MIRROR SYMMETRY: LECTURE 19

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1. Homological Mirror Symmetry

Conjecture 1. *$X, X^\vee$ are mirror Calabi-Yau varieties* $\Leftrightarrow D^\pi \mathrm{Fuk}(X) \cong D^b \mathrm{Coh}(X^\vee)$

Look at T^2 at the level of homology [Polishchuk-Zaslow]: on the symplectic side, $T^2 = \mathbb{R}^2/\mathbb{Z}^2, \omega = \lambda dx \wedge dy$, so $\int_{T^2} \omega = \lambda$. On the complex side, $X^\vee = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}, \tau = i\lambda$. The Lagrangians L in X are Hamiltonian isotopic to straight lines with rational slope, and given a flat connection ∇ on a $U(1)$-bundle over L, we can arrange the connection 1-form to be constant. We will see that families of (L, ∇) in the homology class (p, q) correspond to holomorphic vector bundles over $X^\vee$ of rank p, $c_1 = -q$. For $L \to X^\vee$ a line bundle, the pullback of L to the universal cover $\mathbb{C}$ is holomorphically trivial, and

$$\begin{aligned} L \cong \mathbb{C} \times \mathbb{C}/(z,v) \sim &(z+1, v), (z,v) \sim (z+\tau, \phi(z)v) \\ &\phi \text{ holomorphic}, \phi(z+1) = \phi(z) \end{aligned} \tag{1}$$

Example. $\phi(z) = e^{-2\pi i z} e^{-\pi i \tau}$ determines a degree 1 line bundle $\mathcal{L}$ with a section given by the theta function

$$\theta(\tau, z) = \sum_{m \in \mathbb{Z}} e^{2\pi i (\frac{\tau m^2}{2} + mz)} \tag{2}$$

More generally, set

$$\theta[c', c''](\tau, z) = \sum_{m \in \mathbb{Z}} \exp\left(2\pi i \left[\frac{\tau(m+c')^2}{2} + (m+c')(z+c'')\right]\right) \tag{3}$$

Then

$$\begin{aligned} \theta[c', c''](\tau, z+1) &= e^{2\pi i c'} \theta[c', c''](\tau, z) \\ \theta[c', c''](\tau, z+\tau) &= e^{-\pi i \tau} e^{-2\pi i (z+c'')} \theta[c', c''](\tau, z) \end{aligned} \tag{4}$$

since the interior of exp for the latter formula is

$$\begin{aligned} \frac{\tau(m+c')^2}{2} &+ \tau(m+c') + (z+c'')(m+c') \\ &= \frac{\tau(m+1+c')^2}{2} - \frac{\tau}{2} + (m+1+c')(z+c'') - (z+c'') \end{aligned} \tag{5}$$

Furthermore, sections of $\mathcal{L}^{\otimes n}$ are $\theta[\frac{k}{n},0](n\tau,nz), k\in\mathbb{Z}/n\mathbb{Z}$. By the above

$$\begin{aligned}\theta[\frac{k}{n},0](n\tau,nz+n)&=\theta[\frac{k}{n},0](n\tau,nz)\\ \theta[\frac{k}{n},0](n\tau,nz+n\tau)&=e^{-\pi in\tau}e^{-2\pi inz}\theta[\frac{k}{n},0](n\tau,nz)\end{aligned}\tag{6}$$

as desired. Other line bundles are given by pullback over the translation $z\mapsto z+c''$, and the higher rank bunddles are given by matrices or pushforward by finite covers.

On the mirror, consider the Lagrangian subvarieties

$$\begin{aligned}&L_0=\{(x,0)\},\nabla_0=d\ (\text{mirror to }\mathcal{O}),\\ &L_n=\{(x,-nx)\},\nabla_n=d\ (\text{mirror to }\mathcal{L}^{\otimes n}),\\ &L_p=\{(a,y)\},\nabla_p=d+2\pi ibdy\ (\text{“mirror to }\mathcal{O}_Z, z=b+a\tau\text{”})\end{aligned}\tag{7}$$

For gradings, pick $\arg(dz)|_{L_i}\in[-\frac{\pi}{2},0]$. Then

$$\begin{aligned}s_k&=\left(\frac{k}{n},0\right)\in CF^0(L_0,L_n),\\ e&=(a,-na)\in CF^0(L_n,L_p),\\ e_0&=(a,0)\in CF^0(L_0,L_p)\end{aligned}\tag{8}$$

We want to find the coefficient of e_0 in $m_2(e,s_0)$, i.e. we need to count holomorphic disks in T^2. All these disks lift to the universal cover $\mathbb{C}$, and a Maslov index calculation gives that rigid holomorphic disks are immersed. We obtain an infinite sequence of triangles T_m, $m\in\mathbb{Z}$ in the universal cover. T_m has corners at $(0,0)$, $(a+m,-n(a+m))$, $(a+m,0)$, and the area is $\int_{T_m}\omega=\frac{\lambda n(a+m)^2}{2}$. Taking holonomy on ∂T_m gives

$$\exp(2\pi i\int_{-n(a+m)}^{0}bdy)=\exp(2\pi in(a+m)b)\tag{9}$$

The T_m are regular, and doing sign calculations makes them count positively. Now,

$$m_2(e,s_0)=\left(\sum_{m\in\mathbb{Z}}T^{\lambda\frac{n}{2}(a+m)^2}e^{2\pi in(a+m)b}\right)e_0\tag{10}$$

As usual, set $T=e^{-2\pi}$ (convergence is not an issue here), i.e. $T^{\lambda}=e^{2\pi i\tau}$. Then

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\exp\ 2\pi i\left[\frac{n\tau m^2}{2}+n(\tau a+b)m+(n\tau\frac{a^2}{2}+nab)\right]\\ &\qquad=e^{\pi in\tau a^2}e^{2\pi inab}\theta(n\tau,n(\tau a+b))\end{aligned}\tag{11}$$

What we have computed is the composition $\mathcal{O} \xrightarrow{s_0} \mathcal{L}^n \xrightarrow{\mathrm{ev}_z} \mathcal{O}_z$, where ev_z is obtained by picking a trivialization of the fiber at z. Looking at the coefficient of e_0 in $m_2(e, s_k)$, we obtain

(12)
$$\begin{aligned}\sum_{m\in\mathbb{Z}} \exp\, 2\pi i \left[\frac{n\tau}{2}(a+m-\frac{k}{n})^2 + n(a+m-\frac{k}{n})b\right] \\ &= \sum_{m\in\mathbb{Z}} \exp\, 2\pi i \left[\frac{n\tau}{2}(m-\frac{k}{n})^2 + n(\tau a+b)(m-\frac{k}{n}) + \frac{n\tau}{2}a^2 + nab\right] \\ &= e^{\pi i n\tau a^2} e^{2\pi i nab}\theta[0, \frac{k}{n}](n\tau, n(\tau a+b))\end{aligned}$$

so the ratios $\frac{s_k}{s_0}$ match.

Next, we need to multiply sections. For $s_0^{1\to 2} \in \hom(L_1, L_2), s_0^{0\to 1} \in \hom(L_0, L_1)$, $m_2(s_0^{1\to 2}, s_0^{0\to 1}) = c_0 s_0^{0\to 2} + c_1 s_1^{0\to 2}$ for $s_0^{0\to 2}, s_1^{0\to 2} \in \hom(L_0, L_2)$ and

$$\begin{aligned} c_0 &= \sum_{n\in\mathbb{Z}} T^{n^2\lambda} = \sum_{n\in\mathbb{Z}} e^{2\pi i\tau n^2} \\ c_1 &= \sum_{n\in\mathbb{Z}} e^{2\pi i\tau(n+\frac{1}{2})^2} \end{aligned} \tag{13}$$

This corresponds to $\mathcal{O} \xrightarrow{\theta} \mathcal{L} \xrightarrow{\theta} \mathcal{L}^2$,

$$\theta(\tau, z)\theta(\tau, z) = \underbrace{\theta(2\tau, 0)}_{c_0}\underbrace{\theta(2\tau, 2z)}_{s_0} + \underbrace{\theta[\frac{1}{2}, 0](2\tau, 0)}_{c_1}\underbrace{\theta[\frac{1}{2}, 0](2\tau, 2z)}_{s_1} \tag{14}$$