MIRROR SYMMETRY: LECTURE 6

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1. The Quintic 3-fold and Its Mirror

The simplest Calabi-Yau's are hypersurfaces in toric varieties, especially smooth hypersurfaces X in $\mathbb{CP}^{n+1}$ defined by a polynomial of degree $d = n+2$, i.e. a section of $\mathcal{O}_{\mathbb{P}^{n+1}}(d)$. Smoothness implies that $NX \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^{n+1}}(d)|_X$, defined by $v \mapsto \nabla_v P = dP(v)$, so $T\mathbb{P}^{n+1}|_X = TX \oplus NX = TX \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(d)|_X$ ("adjunction"). Passing to the dual and taking the determinant, we obtain

$$\Omega^{n+1}|_{\mathbb{P}^{n+1}}|_X \cong \Omega^n_X \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-d)|_X \tag{1}$$

Now:

$$T_\ell\mathbb{P}^{n+1} \oplus \mathbb{C} = \mathrm{Hom}(\ell, \ell^\perp) \oplus \mathrm{Hom}(\ell, \ell) = \mathrm{Hom}(\ell, \mathbb{C}^{n+2}) = \mathrm{Hom}(\mathcal{O}(-1)_\ell, \mathbb{C}^{n+2}) \tag{2}$$

implying that $T\mathbb{P}^{n+1} \oplus \mathcal{O} \cong \mathcal{O}(1)^{n+2}$. Again, passing to the dual and taking the determinant, we obtain

$$\Omega^{n+1}_{\mathbb{P}^{n+1}} \otimes \mathcal{O} \cong \mathcal{O}(-1)^{\otimes(n+2)} = \mathcal{O}(-(n+2)) \tag{3}$$

We finally have

$$\mathcal{O}_{\mathbb{P}^{n+1}}(-(n+2))|_X \cong \Omega^n_X \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-d)|_X \implies \Omega^n_X \cong \mathcal{O} \tag{4}$$

if $d = n+2$, i.e. our X is indeed Calabi-Yau.

Example. Cubic curves in $\mathbb{P}^2$ correspond to elliptic curves (genus 1, isomorphic to tori), while quartic surfaces in $\mathbb{P}^3$ are K3 surfaces.

The quintic in $\mathbb{P}^4$ is the world's most studied Calabi-Yau 3-fold. The cohomology of the quintic can be computed via the Lefschetz hyperplane theorem: inclusion induces $i_* : H_r(X) \xrightarrow{\sim} H_r(\mathbb{CP}^4)$ for $r < n = 3$, so $H_1(X) = 0, H_2(X) = H_2(\mathbb{CP}^4) = \mathbb{Z}$. Thus, $h^{1,0} = 0$ and $h^{2,0} = 0$: by argument seen before, $h^{1,1} = 1$. Moreover,

$$\chi(X) = e(TX) \cdot [X] = c_3(TX) \cdot [X] \tag{5}$$

By working out $c(T\mathbb{P}^4)|_X = c(TX)c(\mathcal{O}_{\mathbb{P}^4}(5))|_X$ (from adjunction), we have

$$c(T\mathbb{P}^4) = c(T\mathbb{P}^4 \oplus \mathcal{O}) = c(\mathcal{O}(1)^{\oplus 5}) = (1+h)^5 \tag{6}$$

where $h = c_1(\mathcal{O}(1))$ is the generator of $H_2(\mathbb{CP}^4)$ and is Poincaré dual to the hyperplane. Restricting to X gives

$$(1 + h|_X)^5 = 1 + 5h|_X + 10h^2|_X + 10h^3|_X = (1 + c_1 + c_2 + c_3)(1 + 5h_|X) \tag{7}$$

so $c_1 = 0, c_2 = 10h^2|_X, c_3 = -40h^3|_X$. Thus,

$$\chi(X) = -40h^3 \cdot [X] = -40([\text{line}] \cap [X]) = -40 \cdot 5 = -200 \tag{8}$$

We conclude that

$$h_0 + h_2 - h_3 + h_4 + h_6 = 1 + 1 - \dim\, H_3(X) + 1 + 1 = -200 \tag{9}$$

implying that dim $H_3 = 204$. Since $h^{3,0} = h^{0,3} = 1$, we obtain $h^{1,2} = h^{2,1} = 101$. In fact, $h^{1,1} = 1$, and we have a symplectic parameter given by the area of a generator of $H_2(X)$ (given by the class of a line in $H_2(\mathbb{P}^4)$). We further have $101 = h^{2,1}$ complex parameters: the equation of the quintic gives $h^0(\mathcal{O}_{\mathbb{P}^4}(5)) = \binom{9}{5} = 126$ dimensions, from which we lose one by passing to projective space, and 24 by modding out by $\mathrm{Aut}(\mathbb{CP}^4) = PGL(5, \mathbb{C})$. That is, all complex deformations are still quintics.

Now we construct the mirror of X. Start with a distinguished family of quintic 3-folds

$$X_\psi = \{(x_0 : \cdots : x_4) \in \mathbb{P}^4 \mid f_\psi = x_0^5 + \cdots + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0\} \tag{10}$$

Let $G = \{(a_0, \ldots, a_4) \in (\mathbb{Z}/5\mathbb{Z})^5 \mid \sum a_i = 0\}/(\mathbb{Z}/5\mathbb{Z} = \{(a, a, a, a, a)\})$. Then $G \cong (\mathbb{Z}/5\mathbb{Z})^3$ acts on X_ψ by $(x_j) \mapsto (x_j \xi^{a_j})$ where $\xi = e^{2\pi i/5}$ (f_ψ is G-invariant because $\sum a_j = 0 \mod 5$, and $(1, 1, 1, 1, 1)$ acts trivially because the x_j are homogeneous coordinates). Furthermore, X_ψ is smooth for ψ generic (i.e. $\psi^5 \neq 1$), but X_ψ/G is singular: the action has fixed point $(x_0 : \cdots : x_4) \in X_\psi$ s.t. at least two coordinates are 0. This consists of

- 10 curves C_{ij}, where e.g. $C_{01} = \{x_0 = x_1 = 0, x_2^5 + x_3^5 + x_4^5 = 0\}$ with stabilizer $\mathbb{Z}/5 = \{(a, -a, 0, 0, 0)\}$, so $C_{01}/G \cong \mathbb{P}^1$ is the line $y_2 + y_3 + y_4 = 0$ in $\mathbb{P}^2$, $y_i = x_i^5$, and
- 10 points P_{ijk}, e.g. $P_{0,1,2} = \{x_0 = x_1 = x_2 = 0, x_3^5 + x_4^5 = 0\}$ with stabilizer $(\mathbb{Z}/5\mathbb{Z})^2$, so $P_{012}/G = \{\text{pt}\}$.

The singular locus of X_ψ/G is the 10 curves $\overline{C_{ij}} = C_{ij}/G \cong \mathbb{P}^1$ with $\overline{C}_{ij}, \overline{C}_{jk}, \overline{C}_{ik}$ meeting at the point $\overline{P}_{ijk}$.

Next, let $X_\psi^\vee$ be the resolution of singularities of (X_ψ/G), i.e. $X_\psi^\vee$ smooth and equipped with a map $X_\psi^\vee \xrightarrow{\pi} X_\psi/G$ which is an isomorphism outside $\pi^{-1}(\bigcup C_{ij})$. The explicit construction is complicated, and one can use toric geometry to do it. One can further show that it is a crepant resolution, i.e. the canonical bundle $K_{X_\psi^\vee} = \pi^* K_{X_\psi/G}$, so the Calabi-Yau condition is preserved and $X_\psi^\vee$ is a Calabi-Yau 3-fold.

Along $\overline{C}_{ij}$ (away from $\overline{P}_{ijk}$), X_ψ/G looks like $(\mathbb{C}^2/(\mathbb{Z}/5\mathbb{Z}))\times\mathbb{C}, (x_1,x_1,x_3)\sim(\xi^a x_i,\xi^{-a}x_2,x_3)$. Now $\mathbb{C}^2/(\mathbb{Z}/5\mathbb{Z})\cong\{uv=w^5\}\subset\mathbb{C}^3, [x_1,x_2]\mapsto[x_1^5,x_2^5,x_1x_2]$ is an A_4 singularity, which can be resolved by blowing up twice, getting four exceptional divisors. Doing this for each $\overline{C}_{ij}$ gives 40 divisors. Similarly, resolving each $\overline{p}_{ijk}$ creates six divisors, for a total of 60 divisors. Thus, $X_\psi^\vee$ contains 100 new divisors in addition to the hyperplane section, so indeed $h^{1,1}(X_\psi^\vee)=101$. Similarly, as we were only able to build a one-parameter family, $h^{2,1}(X_\psi^\vee)=1$, giving us mirror symmetric Hodge diamonds:

$$h^{ij}(X)=\begin{pmatrix}1&0&0&1\\0&1&101&0\\0&101&1&0\\1&0&0&1\end{pmatrix}, h^{ij}(X_\psi^\vee)=\begin{pmatrix}1&0&0&1\\0&101&1&0\\0&1&101&0\\1&0&0&1\end{pmatrix} \tag{11}$$

We want to see how mirror symmetry predicts the Gromov-Witten invariants N_d (the "number of rational curves" n_d) of the quintic. For that, we need to understand the mirror map between the Kähler parameter $q=\exp(2\pi i\int_\ell B+i\omega)$ on X and the complex parameter ψ on the mirror $X_\psi^\vee$ (which will also give, by differentiating, an isomorphism $H^{1,1}(X)\xrightarrow{\sim}H^{2,1}(X)$) as well as calculations of the Yukawa coupling on $H^{2,1}(X_\psi^\vee)$.

1.1. **Degenerations and the Mirror Map.** Last time, we saw a basis $\{e_i\}$ of $H^2(X,\mathbb{Z})$ by elements of the Kähler cone gives coordinates on the complexified Kähler moduli space: if $[B+i\omega]=\sum t_ie_i$, the parameter $q_i=\exp(2\pi i t_i)\in\mathbb{C}^*$ gives the large volume limit as $q_i\to 0, \operatorname{Im}(t_i)\to\infty$. Physics predicts that the mirror situation is degeneration of a large complex structure limit and that, near such a limit point, there are "canonical coordinates" on the complex moduli spaces making it possible to describe the mirror map.

- Degeneration: consider a family $\mathcal{X}\xrightarrow{\pi}D^2$ where for $t\neq 0, X_t\cong X$ (with varying J) and for $t=0$, X_0 is typically singular. For instance, consider the camily of elliptic curves $C_t=\{y^2z=x^3+x^2z-tz^3\}\subset\mathbb{CP}^2$ (in affine coordinates, $C_t: y^2=x^3+x^2-t$). C_t is a smooth torus for $t\neq 0$, and nodal at $t=0$, obtained by pinching a loop on the torus.
- Monodromy: follow the family (X_t) as t varies along the loop in $\pi_1(D^2\setminus\{0\},t_0)$ going around the origin. All the X_is are diffeomorphic, and thus induce a monodromy diffeomorphism ϕ of X_{t_0}, defined up to isotopy. This in turn induces $\phi_*\in\operatorname{Aut}(H_n(X_{t_0},\mathbb{Z}))$. In the above example, ϕ acts on $H_1(C_{t_0})=\mathbb{Z}^2$ by $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ (the Dehn twist): observe that $C_t\xrightarrow{2:1}\mathbb{CP}^1=\mathbb{C}\cup\{\infty\}$ by projection to x, and the branch points are ∞ plus the roots of x^3+x^2-t. As $t\to 0$, there is one root near -1 and two near 0, which rotate as t goes around 0. Letting a be the line between the two roots

near 0 and b be between the root near -1 and the closest other root, the monodromy maps a, b to $a, b+a$.

Remark. Note that this complex parameter t is ad hoc. A more natural way to describe the degeneration would be to describe C_t as an abstract elliptic curve $C_t \cong \mathbb{C}/\mathbb{Z} + \tau(t)\mathbb{Z}$. Then $\tau(t)$, or rather $\exp(2\pi i\tau)$, is a better quantity. Equip C_t with a holomorphic volume form Ω_t normalized so $\int_a \Omega_t = 1 \,\forall\, t$. Then let $\tau(t) = \int_b \Omega_t$: as t goes around the origin, $\tau(t) \to \tau(t) + 1$ since $b \mapsto b + a$. Moreover, $q(t) = \exp(2\pi i\tau(t))$ is still single-valued, and as $t \to 0$, we still have $\mathrm{Im}\, \tau(t) \to \infty$ and $q(t) \to 0$. In the former case, we have $\int_a \frac{dx}{y} \in -i\mathbb{R}^+$ tending to 0 and $\int_b \frac{dx}{y} \in \mathbb{R}^+$ tending to a constant value, so the ratio goes to $+i\infty$. In the latter case, $q(t)$ is a holomorphic function of t, and goes around 0 once when t does, i.e. it has a single root at $t = 0$. Thus, q is a local coordinate for the family.

Next time, we will see an analogue of this for a family of Calabi-Yau manifolds.