r>0} \|f(\cdot + ir)\|_p$.

The main point to be established concerning complex spectral parameters z is the convergence of generalized eigenfunctions as $z \to E \in \mathbb{R}$.

Lemma 11.11. *Let $1 \le p < 2$ and $V \in L^p(\mathbb{R})$. For almost every $E \in \mathbb{R}$, for every $n \ge 1$, $T_n(V, V, \dots, V)(x, z) \to T_n(V, V, \dots, V)(x, E)$ uniformly for all $x \ge 0$ as $\mathbb{C}^+ \ni z \to E$ nontangentially.*

By this last clause we mean that for any fixed $\alpha > 0$, the conclusion holds as $z \to E$ through the cone $\Gamma_{\alpha,\delta}(E)$.

Proof. Fix $\alpha, \delta > 0$ and let $N = N_{\alpha,\delta}$. Let $\varepsilon > 0$ be arbitrary, and fix a martingale structure $\{E_j^m\}$ adapted to V in L^p on $\mathbb{R}^+$. Decompose $V = W + (V - W)$ where $W(x) = V(x)\chi_{(0,R]}(x)$, with R chosen so that $\|V - W\|_{L^p} < \varepsilon$ and moreover so that $\|NG(V - W)\|_{L^{p'}(\Lambda)} < \varepsilon$. Such a choice is possible, since

$$\|NG_M(V\chi_{[R,\infty)})\|_{L^{p'}(\Lambda)} \le C \min(2^{-M\delta}\|V\|_{L^p}, \|V\chi_R\|_{L^p}).$$

Then

$$\begin{aligned}(11.29)\quad |S_n(V, V, \dots, V)(x, \zeta) - S_n(V, V, \dots, V)(x, \lambda)| \\ \le |S_n(W, W, \dots, W)(x, \zeta) - S_n(W, W, \dots, W)(x, \lambda)| \\ + |S_n(V, V, \dots, V)(x, \zeta) - S_n(W, W, \dots, W)(x, \zeta)| \\ + |S_n(V, V, \dots, V)(x, \lambda) - S_n(W, W, \dots, W)(x, \lambda)|.\end{aligned}$$

The first term on the right tends to zero, in the sense desired, since W has compact support. The second may be expanded as a telescoping sum and then majorized by

$$(11.30)\qquad \sum_{i=1}^{n} |S_n(V, \dots, V, V - W, W, \dots, W)(z, \zeta)|$$

where in the i-th summand, the argument of S_n has $i - 1$ copies of V and $n - i$ copies of W. Fix any aperture $\alpha \in \mathbb{R}^+$. Thus as established in the proof of Proposition 4.1 of [17],

$$\begin{aligned}(11.31)\quad \sup_{x\ge 0} \sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)} |S_n(V, V, \dots, V)(x, \zeta) - S_n(W, W, \dots, W)(x, \zeta)| \\ \le C_n \sum_{i=1}^{n} \sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)} G(V)^{i-1}(\zeta) G(W)^{n-i}(\zeta) G(V - W)(\zeta) \\ \le C_n \sum_{i=1}^{n} NG(V)^{i-1}(\lambda) NG(W)^{n-i}(\lambda) NG(V - W)(\lambda).\end{aligned}$$

From Chebyshev's inequality and the $L^p \mapsto L^{p'}$ bounds for $V \mapsto NG(V)$, one deduces that

$$\begin{aligned}\big|\{\lambda \in \Lambda : \sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)} \sup_{x\ge 0} |S_n(V, V, \dots, V)(x, \zeta) - S_n(W, W, \dots, W)(x, \zeta)| > \beta\}\big| \\ \le C_n \beta^{-p'/n} \|V\|_{L^p}^{p'(n-1)/n} \varepsilon^{p'/n}\end{aligned}$$

with an analogous bound for $\zeta = \lambda \in \mathbb{R}$, so

$$\Big|\{\lambda \in \Lambda : \limsup_{\Gamma_\alpha(\lambda)\ni\zeta\to\lambda} \sup_{x\geq 0} \big|S_n(V,V,\dots,V)(x,\zeta) - S_n(V,V,\dots,V)(x,\lambda)\big| > \beta\}\Big| \\ \leq C_n \beta^{-p'/n} \|V\|_{L^p}^{p'(n-1)/n} \varepsilon^{p'/n}, \tag{11.32}$$

for all $\beta, \varepsilon \in \mathbb{R}^+$. Letting $\varepsilon \to 0$, we conclude that the lim sup vanishes for almost every λ. □

The last substep is to sum over n. For individual n one has the convergence result of the preceding lemma. On the other hand, one has very strong uniform inequalities, with factors of $1/\sqrt{n!}$. Combining these in a straightforward way gives Theorem 11.7.

11.7. **Resolvents, projection onto $\mathcal{H}_{\text{ac}}$, and spectral resolution.** We continue to discuss the simplest case, H_V on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition. For each $z \in \mathbb{C}$, let $u_1(x,z), u_2(x,z)$ be the unique solutions of $-u'' + Vu = zu$ satisfying the boundary conditions $u_1(0,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $u_2(0,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. These are continuous functions of (x,z), and are entire holomorphic functions of z for each x.

It is easily shown that for $V \in L^1 + L^p$ for any $p < \infty$, for any $z \in \mathbb{C} \setminus \mathbb{R}$ there exists a unique complex number $m(z)$ such that the particular linear combination of solutions

$$f(x,z) = u_1(x,z)m(z) + u_2(x,z) \text{ belongs to } L^2(\mathbb{R}^+); \tag{11.33}$$

this defines f, as well as m. All other generalized eigenfunctions either are scalar multiples of f, or grow exponentially. m is called the Weyl m-function, and depends holomorphically on $z \in \mathbb{C}^+$. m has positive imaginary part in $\mathbb{C}^+$ (such a holomorphic function is called a Herglotz function).

To verify this positivity, consider the Wronskian $W[f,\overline{f}]$, where the Wronskian of two functions is defined to be $W[f,g] = f'g - fg'$. Then $W[f,\overline{f}](0) = 2i \operatorname{Im} m(z)$, $W[f,\overline{f}](\infty) = 0$, and $W'[f,\overline{f}](x) = -2i \operatorname{Im} z |f(x,z)|^2$. Hence

$$\operatorname{Im} m(z) = \operatorname{Im} z \int_0^\infty |f(x,z)|^2 \, dx.$$

By the usual variation of parameters formula which expressions a solution of an inhomogeneous second order ODE in terms of two linearly dependent solutions of the corresponding homogeneous equation, the resolvent $(H_V - z)^{-1}$ for $z \in \mathbb{C}^+$ is given by[22]

$$(H_V - z)^{-1} g(x) = u_1(x,z) \int_x^\infty f(y,z) g(y)\, dy + f(x,z) \int_0^x u_1(y,z) g(y)\, dy. \tag{11.34}$$

The m function (in fact, any Herglotz function which is not too large at infinity)[23] has a representation

$$m(z) = C_1 + C_2 z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t) \tag{11.35}$$

[22]For $\operatorname{Im}(z) > 0$, $H_V - z$ has trivial nullspace in L^2 with the given boundary condition; the bounded operator $(H_V - z)^{-1}$ is well-defined and can be expressed in terms of the homogeneous solutions in only one way.

[23]For Dirichlet boundary condition, as considered here, μ is the spectral measure corresponding to the generalized vector δ_0', defined by $\delta_0'(u) = u'(0)$ for any function u in the domain of H_V. The moment condition above corresponds to the fact that the derivative δ_0' belongs to the Sobolev-like space $H_{-2}(H_V)$ associated to H_V. See e.g. [6] for details on families of Sobolev-like spaces associated with any selfadjoint operator A.

for some positive Borel measure μ satisfying $\int(1+|t|^2)^{-1}d\mu(t)<\infty$, see e.g. [4]. In the m function context, μ is often called the spectral measure.

Since Im (m) equals the Poisson integral of μ, it follows by routine real analysis that Im $m(E+i\epsilon)$ converges weakly to $\pi\mu$ as $\epsilon\to 0^+$. Moreover, Im $m(E+i\epsilon)$ has limiting boundary values for Lebesgue-almost every E, and the density of the absolutely continuous part $\mu_{\rm ac}$ of μ satisfies

$$d\mu_{\rm ac}(E)=\pi^{-1}\,\mathrm{Im}\ m(E+i0)\,dE, \tag{11.36}$$

where $m(E+i0)=\lim_{\varepsilon\to 0^+} m(E+i\varepsilon)$.

Denote by $P_{(a,b)}$ the spectral projection associated to H_V and to the interval (a,b). By the spectral theory (see, e.g. [57], volume 1, Stone formula), for any compactly supported test functions g,h,

$$\begin{aligned}\langle P_{(a,b)}g,h\rangle &= (2\pi i)^{-1}\lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\langle (H_V-z)^{-1}g,h\rangle\,dz \\ &= \int_a^b\int_{\mathbb{R}}\int_{\mathbb{R}} u_1(x,E)u_1(y,E)g(x)\overline{h}(y)\,dx\,dy\,d\mu(E),\end{aligned} \tag{11.37}$$

where γ_ϵ is the rectangular contour consisting of the two horizontal segments $(a\pm i\epsilon, b\pm i\epsilon)$ together with two vertical segments joining $a+i\epsilon$ to $a-i\epsilon$ and $b+i\epsilon$ to $b-i\epsilon$. This follows by combining the various ingredients discussed above. Whereas the contribution of u_2 disappears, u_1 contributes to the final formula because $f=mu_1+u_2$ and m does not continue analytically across the real axis. Similar formulas can be found in [23, 77].

Because $P_{(a,b)}$ is an orthogonal projection, it follows immediately that for each $g\in L^2(\mathbb{R},d\mu)$,

$$U_0g(x)=\lim_{N\to\infty}\int_{-N}^{N}u_1(x,E)g(E)d\mu(E) \tag{11.38}$$

exists in $L^2(\mathbb{R}^+,dx)$ norm, and that the linear operator U_0 thus defined is a unitary bijection from $L^2(\mathbb{R},d\mu)$ to $L^2(\mathbb{R},dx)$ with inverse

$$U_0^{-1}g(E)=\lim_{N\to\infty}\int_{-N}^{N}u_1(x,E)g(x)dx, \tag{11.39}$$

where the limit again exists in L^2 norm.

Via functional calculus, this formula for the spectral resolution leads to expressions for other functions of H_V. To any interval (a,b) is associated a maximal closed subspace of $L^2(\mathbb{R}^+)$ on which H_V has purely absolutely continuous spectrum, contained in (a,b). The projection $P^{\rm ac}_{(a,b)}$ of $L^2(\mathbb{R}^+)$ onto this subspace is

$$P^{\rm ac}_{(a,b)}g(x)=\pi^{-1}\int_a^b u_1(x,E)\Big(\int_{\mathbb{R}}u_1(y,E)g(y)\,dy\Big)\,\mathrm{Im}\ m(E+i0)\,dE. \tag{11.40}$$

Consequently the operator U mapping continuous functions with compact support to $L^2(\mathbb{R},dx)$, defined by

$$Uh(x)=\pi^{-1}\int_{\mathbb{R}}u_1(x,E)h(E)\,\mathrm{Im}\ m(E+i0)\,dE \tag{11.41}$$

extends to an isometry of $L^2(\mathbb{R}, \mathrm{Im}\ m(E+i0)\,dE)$ onto the absolutely continuous subspace $\mathcal{H}_{\mathrm{ac}}$ associated to H_V. Likewise

$$e^{-iH_V t} g(x) = \pi^{-1} \int_{\mathbb{R}} e^{-iEt} u_1(x,E) \tilde{g}(E)\, \mathrm{Im}\ m(E+i0)\, dE, \tag{11.42}$$

where

$$\tilde{g}(E) = \pi^{-1} \int_{\mathbb{R}} u_1(y,E) g(y)\, dy.$$

For comparison, in the free case $V \equiv 0$, these formulas become $u_1(x,E) = \sqrt{E}^{-1} \sin \sqrt{E}x$, $m(z) = \sqrt{z}$, and

$$e^{-iH_0 t} g(x) = \pi^{-1} \int_{\mathbb{R}} e^{-iEt} \sin(\sqrt{E}x) \hat{g}(E)\, E^{-1/2}\, dE$$

where $\hat{g}(E) = \int \sin(\sqrt{E}x) g(x)\, dx$.

11.8. **Scattering coefficients and more concrete spectral resolution.** The formulas of the preceding section express quantities of interest in terms of the Weyl function m, about which we have as yet derived no useful information beyond very general facts. The next task is to reexpress these results in terms of the generalized eigenfunctions associated to almost every (nonnegative) real spectral parameter.

For almost every $E > 0$, define the scattering coefficient $\gamma(E) \in \mathbb{C}$ by

$$\gamma(E) = 1/u(0,E) \tag{11.43}$$

where $u(x,E)$ is the unique generalized eigenfunction asymptotic to $\exp(i\xi(x,E))$ as $x \to +\infty$, whose existence was established in Theorem 11.7. Since we have proved that $\sup_x \log^+(u(x,E+i\varepsilon))$ is locally integrable as a function of E, uniformly in $\varepsilon > 0$, $u(0,E)$ can vanish only for E in a set of Lebesgue measure zero, by a local analogue of the theory of Nevanlinna class functions. Thus $\gamma(E)$ is well-defined almost everywhere.

The following two basic relations hold for almost every $E > 0$:

$$u_1(x,E)m(E+i0) + u_2(x,E) = \frac{u(x,E)}{u(0,E)} = \gamma(E)e^{i\xi(x,E)}(1+o(1)) \text{ as } x \to \infty$$

$$u_1(x,E) = \frac{1}{2i\,\mathrm{Im}\ m(E+i0)} \left(\gamma(E)u(x,E) - \overline{\gamma}(E)\overline{u}(x,E)\right).$$

The first identity is proved by comparing the Wronskian of $u(\cdot,E)$ and $\overline{u(\cdot,E)}$ with the corresponding Wronskian associated to the left-hand side of the equation.

The following limiting absorption principle links the spectral resolution with the generalized eigenfunctions analyzed in §11.6. Let $f(x,z)$ denote the decaying generalized eigenfunction, for $\mathrm{Im}\,(z) > 0$, defined in (11.33).

Proposition 11.12. *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. Then for almost every $E \in \mathbb{R}^+$, the generalized eigenfunction $f(x,E+i0) = u_1(x,E)m(E+i0) + u_2(x,E)$ and scattering coefficient $\gamma(E)$ satisfy*

$$f(x,E+i0) = \gamma(E)e^{i\xi(x,E)}(1+o(1)), \tag{11.44}$$

$$|\gamma(E)|^2 = \mathrm{Im}\ m(E+i0)E^{-1/2} \tag{11.45}$$

(11.12) is just a restatement of the definitions of f, γ. (11.44) follows from (11.12) by computing Wronskians of the two sides of (11.12) with their respective complex conjugates.

Introduce

$$\psi(x,\lambda) = \lambda\overline{\gamma}(\lambda^2)u_1(x,\lambda^2) = (2i)^{-1}\Big(u(x,\lambda^2) - \frac{\overline{\gamma}(\lambda^2)}{\gamma(\lambda^2)}\cdot\overline{u}(x,\lambda^2)\Big), \tag{11.46}$$

where $u(x,E)$ continues to denote the generalized eigenfunction whose existence was established in Theorem 11.7, now for $E\in\mathbb{R}^+$. Then $\psi(0,\lambda)\equiv 0$, by the definition of $\gamma(\lambda^2)$. Then the results of this subsection may be summarized as follows.

Proposition 11.13. *Suppose that $V\in L^1+L^p(\mathbb{R}^+)$ for some $1<p<2$. Then for the associated selfadjoint Schrödinger operator H_V on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition, the spectral projection $P^{\mathrm{ac}}_{(a,b)}$ can be expressed as*

$$P^{\mathrm{ac}}_{(a^2,b^2)}g(x) = 2\pi^{-1}\int_{[a,b]\cap\mathbb{R}^+}\psi(x,\lambda)\tilde{g}(\lambda)\,d\lambda \tag{11.47}$$

for any $g\in L^2(\mathbb{R}^+)$, where the modified Fourier transform $\tilde{g}$ is defined by

$$\tilde{g}(\lambda) = \int_{\mathbb{R}}\overline{\psi}(x,\lambda)g(x)\,dx. \tag{11.48}$$

Similarly, the associated wave group is

$$e^{-itH_V}g(x) = 2\pi^{-1}\int_0^\infty e^{-i\lambda^2 t}\psi(x,\lambda)\tilde{g}(\lambda)\,d\lambda. \tag{11.49}$$

The operator

$$U_V f(x) = \sqrt{2/\pi}\int_0^\infty f(\lambda)\psi(x,\lambda)\,d\lambda \tag{11.50}$$

is a unitary surjection from $L^2(\mathbb{R}^+,d\lambda)$ onto $\mathcal{H}_{ac}$.

11.9. **Multilinear analysis, revisited.** So far, all we have done is to set the problem up in a form amenable to concrete analysis. In this subsection I outline the estimates on which that analysis will rely.

Consider multilinear expressions

$$M_n(f_1,\dots,f_n) = \iint_{x_1\le\cdots\le x_n}\prod_{i=1}^n f_i(x_i)dx_i \tag{11.51}$$

and their maximal variants

$$M_n^*(f_1,\dots,f_n) = \sup_y\Big|\iint_{x_1\le\cdots\le x_n\le y}\prod_{i=1}^n f_i(x_i)dx_i\Big|. \tag{11.52}$$

Recall that by a martingale structure $\{E_j^m\}$ on an interval $I\subset\mathbb{R}$ we mean a collection of subintervals $E_j^m\subset I$, indexed by $m\in\{0,1,2,\dots\}$ and $j\in\{1,2,\dots,2^m\}$, possessing the following three properties. (i) E_j^m lies to the left of E_{j+1}^m for all m,j. (ii) Except for endpoints, $\{E_j^m : 1\le j\le 2^m\}$ is a partition of I, for each m. (iii) $E_j^m = E_{2j-1}^{m+1}\cup E_{2j}^{m+1}$ for all m,j.

To any $f\in L^1$, any $\delta\in\mathbb{R}$, and any martingale structure, we associate

$$g_\delta(f) = \sum_{m=1}^\infty 2^{\delta m}\Big(\sum_{j=1}^{2^m}|\textstyle\int_{E_j^m}f|^2\Big)^{1/2}. \tag{11.53}$$

More generally, define

$$g_\delta(\{f_k\}) = \sum_{m=1}^{\infty} 2^{\delta m} (\sum_{j=1}^{2^m} \sup_k |\textstyle\int_{E_j^m} f_k|^2)^{1/2}. \tag{11.54}$$

Proposition 11.14. *There exists $C < \infty$ such that for any martingale structure $\{E_j^m\}$, any $\delta \geq 0$, any $f_1, \dots, f_n \in L^1(\mathbb{R})$, and any $n \geq 2$,*

$$|M_n(f_1, \dots, f_n)| \leq \frac{C^{n+1}}{\sqrt{n!}} g_{-\delta}(f_1) \cdot g_\delta(\{f_k : k \geq 2\})^{n-1}. \tag{11.55}$$

Moreover for any $\delta' > \delta \geq 0$, there exists $C < \infty$ such that for all $\{f_i\}$ and all $n \geq 2$,

$$|M_n^*(f_1, \dots, f_n)| \leq \frac{C^{n+1}}{\sqrt{n!}} g_{-\delta}(f_1) \cdot g_{\delta'}(\{f_k : k \geq 2\})^{n-1}. \tag{11.56}$$

This is proved by the method already explained above. To see the idea, consider the simplest case, $n = 2$. Proceeding as in (10.15), we seek to show that for any two functions f, g

$$|M_2(f, g)| \leq \sum_{m=1}^{\infty} \Big(\sum_j \Big| \int_{E_j^m} f \Big| \cdot \Big| \int_{E_{j+1}^m} g \Big| \Big),$$

where the sum is taken over all odd $1 \leq j < 2^m$. For we may then replace $\int_{E_j^m} f$ by $2^{-m\delta} \int_{E_j^m} f$ and $\int_{E_{j+1}^m} g$ by $2^{m\delta} \int_{E_{j+1}^m} g$ and apply Cauchy-Schwarz to conclude the proof. Now

$$|M_2(f, g)| \leq |M_2(f_1^1, g_1^1)| + \Big| \int_{E_1^1} f \Big| \cdot \Big| \int_{E_2^1} g \Big| + |M_2(f_2^1, g_2^1)|.$$

The middle term on the last line is one of those we're aiming for. For the first term, we can repeat the reasoning to majorize

$$|M_2(f_1^1, g_1^1)| \leq |M_2(f_1^2, g_1^2)| + \Big| \int_{E_1^2} f \Big| \cdot \Big| \int_{E_2^2} g \Big| + |M_2(f_2^2, g_2^2)|.$$

Repeating this N times we end up with all the desired terms with $m \leq N$, plus the sum over all $j \leq 2^m$ of $|M_2(f_j^N, g_j^N)|$. This remainder tends to zero as $N \to \infty$ under various circumstances, for instance, provided that $f, g \in L^\infty$ have bounded supports and that $\max_j |E_j^N| \to 0$ as $N \to \infty$. For then $|M_2(f_j^N, g_j^N)| \leq C|E_j^N|^2$, so

$$\sum_{j=1}^{2^N} |M_2(f_j^N, g_j^N)| \leq C \max_j |E_j^N| \sum_j |E_j^N| \leq C' \max_j |E_j^N|.$$

For larger n our proof [21] is not so simple (although I suspect that it may be more complicated than necessary), but the basic idea is that f_1, f_2 are eventually separated at some stage m, and one then multiplies f_1 by $2^{-m\delta}$ and f_2 by $2^{m\delta}$ and proceeds from then on as in the proof of Lemma 10.4.

Certain bounds for multilinear operators acting on $L^2 \times (L^p)^{n-1}$ for $1 \leq p < 2$ can be deduced from Proposition 11.14. Our scattering analysis relies on the Proposition itself, so we will not formulate those bounds here.

11.10. **Multilinear expansion meets long-time asymptotics.** We are now in a position to outline the crucial step in the time-dependent scattering analysis. Recall some of the actors introduced so far:

(1) The corrected phase is $\phi(x,\lambda) = \lambda x - (2\lambda)^{-1}\int_0^x V$.
(2) For almost every $\lambda \in \mathbb{R}^+$, $u(x,\lambda)$ denotes the unique generalized eigenfunction associated to the spectral parameter λ, satisfying $u(x,\lambda) = \exp(i\phi(x,\lambda)) + o(1)$ as $x \to +\infty$. (This is a change of notation; these functions were denoted by $u(x,E)$ above, where $E = \lambda^2$.)
(3) The scattering coefficient $\gamma(\lambda^2)$ satisfies $f(x,\lambda^2+i0) = \gamma(\lambda^2)e^{i\phi(x,\lambda)}(1+o(1)$ as $x \to +\infty$, where for $\mathrm{Im}\,(z) > 0$, $f(x,z)$ is the unique solution of the generalized eigenfunction satisfying $f(0,z) = 1$ which is in $L^2(\mathbb{R}^+)$.
(4) The generalized eigenfunctions $\psi(x,\lambda)$ are defined to be $(2i)^{-1}(u(x,\lambda) - e^{i\omega(\lambda)}\overline{u(x,\lambda)})$, where $\exp(i\omega(\lambda)) = \overline{\gamma}(\lambda^2)/\gamma(\lambda^2)$; these satisfy the boundary condition $\psi(0,\lambda) = 0$.
(5) The unitary bijection $U_V : L^2(\mathbb{R}^+, d\lambda) \mapsto \mathcal{H}_{\mathrm{ac}}$ is $U_V f(x) = \sqrt{2/\pi}\int_0^\infty f(\lambda)\psi(x,\lambda)\,d\lambda$.
(6) The wave group is $e^{-iH_V t}(U_V f)(x) = \sqrt{2/\pi}\int_0^\infty e^{-i\lambda^2 t} f(\lambda)\psi(x,\lambda)\,d\lambda$.

For the sake of simplicity I will assume that $V \in L^p$, rather than $L^1 + L^p$. We continue to assume that $1 \le p < 2$. Fix a martingale structure $\{E_j^m\}$ on $\mathbb{R}^+$ that is adapted to V, in the sense that $\int_{E_j^m} |V|^p = 2^{-m}\int_{\mathbb{R}^+} |V|^p$ for all $m \ge 0$ and all j. For any sufficiently small $\delta > 0$, recall the functional

$$g_\delta(f)(\lambda) = \sum_{m=1}^\infty 2^{m\delta}\Big(\sum_{j=1}^{2^m} \Big|\int_{E_j^m} e^{2i\phi(x,\lambda)} f(x)\,dx\Big|^2 + \Big|\int_{E_j^m} e^{-2i\phi(x,\lambda)} f(x)\,dx\Big|^2\Big)^{1/2}.$$

We have shown that for all sufficiently small δ_0, $g_{\delta_0}(V)(\lambda) < \infty$ for almost every $\lambda \in \mathbb{R}^+$. Fix some $\delta < \delta_0$.

Definition. A compact set $\Lambda \Subset (0,\infty)$ is a *set of uniformity* if $g_\delta(V)$ is a bounded function of $\lambda \in \Lambda$, and if $u(x,\lambda) - e^{i\phi(x,\lambda)} \to 0$ as $x \to +\infty$, uniformly for all $\lambda \in \Lambda$.

The first step here is to show that for any $f \in L^2(\mathbb{R}^+)$ and any $R < \infty$, $\|e^{-itH_V}U_V f\|_{L^2([0,R])} \to 0$ as $t \to \infty$. That this holds in the weak topology is a direct consequence of the Riemann-Lebesgue lemma, since $\langle e^{-itH_V}U_V f, h\rangle = \int_\Lambda e^{-i\lambda^2 t} f(\lambda) U_V^* h(\lambda)\,d\lambda$ for any test function h; $f(\lambda)\cdot U_V^* h(\lambda) \in L^2 \cdot L^2 \subset L^1(d\lambda)$.

L^2 norm convergence then follows from compactness, which in turn results from the generalized eigenfunction equation for $\psi(\cdot,\lambda)$ and Rellich's lemma.

Thus we can concentrate on $x \in [R,\infty)$, where $R = R(t) \to +\infty$ as $|t| \to \infty$. The second step, which is the key to the whole analysis, is to show that if $\psi(x,\lambda)$ is expanded in modified Taylor series in V, that is, if $u(x,\lambda)$ is expanded in our usual infinite series of multilinear operators with the WKB phase shift, then in the limit as $t \to \pm\infty$, all terms but the leading one may be dropped. It is reasonable to expect this, since (disregarding the phase shift) on the interval $[R,\infty)$, all such terms depend only on the restriction of V to $[R,\infty)$, which tends to zero in $L^1 + L^p$ norm as $R \to +\infty$.

Define

$$\psi_0(x,\lambda) = (2i)^{-1}\big(e^{i\phi(x,\lambda)} - e^{i\omega(\lambda)}e^{-i\phi(x,\lambda)}\big)$$
$$U_V^\dagger f(x) = (2/\pi)^{1/2}\int_0^\infty f(\lambda)\psi_0(x,\lambda)\,d\lambda,$$

obtained from ψ, U_V by dropping all but the leading terms in the multilinear expansion for the generalized eigenfunctions.

In the next lemma the parameter t does not appear, but in its application, R will depend on t.

Lemma 11.15. *For any set of uniformity Λ and any $R > 0$, there exists $C(R, \Lambda) < \infty$ such that for all $f \in L^\infty(\Lambda)$,*

$$\|(U_V - U_V^\dagger)f\|_{L^2([R,\infty))} \le C(R,\Lambda)\|f\|_{L^\infty}. \tag{11.57}$$

Moreover, $C(R, \Lambda) \to 0$ as $R \to \infty$.

Proof. By pairing $U_V f - U_V^\dagger f$ with an arbitrary test function $h \in L^2$ supported on $[R, \infty)$, this boils down to analyzing

$$\int \sum_{n=1}^\infty \int_R^\infty e^{i\phi(x,\lambda)} h(x) S_{2n}(V, V, \ldots, V)(x, \lambda)\, dx\, f(\lambda)\, d\lambda;$$

to arrive at this I have omitted a similar term involving $e^{i\omega(\lambda)}\bar{u}(x, \lambda^2)$ and a similar sum involving odd indices $2n + 1$. For any single n, the summand in the inner integral in the preceding formula has the form

$$\iint_{R \le t_0 \le t_1 \le \cdots \le t_{2n}} e^{i\phi(t_0,\lambda)} h(t_0)\, dt_0 \prod_{k=1}^{2n} e^{\pm_k 2i\phi(t_k,\lambda)} V(t_k)\, dt_k$$

where $\pm_k$ denotes a plus or minus sign depending on k in some particular manner. By Proposition 11.14,

$$\Big| \iint_{R \le t_0 \le t_1 \le \cdots \le t_{2n}} e^{i\phi(t_0,\lambda)} h(t_0)\, dt_0 \prod_{k=1}^{2n} e^{\pm_k 2i\phi(t_k,\lambda)} V(t_k)\, dt_k \Big| \le \frac{C^{n+1}}{\sqrt{2n!}} g_{-\delta}(h)(\lambda) g_\delta(V_R)^{2n}(\lambda) \tag{11.58}$$

where $V_R(x) = V(x)\chi_{[R,\infty)}(x)$. Now $\sup_R g_\delta(V_R)(\lambda)^2$ may be dominated by $C' g_{\delta'}(V)(\lambda)^2$ for any $\delta' > \delta$, by an argument in §9; to simplify notation we replace δ' again by δ. Summing over n, we find that

$$\begin{aligned} &\Big|\Big\langle \int_\Lambda \int_R^\infty f(\lambda) \left(u(x,\lambda) - e^{i\phi(x,\lambda)}\right) h(x)\, dx\, d\lambda \Big\rangle\Big| \\ &\qquad\qquad \le \int_\Lambda C g_{-\delta}(h)(\lambda) g_\delta(V_R)(\lambda) e^{C g_\delta(V)(\lambda)^2} |f(\lambda)|\, d\lambda. \end{aligned} \tag{11.59}$$

For any $\delta > 0$, $\|g_{-\delta}(h)\|_{L^2(\Lambda)} \le C_{\Lambda,\delta}\|h\|_{L^2} \le C_{\Lambda,\delta} < \infty$, for any compact $\Lambda \Subset (0, \infty)$, uniformly over all martingale structures, *not necessarily adapted to h*. The factor $\exp(C g_\delta(V)(\lambda)^2)$ is bounded uniformly for $\lambda \in \Lambda$, by definition of a set of uniformity. Likewise f is bounded, by assumption. Thus it suffices to show that $\|g_\delta(V_R)(\lambda)\|_{L^2(\Lambda, d\lambda)} \to 0$ as $R \to \infty$, for any arbitrary compact subset Λ of $(0, \infty)$. Now in the sum (11.53) defining $g_\delta(V)$, the ℓ^2 sum over j for fixed m is $\le C 2^{-\varepsilon m}\|V_R\|_{L^2}$ in $L^2(\Lambda)$ for some $\varepsilon > 0$, so it suffices to show that for each m, j, $\int_{E_j^m} e^{2i\phi(x,\lambda)} V_R(x)\, dx \to 0$ in $L^2(\Lambda)$. This holds by Lemma 11.8, since $V_R \to 0$ in $L^1 + L^p$ norm. $\square$

Corollary 11.16. *Let $\rho > 0$. For any $f \in L^2(\mathbb{R}^+, d\lambda)$ supported on $[\rho, \infty)$,*

$$\|e^{-itH_V} U_V f - \sqrt{2/\pi} \int_0^\infty e^{-i\lambda^2 t} f(\lambda) \psi_0(x, \lambda)\, d\lambda\|_{L^2(\mathbb{R}^+)} \to 0 \quad \textit{as } |t| \to \infty. \tag{11.60}$$

This follows from the last two results combined with various approximations. For instance, since U_V and e^{-itH_V} are unitary, we may approximate f by F where $F \in L^\infty$ is supported on some set of uniformity Λ and $\|f - F\|_{L^2(\mathbb{R}^+, d\lambda)}$ is small, without changing $e^{-itH_V} U_V f$ significantly in $L^2(\mathbb{R}^+, dx)$. □

The restriction on the support of f comes about because the factor of λ^{-1} in the definition of ϕ blows up as $\lambda \to 0$. This causes no difficulty in the final result; again, by unitarity, it suffices to prove existence of the limits defining the wave operators for all functions f belonging to some dense subspace of $L^2(\mathbb{R}^+, d\lambda)$; thus we may assume without loss of generality that f is supported in some $[\rho, \infty)$.

11.11. **One last conversion.** The final step is to convert

$$\int_0^\infty e^{-i\lambda^2 t + i\lambda x - i(2\lambda)^{-1}\int_0^x V} f(\lambda)\, d\lambda \quad \text{to} \quad \int_0^\infty e^{-i\lambda^2 t + i\lambda x - i(2\lambda)^{-1}\int_0^{2\lambda|t|} V} f(\lambda)\, d\lambda,$$

that is, to replace $\int_0^x V$ by $\int_0^{2\lambda|t|} V$, so that we have a Fourier multiplier operator; of course, this is to be valid only in the limit as $|t| \to \infty$. Physically, this expresses the principle that a wave packet with frequency λ propagates with velocity $\pm 2\lambda$. Mathematically, it is the principle of stationary phase — except that V is not assumed to have any regularity at all, and of course may decay rather slowly, so that this standard approximation must be justified.

The justification is elementary but tedious, and the reader will not be subjected to the details. See [21]. It is curious that, at least for our analysis, the condition $V \in L^1 + L^2$ again seems to arise naturally; the argument breaks down for L^p with $p > 2$.

11.12. **Results for $L^2(\mathbb{R})$.** Consider $-d^2/dx^2 + V(x)$ as an essentially selfadjoint operator on $L^2(\mathbb{R})$. The time-independent asymptotic analysis used above for the half-line case works equally well as $x \to -\infty$. There are two main differences: whereas it was necessary in the half-line case to relate asymptotics at $+\infty$ to boundary conditions at $x = 0$, now asymptotics at $+\infty$ must be related to those at $-\infty$. And the absolutely continuous spectrum now has multiplicity two, complicating some of the formulae.

Let $z \in \mathbb{C}^+$. Solutions $u_1(x, z), u_2(x, z)$ are defined precisely as before, with the same initial conditions at $x = 0$; now they are considered as global solutions on $\mathbb{R}$ rather than merely on $[0, \infty)$. Introduce two solutions $f_\pm(x, z) = u_1(x, z)m_\pm(z) + u_2(x, z)$, so that $f_+ \in L^2(\mathbb{R}^+, dx)$ and $f_- \in L^2(\mathbb{R}^-, dx)$ for each $z \in \mathbb{C}^+$. In terms of these, the resolvent is given by
(11.61)

$$(H_V - z)^{-1} g(x) = \frac{-1}{W[f_+, f_-]}\Big(f_+(x, z)\int_{-\infty}^{x} f_-(y, z) g(y)\, dy + f_-(x, z)\int_{x}^{\infty} f_+(y, z) g(y)\, dy\Big).$$

Now the Wronskian of f_+, f_- is $W[f_+, f_-] = m_+ - m_-$.

In the general case, we introduce scattered waves, defining $\psi_\pm(x, \lambda)$ for almost every $\lambda \in \mathbb{R}^+$ to be the unique generalized eigenfunctions with the asymptotic behavior

$$\psi_+(x, \lambda) = \begin{cases} t_1(\lambda) e^{i\phi(x,\lambda)}(1 + o(1)), & x \to +\infty, \\ \left(e^{i\phi(x,\lambda)} + r_1(\lambda) e^{-i\phi(x,\lambda)}\right) + o(1), & x \to -\infty \end{cases} \tag{11.62}$$

$$\psi_-(x, \lambda) = \begin{cases} t_2(\lambda) e^{-i\phi(x,\lambda)}(1 + o(1)), & x \to -\infty, \\ \left(e^{-i\phi(x,\lambda)} + r_2(\lambda) e^{i\phi(x,\lambda)}\right) + o(1), & x \to +\infty, \end{cases} \tag{11.63}$$

where we recall that $\phi(x,\lambda)=\lambda x-(2\lambda)^{-1}\int_0^x V(s)\,ds$. The asymptotics in (11.62), (11.63) can be differentiated, in the sense that $d\psi_+(x,\lambda)/dx = i\lambda t_1(\lambda)\exp(i\xi(x,\lambda^2))+o(1)$ as $x\to+\infty$, with analogous formulae as $x\to-\infty$ and for ψ_-.

Such solutions $\psi_\pm$ do exist. First of all, we have explained how, for almost every λ, one constructs a solution which is asymptotic to $e^{i\phi(x,\lambda)}$ as $x\to+\infty$. For almost every λ, this solution is asymptotic to $a(\lambda)e^{i\phi(x,\lambda)}+b(\lambda)e^{-i\phi(x,\lambda)}$ as $x\to-\infty$, for some a,b. By a standard Wronskian argument, $|a(\lambda)|^2\equiv 1+|b(\lambda)|^2$ for compactly supported potentials, and a limiting argument extends this to general $V\in L^1+L^p$. Dividing through by $a(\lambda)$ then produces ψ_+.

As in the half-line case, there exist coefficients $\gamma_\pm(\lambda)$ such that

$$\psi_\pm(x,\lambda)=\gamma_\pm^{-1}(\lambda)(u_1(x,\lambda^2)m_\pm(\lambda^2+i0)+u_2(x,\lambda^2)) \tag{11.64}$$

for almost every λ. The following formulae relate the transmission and reflection coefficients t_i, r_i with one another, and with the coefficients $\gamma_\pm, m_\pm$:

$$|r_i|^2+|t_i|^2=1 \text{ for } i=1,2, \tag{11.65}$$

$$t_1=t_2, \tag{11.66}$$

$$r_2=-(t_1/\bar{t}_1)\,\bar{r}_1, \tag{11.67}$$

$$|\gamma_\pm|^2|t_1|^2\lambda=\pm\,\mathrm{Im}\ m_\pm(\lambda^2+i0). \tag{11.68}$$

The spectral projections associated to the absolutely continuous spectrum are
(11.69)

$$P^{\mathrm{ac}}_{(a^2,b^2)}g(y)=\frac{1}{2\pi}\int_a^b \chi_{\mathbb{R}^+}(\lambda)\Big(\psi_+(x,\lambda)\int_{\mathbb{R}}\overline{\psi}_+(y,\lambda)g(y)\,dy+\psi_-(x,\lambda)\int_{\mathbb{R}}\overline{\psi}_-(y,\lambda)g(y)\,dy\Big)\,d\lambda.$$

The wave group restricted to $\mathcal{H}_{\mathrm{ac}}$ is thus

$$e^{-iH_V t}g(x)=\frac{1}{2\pi}\int_0^\infty e^{-i\lambda^2 t}\Big(\psi_+(x,\lambda)\tilde{g}_+(\lambda)+\psi_-(x,\lambda)\tilde{g}_-(\lambda)\Big)\,d\lambda, \tag{11.70}$$

where the transforms $\tilde{g}_\pm$ are defined by

$$\tilde{g}_\pm(\lambda)=\int_{\mathbb{R}}\overline{\psi}_\pm(y,\lambda)g(y)dy. \tag{11.71}$$

Modified wave operators are defined by

$$\Omega^m_\pm f=\lim_{t\to\mp\infty}e^{itH_V}e^{-iW_a(-i\partial_x,t)}f, \tag{11.72}$$

where $\partial_x=d/dx$, and

$$W_a(\lambda,t)=\lambda^2+\frac{1}{2\lambda}\int_0^{2\lambda t}V(s)\,ds. \tag{11.73}$$

The modified free evolution operator can be written as

$$e^{-iW_a(-i\partial_x,t)}f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\lambda^2 t+i\lambda x-\frac{i}{2\lambda}\int_0^{2\lambda t}V(s)\,ds}\hat{f}(\lambda)\,d\lambda, \tag{11.74}$$

where $\hat{f}(\lambda) = \int \exp(-i\lambda x) f(x)\, dx$ is the Fourier transform of f. Writing $E = \lambda^2$, the perturbed evolution may be expressed as

$$e^{-iH_V t} f(x) = \frac{1}{2\pi} \int_0^\infty e^{-i\lambda^2 t} (\psi_+(x,\lambda)\tilde{f}_+(\lambda) + \psi_-(x,\lambda)\tilde{f}_-(\lambda))\, d\lambda. \tag{11.75}$$

Denote by U_V the operator $f \mapsto (f_+, f_-)$, where $f_\pm(\lambda) = \int_{\mathbb{R}} \overline{\psi}_\pm(x,\lambda) f(x)\, dx$.

Theorem 11.17. *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. Then for every $f \in L^2$, the limits in (11.72) exist in $L^2(\mathbb{R})$ norm as $t \to \mp\infty$. The modified wave operators $\Omega_\pm^m$ thus defined are surjective and unitary from $L^2(\mathbb{R})$ to $\mathcal{H}_{ac}$. One has*

$$\begin{aligned} \Omega_+^m &= U_V^{-1} U_0, \\ \Omega_-^m &= U_V^{-1} S(\lambda)^{-1} U_0, \\ S^m &= U_0{}^{-1} S(\lambda) U_0 \end{aligned} \tag{11.76}$$

where $S(\lambda)$ denotes multiplication by the scattering matrix

$$S(\lambda) = \begin{pmatrix} t_1(\lambda) & -\overline{r}_1(\lambda)\frac{t_1(\lambda)}{\overline{t}_1(\lambda)} \\ r_1(\lambda) & t_1(\lambda) \end{pmatrix}. \tag{11.77}$$

As in the half-line case, the wave operators could be made to appear more symmetric by modifying the definition of U_V. Let $A = \sqrt{S}$ be any matrix square root of the unitary operator S. Then in terms of $\tilde{U}_V = AU_V$, the wave operators become $\Omega_+^m = \tilde{U}_V{}^{-1} A^* U_0$, and $\Omega_-^m = \tilde{U}_V{}^{-1} A U_0$.

12. Slowly varying and power-decaying potentials

The Fourier transform has the following properties. (i) If $\partial_x^k f \in L^p$ for some $1 \le p \le 2$, then $\hat{f}$ is almost everywhere finite. (ii) If $\hat{f}$, $\hat{g}$ are both almost everywhere finite, then so is $\widehat{f+g}$.

We regard the mapping $V \mapsto u(x,\lambda)$, from the potential to the unique generalized eigenfunction with appropriate asymptotics at $+\infty$, as a nonlinear variant of the Fourier transform. Thus it is natural to ask whether basic properties of the ordinary Fourier transform are shared. The above two properties are of interest in idealized quantum mechanics; for instance, a potential could easily arise as the sum of contributions from different types of effects, so we would like to handle sums of potentials. This is potentially troublesome in a nonlinear situation, if different arguments are required for different classes of potentials.

Throughout this section, we assume the following conditions. Let $n \ge 0$ be a non-negative integer, and let $p \in [1,2)$ be an exponent. Let V be a measurable, real-valued function defined on the real line $\mathbb{R}$. We assume[24] that $V \to 0$ in L^1_{loc} at $\pm\infty$, that is, that $\int_{|y-x|\le 1} |V| \to 0$ as $x \to \pm\infty$. Suppose that V admits a decomposition $V = V_0 + V_n$ where[25] $V_0 \in L^p + L^1$, V_n is continuous and tends to zero, and $d^n V_n/dx^n \in L^p + L^1$, in the sense of distributions. Note that under these hypotheses, V can tend to zero arbitrarily slowly in L^1_{loc}. We continue to write $H = H_0 + V = -\partial_x^2 + V$.

A classical theorem of Weidmann [79] asserts that if $V = V_0 + V_1$ with V_0 and $dV_1/dx \in L^1(\mathbb{R})$, and if $V_1(x) \to 0$ as $|x| \to \infty$, then $\mathbb{R}^+$ is an essential support of the absolutely

[24]For the generalization to the case where V need not tend in any sense to zero, but is merely uniformly in L^1_{loc}, see [19].

[25]This includes any potential decomposable as $\sum_{k=0}^n V_k$ where $d^k V_k/dx^k \in (L^p + L^1)(\mathbb{R})$ for each $k \ge 0$, and where $\sum_{k=1}^n V_k \to 0$ in L^1_{loc}.

continuous spectrum (moreover, at positive energies, $H_0 + V$ is unitarily equivalent to H_0). For higher derivatives, L^1 results were obtained by Behncke [5] and Stolz [73]. We extend this to L^p, $p < 2$, with a (necessarily) weaker form of the conclusion.

Theorem 12.1. [19] *Under the above hypotheses, for almost every $\lambda \in \mathbb{R}$, each solution of the generalized eigenfunction equation $Hu = \lambda^2 u$ is a bounded function of $x \in \mathbb{R}$. An essential support for the absolutely continuous spectrum of H is $\mathbb{R}^+$.*

Moreover, suitably generalized WKB asymptotics are valid for almost every λ; there exists a solution satisfying $u(x, \lambda) = \exp(i\Psi(x, \lambda)) + o(1)$ as $x \to +\infty$, where Ψ (which depends in a much more complicated way on n, V) has bounded imaginary part and may in principle be computed in terms of V by a recipe described below.

A result of Molchanov, Novitskii and Vainberg [49], in the spirit of the work of Deift and Killip based on trace identities, asserts existence of absolutely continuous spectrum for potentials satisfying $d^n V/dx^n \in L^2$, under the supplementary hypothesis that $V \in L^{n+1}$.

For potentials with more rapidly decaying derivatives, our conclusions can be strengthened. Define $p' = p/(p-1)$.

Theorem 12.2. [19] *Suppose that $n \geq 0$, $1 \leq p \leq 2$, $0 < \gamma$, and $\gamma p' \leq 1$. Let V be a measurable, real-valued function defined on $\mathbb{R}$. Suppose that $V = V_0 + V_n$ where V_n is bounded and continuous, and both $(1+|x|)^\gamma V_0$ and $(1+|x|)^\gamma d^n V_n/dx^n$ belong to $L^p + L^1$. Then every solution of $Hu = Eu$ is a bounded function of $x \in \mathbb{R}$, for all $E > 0$, except for a set of values of E having Hausdorff dimension $\leq 1 - \gamma p'$.*

For $n = 0$ with stronger power decay hypotheses $V(x) = O(|x|^{-r})$ for $r > 1/2$, this result is due to Remling [59].

Again, generalized WKB asymptotics hold on the complement of the lower-dimensional exceptional set. In the case $n = 0$, Remling and Kriecherbauer [44, 60] have constructed examples demonstrating that WKB asymptotics can indeed fail to hold on sets of the stated dimension. The question of behavior for the exceptional energies is of considerable interest, firstly because it determines to what extent these energies contribute to the spectrum, and in particular whether singular continuous spectrum can arise, and secondly because it is connected with asymptotic completeness for the associated time-dependent Schrödinger evolution; see §11 below.

To see how to control the Hausdorff dimension of the exceptional set, let us first see how to do so for the Fourier transform itself.

Observation 12.3. *If $1 \leq p \leq 2$, $\gamma > 0$, and $(1+|x|)^\gamma f(x) \in L^p(\mathbb{R})$ then*

$$\lim_{x \to +\infty} \int_0^x e^{-i\lambda y} f(y)\, dy \tag{12.1}$$

exists for all $\lambda \in \mathbb{R} \backslash S$, where S has Hausdorff dimension $\leq 1 - \gamma p'$.

For the proof, let B be the Banach space consisting of all doubly indexed sequences $\{a_{m,j}\}$ for which $\sum_{m \geq 0} m \big(\sum_j |a_{m,j}|^2 \big)^{1/2}$ is finite. Consider the linear operator mapping f to $\{\int_{\mathbb{R}} e^{-i\lambda y} f_j^m(y)\, dy\}$, a function $g(f)(\lambda)$ taking values in B. The hypothesis $|x|^\gamma f \in L^p$ implies that $\hat{f}$ belongs to the Sobolev space of functions possessing γ derivatives in $L^{p'}$, and as is well known, a simple potential-theoretic argument shows that such a Sobolev function is well defined outside a set of the desired dimension. The same reasoning, coupled with the analysis outlined in earlier sections of these notes, shows that $g(f)$ is (on compact subsets of $\mathbb{R} \backslash \{0\}$) a B-valued function in this same Sobolev space. The potential-theoretic argument then applies as before.

This analysis can be adapted to the "Taylor series" representation of generalized eigenfunctions, by following the arguments outlined for the case $\gamma = 0$ in preceding sections of these notes.

The principal change needed to adapt our machinery to the slowly varying case is a substantially modified WKB approximation. To analyze the Fourier transform of a function possessing some smoothness, one typically integrates by parts; in our formalism, this integration by parts is implicitly incorporated when the modified WKB approximation is inserted into the analysis of the first-order system $y' = \begin{pmatrix} 0 & 1 \\ V - \lambda^2 & 0 \end{pmatrix} y$.

To begin, we decompose[26] $V = W + \tilde{V}$ via a partition of unity on the Fourier transform side; W is the low-frequency part of V in the sense that $\hat{V}(\xi) \equiv \hat{W}(\xi)$ in a neighborhood of $\xi = 0$, and $\hat{W}$ has compact support.

In step 2, we seek an approximation $\exp(i\Psi(x,\lambda))$ to a generalized eigenfunction $u(x,\lambda)$. Replacing V by W and Ψ' by an unknown Φ, the equation $(-\partial_x^2 + W - \lambda^2)\exp(i\int\Phi) \approx 0$ becomes the eikonal equation

$$\Phi^2 - i\Phi' + W - \lambda^2 \approx 0 \; . \tag{12.2}$$

We solve the recursion

$$\Phi_{k+1} = \sqrt{\lambda^2 - W + i\Phi_k'} \tag{12.3}$$

by induction on k, with $\Phi_0 \equiv \lambda$. Derivatives of W up to order $k-1$ appear in Φ_k; this is why we are led to decompose $V = W + \tilde{V}$ with $W \in C^\infty$, and to omit the nonsmooth part, $\tilde{V}$, in this WKB part of the analysis. Since $W \to 0$ as $|x| \to \infty$, together with all its derivatives, there is no difficulty in carrying out this recursion for all sufficiently large x.

The error

$$E_k = \Phi_k^2 - i\Phi_k' + W - \lambda^2 \tag{12.4}$$

satisfies the useful recursion

$$E_{k+1} = i\frac{d}{dx}\frac{E_k}{\Phi_k + \sqrt{\Phi_k^2 - E_k}} \; , \tag{12.5}$$

so that needed properties of Φ_k, E_k can be deduced by induction. Set $\Phi = \Phi_n$ where n is the index in the hypothesis of the theorem, and set

$$\Psi(x,\lambda) = \int_0^x \Big(\Phi_n - \frac{\tilde{V} - E_n}{2\,\mathrm{Re}\,\Phi_n}\Big)(y,\lambda)\,dy \; . \tag{12.6}$$

Finally, set

$$\mathcal{E}(x,\lambda) = -E_n - \tilde{V}. \tag{12.7}$$

The recursions for Φ_k, E_k, along with standard Sobolev embedding estimates, can be used to show that $\mathcal{E}(x,\lambda) \in L^1 + L^p(\mathbb{R}, dx)$, and the same holds for all its derivatives with respect to λ.

[26] Observe that for the WKB approximation $\phi(x,\lambda) = \lambda x - (2\lambda)^{-1}\int_0^x V$, replacing V by W makes no essential difference since $\int_0^x \tilde{V} \to 0$ as $x \to \infty$.

In step 3, to solve the first-order system $y' = \begin{pmatrix} 0 & 1 \\ V - \lambda^2 & 0 \end{pmatrix} y$, we set[27]

$$y = \begin{pmatrix} e^{i\Psi} & e^{-i\bar{\Psi}} \\ i\Phi e^{i\Psi} & -i\bar{\Phi}e^{-i\bar{\Psi}} \end{pmatrix} z \; ; \tag{12.8}$$

Ψ is not in general real-valued. Under our hypotheses, it can be shown to have bounded real part, which need not tend to a limit as $x \to +\infty$ and hence is not negligible in the asymptotics. The upshot of all these algebraic manipulations is a simplified first-order evolution:

$$z' = \begin{pmatrix} 0 & \frac{i\bar{\mathcal{E}}}{2\,\mathrm{Re}\,\Phi'} e^{-i\psi} \\ \frac{-i\mathcal{E}}{2\,\mathrm{Re}\,\Phi'} e^{i\psi} & 0 \end{pmatrix} z \; . \tag{12.9}$$

where

$$\psi = 2\,\mathrm{Re}\,\Psi. \tag{12.10}$$

This is like the system in (8.3), with the potential replaced by $-i\mathcal{E}/2\,\mathrm{Re}\,\Phi'$. The denominator Re Φ' turns out to be relatively harmless; the main new complication is that the "effective potential" $\mathcal{E}/$ Re Φ depends strongly, though smoothly, on λ. The method applies, after relatively minor modifications.

Here is a typical result concerning energy-dependent potentials.[28] Its proof, rather than the result itself, is what is required to complete the proof of Theorem 12.1.

Theorem 12.4. [17] *Let J be a compact subinterval of $\mathbb{R}\backslash\{0\}$. Suppose that $p < 2$, that $W(x,\lambda)$ is real-valued, and that*

$$\partial^j W(x,\lambda)/\partial\lambda^j \in L^p(\mathbb{R})$$

uniformly in $\lambda \in J$ for $j = 0, 1$. Suppose further that the derivatives $\partial^j W(x,\lambda)/\partial\lambda^j \to 0$ as $|x| \to \infty$, uniformly in $\lambda \in J$, for $j = 2, 3$. Then for almost every $\lambda \in J$, there exist linearly independent, bounded solutions $u_\pm(x,\lambda)$ of

$$-u'' + W(x,\lambda)u = \lambda^2 u$$

with WKB asymptotic behavior as $x \to +\infty$.

The number of derivatives hypothesized here may not be optimal.

The main idea in the proof is quite standard. To estimate for example $\int_{\mathbb{R}} e^{-i\lambda x} V(x,\lambda)\,dx$ for λ in some compact interval, consider more generally $g(\lambda,\rho) = \int_{\mathbb{R}} e^{-i\lambda x} V(x,\rho)\,dx$. If $\partial^k V/\partial\lambda^k \in L^p$ for $k = 0, 1$, for some $1 \le p \le 2$, then $\partial^k g/\partial\rho^k \in L^{p'}(d\lambda)$, uniformly in ρ in an interval. The Sobolev embedding theorem then controls the restriction of g to $\rho = \lambda$.

13. Perturbations of Stark operators

A single electron in a uniform external electrical field (independent of space and time) is modeled quantum mechanically by the Stark Hamiltonian $H(u) = -u'' - xu$, the factor x representing the electrical potential. We consider perturbations

$$H_q(u) = -u'' - xu + qu, \tag{13.1}$$

[27]The presence of Φ in the second row of the coefficient matrix, where one might expect to see instead Ψ', is not a typo.

[28]One could try to eliminate the WKB phase correction factor, $\exp(-i(2\lambda)^{-1}\int_0^x V)$ in the case $n = 0$, by incorporating it into the potential as well, but that would not work because its derivative with respect to λ is in general unbounded.

where q represents some perturbing electrical potential. Physical intuition suggests, and earlier results in the literature confirm, that weaker hypotheses on q suffice to guarantee the presence of absolutely continuous spectrum than are needed without the background field; the force exerted by the field tends to push everything off to infinity, making it more difficult for bound states to exist. The following theorems refine various earlier results, which required faster decay or more smoothness of the perturbation.

For convenience we assume always that q is uniformly in L^1_{loc} as $x \to -\infty$; much weaker hypotheses would suffice there because the external potential $-x$ is so large.

Theorem 13.1. [20] *Consider a Stark operator H_q on $\mathbb{R}^1$. Assume that the potential $q(x)$ admits a decomposition $q = q_1 + q_2$, where both $q_1(x^2)$ and $x^{-1}q_2'(x^2)$ belong to $(L^1 + L^p)(\mathbb{R}, dx)$ for some $1 < p < 2$. Assume further that there exists $\zeta < 1$ such that $|q_2(x)| \le \zeta|x|$ for sufficiently large $|x|$. Then for almost every energy $E \in \mathbb{R}$ there exists a generalized eigenfunction $u_+(x, E)$ satisfying $H_q u_+ = Eu_+$, with asymptotic behavior*

$$u_+(x, E) = (x - q_2(x) + E)^{-1/4} e^{i\phi(x,E)} (1 + o(1)) \tag{13.2}$$

as $x \to +\infty$, where

$$\phi(x, E) = \int_0^x \Big[\sqrt{t - q_2(t) + E} - \frac{q_1(t)}{2\sqrt{x - q_2(t) + E}}\Big]\, dt$$

An essential support for the absolutely continuous spectrum of H_q is the entire line $\mathbb{R}$.

Corollary 13.2. *If q is Hölder continuous of order $\alpha > 1/2$, or if $q(x) = O(|x|^{-\delta})$ for some $\delta > 1/4$, then $\mathbb{R}$ is an essential support for the absolutely continuous spectrum of H_q. For almost every $E \in \mathbb{R}$, all generalized eigenfunctions satisfy $u(x) = O(|x|^{-1/4})$ and $u'(x) = O(|x|^{+1/4})$ as $x \to +\infty$.*

The corollary is deduced from the theorem by verifying that any function Hölder continuous of order $> 1/2$ can be decomposed as a sum of two functions satisfying the hypotheses of Theorem 13.1. The endpoint case $p = 2$ of the theorem remains open, but otherwise the result is rather sharp:

Theorem 13.3. [20] *There exists a potential q which is $O(|x|^{-1/4})$ and is also Hölder continuous of order $1/2$, for which the spectrum of H_q is purely singular.*

What is actually shown is that the spectrum is almost surely purely singular, for a certain family of random potentials satisfying both these restrictions. The analysis is closely based on a similar result of Kiselev, Last, and Simon [41].

The method of proof of Theorem 13.1 is in outline the same as that for perturbations of the vacuum. We convert to a first-order system, and diagonalize it modulo small errors. Then we reformulate as an integral equation and iterate to obtain an expansion of the generalized eigenfunctions in WKB phase-modified power series in q. After making the change of variables $x \mapsto \sqrt{x}$ for $x \gg 1$, we invoke the multilinear maximal operator machinery.

We will not give the relevant formulae in detail. A caricature for the "linear" term in the "Taylor" expansion for the generalized eigenfunctions is

$$\int_{x^{1/2}}^{\infty} e^{i\lambda s + is^3} q(s^2)\, ds; \tag{13.3}$$

the higher-order multilinear operators may be similarly caricatured. Numerous simplifications have been made to arrive here. From (13.3) one sees the relevance of the hypothesis $q(x^2) \in L^p(dx)$. This also indicates why hypotheses such as Hölder continuity, or

$x^{-1}q'(x^2) \in L^p$, are relevant: integration by parts allows one to exploit the term s^3 in the exponent, for large s, to substantial advantage.

Theorem 13.3 is a straightforward adaptation of the analysis by Kiselev, Last, and Simon of $-\partial_x^2 + V_\omega(x)$, where V_ω is defined by (2.9). For the Stark case, we modify the perturbing potentials, as follows. Fix $f \in C_0^\infty((0,1))$, not identically zero, and let $a_n(\omega)$ be independent, identically distributed random variables with uniform distribution in $[0, 2\pi]$. Define

$$q_\omega(x) = \sum_{n=1}^{\infty} n^{-1/2} f(\sqrt{x} - n) \sin(\tfrac{4}{3} x^{\frac{3}{2}} + a_n(\omega)) \ . \tag{13.4}$$

Then [20] for almost every ω, the spectrum of the corresponding perturbed Stark operator $-\partial_x^2 - x + q_\omega$ is purely singular on the whole real line.

14. Dirac operators

Another case that can be treated by our methods is that of certain Dirac-type operators, which arise in the inverse scattering method for the nonlinear Schrödinger equation. The unperturbed Hamiltonian is now

$$H_0 y = \begin{pmatrix} -i\partial_x & 0 \\ 0 & i\partial_x \end{pmatrix} y \tag{14.1}$$

where y takes values in the space $\mathbb{C}^2$ of column vectors, and $\partial_x = d/dx$. The perturbed Hamiltonian is

$$H = H_0 + \begin{pmatrix} 0 & V \\ \bar{V} & 0 \end{pmatrix} \tag{14.2}$$

where V is complex-valued. We assume that $V \in L^1 + L^p(\mathbb{R})$ for some $p < 2$.

The theory for this equation is closely parallel to that for $-\partial_x^2 + V$, the main difference being a simplification: no WKB phase shift term $(2\lambda)^{-1}\int_0^x V$ appears in the exponentials.

Let V be a real-valued potential. Consider the operator D_V, acting on $\mathbb{C}^2$–valued functions, defined by

$$D_V = \begin{pmatrix} V & -\partial_x \\ \partial_x & V \end{pmatrix}, \tag{14.3}$$

regarded as symmetric operators on $L^2(\mathbb{R}, \mathbb{C}^2)$.

If $V \in L^1 + L^p$ for some $1 < p < 2$ then there exist solutions $\eta_\pm(x,z)$ of $D_V\eta_\pm(x,z) = z\eta_\pm(x,z)$, for $z \in \mathbb{C}^+ \cup \mathbb{R}$, of the form

$$\begin{aligned} \eta_+(x,z) &= e^{izx}\left(\begin{pmatrix} 1 \\ i \end{pmatrix} + o(1)\right), \qquad x \to +\infty \\ \eta_-(x,z) &= e^{-izx}\left(\begin{pmatrix} 1 \\ -i \end{pmatrix} + o(1)\right), \quad x \to -\infty. \end{aligned} \tag{14.4}$$

This holds for all $z \in \mathbb{C}^+$, and for almost every $z \in \mathbb{R}$. It is proved by the method of Taylor expansion and multilinear operator estimation. Note that for this type of Dirac operator, the exponents are simply $\pm izx$; no phase corrections arise.

Some notation: If $f = (f_1, f_2)^T$ and $g = (g_1, g_2)^T$ are $\mathbb{C}^2$–valued functions (the superscripts denoting transposes, so that these are regarded as column vectors), let $W[f, g] =$

$f_2 g_1 - f_1 g_2$, and

$$\langle f, g\rangle = \int_{\mathbb{R}} (f_1(x) g_1(x) + f_2(x) g_2(x))\, dx.$$

Denote by $u_{1,2}(x,z)$ solutions satisfying $u_1(0,z) = (0,1)^T$, $u_2(0,z) = (1,0)^T$. For $z \in \mathbb{C}^+$ let $f_\pm(x,z) = u_1(x,z) m_\pm(z) + u_2(x,z)$ be global solutions of the generalized eigenfunction equation $D_V f_\pm = z f_\pm$ which belong to L^2 on $R^\pm$, respectively; this defines $m_\pm(z)$.

The spectral representation of the absolutely continuous part of the spectral measure is given by the map $U : L^2(\mathbb{C}^2, dx) \mapsto L^2(\mathbb{C}^2, M dE)$ sending $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ to $\begin{pmatrix} \tilde g_1 \\ \tilde g_2 \end{pmatrix}$ where

$$\tilde g_i(E) = \langle u_i(x,E), g\rangle$$

and

$$M(E) = \mathrm{Im} \begin{pmatrix} \frac{m_+ m_-}{m_- - m_+} & \frac{m_-}{m_- - m_+} \\ \frac{m_-}{m_- - m_+} & \frac{1}{m_- - m_+} \end{pmatrix} (E + i0).$$

As usual, $m(E+i0)$ means $\lim_{\varepsilon \to 0^+} m(E + i\varepsilon)$.

For almost every $E \in \mathbb{R}$ there exist solutions $\zeta_\pm(x,E)$ of $D_V \zeta_\pm = E \zeta_\pm$ with asymptotic behavior

$$\zeta_+(x,E) = \begin{cases} t_1(E) e^{iEx} \begin{pmatrix} 1 \\ i \end{pmatrix} + o(1), & x \to +\infty, \\ \left(e^{iEx} \begin{pmatrix} 1 \\ i \end{pmatrix} + r_1(E) e^{-iEx} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right) + o(1), & x \to -\infty \end{cases} \tag{14.5}$$

$$\zeta_-(x,E) = \begin{cases} t_2(E) e^{-iEx} \begin{pmatrix} 1 \\ -i \end{pmatrix} + o(1), & x \to -\infty, \\ \left(e^{-iEx} \begin{pmatrix} 1 \\ -i \end{pmatrix} + r_2(E) e^{iEx} \begin{pmatrix} 1 \\ i \end{pmatrix} \right) + o(1), & x \to +\infty \end{cases} \tag{14.6}$$

for certain scattering coefficients $t_j(E), r_j(E)$; these relations define those coefficients as well as specifying $\zeta_\pm$. This is just a rephrasing of (14.4).

A computation parallel to the whole axis Schrödinger operator case allows us to rewrite the dynamics as

$$e^{-iD_V t} g(x) = \frac{1}{4} \int_{\mathbb{R}} e^{-iEt} \left(\zeta_+(x,E) \langle \overline{\zeta}_+, g\rangle + \zeta_-(x,E) \langle \overline{\zeta}_-, g\rangle \right). \tag{14.7}$$

The final result is obtained from the same method outlined above.

Theorem 14.1. *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. Then for each $f \in L^2(\mathbb{R})$, both of the limits*

$$\Omega_\pm f = \lim_{t \to \mp\infty} e^{-itD_V} e^{itD_0} f \tag{14.8}$$

exist in $L^2(\mathbb{R})$ norm, and the wave operators thus defined are surjective and unitary operators to $\mathcal{H}_{ac}$. Moreover $\Omega_+ = U_V^{-1} U_0$ and $\Omega_- = U_V^{-1} S(E)^{-1} U_0$, where

$$S(E) = \begin{pmatrix} t_1(E) & -\overline{r}_1(E) \frac{t_1(E)}{\overline{t}_1(E)} \\ r_1(E) & t_1(E) \end{pmatrix}.$$

The scattering operator S is equal to $U_0^{-1} S(E) U_0$.

15. Three variations on a theme of Strichartz

In this section we briefly discuss three different ways in which estimates of Strichartz type are relevant to our subject matter. The first was alluded to earlier: the linear maximal function theory allows one to deduce one Strichartz estimate from another, an application first observed by Tao [76]. For the free Laplacian H_0, the following three inequalities are all valid for all $f \in L^2(\mathbb{R})$, $g \in L^{6/5}(\mathbb{R}^{1+1})$:

$$\|e^{itH_0} f(x)\|_{L^6_{x,t}} \le C\|f\|_{L^2_x} \tag{15.1}$$

$$\|\int_{-\infty}^{\infty} e^{i(t-t')H_0} g(t')\, dt'\|_{L^6_{x,t}} \le C\|g\|_{L^{6/5}_{x,t}} \tag{15.2}$$

$$\|\int_0^t e^{i(t-t')H_0} g(t')\, dt'\|_{L^6_{x,t}} \le C\|g\|_{L^{6/5}_{x,t}}. \tag{15.3}$$

In the latter two lines, $g(t')$ denotes a function of $x' \in \mathbb{R}$, and $e^{i(t-t')H_0} g(t')$ is what one gets by applying the indicated operator to that function, and evaluating at x. The first inequality implies the second, by dualizing and then composing an operator with its adjoint. The third is of interest, because the quantity whose norm appears on the left-hand side appears in Duhamel's formula.

It is in deducing (15.3) from (15.2) that Corollary 9.2 is useful. Regard functions of (x,t) as being functions of $t \in \mathbb{R}$, taking values in auxiliary Banach spaces $L^p(\mathbb{R}, dx)$. The left-hand side of (15.3), evaluated at t, is obtained by applying the operator $Tg(t) = \int_{\mathbb{R}} \exp(i(t-t')H_0) g(t')\, dt'$ to g times the characteristic function of $[0,t]$. (15.2) asserts that T is bounded from the space $L^p_t(X)$ of X-valued functions in L^p to $L^q_t(Y)$, where X, Y equal $L^{6/5}(\mathbb{R}), L^6(\mathbb{R})$, respectively, and $p = 6/5 < q = 6$. Thus Corollary 9.2, extended to Banach space-valued functions, says that (15.2) directly implies (15.3). This extension to Banach spaces follows from the same proof as in the scalar case.

A second way to bring Strichartz and Fourier restriction inequalities into the subject is to consider the following situation, which may have no physical relevance. Consider a one-parameter family of potentials

$$V_s(x) = W(x)\cos(sx^2) \tag{15.4}$$

where W is real-valued and fixed. Let $H_s = -\partial_x^2 + V_s$.

Theorem 15.1. *Suppose that $W \in L^p + L^1(\mathbb{R})$ for some $p < 4$. Then for almost every $s \in \mathbb{R}$, an essential support for the absolutely continuous spectrum of H_s is $\mathbb{R}^+$. For almost every pair (s, λ), all generalized eigenfunctions of H_s with spectral parameter λ^2 are bounded and have WKB asymptotic behavior.*

The basic point here is that the operator $f \mapsto \int_{\mathbb{R}} \exp(-i\lambda x + isx^2) f(x)\, dx$ maps L^p to L^q for all $p < 4$, with $q = q(p) > 4$. This can be generalized to incorporate the WKB phase correction. Otherwise the analysis is essentially the same as in the proof of Theorem 3.1. We have not established the presence of a negative power of $n!$ in the analogue of Theorem 10.2, but as explained in §10, these conclusions can be obtained without it.

One cannot expect to have the Strichartz estimate (15.1) with the free Laplacian replaced by $H = H_0 + V$ for general $V \in L^1 + L^p$, $1 < p < 2$, for two reasons. Firstly, as already pointed out, bound states can occur, indeed the point spectrum can be dense in $\mathbb{R}^+$, and they destroy any such dispersion inequality. Secondly, although one could ask for such an estimate only for all $f \in \mathcal{H}_{\mathrm{ac}}$, that is unlikely to hold. The problem is that our estimates are far from uniform in the spectral parameter λ, and are very weak; we know only that

$\log \sup_x |u(x,\lambda)|$ is locally integrable in λ. The following seems nearly the best that is likely to be true.

Problem 1. Suppose that $V \in L^1 + L^p$. Show that there exists a nonnegative function w, strictly positive almost everywhere, such that for any function f satisfying $\int_{\mathbb{R}} |f(\lambda)|^2 w(\lambda)\, d\lambda < \infty$, the function $g(y) = \int f(\lambda) u(y,\lambda)\, d\lambda$ satisfies $\exp(itH)g(x) \in L^6_{x,t}$.

Here $u(y,\lambda)$ denotes a generalized eigenfunction with WKB asymptotics at $y = +\infty$.

I believe that this is essentially an exercise which can be done by combining ingredients from our analysis of wave operators with the usual derivation of the $L^{4-\delta}$ restriction theorem in $\mathbb{R}^2$. However, I have not worked out the details.

16. Stability for nonlinear ODE perturbations

Recall the Prüfer variables reformulation (7.6) of the generalized eigenfunction equation $-u'' + Vu = \lambda^2 u$:

$$\begin{cases} \rho' = (2\lambda)^{-1} V \sin(2\theta) \\ \theta' = \lambda - (2\lambda)^{-1} V + (2\lambda)^{-1} V \cos(2\theta). \end{cases}$$

where $u = R\sin(\theta)$, $u' = \lambda R \cos(\theta)$, and $\rho = \ln(R)$. This system is nonlinear, but does enjoy a special property: the equation for θ does not involve ρ. Moreover, for our purposes, it is relatively simple to show that the equation for ρ has bounded solutions for almost every λ, once the solutions θ of (7.6) have been successfully analyzed. The recent construction by Kiselev [40] of potentials in $\cap_{p>1} L^p$ with nonempty singular continuous spectrum exploits this formulation, for instance.

Thus it is natural to ask to what extent our methods apply to nonlinear equations. In this section we outline a preliminary result in that direction.

Consider a *scalar* one-parameter family of ordinary differential equations of the form

$$\frac{d\theta}{dt} = \lambda + \mathcal{V}(t,\theta,\lambda), \tag{16.1}$$

where λ ranges over some bounded interval. We think of θ as parametrizing a motion in $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, so assume that $\mathcal{V}$ is periodic with respect to θ, with period 2π. Regard this as a perturbation of the simple family of motions $t \mapsto \lambda t_c$. If for some parameter λ_0 $\sup_{\theta\in\mathbb{T}} |\mathcal{V}(t,\theta,\lambda_0)| \in L^1(dt)$, then for each $c \in \mathbb{R}$ there exists a solution of $\dot\theta = \mathcal{V}(t,\theta,\lambda_0)$ of the form $\theta(t) = \lambda_0 t + c + o(1)$ as $t \to +\infty$, and every solution takes this asymptotic form. We wish to assert that families of ODE are generically stable under significantly strong perturbations, in the sense that if $\mathcal{V}$ decays more slowly, then a similar conclusion holds for Lebesgue–almost every parameter λ.

Assume that $\mathcal{V} = O(|t|^{-\gamma})$ for some γ, and moreover that

$$\frac{\partial^{\alpha+\beta}\mathcal{V}}{\partial\theta^\alpha \partial\lambda^\beta} = O(|t|^{-\gamma}) \quad \text{for all } \alpha, \beta.$$

No smoothness with respect to λ is assumed. Define

$$V_0(t,\lambda) = (2\pi)^{-1} \int_0^{2\pi} \mathcal{V}(t,\theta,\lambda)\, d\theta.$$

Theorem 16.1. *If $\gamma > \frac{1}{2}$ then there exists $\delta > 0$ such that for almost every λ, for each $c \in \mathbb{R}$ there exists a solution θ of* (16.1) *with asymptotic behavior*

$$\theta(t) = \lambda t + \textstyle\int_0^t V_0(s,\lambda)\, ds + c + O(t^{-\delta})$$

as $t \to +\infty$. Moreover, for almost every λ, any solution of (16.1) *has this form for some $c \in \mathbb{R}$.*

Presumably this holds under the weaker hypothesis that $\sup_{\theta,\lambda} \partial_{\theta,\lambda}^{\alpha\beta} \mathcal{V} \in L^1 + L^p(\mathbb{R}, dt)$ for some $1 \le p < 2$, but I have not yet carried out the more refined analysis required to prove that.[29]

There are generalizations to systems of arbitrary finite dimension. A key is to have sufficiently effective dependence on parameters. Let θ take values in $\mathbb{R}^n$, let $\mathcal{V}$ be $\mathbb{R}^n$–valued, and likewise let λ take values in a rectangle in $I \subset \mathbb{R}^n$. Then a corresponding conclusion holds for almost every $\lambda \in I$, with respect now to n-dimensional Lebesgue measure.[30]

The theorem is proved via the machinery of multilinear expansion and Fourier analysis of iterated Volterra-type integrals developed in these lectures. We may without loss of generality restrict to $t \ge 1$. Let $W = \mathcal{V} - V_0$, and expand

$$W(t, \beta, \lambda) = \sum_{k \neq 0} a_k(t, \lambda) e^{ik\beta};$$

thus $|a_k(t, \lambda)| \le C_N |k|^{-N} t^{-\gamma}$ for all N, and likewise for all its derivatives with respect to λ. Write

$$\theta = \theta_0 + \varphi$$
$$\theta_0 = \lambda t + \int_0^t V_0(s, \lambda)\, ds + c.$$

In terms of φ, our equation is

$$d\varphi/dt = \sum_{m=0}^{\infty} b_m(t, \lambda) \varphi^m$$

where

$$b_m(t, \lambda) = \sum_{k \neq 0} a_k e^{ik\theta_0} (ik)^m / m!.$$

For almost all λ, we seek a solution which is $O(t^{-\delta})$ as $t \to +\infty$.

The main step is to solve

$$d\psi/dt = \sum_{m=0}^{N} b_m(t, \lambda) \psi^m + R(t, \lambda) \tag{16.2}$$

where N is to be determined, and where the remainder R is bounded by $C_\lambda t^{-1-\delta}$ for some $\delta > 0$ and C_λ is finite for almost all λ, with $|\psi(t, \lambda)|$ likewise bounded by $C_\lambda t^{-\varepsilon}$, with ε independent of N. The parameter N is then chosen so that $\sum_{m=N+1}^{\infty} b_m \psi^m \in L^1(dt)$ for

[29] Under the power-decay hypothesis, it is necessary to iterate the multilinear expansion only to some suitably high order, whereas handling $L^1 + L^p$ would require summation of an infinite series.

[30] It appears that the proof works for one-parameter systems, as well, subject to a Diophantine condition: replace λt by $\lambda t \omega$ where λ is a scalar-valued parameter, and the vector $\omega \in \mathbb{R}^n$ satisfies $|k \cdot \omega| \ge c|k|^{-N}$ for all $k \in \mathbb{Z}^n \backslash \{0\}$, for some c, N. Then the same conclusion holds for almost every λ, with ω fixed. However, this is work in progress; I haven't yet completed verification of the details.

almost all λ. The equation for $\zeta = \varphi - \psi$ is then

$$\begin{aligned} d\zeta/dt &= R + \sum_{m=1}^{N} b_m((\psi + \zeta)^m - \psi^m) + \sum_{m=N+1}^{\infty} b_m(\psi+\zeta)^m \\ &= B_0 + B_1\zeta + O(t^{-\varepsilon}\zeta^2). \end{aligned}$$

Here $B_0 = O(t^{-1-\varepsilon})$ for almost every λ. This equation can be solved by the fixed-point method to obtain a solution which is $O(t^{-\varepsilon})$, by using the method of integrating factors to remove the term $B_1\zeta$, provided that $\int_0^t B_1(s,\lambda)\, ds$ remains bounded as $t \to +\infty$. This boundedness does hold, and turns out to be a variant of the fact that $\int_0^t e^{ik\theta_0(s,\lambda)} a_l(s,\lambda)\, ds$ is bounded (for almost all λ, for $k \neq 0$), which is a variant of the almost everywhere boundedness of $\int_0^t e^{i\lambda s} f(s)\, ds$ for $f = O(t^{-\gamma})$, $\gamma > \frac{1}{2}$. But until ψ is determined by (approximately) solving (16.2), B_1 is not even defined.

To solve the equation for ψ, rewrite it as an integral equation[31]

$$\psi(t,\lambda) = -\int_t^{\infty} b_0(s,\lambda)\, ds - \sum_{m=1}^{N} \int_t^{\infty} b_m(s,\lambda)\psi^m(s,\lambda)\, ds.$$

By repeatedly feeding this equation into itself, we obtain an expansion for ψ as an infinite sum of iterated integrals, involving the coefficients b_j but not ψ. We truncate this sum by dropping all terms involving more than some large number K of integrations.

The key point now is that because the coefficients a_k decay at least like $t^{-\gamma}$ with $\gamma > 1/2$, one has not only that $\lim_{T\to\infty} \int_0^T e^{i\lambda s} a_k(s)\, ds$ exists for almost every λ, but $\int_T^{\infty} e^{i\lambda s} a_k(s)\, ds = O(T^{-\varepsilon})$ for all $\varepsilon < \gamma - \frac{1}{2}$, for almost every λ. The same holds with the phase replaced by $il\theta_0(s,\lambda)$ for $l \neq 0$. A multilinear generalization is that

$$\int_{T \le s_1 \le s_2 \le \cdots \le s_n} e^{i\sum_{j=1}^n l_j\theta_0(s_j,\lambda)} \prod_{j=1}^{n} a_{k_j}(s_j,\lambda)\, ds_j = O(T^{-n\varepsilon}),$$

for almost every λ, provided that each integer l_j is nonzero. Thus the error committed by truncating after K integrations in the solution series is $O(t^{-r})$ for some $r > 1$, provided that K is chosen sufficiently large relative to $\gamma - \frac{1}{2}$.

However, because the ODE for ψ is nonlinear, more complicated multiple integrals arise; the variables of integration s_j are not in general linearly ordered, but instead are partially ordered; this partial order has a tree structure, with anywhere from 1 to N edges branching from any node. To illustrate, a very simple trilinear operator with branching is

$$\int_{\Omega} e^{i\phi(t,\lambda)} \prod_{j=1}^{3} f_j(t_j)\, dt_j$$

where $\mathbb{R}^3 \ni t = (t_1,t_2,t_3)$, $\phi = \lambda \cdot (\sum_j k_j t_j)$, each $k_j \in \mathbb{Z}\backslash\{0\}$, and $\Omega \subset \mathbb{R}^3$ is the set of all t satisfying $t_1 < \min(t_2,t_3)$.

Thus in order to complete the argument outlined, it is necessary to extend our machinery to handle multilinear operators with this structure. Such an extension is straightforward;[32] in fact, such integrals were already treated in [17], albeit in a somewhat clumsy way.

[31]One could use the method of integrating factors to first remove the linear term $b_1\psi^1$ from the right-hand side, but this does not result in any significant simplification.

[32]It is straightforward for fixed n, but I have not yet examined the problem of summability of the resulting infinite series.

It is likely that Theorem 16.1 and the proof outlined here extend to the case where $\sup_{\beta,\lambda} |\mathcal{V}(t,\beta,\lambda)| \in L^1 + L^p$ for $1 < p < 2$ and likewise for all partial derivatives with respect to β, λ, but I have not yet verified this.

17. Questions

The following are some of the principal open problems, for the one-dimensional case, related to the results discussed in these notes.

Problem 2. *Square integrable potentials.* Extend all results, both stationary and time-dependent, from $L^1 + L^p$, $p < 2$, to $L^1 + L^2$. As is clear from the discussion, this amounts to a nonlinear extension of Carleson's theorem on almost everywhere convergence of Fourier transforms and series.

Carleson showed[33] that the map

$$f \mapsto \sup_y \Big| \int_{-\infty}^{y} e^{ix\xi} \hat{f}(\xi)\, d\xi \Big| \tag{17.1}$$

maps $L^2(\mathbb{R})$ to weak[34] L^2. Since the Fourier transform is an invertible isometry on L^2, by setting $f = \hat{V}$ we deduce that $V \mapsto \sup_y \big| \int_{-\infty}^{y} e^{ix\xi} V(\xi)\, d\xi \big|$ is bounded. The first-order term in our expansion[35] is this, with the added complication that the phase $x\xi$ is replaced by $x\xi - (2x)^{-1} \int_0^\xi V(t)\, dt$.

Trilinear operators $\int_{t_1 \le t_2 \le t_2} e^{i(\sum_j c_j t_j)\lambda} \prod_{j=1}^{3} f_j(t_j)\, dt_j$ have been successfully analyzed by Muscalu, Tao, and Thiele [50], for generic (c_2, c_2, c_3), but the case $(1, -1, 1)$ arising in our application is degenerate from their point of view and is not included in that result. More recently, those authors have constructed counterexamples [51] showing that the trilinear operators with phases $\lambda(t_1 - t_2 + t_3)$ are actually unbounded at the $(L^2)^3$ endpoint, and likewise for the maximal binilinear operators. However, counterexamples for the individual terms in these expansions do not translate into counterexamples for the sums of the series, and questions about the reflection/transmission coefficients for potentials in L^2 remain unresolved.

Problem 3. *Existence of singular continuous spectrum of positive dimension.* For potentials which are $O(|x|^{-r})$ for some $r > 1/2$, or more generally, for potentials in L^2, can the spectral measure have singular components of dimension $0 < \alpha < 1$?

If $(1 + |x|)^\gamma V \in L^p$ and $1 \le p \le 2$, then we have shown that WKB asymptotics hold for all energies except an exceptional set of Hausdorff dimension $\le 1 - \gamma p'$ (provided this quantity is ≥ 0). On the other hand, Remling and Kriecherbauer [44] have shown that WKB asymptotics can indeed fail for a set of energies of precisely this dimension. However, in order to obtain spectrum of this dimension, according to an analogue of the criterion (4.9), one needs to construct sufficiently many generalized eigenfunctions with appropriate decay.

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[33] Subsequently extended to $1 < p < 2$ by Hunt, with further refinements near L^1 by Sjölin, an influential second proof by C. Fefferman, and recently a superb short analysis by Lacey and Thiele [45].

[34] It actually is bounded from L^2 to L^2, as was shown by Rubio de Francia via a simple application of weighted norm inequalities and extrapolation.

[35] One should beware the perils of reductionism; we have seen that certain fundamental properties of the generalized eigenfunctions and scattering coefficients are obscured when individual terms of this multilinear expansion are examined in isolation.

Essentially, in order to produce spectrum of dimension α one needs

$$\limsup_{R\to\infty} R^{-\alpha} \int_{|x|\le R} |u_E(x)|^2\, dx < \infty. \tag{17.2}$$

Kiselev [40] has produced potentials decaying at any preassigned rate slower than $|x|^{-1}$, for which there is singular continuous spectrum, but his construction (so far) gives zero-dimensional spectrum (as it must, for potentials decaying more rapidly than $|x|^{-1+\varepsilon}$ for all $\varepsilon > 0$).

Problem 4. *Asymptotic completeness.* Under what circumstances are Schrödinger operators with L^p potentials necessarily asymptotically complete?

Problem 5. Analyze the wave and scattering operators for slowly varying and Stark type potentials.

These variants should be amenable to analysis by the same method; we have not looked into the details.

Problem 6. *Stability of dynamical systems* under time-dependent perturbations. What is the optimal formulation of Theorem 16.1? What can one say when there is a single (that is, scalar) external parameter λ, and the dependence of the system on λ is somehow degenerate?

Michael Goldberg [33] has some interesting observations, a counterexample, and positive results in this direction.

Problem 7. *Higher dimensions.* (i) Consider $H_V = -\Delta + V$ in $\mathbb{R}^d$ for $d \geq 2$. Assume that $|V(x)| \lesssim (1+|x|)^{-\alpha}$ for some $\alpha > 1/2$. Must H_V have nonempty absolutely continuous spectrum?

(ii) Same question, on $\mathbb{Z}^d$, with $-\Delta$ replaced by its usual discrete analogue.

Bourgain [8] has established this for *random* potentials, in the usual almost sure sense, for $\mathbb{Z}^2$; Schlag [61] has explained how the analysis can be modified to apply in all dimensions. The Agmon-Kato-Kuroda theorem [57] says that the conclusion holds, in a stronger form (no sc spectrum, and only discrete point spectrum in $(0,\infty)$) when $\alpha > 1$, in all dimensions, but no improvement to $\alpha < 1$ is known.

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