MIRROR SYMMETRY: LECTURE 22

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1. SYZ Conjecture (cntd.)

Recall:

Proposition 1. *First order deformations of special Lagrangian L in a strict (resp. almost) Calabi-Yau manifold are given by $\mathcal{H}^1(L,\mathbb{R})$ (resp. $\mathcal{H}^1_\psi(L,\mathbb{R})$), where*

$$H^1_\psi(L,\mathbb{R}) = \{\beta \in \Omega^1(L,\mathbb{R}) \,|\, d\beta = 0, d^*(\psi\beta) = 0\} \tag{1}$$

It is still true that $\mathcal{H}^1_\psi(L,\mathbb{R}) \cong H^1(L,\mathbb{R})$.

Theorem 1 (McLean, Joyce)**.** *Deformations of special Lagrangians are unobstructed, i.e. the moduli space of special Lagrangians is a smooth manifold B with $T_LB \cong \mathcal{H}^1_\psi(L,\mathbb{R}) \cong H^1(L,\mathbb{R})$.*

There are two canonical isomorphisms $T_LB \xrightarrow{\sim} H^1(L,\mathbb{R}), v \mapsto [-\iota_v\omega]$ ("symplectic") and $T_LB \xrightarrow{\sim} H^{n-1}(L,\mathbb{R}), v \mapsto [\iota_v \mathrm{Im}\,\Omega]$ "complex".

Definition 1. *An* affine structure *on a manifold N is a set of coordinate charts with transition functions in $GL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$.*

Corollary 1. *B carries two affine structures.*

For affine manifolds, mirror symmetry exchanges the two affine structures. Our particular case of interest is that of special Lagrangian tori, so dim $H^1 = n$. The usual harmonic 1-forms on flat T^n have no zeroes, and give a pointwise basis of T^*L. We will make a standing assumptions that ψ-harmonic 1-forms for $g|_L$ have no zeroes (at least ok for $n \leq 2$). Then a neighborhood of L is fibered by special Lagrangian deformations of L: locally,

$$\begin{tikzcd} T^n \arrow[r] & U \subset X \arrow[d, "\pi"] \\ & V \subset B \end{tikzcd} \tag{2}$$

In local affine coordinates, we pick a basis $\gamma_1, \dots, \gamma_n \in H_1(L,\mathbb{Z})$: deforming from L to L', the deformation of γ_i gives a cylinder Γ_i, and we set $x_i = \int_{\Gamma_i}\omega$ (the *flux* of the deformation $L \to L'$). These are affine coordinates on the symplectic side. On the complex side, pick a basis $\gamma_1^*, \dots, \gamma_n^* \in H_{n-1}(L,\mathbb{Z})$, construct the associated Γ_i^*, and set $x_i^* = \int_{\Gamma_i^*} \mathrm{Im}\,\Omega$. Globally, there is a monodromy $\pi_1(B,*) \to \mathrm{Aut} H^*(L,\mathbb{Z})$. In our case, the monodromies in $GL(H^1(L,\mathbb{Z})), GL(H^{n-1}(L,\mathbb{Z}))$ are transposes of each other.

1.1. **Prototype construction of a mirror pair.** Let B be an affine manifold, $\Lambda \subset TB$ the lattice of integer vectors. Then TB/Λ is a torus bundle over B, and carries a natural complex structure, e.g.

$$T(\mathbb{R}^n) \cong \mathbb{C}^n, \mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n, GL(n,\mathbb{Z}) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \tag{3}$$

Setting $\Lambda^* = \{p \in T^*B \,|\, p(\Lambda) \subset \mathbb{Z}\}$ to be the dual lattice of integer covectors, we find that T^*B/Λ^* has a natural symplectic structure since $GL(n,\mathbb{Z}) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix} \in \mathrm{Sp}(2n)$.

In our case, we have two affine structures with dual monodromies

$$\begin{tikzcd} TB \arrow[r, "\sim"] \arrow[d, "cx", "\sim"'] & T^*B \arrow[d, "\sim", "symp"'] \\ H^{n-1}(L,\mathbb{R}) \arrow[r, "\sim", "PD"'] & H_1(L,\mathbb{R}) \\ \Lambda_c = H^{n-1}(L,\mathbb{Z}) \arrow[u, hook] \arrow[r, "\sim"'] & H_1(L,\mathbb{Z}) = \Lambda_s^* \arrow[u, hook] \end{tikzcd} \tag{4}$$

so the complex manifold TB/Λ_c is diffeomorphic to the symplectic manifold T^*B/Λ_S^*. Dually, $X^\vee \cong T^*B/\Lambda_c^* \cong TB/\Lambda_s$.

1.2. **More explicit constructions [cf. Hitchin].** Let

$$\begin{aligned} M = \{(L,\nabla) \,|\, & L \text{ a special Lagrangian torus in } X, \\ & \nabla \text{ flat } U(1) - \text{conn on } \mathbb{C} \times L \text{ mod gauge}\} \end{aligned} \tag{5}$$

i.e. $\nabla = d + iA, iA \in \Omega^1(L, i\mathbb{R}), dA = 0$ mod exact forms.

(6)

$$\begin{aligned} T_{(L,\nabla)}M &= \{(v, i\alpha) \in C^\infty(NL) \oplus \Omega^1(L; i\mathbb{R}) \,|\, -\iota_v\omega \in \mathcal{H}^1_\psi(L,\mathbb{R}), d\alpha = 0 \text{ mod Im } (d)\} \\ &= \{(v, i\alpha) \in C^\infty(NL) \oplus \Omega^1(L; i\mathbb{R}) \,|\, -\iota_v\omega + i\alpha \in \mathcal{H}^1_\psi(L;\mathbb{C})\} \\ &= H^1_\psi(L,\mathbb{C}) \end{aligned}$$

which is a complex vector space, and $J^\vee$ is an almost-complex structure.

Proposition 2. *$J^\vee$ is integrable.*

Proof. We build local holomorphic coordinates. Let $\gamma_1, \dots, \gamma_n$ be a basis of $H_1(L, \mathbb{Z})$, and assume $\gamma_i = \partial\beta_i, \beta_i \in H_2(X, L)$. Set

$$z_i(L, \nabla) = \underbrace{\exp(-\int_{\beta_i} \omega)}_{\mathbb{R}_+} \underbrace{\mathrm{hol}_\nabla(\gamma_i)}_{U(1)} \in \mathbb{C}^* \tag{7}$$

Then

$$\mathrm{dlog}\, z_i : (v, i\alpha) \mapsto -\int_{\gamma_i} \iota_v\omega + \int_{\gamma_i} i\alpha = \langle \underbrace{[-\iota_v\omega + i\alpha]}_{H^1(L,\mathbb{C})}, \gamma_i\rangle \tag{8}$$

is $\mathbb{C}$-linear. If there are no such β_i, we instead use a deformation tube as constructed earlier. Warning: all of our formulas are up to (i.e. may be missing) a factor of 2π. □

Next, consider the holomorphic $(n, 0)$-form on M

$$\Omega^\vee((v_1, i\alpha_1), \dots, (v_n, i\alpha_n)) = \int_L (-\iota_{v_1}\omega + i\alpha_1) \wedge \cdots \wedge (-\iota_{v_n}\omega + i\alpha_n) \tag{9}$$

After normalizing $\int_L \Omega = 1$, we have a Kähler form

$$\omega^\vee((v_1, i\alpha_1), (v_2, i\alpha_2)) = \int_L \alpha_2 \wedge (\iota_{v_1}\mathrm{Im}\ \Omega) - \alpha_1 \wedge (\iota_{v_2}\mathrm{Im}\ \Omega) \tag{10}$$

Proposition 3. *$\omega^\vee$ is a Kähler form compatible with $J^\vee$.*

Proof. Pick a basis $[\gamma_i]$ of $H_{n-1}(L, \mathbb{Z})$ with a dual basis $[e_i]$ of $H_1(L, \mathbb{Z})$, i.e. $e_i \cap \gamma_j = \delta_{ij}$. For all $a \in H^1(L), b \in H^{n-1}(L)$

$$()\langle a \cup b, [L]\rangle = \sum_i \langle a, e_i\rangle\langle b, \gamma_i\rangle \tag{11}$$

Letting $a = \sum a_i dx_i, b = \sum b_i(-1)^{i-1}(dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n)$, $\int_{T^n} a\wedge b = \sum a_i b_i$. Again, take a deformation from L_0 to L', C_i the tube (an n-chain) formed by the deformation of γ_i, and set $p_i = \int_{C_i} \mathrm{Im}\ \Omega, \theta_i = \int_{e_i} A$ for A the connection 1-form (i.e. $\mathrm{hol}_{e_i}(\nabla) = \exp(i\theta_i)$). Then

$$\begin{aligned} dp_i : (v, i\alpha) &\mapsto \int_{\gamma_i} \iota_v \mathrm{Im}\ \Omega = \langle [\iota_v \mathrm{Im}\ \Omega], \gamma_i\rangle \\ d\theta_i : (v, i\alpha) &\mapsto \int_{e_i} \alpha = \langle [\alpha], e_i\rangle \end{aligned} \tag{12}$$

By (11), $\omega^\vee = \sum dp_i \wedge d\theta_i$, implying that $\omega^\vee$ is closed, and

$$\begin{aligned}
\omega^\vee((v_1,\alpha_1),(v_2,\alpha_2)) &= \int_L \alpha_2 \wedge (-\psi *_g \iota_{v_1}\omega) - \alpha_1 \wedge (-\psi *_g \iota_{v_2}\omega) \\
&= \int_L \psi \cdot (\langle \alpha_1, \iota_{v_2}\omega\rangle_g - \langle \alpha_2, \iota_{v_1}\omega\rangle_g)\mathrm{vol}_g \qquad (13) \\
\omega^\vee((v_1,\alpha_1),J^\vee(v_2,\alpha_2)) &= \int_L \psi \cdot (\langle \alpha_1, \alpha_2\rangle_g + \langle \iota_v\omega, \iota_{v_2}\omega\rangle_g)\mathrm{vol}_g
\end{aligned}$$

which is clearly a Riemannian metric. □