SPACING BETWEEN PHASE SHIFTS IN A SIMPLE SCATTERING PROBLEM

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1. Introduction and statement of the result

The Berry-Tabor conjecture [2] for quantum integrable systems with discrete spectra asserts that the spacings between normalized eigenvalues of a quantum integrable system should exhibit Poisson statistics in the semi-classical limit. In particular, when the eigenvalues are scaled to have *unit mean level spacing*, the distribution of their differences should be uniform. This conjecture has been verified numerically in many cases [4] and has been rigorously proved in an almost everywhere sense for flat 2-tori (Sarnak [13]), flat 4-tori (Vanderkam [16]), and for certain integrable quantum maps in one degree of freedom (Rudnick-Sarnak [11], Zelditch [1]). Smilansky [15] has more recently posed an analogous conjecture for scattering systems with continuous spectra. Since the scattering matrix $S(E)$ at energy level E is, at least heuristically, the quantization of the classical scattering map, he argues that when the scattering map is integrable the eigenvalues of $S(E)$ (known as *phase shifts*) should exhibit Poisson statistics. In particular, he proposed that the pair correlation function of scaled phase shifts should be uniform for surfaces of revolution with a cylindrical end (see figure 1). The purpose of this paper is to prove (a somewhat modified form of) this conjecture for almost all surfaces in an infinite dimensional family of (pairs of) such surfaces.

To explain the modifications and state our results, we need to introduce some notation. The surfaces we consider are topological discs X on which S^1 acts freely except for a unique fixed point m. The metrics g we consider are invariant under the S^1 action and in geodesic polar coordinates centered at m have the form $g = dr^2 + a(r)^2 d\theta^2$ where $a(r)$ defines a short range *cylindrical end* metric (see Sect.2). For technical reasons, we are only able to analyse the phase shifts at this time in the case where g has a conic singularity at m.

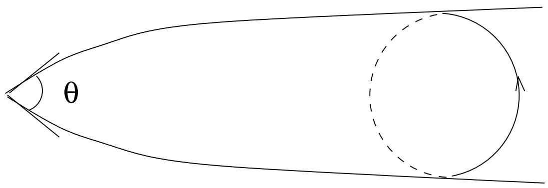

Figure 1. A surface of revolution with a conic singularity and a cylindrical end.

To define the pair correlation measure, we recall that at energy λ^2 the scattering matrix for a surface of revolution with a cylindrical end is given by a diagonal $(2[\lambda]+1)\times(2[\lambda]+1)$ matrix with entries $\exp(2\pi i \delta_k(\lambda))$, $|k| \leq [\lambda]$ – see Sect.3 for a detailed presentation. The phase shifts are given by $\delta_k(\lambda)$ and are well defined modulo $\mathbb{Z}$. The parameter k corresponds to the angular momentum or in other words to the eigenvalues of the Laplacian on the cross-section (the circle in our case). When $|k|$ is close to λ we expect no scattering phenomena as the classical motion is close to the bounded motion along the cross-sections (see Fig.2). At the opposite extreme, when $|k|/|\lambda|$ is close to 0, the classical motion is along geodesics approaching the singularity on the surface.

Since the properties of the pair correlation measure are supposed to correspond to the properties of smooth classical motion it is natural, at least at this early stage, to delete the angular momenta corresponding to the neighbourhoods of the singularities.

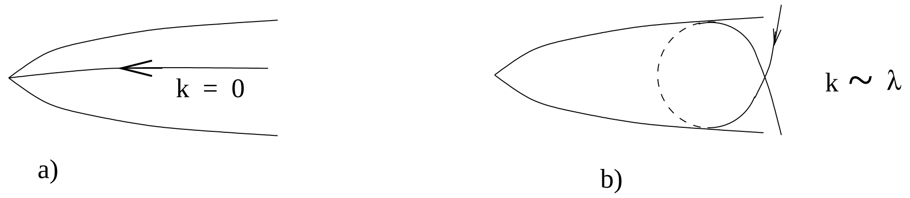

FIGURE 2. a) Geodesics approaching the singularity; b) Geodesics close to the cross-sections.

Based on this discussion we define for any $\epsilon > 0$ the following measure

$$\begin{aligned}\rho_\lambda^\epsilon([a,b]) \stackrel{\text{def}}{=} \frac{1}{(1-2\epsilon)}\frac{1}{2\lambda+1}\,\sharp\,\{\,(l,m,k)\ :\ l,m,k\in\mathbb{Z}\,,\ \epsilon<|l/\lambda|\,,|m/\lambda|<1-\epsilon\,,\\ (2\lambda+1)(1-2\epsilon)(\delta_l(\lambda)-\delta_m(\lambda)+k)\in[a,b]\,\}\ .\end{aligned} \tag{1.1}$$

In other words for $f\in\mathcal{S}(\mathbb{R})$,

$$\int f(x)\rho_\lambda^\epsilon(dx) = \frac{1}{(1-2\epsilon)}\frac{1}{2\lambda+1}\sum_{k\in\mathbb{Z}}\sum_{\substack{m,l\in\mathbb{Z}\\ \epsilon<|m/\lambda|,|l/\lambda|<1-\epsilon}} f\left((1-2\epsilon)(1+2\lambda)(\delta_l(\lambda)-\delta_m(\lambda)+k)\right)\ . \tag{1.2}$$

Although rather cumbersome, this definition follows the standard procedure for defining pair correlation measures – see the references listed above.

Our main result concerns the (modified) pair correlation function for an infinite dimensional set $\mathcal{G}$ of 2-parameter families of surfaces of revolution

$$(X, g^{\alpha,\beta})\,,\quad (\alpha,\beta)\in(\alpha_0-\delta,\alpha_0+\delta)\times(-\delta,\delta)\subset\mathbb{R}^2\,,$$

with cylindrical ends. The precise definition of $\mathcal{G}$ will be given in Proposition 6 in Sect.5. The key property of the metrics is that the leading parts of the phase shifts, $\psi^{\alpha,\beta}$, of the 2-parameter families $(X, g^{\alpha,\beta})$ depend linearly on the parameters (α,β). This feature allows us to prove:

Theorem. *Let $\{g_{\alpha,\beta}\}\in\mathcal{G}$. Then for almost every pair (α,β) (in the sense of Lesbesgue measure) and for any sequence $\{\lambda_m\}_{m=0}^\infty$ satisfying*

$$\sum_{m=0}^{\infty}\frac{\log^3\lambda_m}{\lambda_m}<\infty\,,$$

we have

$$\lim_{m\to\infty}\rho_{\lambda_m}^\epsilon(f) = \int f(x)dx + f(0)\,,\quad f\in\mathcal{S}(\mathbb{R})\,,\ \ \epsilon>0\,. \tag{1.3}$$

This Theorem proves the Berry-Tabor-Smilansky conjecture for phase shifts for our class of surfaces. The statement can only hold almost everywhere as we can produce one parameter families of surfaces for which the pair correlation measure is *not* uniform. The proof of the Theorem is based on Proposition 5 below which is a somewhat stronger and more precise result.

ACKNOWLEDGMENTS. This note originated in a discussion between U.Smilansky and the first author at the Newton Institute during the program on Quantum Chaos and Disordered Systems in 1997 following a talk on the results of [17]. Smilansky pointed out that the formal WKB formula for phase shifts of a surface of revolution with cylindrical end closely resembled the WKB formula for eigenvalues of an integrable quantum map on S^2 and proposed the problem of proving that the scaled phase shifts exhibit Poisson statistics. We would like thank U.Smilansky for bringing this to our attention.

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2. Surfaces of revolution with cylindrical ends

We consider a class of incomplete two dimensional smooth manifolds denoted by $X \setminus \{m\}$, such that X is a topological completion of $X \setminus \{m\}$. We can then consider X as a manifold with a conic singularity. The manifold $X \setminus \{m\}$ is globally parametrized by $(0, \infty) \times \mathbb{S}^1$ and we put on it metrics of revolution:

$$g = dr^2 + a(r)^2 d\theta^2 \,, \quad r \in (0, \infty) \,, \ \theta \in \mathbb{S}^1 \,. \tag{2.1}$$

The metric is assumed to be a short range *cylindrical end* metric, that is, we require that

$$\left| \partial_r^k \left(a(r)^2 - 1 \right) \right| \leq C_k r^{-2-k} \,, \quad r \longrightarrow \infty \,. \tag{2.2}$$

At m we assume a conic structure:

$$a(0) = 0 \,, \quad a'(0) \neq 0 \,. \tag{2.3}$$

We also make a convexity assumption by demanding that

$$a'(r) > 0 \,. \tag{2.4}$$

The metric can be extended to a smooth metric on X (endowed with a natural $\mathcal{C}^\infty$ structure coming from polar coordinates, (r, θ)) if and only if

$$a'(0) = 1 \,, \quad a^{2p}(0) = 0 \,, \ p \geq 0 \,, \tag{2.5}$$

see for instance [3]. We will not assume (2.5) and consequently we allow *bullet like* surfaces shown in Fig.1.

The classical dynamics is given by the Hamiltonian flow of the metric:

$$p = |\xi|_{g_x}^2 = \rho^2 + a(r)^{-2} t^2 \,, \tag{2.6}$$

where we parametrized $T^*(X \setminus \{m\})$ by $(x, \xi) = (r, \theta; \rho, t)$, with ρ and t dual to r and θ respectively. As is well known this flow is completely integrable:

$$\{p, t\} = 0 \,,$$

and $t = \xi(\partial_\theta)$ is called the Clairaut integral. Abstractly, ∂_θ is the vector field generating the $\mathbb{S}^1$ action on $X \setminus \{m\}$. As in the case of compact simple surfaces of revolution (see [1],[5],[3]) we have as stronger statement:

Proposition 1. *For* $(X \setminus \{m\}, g)$ *with the metric* g *satisfying* (2.1)-(2.4) *there exist* global *action angle variables on* $T^*(X \setminus \{m\})$.

Although it plays no part in the proof, this is worth presenting here as the global action variables are closely related to the asymptotics of the phase shifts.

Proof. The moment map

$$T^*(X \setminus \{m\}) \ni (x, \xi) \stackrel{P}{\longmapsto} (|\xi|_{g_x}, \xi(\partial_\theta)) \in \mathbb{R}_+ \times \mathbb{R}$$

has the range given by the open set $B = \{(b_1, b_2) : |b_2| < b_1\}$. For any $(b_1, b_2) \in B$, $P^{-1}(b_1, b_2)$ consists of a $\mathbb{R} \times \mathbb{S}^1$ orbit of a single geodesic in $T^*(X \setminus \{m\}$ (the $\mathbb{R}$-action corresponds to the geodesic flow and the $\mathbb{S}^1$-action to the θ-rotation).

In the case of case of a *simple* surface of revolution, the global action variables, (I_1, I_2), are defined by

$$I_j(b) = \frac{1}{2\pi} \int_{\gamma_j(b)} \alpha \,, \quad \alpha = \xi \cdot dx \,, \tag{2.7}$$

where $(\gamma_1(b), \gamma_2(b))$ is a global trivialization of the bundle $H_1(P^{-1}(b), \mathbb{Z})$ of the homology groups along the the fibers of P. When $\gamma_1(b)$ is chosen as the orbit of the $\mathbb{S}^1$-action, then $I_1 = \xi(\partial_\theta)$. In the case of non-compact surfaces discussed here the fibers are given by $\mathbb{R} \times \mathbb{S}^1$ and not by $\mathbb{S}^1 \times \mathbb{S}^1$ (except for the degenerate case of the meridians, $t = 0$ where the fiber is $(\mathbb{R} \setminus \{0\}) \times \mathbb{S}^1$ where 0 corresponds to the point m). Consequently the integral for I_2 given by (2.7) diverges (for γ_1 we can still take the compact orbit of the $\mathbb{S}^1$-action). Hence we have to normalize the integral using the fact that the surface is asymptotic to a cylinder with $a(r) \equiv 1$. If we take $\gamma_2(b)$ to correspond to a geodesic in $P^{-1}(b)$, then outside of the turning point $\rho = 0$ (or $r = 0$ for the degenerate case of the meridians) it can be parametrized by r. Then $\xi \cdot dx$ becomes ρdr and we can put

$$I_2(b) = \frac{1}{\pi} \lim_{R \to \infty} \int_0^R \left(b_1^2 - \frac{{b_2}^2}{a(r)^2} \right)_+^{\frac{1}{2}} dr - \int_0^R (b_1^2 - b_2^2)^{\frac{1}{2}} dr \,, \tag{2.8}$$

that is we normalize by subtracting the "free" ρdr defined by $\rho^2 + b_2^2 = b_1^2$. From this we find the angle variables as in [1],[5]. □

3. Review of scattering theory

There are many ways of introducing the scattering matrix on a manifold of the type we consider. Since we only assume (2.2), X is not a b-manifold in the sense of Melrose – see [8], [7]. It is a manifold with a cusp metric at one end and a conic metric at the other – see [8]. We shall not however use this point of view here. Instead we will proceed more classically and we will define the scattering matrix using the wave operators – see [7] for an indication of the relation between the two approaches. As in the proof of Proposition 1 we need a free reference problem

$$X_0 \simeq \mathbb{R} \times \mathbb{S}^1 \,, \quad g_0 = dr^2 + d\theta^2 \,. \tag{3.1}$$

On X and X_0 we define the wave groups, $U(t)$ and $U_0(t)$:

$$U(t) \ : \ \mathcal{C}^\infty_c(X) \times \mathcal{C}^\infty_c(X) \ni (u_0, u_1) \longmapsto (u(t), D_t u(t))$$

where

$$(D_t^2 - \Delta_g) u = 0 \,, \quad u\restriction_{t=0} = u_0 \,, \quad D_t u \restriction_{t=0} = u_1 \,.$$

The operators $U(t)$ extend as a unitary group to the energy space, $\mathcal{H}(X)$, obtained by taking the closure of $\mathcal{C}^\infty_c(X) \times \mathcal{C}^\infty_c(X)$ with respect to the norm

$$\|(u_0, u_1)\|_E^2 = \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 \,.$$

The definition and properties of $U_0(t)$ are analogous.

We then define the Møller wave operators

$$W_\pm \ : \ \mathcal{H}(X_0) \longrightarrow \mathcal{H}(X) \,,$$

by

$$W_\pm[w] = \lim_{t \to \pm\infty} U(-t) \chi(r) U_0(t) w \,, \quad w \in \mathcal{H}(X_0) \,,$$

where $\chi \in \mathcal{C}^\infty([0, \infty); [0, 1])$, $\chi(r) \equiv 0$ for $r < 1$ and $\chi(r) \equiv 1$ for $r > 2$, and where for $r > 1$ we used the obvious idenitification of the corresponding subsets of X and X_0. In the situation we consider the existence of $W_\pm$ is quite straightforward and we choose the wave rather than the Schrödinger picture just for variety. The scattering operator is

$$S \stackrel{\text{def}}{=} W_-^* W_+ \ : \ \mathcal{H}(X_0) \longrightarrow \mathcal{H}(X_0) \tag{3.2}$$

and, as we will see below it is a unitary operator.

When there is no pure point spectrum then the wave operators $W_\pm$ are themselves unitary. In all situations they are partial isometries and $W_\pm^* = \lim_{t \to \pm\infty} U_0(-t) \chi(r) U(t)$. The null space of $W_\pm^*$ is the span of the L^2 eigenfunctions of Δ. Under our assumptions there could only be finitely many such eigenfunctions.

The wave operators have the intertwining properties:

$$W_\pm \begin{pmatrix} 0 & I \\ \Delta_g & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta_{g_0} & 0 \end{pmatrix} W_\pm \Longrightarrow \left[S, \begin{pmatrix} 0 & I \\ \Delta_{g_0} & 0 \end{pmatrix} \right] = 0 \,.$$

Since all operators commute with the generator of the $\mathbb{S}^1$ action, ∂_θ, we decompose S using the spectral decompositions of Δ_{g_0} and of ∂_θ. It is easy to check that

$$\begin{pmatrix} 0 & I \\ \Delta_{g_0} & 0 \end{pmatrix} = \int_{-\infty}^{\infty} \lambda dE^0_\lambda$$

where the Schwartz kernel of dE^0_λ is given by

$$dE^0_\lambda(r,\theta;r',\theta') = \frac{\operatorname{sgn}(\lambda)}{(2\pi)^2}\sum_{n\in\mathbb{Z}} e^{in(\theta-\theta')}\begin{pmatrix} 0 & I \\ \lambda^2 & 0 \end{pmatrix} e^{i\operatorname{sgn}(\lambda)(\lambda^2-n^2)^{\frac12}(r-r')}(\lambda^2-n^2)_+^{-\frac12}d\lambda\,.$$

Because S commutes with the generator of the free propagator, $U_0(t)$ we obtain the scattering matrix at fixed energy using the above spectral decomposition:

$$S = \int S(\lambda)dE^0_\lambda$$

and then the decomposition corresponding to the eigenvalues of ∂_θ:

$$S(\lambda) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} S_n(\lambda)e^{in(\theta-\theta')}\,.$$

From the structure of dE^0_λ it is clear that $S_n(\lambda)\equiv 0$ for $|n|>|\lambda|$. For $|\lambda|=|n|$ we follow [7] and put

$$S_n(\lambda) = \lim_{\tau\to|\lambda|+} S_n(\operatorname{sgn}(\lambda)\tau)\,.$$

We also note that $S_n(\lambda) = S_{-n}(\lambda)$.

For $|n|\le|\lambda|$, $S_n(\lambda)$ is a unitary operator, that is, it is given by multiplication by a complex number of unit length:

$$S_n(\lambda) = e^{2\pi i\delta_n(\lambda)} \tag{3.3}$$

and the number $\delta_n(\lambda)$ is the nth *phase shift* at energy λ^2. Another way to think about $S(\lambda)$ is as a diagonal unitary $(2n+1)\times(2n+1)$ matrix, where $n=[|\lambda|]$:

$$S(\lambda) = \left(e^{2\pi i\delta_k(\lambda)}\delta_{kj}\right)_{-n\le k,j\le n}\,.$$

A more “down-to-earth” definition, following the traditional way of introducing phase shifts in one dimensional scattering, is given through asymptotic expansions in (3.6) below.

The uniform behaviour as k and λ go to infinity and $k\ll\lambda$ is a well understood semi-classical problem. To describe it we separate variables in the eigen-equation of the Laplacian. We remark that this procedure can also provide direct proofs of the general scattering theoretical statements above.

The Laplace operator is given by

$$\Delta_g = D_r^2 - i\frac{a'(r)}{a(r)}D_r + \frac{1}{a(r)^2}D_\theta^2$$

and on the eigenspaces of D_θ it acts as

$$\begin{aligned}\Delta_n &= D_r^2 - -i\frac{a'(r)}{a(r)}D_r + \frac{1}{a(r)^2}n^2\\ &= a(r)^{-\frac12}\left(D_r^2 + \frac{n^2}{a(r)^2} - \frac{2a''(r)a(r)-(a'(r))^2}{4a(r)^2}\right)a(r)^{\frac12}\,.\end{aligned} \tag{3.4}$$

The reduced operator appearing in brackets in the second line above has a self-adjoint realization on $L^2((0,\infty)_r)$ and for large λ it can be considered semi-classically:

$$\begin{gathered}\Delta_n - \lambda^2 = \lambda^2 a(r)^{-\frac{1}{2}} P(x,h) a(r)^{\frac{1}{2}} \,, \quad h = \frac{1}{|\lambda|} \,, \; x = \frac{n}{\lambda} \,, \\ P(x,h) = (hD_r)^2 + V(r;x,h) - 1 \,, \\ V(r;x,h) = \frac{x^2}{a(r)^2} - h^2 \frac{2a''(r)a(r) - (a'(r))^2}{4a(r)^2} \,, \quad V_0(r;x) \stackrel{\text{def}}{=} V(r;x,0) \,. \end{gathered} \tag{3.5}$$

The principal symbol of $P(x,h)$ is given by $p = \rho^2 + x^2/a(r)^2 - 1$ and the natural range of x for which semi-classical methods are applicable is given by $0 < \epsilon < |x| < 1 - \epsilon$. In fact, since $a(r)$ is one at infinity, we approach zero energy when x^2 is close to 1. On the other hand when $x \to 0$ the characteristic variety of p has a singular limit – see Fig.3. A detailed analysis of the $x \to 0$ limit has to involve the lower order terms in $V(r;x,h)$. In particular, miraculous cancellations in the expansions due to the interaction between the leading and lower order terms occur when we have *product type* conic singularities since we can then use the theory of Bessel functions. The general situation is, at least to the authors, unclear at the moment. What is quite clear is that we have a uniform expansion in h/x.

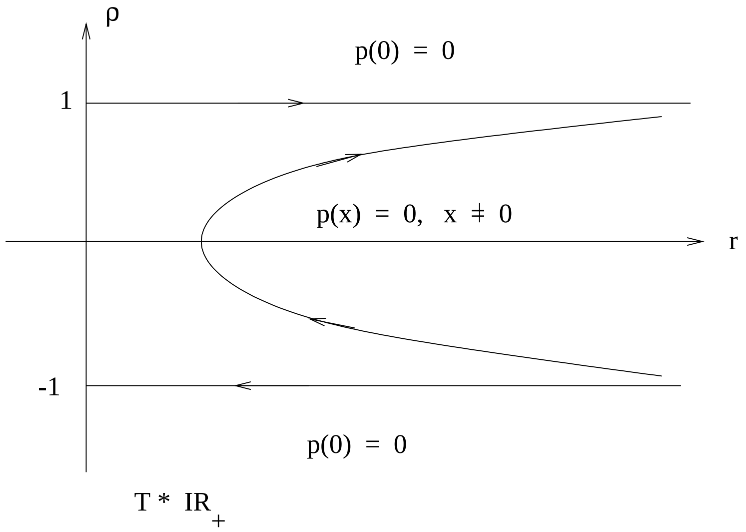

FIGURE 3. The characteristic variety of $P(x,h)$.

The phase shifts $\delta_n(\lambda)$ are related to the semi-classical phase shifts of the operator $P(x,h)$, $\psi(x,h)$ which are defined by asymptotics of solutions:

$$\begin{gathered} P(x,h)u = 0 \,, \quad u(r) = e^{\frac{i}{h}\sqrt{1-x^2}r} + e^{\frac{i}{h}\psi(x,h)} e^{-\frac{i}{h}\sqrt{1-x^2}r} + \mathcal{O}\left(\frac{1}{r}\right) \,, \quad r \to \infty \,, \\ \delta_n(\lambda) = \frac{|\lambda|}{2\pi} \psi\left(\frac{n}{\lambda}, \frac{1}{|\lambda|}\right) \,. \end{gathered} \tag{3.6}$$

We recall now the essentially standard asymptotic properties of ψ – see [9], Chapter 6 and for a more microlocal discussion [10].

Proposition 2. *As $h \to 0$, $\psi(x,h)$ defined by (3.6) has an asymptotic expansion uniform in $\epsilon < |x| < 1 - \epsilon$ for any fixed $\epsilon > 0$:*

$$\psi(x,h) \sim \psi(x) + h\frac{\pi}{2} + h^2 \psi_2(x) + \cdots \tag{3.7}$$

where

$$\psi(x) = \int_0^\infty \left((1 - V_0(r,x))_+^{\frac{1}{2}} - (1-x^2)^{\frac{1}{2}} \right) dr \,. \tag{3.8}$$

□

We remark that when we translate the asymptotics to the coordinates on $T^*(X \setminus \{m\})$: $x = t/\lambda$, $\lambda^2 = \rho^2 + t^2/a(r)^2$ we obtain the second action variable defined in the proof of Proposition 1.

As mentioned in the introduction, we can describe this connection between the phase shifts and action variables by saying that $S(\lambda)$ is a quantum map on $G(X,g)$, the space of geodesics. We now digress to explain this statement in more detail. For simplicity we will consider S at integral values of λ and denote them by N.

Since it is not needed in the calculation of the limit pair correlation function we give a somewhat sketchy discussion and refer to [17] for background on Toeplitz quantization. See also [14] for a related discussion from a physicist's point of view.

We can identify $G(X,g)$ with the set $S^*_{\text{in}}(X_{r_0})$ of incoming vectors at the parallel

$$X_{r_0} \stackrel{\text{def}}{=} X \cap \{r = r_0\} \,.$$

As in Proposition 2 we have to to delete ϵ-neighborhoods of the singular set, given by $\{|t/\lambda| < \epsilon\}$ where $t = I_1 = \xi(\partial_\theta)$ is the first action variable and $\lambda^2 = \rho^2 + t^2 a(r)^{-1}$ is the energy.

We denote the deleted space of geodesics by $G_\epsilon(X,g)$ and identify it with the set $S^*_{\text{in},\epsilon}(X_{r_0})$ of incoming vectors at X_{r_0} with incoming angle satisfying $|\theta| > \epsilon$ and $|\theta - \pi| > \epsilon$. If we consider the deleted space of geodesics as a phase space then, on the quantum level, it corresponds to the sequence of truncated Hilbert spaces $\mathcal{H}_{N,\epsilon}$ spanned by the eigenfunctions $\{e^{in\theta}\}$ of the quantum action $\frac{1}{N}\hat{I}_1$ with $\epsilon < |\frac{n}{N}| < 1 - \epsilon$ where $\hat{I}_1 = -i\partial_\theta$. Here, $1/N$ plays the role of the Planck constant and we restrict it to integral values. Since $\mathcal{H}_{N,\epsilon}$ is invariant under $S(N)$ we may restrict the latter to a unitary scattering matrix $S_\epsilon(N)$ on $\mathcal{H}_{N,\epsilon}$.

We now state the somewhat informal:

Proposition 3. *The sequence $\{S_\epsilon(N)\}$ is a semiclassical quantum map over $G_\epsilon(X,g)$ associated to the classical scattering map*

$$\beta : S^*_{\text{in},\epsilon}(X_{r_0}) \to S^*_{\text{in},\epsilon}(X_{r_0})$$

where $\beta(x,\xi)$ is obtained by following the geodesic $\gamma_{(x,\xi)}$ through (x,ξ) until it intersects X_{r_0} for the last time and reflecting the outgoing tangent vector inward.

Proof. From the explicit formula

$$S_\epsilon(N) = (e^{2\pi i \delta_k(N)} \delta_{kj})_{\epsilon \le |k/N|, |j/N| \le (1-\epsilon)}, \quad \delta_k(N) = \frac{|N|}{2\pi} \psi\left(\frac{k}{N}, \frac{1}{N}\right) \tag{3.9}$$

we see that $S_\epsilon(N)$ is the exponential of N times the Hamiltonian

$$\widehat{H}_{N,\epsilon} = \chi_\epsilon \left(\frac{\widehat{I}_1}{N} \right) \psi \left(\frac{\widehat{I}_1}{N}, \frac{1}{N} \right)$$

on $\mathcal{H}_{N,\epsilon}$ where χ_ϵ is a smooth cutoff function defining the truncated Hilbert space.

The truncated phase space $G_\epsilon(X,g)$ is symplectically equivalent to a truncated S^2, equipped with its standard area form, with neighbourhoods of the poles and of the equator deleted. Indeed, the equivalence is defined by the identity map between global action-angle charts on the surfaces. This map intertwines the obvious S^1 actions which rotate the spaces. The quantization of this chart then defines a unitary equivalence on the quantum level which intertwines the operators ∂_θ on cylinder and sphere (they can be considered as the angular momentum operators). The equivalence is specified up to a choice of $2N+1$ phases by mapping the spherical harmonic of degree N which transforms under rotation by θ on S^2 by $e^{ik\theta}$ to the exponential $e^{ik\theta}$ with $k \in [-N, \dots, N]$. The map is completely specified by requiring that the spherical harmonic be real valued along $\theta = 0$.

Thus we may identify $\widehat{H}_{N,\epsilon}$ with a Hamiltonian over the compact phase space S^2. Since it is a function of the (Toeplitz) action operator $\widehat{I}_1$ it is necessarily a semiclassical Toeplitz operator of order zero with principal symbol $\chi_\epsilon(I_1/E)\psi(I_1/E)$ on S^2. The semiclassical parameter N is identified in the Toeplitz theory with a first order positive elliptic Toeplitz operator with eigenvalue N in $\mathcal{H}_N$ – see [17] and references given there. Hence $N\widehat{H}_{N,\epsilon}$ is a first order Toeplitz operator of real principal type. As in the essentially analogous case of pseudodifferential operators, the exponential of a first order Toeplitz operator of real principal type is a Fourier integral Toeplitz operator whose underlying classical map is the Hamilton flow generated by ψ.

We now wish to identify this map at time one with the classical scattering (or billiard ball) map on $S^*_{\text{in},\epsilon}(X_{r_0})$, that is, we wish to prove that $\beta = \exp \Xi_\psi$ where Ξ_ψ is the Hamilton vector field of ψ.

Indeed, let us work in the symplectic action-angle coordinates (θ, I_1) where θ is the angle along X_{r_0}. The Hamilton flow of ψ then takes the form

$$\exp t\Xi_\psi(\theta, I_1) = (\theta + t\omega, I_1), \quad \omega = \partial_{I_1}\psi. \tag{3.10}$$

At time $t = 1$ the angle along the parallel X_{r_0} changes by ω. We claim that ω is also the change in angle along the incoming geodesic through $\theta \in X_{r_0}$ in the direction I_1 as it scatters in the bullet head before exiting again along X_{r_0}.

To see this, we use Proposition 2 which shows that ψ is closely related to the second action variable: in the notation of the proof of Proposition 1

$$I_2(b_1, b_2) = b_1\psi\left(\frac{b_2}{b_1}\right), \quad I_1(b_1, b_2) = b_2 \,.$$

Since $b_1 = \lambda$ is preserved by the flow we can fix it at $\lambda = 1$ and then

$$\partial_{I_1}\psi(I_1) = 2I_1 \int \frac{dr}{a(r)^2}\left(1 - \frac{I_1^2}{a(r)^2}\right)_+^{-\frac{1}{2}}. \tag{3.11}$$

On the other hand the equations of motion show that

$$\frac{d\theta}{dr} = \frac{\dot\theta}{\dot r} = \frac{2ta(r)^{-2}}{2\rho} = \frac{I_1}{a(r)^2}\left(1 - \frac{I_1^2}{a(r)^2}\right)^{-\frac{1}{2}}. \tag{3.12}$$

It follows that $\omega(I_1)$ is twice the change in angle as the radial distance changes from r_0 to its minimum along the geodesic.

The piece of the geodesic lying in the bullet-head consists of two segments: the initial segment beginning on S_{r_0} and ending upon its tangential intersection with the parallel $X_{r_-(I_1)}$ closest to m, and the segment beginning at this intersection and ending on X_{r_0}. The change in θ-angle along both segments is the same, so that the total change in angle during the scattering is given by the integral (3.12) above. This shows that β and $\exp \Xi_\psi$ have precisely the same formula in action-angle variables and completes the proof of the proposition. $\square$

4. Exponential sums

Following [17] we will reduce the study of (1.2) to a study of certain exponential sums. We first remark that because of symmetries of $\delta_k(\lambda)$ we can study a slightly simpler expression

$$\tilde\rho_\lambda^\epsilon(f) = \frac{1}{(1-2\epsilon)\lambda}\sum_{m\in\mathbb{Z}}\sum_{\epsilon<j/\lambda,k/\lambda<1-\epsilon} f\left((1-\epsilon)\lambda(\delta_j(\lambda) - \delta_k(\lambda) + m)\right)$$

as one easily checks that

$$\tilde\rho_\lambda^\epsilon(f) = \frac{1}{2}\rho_\lambda^\epsilon\left(f\left(\frac{1}{2}\bullet\right)\right).$$

The reduction to exponential sums follows from an application of the Poisson summation formula in m:

$$\tilde{\rho}^{\epsilon}_{\lambda}(f) = \frac{1}{[(1-2\epsilon)\lambda]^2} \sum_{\ell\in\mathbb{Z}} \hat{f}\left(\frac{2\pi\ell}{(1-2\epsilon)\lambda}\right) \left| \sum_{\epsilon<k/\lambda<1-\epsilon} e^{i\ell\delta_k(\lambda)} \right|^2 .$$

From this we see that

$$\tilde{\rho}^{\epsilon}_{\lambda}(f) = \hat{f}(0) + \int \hat{f}(2\pi\xi)d\xi + o_{\lambda\to\infty}(1) + E^{\epsilon}_{\lambda}(f) ,$$

$$E^{\epsilon}_{\lambda}(f) \stackrel{\text{def}}{=} \frac{1}{[(1-2\epsilon)\lambda]^2} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \hat{f}\left(\frac{2\pi\ell}{(1-2\epsilon)\lambda}\right) \sum_{\substack{\epsilon<k/\lambda,j/\lambda<1-\epsilon \\ j\neq k}} e^{i\ell(\delta_k(\lambda)-\delta_j(\lambda))} . \tag{4.1}$$

Ideally, we would like to show that $E^{\epsilon}_{\lambda}(f) \to 0$ as $\lambda \to 0$. That however seems very hard and for some surfaces is simply not true. As in [13],[11],[17] we will instead consider families of surfaces and resulting families of scattering phase shifts:

$$(\alpha_0-\gamma,\alpha_0+\gamma)\times(-\gamma,\gamma) \ni (\alpha,\beta) \longmapsto \delta^{\alpha,\beta}_k(\lambda) .$$

Replacing $\delta_k(\lambda)$ by $\delta^{\alpha,\beta}_k(\lambda)$ in (4.1) we define

$$(\alpha_0-\gamma,\alpha_0+\gamma)\times(-\gamma,\gamma) \ni (\alpha,\beta) \longmapsto E^{\epsilon}_{\lambda}(f;\alpha,\beta) . \tag{4.2}$$

To see the point of doing this we recall from [13],[11] and [17] the following simple

Lemma 1. *If for any $f\in\mathcal{S}(\mathbb{R})$*

$$\int_{\alpha_0-\gamma}^{\alpha_0+\gamma}\int_{-\gamma}^{\gamma} |E^{\epsilon}_{\lambda}(f;\alpha,\beta)|^2 d\alpha d\beta \leq C_{\epsilon,f}F(\lambda) ,$$

then for any sequence $\{\lambda_m\}_{m=0}^{\infty}$ such that

$$\sum_{m=0}^{\infty} F(\lambda_m) < \infty$$

we have

$$E^{\epsilon}_{\lambda_m}(f;\alpha,\beta) \longrightarrow 0 ,\ m\longrightarrow\infty ,\quad \forall\, f\in\mathcal{S}(\mathbb{R})$$

almost everywhere in $(\alpha,\beta)\in(\alpha_0-\gamma,\alpha_0+\gamma)\times(-\gamma,\gamma)$.

□

When $\delta^{\alpha,\beta}_k(\lambda)$ have a somewhat idealized form, the crucial estimate comes from [17], Theorem 5.1.1 where it is loosely based on the *Vinogradov method.* Since we will need a further development of these estimates we present a slightly modified proof.

Proposition 4. *If in* (4.1) *and* (4.2)

$$\delta^{\alpha,\beta}_k(\lambda) = \alpha k + \beta\lambda\Phi\left(\frac{k}{\lambda}\right) , \quad \Phi\in\mathcal{C}^{\infty}((0,1)) , \quad |\Phi''\restriction_{(\epsilon,1-\epsilon)}| \geq C_{\epsilon} > 0 ,$$

then for any $f\in\mathcal{S}(\mathbb{R})$

$$\int_{-1}^{1}\int_{-1}^{1} |E^{\epsilon}_{\lambda}(f;\alpha,\beta)|^2 d\alpha d\beta = \mathcal{O}_{f,\epsilon}\left(\frac{\log^3\lambda}{\lambda}\right) , \quad \lambda\longrightarrow\infty .$$

Proof. Let $\rho_\delta\in\mathcal{C}^{\infty}(\mathbb{R})$ have the following properties

$$\rho_\delta(t) \geq 1\!\!1_{[-1,1]}(t) , \quad \operatorname{supp}\hat{\rho}_\delta \subset (-\delta,\delta) .$$

The estimate of the lemma will clearly follow from

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \rho_\delta(\alpha)\rho_\delta(\beta)\left|E^\epsilon_\lambda(f;\alpha,\beta)\right|^2 d\alpha d\beta = \mathcal{O}\left(\frac{\log^3\lambda}{\lambda}\right). \tag{4.3}$$

Using the representation of E^ϵ_λ, (4.1), the left hand side of (4.3) can be rewritten as

$$\frac{1}{\lambda^4}\sum_{\ell_1\neq 0}\sum_{\ell_2\neq 0} g\left(\frac{\ell_1}{\lambda}\right)\overline{g\left(\frac{\ell_2}{\lambda}\right)} \sum_{\substack{\epsilon<k_1/\lambda, j_1/\lambda<1-\epsilon\\ k_1\neq j_1}}\ \sum_{\substack{\epsilon<k_2/\lambda, j_2/\lambda<1-\epsilon\\ k_2\neq j_2}}$$
$$\hat\rho_\delta\left(\ell_1(j_1-k_1)-\ell_2(j_2-k_2)\right)\hat\rho_\delta\left(\lambda\left(\ell_1\left(\Phi\left(\frac{j_1}{\lambda}\right)-\Phi\left(\frac{k_1}{\lambda}\right)\right)-\ell_2\left(\Phi\left(\frac{j_2}{\lambda}\right)-\Phi\left(\frac{k_2}{\lambda}\right)\right)\right)\right), \tag{4.4}$$

where we dropped the overall factor of $(1-2\epsilon)^{-2}$ and put $g(\xi)\stackrel{\mathrm{def}}{=}f(2\pi(1-2\epsilon)^{-1}\xi)$. From now on we will drop the parameter ϵ altogether: we can for instance extend Φ as a strictly convex or concave function to $[0,1]$ adding additional positive terms to the sums which are being estimated or we can shift and rescale the variables.

The support condition on $\hat\rho_\delta$ implies that

$$\begin{gathered}\ell_1(j_1-k_1)-\ell_2(j_2-k_2)=0\\ \left|\ell_1\left(\Phi\left(\frac{j_1}{\lambda}\right)-\Phi\left(\frac{k_1}{\lambda}\right)\right)-\ell_2\left(\Phi\left(\frac{j_2}{\lambda}\right)-\Phi\left(\frac{k_2}{\lambda}\right)\right)\right|\leq\frac{\delta}{\lambda}.\end{gathered} \tag{4.5}$$

To understand the second expression we apply the mean value theorem twice to the difference of Φ's. For that we make a simple observation: if ϕ'' has a fixed sign on $[a-\epsilon,b+\epsilon]$ then if $(m+h)/2,(m-h)/2\in[a,b]$

$$\phi\left(\frac{m+h}{2}\right)-\phi\left(\frac{m-h}{2}\right)=h\phi'(\Xi(m,h)),$$
$$\frac{1}{2}\frac{\min|\phi''|}{\max|\phi''|}\leq\frac{\partial\Xi}{\partial m}(m,h)\leq\frac{1}{2}\frac{\max|\phi''|}{\min|\phi''|}.$$

In our case we put $\phi=\Phi(\bullet/\lambda)$ and $h_i=j_i-k_i\neq 0$ and $m_i=j_i+k_i$. Then with

$$\xi_{\lambda,h}(m)\stackrel{\mathrm{def}}{=}\Xi(m,h)$$

we have

$$1/C<\partial_m\xi_{\lambda,h}(m)<C, \tag{4.6}$$

and (4.5) implies

$$\begin{gathered}\ell_1h_1=\ell_2h_2,\quad 0<|h_i|\leq\lambda\\ \left|\ell_1h_1\left(m_2-\xi^{-1}_{\lambda,h_2}(\xi_{\lambda,h_1}(m_1))\right)\right|\leq C\delta\lambda,\quad 0\leq m_i\leq 2\lambda,\end{gathered} \tag{4.7}$$

where we can invert ξ_{λ,h_2} in view of (4.6).

Thus we want study the sets of six integers (h_i,m_i,l_i), $i=1,2$, satisfying (4.7). We first note that, say, h_2 is determined by h_1,ℓ_1,ℓ_2. Then we see from the second inequality in (4.7) and from (4.6) that for fixed (h_1,h_2,m_1,ℓ_1) there are

$$\mathcal{O}(1)\max\left(1,\frac{\lambda}{|\ell_1h_1|}\right)\quad m_2\text{'s satisfying (4.7).}$$

When $\lambda>|\ell_1h_1|$ the contribution to (4.4) is estimated by

$$\frac{1}{\lambda^4}\sum_{1\leq m_1\leq 2\lambda}\ \sum_{\substack{-\lambda\leq h_1\leq\lambda\\ h_1\neq 0}}\ \sum_{\ell_1\neq 0}\sum_{\ell_2\neq 0}\frac{\lambda}{|\ell_1h_1|}\left|g\left(\frac{\ell_1}{\lambda}\right)\right|\left|g\left(\frac{\ell_2}{\lambda}\right)\right|\leq C\delta\frac{\log\lambda}{\lambda}\int_{|\xi|>\lambda}|g(\xi)|\frac{d\xi}{|\xi|}\int|g(\xi)|d\xi$$
$$\leq C_g\delta\frac{\log^2\lambda}{\lambda}.$$

When $\lambda \leq |\ell_1 h_1|$ then the number of m_2's is uniformly bounded for each choice of the other variables. We want to count the triples (h_1, h_2, ℓ_1) satisfying the first equation of (4.7) as a function of ℓ_2 and λ. Let $F(\lambda, \ell_2)$ denote that number. If $d(n)$ denotes the number of divisors of $n \neq 0$ then

$$F(\lambda, \ell_2) \leq 8 \sum_{0<h_2\leq\lambda} d(h_2|\ell_2|)$$

since $\ell_1 h_1 = \ell_2 h_2$ and each factorization into a product has to be counted twice since ℓ_1 and h_1 can be interchanged. Then

$$\begin{aligned} G(\lambda, N) &\stackrel{\text{def}}{=} \sum_{0\neq|\ell_2|\leq N} F(\lambda, \ell_2) \leq 8 \sum_{\ell_2\neq 0} \sum_{0<h_2\leq\lambda} d(|\ell_2|h_2) \\ &\leq C \sum_{1\leq n\leq N\lambda} d(n)^2 \leq C'\lambda N(\log\lambda + \log N)^3 \,, \end{aligned}$$

by a theorem of Ramanujan - see [6], Sect.18.2 and references given there. Hence the part of (4.4) corresponding to the bounded number of m_2's is bounded by

$$\begin{aligned} C\lambda^{-4} \sum_{1\leq m_1\leq 2\lambda} \max|g| \sum_{\ell_2\neq 0} F(\lambda, \ell_2) \left| g\left(\frac{\ell_2}{\lambda}\right)\right| &\leq C_g'\lambda^{-3} \int (G(\lambda, \lambda|\xi|) + 1)\, |g'(\xi)| d\xi \\ &\leq C_g \frac{\log^3\lambda}{\lambda} \,, \end{aligned}$$

where we used summation by parts and then approximation by the Riemann integral. This completes the proof of the proposition. □

We now recall from Proposition 2 that for surfaces we consider we have

$$\delta_k(\lambda) = \lambda\psi\left(\frac{k}{\lambda}\right) + \frac{1}{4} + \frac{1}{\lambda}\psi_2\left(\frac{k}{\lambda}, \frac{1}{\lambda}\right) \,, \quad \epsilon < \frac{k}{\lambda} < 1 - \epsilon \,, \quad \lambda \to \infty \,, \tag{4.8}$$

where for $\epsilon < x < 1 - \epsilon$, $\partial_x^k \psi_2 = \mathcal{O}_{k,\epsilon}(1)$.

When the family of surfaces depends on two parameters, (α, β), so that (2.1)-(2.4) hold uniformly, then we also have (4.8) uniformly with respect to the parameters. Hence to apply Proposition 4 to our case we need to estimate the contribution of the error terms coming from $\psi_2^{\alpha,\beta}$. That is given in

Proposition 5. *Let $\delta_k^{\alpha,\beta}(\lambda)$ be given by (4.8) uniformly in $(\alpha, \beta) \in (\alpha_0 - \gamma, \alpha_0 + \gamma) \times (-\gamma, \gamma)$ with*

$$\psi^{\alpha,\beta}(x) = \alpha x + \beta\Phi(x) \,, \quad \Phi \in \mathcal{C}^\infty((0,1)) \,, \quad |\Phi''\restriction_{(\epsilon,1-\epsilon)}| \geq C_\epsilon > 0 \,.$$

Then

$$\int_{\alpha_0-\gamma}^{\alpha_0+\gamma} \int_{-\gamma}^{\gamma} |E_\lambda^\epsilon(f; \alpha, \beta)|^2 d\alpha d\beta = \mathcal{O}_{f,\epsilon}\left(\frac{\log^3\lambda}{\lambda}\right) \,.$$

Proof. We observe that $\psi^{\alpha,\beta}$ is defined for all (α, β) and that we can extend $\delta_k^{\alpha,\beta}$ to all (α, β) by smoothly cutting off the lower order terms for $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} \setminus (-\gamma - \epsilon, \gamma + \epsilon) \times (\alpha_0 - \gamma - \epsilon, \alpha_0 + \gamma + \epsilon)$. Using the inequality $|x|^2 \leq 2|y|^2 + 2|x - y|^2$ and Proposition 4 we see that we need to estimate

$$\int\int \left| \frac{1}{\lambda^2} \sum_{\ell\neq 0} g\left(\frac{\ell}{\lambda}\right) \sum_{\substack{\epsilon<k/\lambda, j/\lambda<1-\epsilon \\ k\neq j}} \left(e^{i\ell(\tilde\delta_k^{\alpha,\beta}(\lambda) - \tilde\delta_j^{\alpha,\beta}(\lambda))} - e^{i\ell(\delta_k^{\alpha,\beta}(\lambda) - \delta_j^{\alpha,\beta}(\lambda))} \right) \right|^2 \rho_\delta(\alpha)\rho_\delta(\beta) d\alpha d\beta \,,$$

$$\tilde\delta_k^{\alpha,\beta}(\lambda) \stackrel{\text{def}}{=} \lambda\psi^{\alpha,\beta}(k/\lambda) \,,$$

where we use the notation of the proof of Proposition 4. We now introduce

$$\tau(z) \stackrel{\text{def}}{=} 2i\sin\frac{z}{2}\exp\left(-i\frac{z}{2}\right) \,, \quad e^{ix} - e^{iy} = e^{ix}\tau(x - y) \,, \quad \frac{\tau(z)}{z} \in \mathcal{C}^\infty \,,$$

and put

$$\psi_{\ell,j,k,\lambda}(\alpha,\beta) = \frac{\lambda}{\ell}\tau\left(\ell\left(\delta_k^{\alpha,\beta}(\lambda) - \delta_j^{\alpha,\beta}(\lambda) - \tilde{\delta}_k^{\alpha,\beta}(\lambda) - \tilde{\delta}_j^{\alpha,\beta}(\lambda)\right)\right)$$
$$= \frac{\lambda}{\ell}\tau\left(\frac{\ell}{\lambda}\left(\psi_2^{\alpha,\beta}\left(\frac{k}{\lambda},\frac{1}{\lambda}\right) - \psi_2^{\alpha,\beta}\left(\frac{j}{\lambda},\frac{1}{\lambda}\right)\right)\right).$$

From Proposition 2 and the obvious properties of τ we see that $\psi_{\ell,j,k,\lambda}$ is $\mathcal{C}^\infty$ and that it satisfies the following estimates

$$\left|\partial_\alpha^{p_\alpha}\partial_\beta^{p_\beta}\psi_{\ell,j,k,\lambda}(\alpha,\beta)\right| \leq C_p\left(1+\left|\frac{\ell}{\lambda}\right|\right)^{p_\alpha+p_\beta}. \tag{4.9}$$

Hence we have to look at

$$\int\int\left|\frac{1}{\lambda^2}\sum_{\ell\neq0}\left(\frac{\ell}{\lambda}\right)g\left(\frac{\ell}{\lambda}\right)\sum_{\substack{\epsilon<k/\lambda,j/\lambda<1-\epsilon\\ k\neq j}}\psi_{\ell,j,k,\lambda}(\alpha,\beta)e^{i\ell(\tilde{\delta}_k^{\alpha,\beta}(\lambda)-\tilde{\delta}_j^{\alpha,\beta}(\lambda))}\right|^2\rho_\delta(\alpha)\rho_\delta(\beta)d\alpha d\beta\,.$$

We would like to proceed as in the proof of Proposition 4 but now taking of Fourier transforms has to be replaced by integration by parts. Using the analysis of the differences of $\tilde{\delta}_k^{\alpha,\beta}$ presented there we need to estimate

$$\frac{1}{\lambda^2}\sum_{\ell_1\neq0}\sum_{\ell_2\neq0}\left|\frac{\ell_1}{\lambda}g\left(\frac{\ell_1}{\lambda}\right)\right|\left|\frac{\ell_2}{\lambda}g\left(\frac{\ell_2}{\lambda}\right)\right|\langle\ell_1/\lambda\rangle^2\langle\ell_2/\lambda\rangle^2 I(\lambda,\ell_1,\ell_2)\,, \tag{4.10}$$

where

$$I(\lambda,\ell_1,\ell_2) = \sum_{m_1=1}^{2[\lambda]}\sum_{m_2=1}^{2[\lambda]}\sum_{\substack{h_1=-[\lambda]\\ h_1\neq0}}^{[\lambda]}\sum_{\substack{h_2=-[\lambda]\\ h_2\neq0}}^{[\lambda]}\langle\ell_1h_1-\ell_2h_2\rangle^{-2}\langle\lambda^{-1}(\ell_2h_2\xi_{\lambda,h_2}(m_2)-\ell_1h_1\xi_{\lambda,h_1}(m_1))\rangle^{-2}\,,$$

and where, as is usual, we write $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$. We note that we can absorb the terms $\langle\ell_1/\lambda\rangle^2$ and $\langle\ell_2/\lambda\rangle^2$ into the g terms.

We first observe that

$$\sum_{m\in\mathbb{Z}}\langle A-Bm\rangle^{-2} = \mathcal{O}(1)\max\left(1,\frac{1}{B}\right),$$

uniformly in $A\in\mathbb{R}$. In fact, for $|B|\leq1$ this follows from the comparison with the integral using the Euler-MacLaurin formula

$$\sum_{-\infty}^{\infty}f(n) = \int_{-\infty}^{\infty}f(x)dx + \mathcal{O}\left(\int_{-\infty}^{\infty}|f''(x)|dx\right),$$

and for $|B|>1$ we can write the sum as $\sum_k(1+B^2(A/B-[A/B]-k)^2)^{-1} = \mathcal{O}(1)$.

Using this and (4.6) we see that

$$\sum_{m\geq1}\langle A-B\xi_{\lambda,h}(m)\rangle^{-2} \leq C\sum_{m\geq1}\langle Bm-B\xi_{\lambda,h}^{-1}(A/B)\rangle^{-2} \leq C\max\left(1,\frac{1}{B}\right).$$

Hence, uniformly in ℓ_2, h_2

$$\sum_{m_1}\langle\lambda^{-1}(\ell_2h_2\xi_{\lambda,h_2}(m_2)-\ell_1h_1\xi_{\lambda,h_1}(m_1))\rangle^{-2} = \mathcal{O}(1)\max\left(1,\frac{\lambda}{|h_1\ell_1|}\right). \tag{4.11}$$

Proceeding as in the proof of Proposition 4 we introduce $\widetilde{F}(\lambda, \ell_2, p)$ as the number of (ℓ_1, h_1, h_2) satisfying $\ell_1 h_1 = \ell_2 h_2 + p$. We now have

$$\widetilde{F}(\lambda, \ell_2, p) \leq 4 \sum_{\substack{0 \neq |h_2| \leq \lambda \\ \ell_1 h_2 + p \neq 0}} d(|\ell_2 h_2 + p|) \,,$$

and

$$\begin{aligned}
\widetilde{G}(\lambda, N) &\stackrel{\text{def}}{=} \sum_{p=-\infty}^{\infty} \sum_{|\ell_2| \leq N} \widetilde{F}(\lambda, \ell_2, p) \langle p \rangle^{-2} \\
&\leq C_1 \sum_{p=-\infty}^{\infty} \sum_{\substack{1 \leq |n| \leq N\lambda \\ n+p \neq 0}} d(|n|) d(|n+p|) \langle p \rangle^{-2} \\
&\leq C_2 \left(\sum_{n=1}^{N\lambda} d(n)^2 \right)^{\frac{1}{2}} \sum_{p=0}^{\infty} \langle p \rangle^{-2} \left(\sum_{m=1}^{N\lambda + p} d(m)^2 \right)^{\frac{1}{2}} \\
&\leq C_3 \left(\lambda N (\log \lambda + \log N)^3 \right)^{\frac{1}{2}} \sum_{p=0}^{\infty} \langle p \rangle^{-2} \left((\lambda N + p) \log^3 (\lambda N + p) \right)^{\frac{1}{2}} \\
&\leq C \lambda N (\log \lambda + \log N)^3 \,.
\end{aligned}$$

Using (4.11) we can estimate (4.10) by:

$$C \max(1 + |\xi|) |g(\xi)| \lambda^{-3} \int \left(\widetilde{G}(\lambda, \lambda |\xi|) + 1 \right) |\partial_\xi (\xi g)(\xi)| d\xi \leq C_g \frac{\log^3 \lambda}{\lambda} \,,$$

completing the proof of the proposition. $\square$

Remark. As was emphasized by the referee, Propositions 4 and 5 are essentially sharp. The "diagonal" solutions $j_1 = j_2$, $k_1 = k_2$, $l_1 = l_2$ give a lower bound $1/\lambda$. Hence, no essential improvement of the Theorem in Sect.1 is possible by this method.

5. Construction of the family of surfaces

To prove the main theorem we need to construct a family of surfaces, $\mathcal{G} = \{g^{\alpha,\beta}\}$, for which in the expansion of the phase shifts (4.8),

$$\psi^{\alpha,\beta}(x) = \alpha x + \beta \Phi(x) \,, \quad |\Phi'' \restriction_{(\epsilon, 1-\epsilon)}| > C_\epsilon > 0 \,, \quad (\alpha, \beta) \in (\alpha_0 - \gamma, \alpha_0 + \gamma) \times (-\gamma, \gamma) \,, \tag{5.1}$$

so that we can apply Propositions 4 and 5. Recalling Proposition 2, this means that we want to find $a^{\alpha,\beta}$ satisfying (2.1)-(2.4), and such that

$$\psi^{\alpha,\beta}(x) = \frac{1}{\pi} \int_0^\infty \left(\left(1 - \frac{x^2}{a^{\alpha,\beta}(r)^2} \right)_+^{\frac{1}{2}} - (1 - x^2)^{\frac{1}{2}} \right) dr = \alpha x + \beta \Phi(x) \,, \tag{5.2}$$

with Φ convex or concave.

We will now skip the indices α and β. If we write

$$W(r) \stackrel{\text{def}}{=} \frac{1}{a(r)^2} - 1 \,,$$

and

$$\phi(x) \stackrel{\text{def}}{=} \int_0^\infty \left(\left(1 - \frac{x^2}{W(r)} \right)_+^{\frac{1}{2}} - 1 \right) dr \,, \tag{5.3}$$

then

$$\psi(x) = \frac{1}{\pi}\sqrt{1-x^2}\phi\left(\frac{x}{\sqrt{1-x^2}}\right),$$

and we might study the simpler function ϕ instead. From the assumptions on a, W is monotonically decreasing and $r^2W(r)$ is smooth and non-zero at $r=0$. Hence there exists a smooth monotonically increasing function, $y(r)$, such that

$$W(r) = \frac{1}{y(r)^2}.$$

Since we can write r as a function of y we define

$$F(y) \stackrel{\text{def}}{=} \frac{dr}{dy}(y).$$

That way we can express $\phi(x)$ as a linear transform of F:

$$\phi(x) = xI(F)(x), \quad I(F)(x) = \int_0^\infty \left(\left(1-\frac{1}{y^2}\right)_+^{\frac{1}{2}} - 1\right) F(xy)dy. \tag{5.4}$$

From this we immediately get a *linear model* correspoding to $F(x) \equiv t > 0$:

$$a(r) = \frac{r^2}{t^2+r^2} \quad \Longrightarrow \quad \psi(x) = \frac{\sqrt{1-x^2}}{\pi}\phi\left(\frac{x}{\sqrt{1-x^2}}\right) = -\frac{1}{2}tx,$$

since

$$\int_0^\infty \left((1-y^{-2})_+^{\frac{1}{2}} - 1\right) dy = -\pi/2.$$

Remark. The surfaces defined using the linear model do *not* have uniform pair correlations measures. In that case we can compute the leading contribution to $E_\epsilon^\lambda(f)$ directly. To apply Propositions 4 and 5 we need to have the linear term in ψ and that forces the singularity at 0 for our surfaces: only one value of α corresponds to a smooth surface.

We want to introduce the convex or concave term in ψ by perturbing the case $F \equiv \text{const}$. For that let us first establish some simple properties of the transform $F \mapsto I(F)$. We denote by $\mathcal{C}_{\mathrm{b}}^\infty$ smooth functions with bounded derivatives and by S^k_{phg} spaces of poly-homogeneous (classical) symbols.

Lemma 2. *For $g \in \mathcal{C}_{\mathrm{b}}^\infty([0,\infty))$*

$$I(g)(x) \in \mathcal{C}_{\mathrm{b}}^\infty([0,\infty)) + x\log x\,\mathcal{C}_{\mathrm{b}}^\infty([0,\infty)).$$

When $g \in S^{-2}_{\mathrm{phg}}([0,\infty))$ then

$$I(g)\restriction_{[1,\infty)} \in S^{-1}_{\mathrm{phg}}([1,\infty)), \quad I(g)(x) \sim \left(-\int f(y)dy\right)\frac{1}{x} + \sum_{k=2}^\infty \frac{g_k}{x^k}, \quad x \longrightarrow \infty.$$

Proof. To prove the first part of the lemma we write

$$I(g)(x) = \int_0^C \left((1-y^{-2})_+^{\frac{1}{2}} - 1\right) g(xy)dy + \int_C^\infty \left((1-y^{-2})^{\frac{1}{2}} - 1\right) g(xy)dy, \quad C > 1,$$

where the first term on the right hand side is clearly in $\mathcal{C}_{\mathrm{b}}^\infty([0,\infty))$. In the second term, the integrand can be rewritten as

$$\frac{1}{y^2}\left(-\frac{1}{2} + \frac{1}{8}\frac{1}{y^2} - \cdots\right) g(xy)dy.$$

Thus we are concerned with integrals of the form

$$\int_C^\infty \frac{1}{y^k} g(xy)dy = x^{k-1}\int_1^\infty Y^{-k}g(Y)dY + x^{k-1}\int_{Cx}^1 Y^{-k}g(Y)dY.$$

where the first term is smooth and uniformly bounded in k. To study the second term we write

$$g(Y) = g_0 + g_1 Y + \cdots g_{l-1} Y^{l-1} + \tilde{g}_l(Y) Y^l$$

which gives

$$\frac{1}{k-1} C^{-k+1} g_0 + \frac{1}{k-2} C^{-k+2} g_1 x + \cdots + \frac{C^{-k+l}}{k-l} g_{l-1} x^{l-1} + x^l F_{k,l,C}(x)\,,$$

for $k > l$ and

$$\frac{1}{k-1} C^{-k+1} g_0 + \frac{1}{k-2} C^{-k+2} g_1 x + \cdots + g_{k-1} x^{k-1} \log x + x^{k-1} G_{k,l,C}(x)\,,$$

for finitely many $k \leq l$. Since we check that

$$G_{k,l,C}(x) \in \mathcal{C}^\infty([0, 1/C))\,, \quad k \leq l\,,$$
$$|F_{k,l,C}(x)| \leq G_l(C^{-k+l+1} + |x|^{k-l-1})\,, \quad k > l\,,$$

we can sum up the contributions from different k's ($C > 1$, $|x| \ll 1$, and we use the uniform convergence of $(1-z)^{\frac{1}{2}}$). Thus for every l we obtain

$$I(g)(x) = h_{1,l}(x) + x \log x h_{2,l}(x) + \mathcal{O}(x^l)\,, \quad h_{1,l}, h_{2,l} \in \mathcal{C}^\infty\,,$$

and consequently $I(g) \in \mathcal{C}^\infty_{\rm b}([0,\infty)) + x \log x \mathcal{C}^\infty_{\rm b}([0,\infty))$.

The second part of the lemma is even more clear. If for large Y, $g(Y) \sim \sum_{k=2}^\infty G_k Y^{-k}$, then then

$$\int_0^\infty \left((1-y^{-2})_+^{\frac{1}{2}} - 1 \right) g(xy) \ \sim_{x\to\infty} \ -\frac{1}{x} \int g(Y) dY + \sum_{k=2}^\infty \frac{1}{x^k} \left(\int_1^\infty \left(1 - \frac{1}{y^2} \right)^{\frac{1}{2}} \frac{f_k}{y^k} dy \right)\,.$$

□

The lemma shows that we cannot expect smoothness of $\psi(x)$ at the end points $x = 0, 1$ but that the function is very well behaved in the interior, as in any case is implicit in Proposition 2.

Having discussed the general properties of the transform I we now state a straightforward

Lemma 3. *For $\Phi(x) = x I(f)(x/\sqrt{1-x^2})$ and $0 < x(z) = z/\sqrt{1+z^2} < 1$ we have*

$$\Phi''(x(z)) = (1+z^2)^{\frac{3}{2}} \left[\int_0^\infty \left((1-y^{-2})_+^{\frac{1}{2}} - 1 \right) \left((2+3z^2) y f'(yz) + z y^2 (1+z^2) f''(zy) \right) dy \right]\,.$$

□

Guided by the two lemmas we can easily construct a family of surfaces for which ψ has the needed properties. We want to find $f \in \mathcal{C}^\infty_{\rm b}(\mathbb{R})$ such that for β small enough and α close to α_0, $F^{\alpha,\beta}(y) = \alpha + \beta f(y) > 0$, and so that $a(r)$ obtained from inverting the process described above has the properties (2.1)-(2.4). This is easily achieved by demanding that f is a symbol of order -2 on $[0,\infty)$.

We also want $\Phi''(x)$ described in Lemma 3 to have a fixed sign. From the formula we see that that Φ is concave if

$$a f'(y) + y f''(y) > 0\,, \quad y > 0\,, \quad a = 2, 3\,. \tag{5.5}$$

In fact, the integrand has the same sign as

$$(2+3z^2) f'(Y) + (1+z^2) Y f''(Y) \restriction_{Y=yz}$$

and that is positive for any z if (5.5) holds.

We can summarize this discussion in

Proposition 6. *For α in a neighbourhood of a fixed $\alpha_0 < 0$ and for β small enough, let $a_f^{\alpha,\beta}(r)$ be obtained from $f \in S^{-2}_{\mathrm{phg}}([0,\infty))$ by the following procedure:*

$$F^{\alpha,\beta}(y) = -2\alpha - \beta f(y)\,,$$

$$\frac{dy^{\alpha,\beta}}{dr} = (F^{\alpha,\beta}(y^{\alpha,\beta}(r))^{-1}\,,\ y^{\alpha,\beta}(0) = 0\,,\ a^{\alpha,\beta}(r) = y^{\alpha,\beta}(r)(1 + y^{\alpha,\beta}(r)^2)^{-\frac{1}{2}}\,.$$

Then $a_f^{\alpha,\beta}$ has the properties (2.1)-(2.5) and for the set of two parameter families of surfaces

$$\mathcal{G} = \{(X, g_f^{\alpha,\beta})\ :\ g_f^{\alpha,\beta} = dr^2 + a_f^{\alpha,\beta}(r)d\theta^2\,,\quad f \in S^{-2}_{\mathrm{phg}}([0,\infty)) \text{ satisfies } (5.5),\ |\alpha - \alpha_0| < \delta_f\,,\ |\beta| < \epsilon_f\,\}$$

the leading part of the phase shifts depends linearly on α and β and (5.1) holds.

Combined with Propositions 2 and 5 this provides an infinite dimensional family of perturbations of the linear model each giving a two parameter family of surfaces for which the Theorem of Sect.1 holds. Perhaps the simplest example is obtained by putting

$$F^{\alpha,\beta}(y) = -2\alpha - \beta\frac{1}{1+\rho y^2}\,,\quad 0 < \rho \le \frac{2}{3}\,.$$

Proof of the Main Theorem. Proposition 6 guarantees that the leading parts of the expansions of the phase shifts of $(X, g^{\alpha,\beta})$ satisfy the assumptions of Proposition 5. Let $\rho_\lambda^{\epsilon,\alpha,\beta}$ denote the pair correlation measure for $(X, g^{\alpha,\beta})$ given by (1.2). Recalling (4.1) and the discussion preceding it we see that all for $f \in \mathcal{S}(\mathbb{R})$

$$\rho_\lambda^{\epsilon,\alpha,\beta}(f) = \hat{f}(0) + f(0) + o_{\lambda\to\infty}(1) + E_\lambda^\epsilon(f;\alpha,\beta)\,.$$

Proposition 5 now shows that the assumptions of Lemma 1 are satisfied with $F(\lambda) = \log^3\lambda/\lambda$ and that lemma gives the statement of the Main Theorem. $\square$

References

[1] V.I. Arnold *Mathematical Methods of Classical Mechanics,* Graduate Texts in Mathematics **60**, Springer-Verlag, 1989.

[2] M.V. Berry and M. Tabor, *Level clustering in regular spectrum.* Proc. Roy. Soc. Lond. Ser. A **356** (1977), 375-394.

[3] A. Besse, *Manifolds all of whose geodesics are closed.* Springer Verlag, 1978.

[4] O.Bohigas, *Random matrix theories and chaotic dynamics,* in *Chaos et Physique Quantique*, Les Houches LII, M.-J.Giannoni, A.Voros and J.Zinn-Justin eds., Elsevier 1991, 87-199.

[5] Y. Colin de Verdière, *Spectre conjoint d'operateurs pseudo-differentiels qui commutent I: Le cas integrable.* Math. Zeit. **171** (1980), 51-73.

[6] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers.* Oxford University Press, 1979.

[7] T. Christiansen, *Scattering theory for manifolds with asymptotically cylindrical ends.* J. Func. Anal. **131** (1995), 499-530.

[8] R.B. Melrose, *Geometric scattering theory.* Cambridge University Press, 1995.

[9] F.W.J. Olver, *Asymptotics and Special Functions.* A K Peters, Wellesley, MA, 1997.

[10] Th. Rammond, *Semiclassical study of quantum scattering on the line.* Comm. Math. Phys. **177**(1996), 221-254.

[11] Z. Rudnick and P. Sarnak, *The pair correlation function of fractional parts of polynomials.* preprint, 1997.

[12] P. Sarnak, *Arithmetic quantum chaos.* Schur lectures, Israel Math. Conf.Proc. **8**(1995).

[13] P. Sarnak, *Values at integers of binary quadratic forms.* C.M.S. Conf. Proc. **21**, A.M.S. (1997), 181-203.

[14] U. Smilansky, *The classical and quantum theory of chaotic scattering.* in *Chaos et Physique Quantique*, Les Houches LII, M.-J.Giannoni, A.Voros and J.Zinn-Justin eds., Elsevier 1991, 371-441.

[15] U. Smilansky, *Private communication.* 1997.

[16] J.Vanderkam, *Pair correlation of four-dimensional flat tori.* to appear in the Duke Math. J.

[17] S. Zelditch, *Level spacings for integrable quantum maps in genus zero.* to appear in Comm. Math. Phys.

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