MIRROR SYMMETRY: LECTURE 4

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1. Pseudoholomorphic Curves

For (X^{2n}, ω) symplectic, J a compatible a.c.s. $\in \mathcal{J}(X, \omega)$,

$$\mathcal{M}_{g,k} = \{(\Sigma, j) \text{ genus } g, z_1, \ldots, z_k \in \Sigma \text{ marked points}\} \tag{1}$$

Definition 1. *$u : \Sigma \to X$ is J-holomorphic if $\overline{\partial}_J u = \frac{1}{2}(du + J du \cdot j) = 0$, and for $\beta \in H_2(X, \mathbb{Z})$, we have a moduli space*

$$\mathcal{M}_{g,k}(X, J, \beta) = \{(\Sigma, j, z_1, \ldots, z_k), u | u_*[\Sigma] = \beta, \overline{\partial}_J u = 0\}/\sim \tag{2}$$

$u : \Sigma \to X$ is simple if it doesn't factor $\Sigma \to \Sigma' \to X$. We can define a linearized operator

$$\begin{gathered} D_{\overline{\partial}} : W^{r+1,p}(\Sigma, u^*TX) \times T\mathcal{M}_{g,k} \to W^{r,p}(\Sigma, \Omega^{0,1}_\Sigma \otimes U^*TX) \\ D_{\overline{\partial}}(v, j') = \overline{\partial} v + \frac{1}{2}(\nabla_v J) du \cdot j + \frac{1}{2} J \cdot du \cdot j' \end{gathered} \tag{3}$$

This operator is Fredholm, with real index

$$\text{index}_{\mathbb{R}} D_{\overline{\partial}} := 2d = 2\langle c_1(TX), \beta\rangle + n(2 - 2g) + (6g - 6 + 2k) \tag{4}$$

u is *regular* if $D_{\overline{\partial}}$ is onto.

Theorem 1. *The set $\mathcal{J}^{reg}(X, \beta)$ of $J \in \mathcal{J}(X, \omega)$ s.t. every simple J-holomorphic curve in class β is regular is a Baire subset. For $J \in \mathcal{J}^{reg}(X, \beta)$, the subset of simple maps $\mathcal{M}^*_{g,k}(X, J, \beta) \subset \mathcal{M}_{g,k}(X, J, \beta)$ is smooth and oriented of dimension $2d$.*

Let $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ be the associated Riemannian metric.

Theorem 2 (Gromov Compactness)**.** *If $u_n : \Sigma_n \to X$ is a sequence of J-holomorphic curves, $J \in \mathcal{J}(X, \omega)$, $E(u_n) = \int_{\Sigma_n} u_n^* \omega = \langle [\omega], u_{n*}[\Sigma_n]\rangle$ bounded $\leq E_0 < \infty$, then $\exists$ a subsequence which converges to a stable map $u_\infty : \Sigma_\infty \to X$.*

Here Σ_∞ is a union of nodal Riemann surfaces.

Remark. The phenomenon occuring here (besides the degeneration of Σ_n comes from the *bubbling of spheres*. For instance, for $u_n : S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \infty \to \mathbb{CP}^1 \times \mathbb{CP}^1, (x_0 : x_1) \mapsto (x_0 : x_1), (nx_1 : x_0)$, we see that, in the affine chart

$x = x_1/x_0$, $\mathbb{C}^* \to \mathbb{C}^2, x \mapsto x, \frac{1}{nx}$ which extends at $0, \infty$ to $\mathbb{CP}^1 \times \mathbb{CP}^1$. Away from $x = 0$, it converges uniformly to $x \mapsto (x, 0)$. But if you reparameterize to $\tilde{x} = nx$, $\tilde{x} \mapsto (\frac{1}{n}\tilde{x}, \frac{1}{\tilde{x}})$ and away from $x = \infty$, it converges uniformly to $\tilde{x} \to (0, \frac{1}{\tilde{x}})$.

The general idea is:

- Identify bubbling regions where $\sup |du_n| \to \infty$.
- Away from those, $\exists$ convergent subsequences.
- Near them, we can rescale the doim to $v_n(z) = u_n(z_n^0 + \epsilon_n z), \epsilon_n \to 0$ so $\sup |dv_n|$ does not tend to ∞ and there is a subsequence converging to v_∞.
- The process is finite because $\forall u$ nonconstant holomorphic curves (closed domain),

$$E(u) = \int |du|^2 = \int u^*\omega \geq \hbar > 0 \tag{5}$$

Assuming we can achieve transversality for all stable J-holomorphic curves in the class β, then

(6)
$$\overline{\mathcal{M}}_{g,k}(X, J, \beta) = \{\text{(nodal) } J\text{-holomorphic curves of genus } g \text{ representing } \beta\}/\sim$$

is compact and oriented of real dimension $2d = 2\langle c_1(TX), \beta\rangle + 2(n-3)(1-g) + 2k$, with a fundamental class

$$[\overline{M}_{g,k}(X, J, \beta)] \in H_{2d}(\overline{M}_{g,k}(X, J, \beta), \mathbb{Q}) \tag{7}$$

This moduli space is equipped with evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,k}(X, J, \beta) \to X, (\Sigma, j, z_1, \ldots, z_k, u) \mapsto u(z_i)$ for $1 \leq i \leq k$. The *Gromov-Witten invariants* are defined as follows: given $\alpha_1, \ldots, \alpha_k \in H^*(X), \sum \deg\,(\alpha_i) = 2d$,

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{g,\beta} = \int_{[\overline{M}_{g,k}(X,J,\beta)]} \mathrm{ev}_1^*\alpha_1 \wedge \cdots \wedge \mathrm{ev}_k^*\alpha_k \in \mathbb{Q} \tag{8}$$

Equivalently, if we represent $PD(\alpha_i)$ by a cycle $C_i \subset X$ (choose C_i transverse to the evaluation map), then the pairing is simply $\#([\overline{M}_{g,k}(X, J, \beta)] \cap \bigcap_i \mathrm{ev}_i^{-1}(C_i))$ (or rather $\#(\mathrm{ev}_*[\overline{M}_{g,k}(X, J, \beta)] \cap (C_1 \times \cdots \times C_k))$ in X^k, where $\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_n)$). That is, we are asking how many curves in the homology class β pass through the cycles C_i. Note that, for a Calabi-Yau 3-fold, $2d = 2k$, so regular curves are isolated: $\#[\overline{M}_{g,0}(X, J, \beta)] \in \mathbb{Q}$.

1.1. **More on the case of Calabi-Yau 3-folds,** $g = 0$**.** For the symplectic geometer, note that we have transversality for simple curves, by taking J generic. However, multiple covers $\Sigma' \xrightarrow{d:1} \Sigma \xrightarrow{\beta} X$ always occur with excess dimension $\forall J$. Also, these have automorphisms (deck transformations of the covering). Thus, $\mathcal{M}_{0,k}(X, J, d\beta)$ are orbifolds (strata of multiply-covered maps are orbifolded). We can restore transversality by taking domain-dependent Js. More precisely, there is a universal curve $\mathcal{C} \to \overline{\mathcal{M}}_{0,k}$ (the fiber over a point is the corresponding curve), and J is now given by a map $\mathcal{C} \to \mathcal{J}(X, \omega)$. The holomorphic curve equation becomes $u : (\Sigma, j) \to X, du + J(u(z), z)du \cdot j = 0$. We choose a superposition of a finite number perturbations, which break the symmetry of the multiple covers and give us transversality.

In order to obtain compactness, we need to include stable maps (i.e. chains of nodal Riemann surfaces). If we have transversality, these have real codimension ≥ 2 in $\overline{\mathcal{M}}_{g,k}(X, J, \beta)$, i.e. $\mathcal{M}_{0,k}(X, J, \beta)$ defines a pseudocycle, so we can still define the fundamental class. The point is that, in a Calabi-Yau 3-fold, for a generic J, we get transversality for all simple curves. Given $\beta \in H_2(X, \mathbb{Z})$, $\exists$ finitely many classes β_i with $E(\beta_1) \leq \cdots \leq E(\beta_n) = \beta$ containing J-holomorphic curves. The simple curves are isolated, and for a generic J, the simple curves are disjoint. So in a stable map $\Sigma_\infty \to X$, all nonconstant components have the same image, so we treat this as a multiple cover.

For an algebraic geometer, one needs to keep J integrable so X remains an algebraic variety. The moduli space $\overline{\mathcal{M}}_{g,k}$ is an *algebraic stack*, as is $\overline{\mathcal{M}}_{g,k}(X, J, \beta)$. For an integrable J and fixed j, we have a $\overline{\partial}$-operator on sections of u^*TX, and the cokernel of this operator is precisely $H^1(\Sigma, u^*TX)$. Where $du \neq 0$, we have $u^*TX = T\Sigma \oplus u^*N$ and $H^1(\Sigma, T\Sigma)$ is simply the deformations of j. There is also an obstruction bundle $\underline{\mathrm{Obs}}_u = H^1(\Sigma, u^*N_\Sigma)$ if u is an immersion. We claim that we can define an obstruction sheaf $\underline{\mathrm{Obs}} \to \overline{\mathcal{M}}_{g,k}(X, J, \beta)$, and perturbing our equation to $\overline{\partial}_J u = \nu$ yields a section $\pi_{\mathrm{Coker}}\ (\nu)$ of $\underline{\mathrm{Obs}}$. We can obtain a "virtual" fundamental class $[\overline{\mathcal{M}}_{g,k}(X, J, \beta)]^{virt} \in H_{2d}(\overline{\mathcal{M}}_{g,k}(X, J, \beta), \mathbb{Q})$, and if $\underline{\mathrm{Obs}}$ is a bundle, this virtual fundamental class is Poincaré dual to Euler class of the obstruction bundle.