FRACTAL UPPER BOUNDS ON THE DENSITY OF SEMICLASSICAL RESONANCES

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Abstract. We consider bounds on the number of semiclassical resonances in neighbourhoods of the size of the semiclassical parameter, h, around energy levels at which the flow is hyperbolic. We show that the number of resonances is bounded by $h^{-\nu}$ where $2\nu+1$ is essentially the dimension of the trapped on the energy surface. We note that in a confined setting this dimension equals to $2n-1$, where n is the dimension of the physical space, and the bound, h^{1-n}, corresponds to the optimal bound on the number of eigenvalues. Although no lower bounds of this type are rigorously known in the setting of semiclassical differential operators, the corresponding bound is optimal for certain models based on open quantum maps [26].

1. Introduction

Let $P=-h^2\Delta_g+V(x)$ be a self-adjoint Schrödinger operator on a compact Riemannian n-manifold, (X,g), $V\in\mathcal{C}^\infty(X;\mathbb{R})$. The spectral asymptotics as $h\to 0$ are given by the celebrated *Weyl law* – see [10] and [17] for recent advances and numerous references. If we assume that the zero energy surface is nondegenerate,

$$p\stackrel{\rm def}{=}|\xi|_g^2+V(x)=0\implies dp\neq 0\,,$$

then

$$|\mathrm{Spec(P)}\cap D(0,Ch)|=\mathcal{O}(h^{-n+1})\,, \tag{1.1}$$

where $D(0,r)=\{z\in\mathbb{C}\ :\ |z|<r\}$, though of course in this case the eigenvalues are all real – see §6.1 for yet another proof of this well known result.

Let H_p be the Hamilton vector field of p on T^*X, locally given by

$$H_p=\sum_{j=1}^n\frac{\partial p}{\partial\xi_j}\partial_{x_j}-\frac{\partial p}{\partial x_j}\partial_{\xi_j}\,,\quad (x,\xi)\in T^*\mathbb{R}^n\,.$$

When the flow, $\exp tH_p\ :\ p^{-1}(E)\to p^{-1}(E)$, has the property that the set of its closed orbits has Liouville measure zero on $p=0$, then we have the infinitesimal version of the Weyl law:

$$|\mathrm{Spec(P)}\cap D(0,Ch)|=\frac{2Ch}{(2\pi h)^n}\int_{p(x,\xi)=0}d\mathcal{L}(x,\xi)+o(h^{-n+1})\,, \tag{1.2}$$

where $d\mathcal{L}$ is the Liouville measure on $p=0$, that is $d\mathcal{L}dp = dxd\xi$. This result, due to Duistermaat-Guillemin, Petkov-Robert, and Ivrii (see [10, Chapter 11]), is the mathematical starting point of many recent investigations, mostly in physical literature, of the finer structure of the spectrum and its relation to classical dynamics – see [1] and references given there.

When the manifold is non-compact the situation is dramatically different. The simplest case is that of a manifold which is Euclidean outside of a compact set and $V+1 \in \mathcal{C}^\infty_{\rm c}(X;\mathbb{R})$. The discrete eigenvalues of P are replaced by *quantum resonances* which are defined as the poles of the meromorphic continuation of

$$(P-z)^{-1} \,:\, \mathcal{C}^\infty_{\rm c}(X) \;\longrightarrow\; \mathcal{C}^\infty(X)\,,\quad \mathrm{Im}\, z>0\,,$$

and we denote the set of resonances by $\mathrm{Res}(P(h))$. The basic physical interpretation is that a resonance at $z=E_0-i\Gamma/2$ corresponds to a state with time evolution given by $\exp(-itE_0/h-\Gamma t/2h)$, and to a Breit-Wigner peak in energy density, $\Gamma/((E-E_0)^2+\Gamma^2/4)$. This intuitive picture becomes however complicated when many resonances are present which is natural for h is small – see [25] and [3],[5],[28] for recent results and references.

Here we provide upper bounds for the number of resonances of P in $D(0,Ch)$. The main result (Theorem 3) states that for classical Hamiltonians p with hyperbolic flow on $p=0$,

$$|\,\mathrm{Res}\, P(h)\cap D(0,Ch)| = \mathcal{O}(h^{-\nu})\,, \tag{1.3}$$

where $2\nu+1$ is essentially the dimension of the trapped (non-wandering) set in $p^{-1}(0)$,

$$K \stackrel{\rm def}{=} \{(x,\xi)\in T^*X \,:\, p(x,\xi)=0\,,\ \exp(tH_p)(x,\xi)\not\to\infty\,,\quad t\to\pm\infty\}\,.$$

In the case of a compact manifold $\nu=n-1$ so that (1.3) reduces to (1.1). By dimension we always mean the Minkowski dimension

$$m_0 = 2n-1-\sup\{d \,:\, \limsup_{\epsilon\to 0}\epsilon^{-d}\,\mathrm{vol}(\{\rho\in p^{-1}(0)\,:\, d(\rho,K)<\epsilon\})<\infty\}\,.$$

A simple example is provided by a three bump potential shown in Fig.1.

The first estimate of this form was proved by the first author in [30, Theorems 4.6, 5.5, and 5.7]: there exists constants $C_0, C_1>0$, such that for $\delta_0>0$ fixed and small enough

$$\begin{gathered}|\,\mathrm{Res}(P(h))\cap\{z\,:\,\ |z|<\delta\,,\ \ \mathrm{Im}\, z>-\mu\}| \le C_1\delta\left(\frac{h}{\mu}\right)^{-n}\mu^{-\frac{1}{2}\widetilde{m}}\,,\\ C_0h\le\mu\le 1/C_0\,,\ \ C_0h^{\frac12}\le\delta\le\delta_0\,,\ \ 0<h<1/C_0\,,\end{gathered} \tag{1.4}$$

where now $\widetilde{m}$ is any number greater than the dimension of the trapped set in

$$p^{-1}([-2\delta_0,2\delta_0])\,.$$

In homogeneous situations, such as for instance obstacle scattering, $\widetilde{m}=m+1$. When $\mu=C_0h$, the improvement in Theorem 3 lies in allowing $\delta\simeq h$, which is the natural limit for this type of spectral estimates.

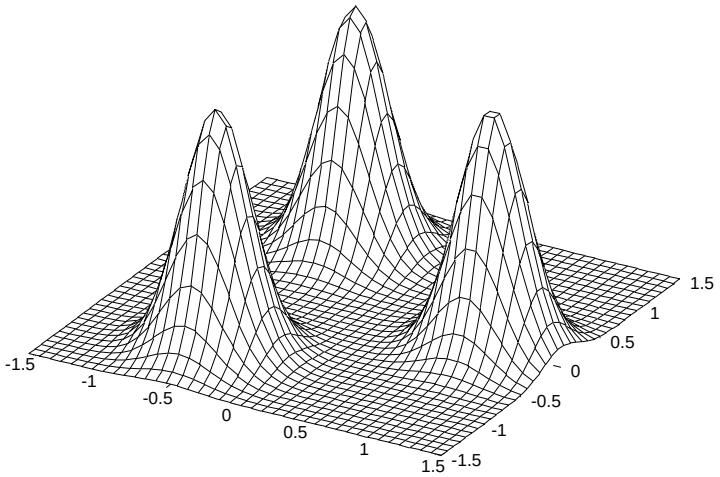

FIGURE 1. A three bump potential exhibiting hyperbolic dynamics at an interval of energies.

Earlier, non-geometric, bounds on the number of resonances (scattering poles) were obtained by Melrose [22],[23] and the second author [40],[41]. In the case of convex co-compact Schottky quotients (and any convex co-compact quotients in dimension two) the analogue of (1.3) was proved in [12] using zeta function techniques, improving earlier estimates of [42] the proof of which was largely based on [30]. These technique gave similar results for the zeros of zeta functions of rational maps [8],[36], in which case the dimension of the trapped set becomes essentially the dimension of the Julia set.

Numerical investigations in different settings of semiclassical three bump potentials [18],[19] (see Fig.2), Schottky quotients [12], three disc scattering [20], and Cantor-like Julia sets for $z \mapsto z^2 + c$, $c < -2$ [36], suggest that for $\mu \simeq Ch$ and $\delta \simeq 1$ the estimate (1.4) is optimal. A different model was recently considered in [26]: quantum resonances were defined using an open quantum map with a classical "trapped set" corresponding to K intersected with a hypersurface transversal to the flow. The numerical results and a simple linear algebraic toy model suggest that the fine estimate (1.3) is optimal. A similar model was also used in [29] where the fractal Weyl law gave corrections to the applications of random matrix theory to open quantum systems.

We now state the general assumptions on the operator P. We reiterate that the simplest case to keep in mind is

$$P = -h^2\Delta + V(x) - 1\,, \quad V \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)\,.$$

In general we consider

$$P(h) \in \Psi^{0,2}(X)\,, \quad P(h) = P(h)^*\,,$$

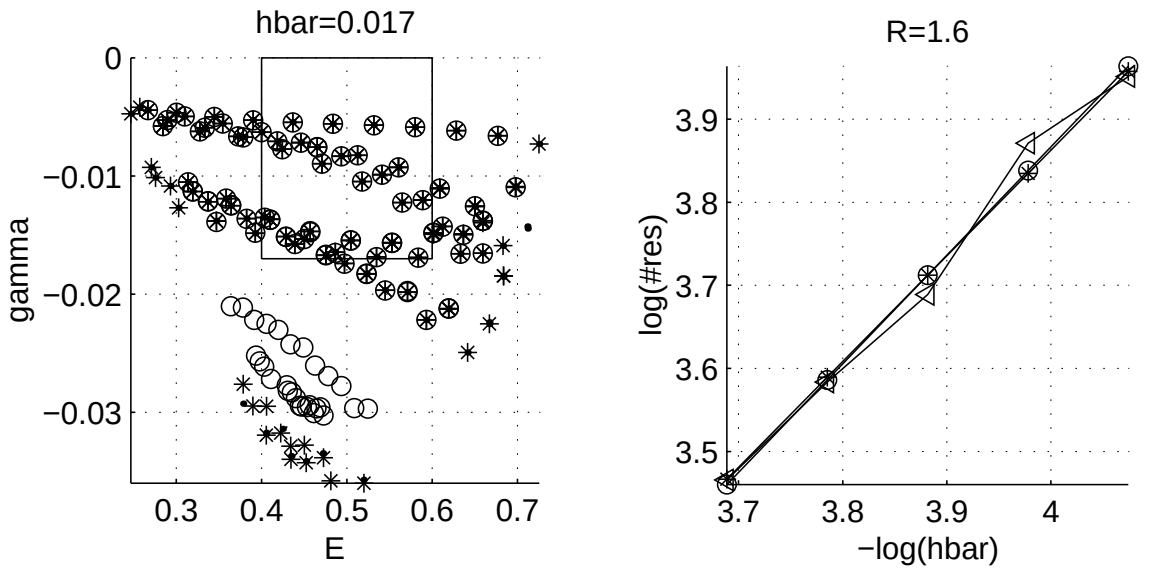

FIGURE 2. A sample of results of [18]: the plot on the left shows resonances for $h = 0.017$, and the plot on the right is the log-log plot of the number of resonances vs. h with $\triangle$ denoting numerical data, $*$ the density predicted by the fractal Weyl law, and $\circ$, the least square interpolants.

where the calculus of semiclassical pseudodifferential operators is reviewed in §3.1,

$$\begin{aligned} &P(h) = p^w(x,hD) + hp_1^w(x,hD;h)\,,\quad p_1 \in S^{0,2}(T^*X)\,,\\ &|\xi| \geq C \implies p(x,\xi) \geq \langle\xi\rangle^2/C\,,\quad p = 0 \implies dp \neq 0\,,\\ &\exists\, R,\ \forall\, u \in \mathcal{C}^\infty(X \setminus B(0,R))\,,\quad P(h)u(x) = Q(h)u(x)\,, \end{aligned} \tag{1.5}$$

where

$$Q(h) = \sum_{|\alpha|\leq 2} a_\alpha(x;h)(hD_x)^\alpha\,,$$

with $a_\alpha(x;h) = a_\alpha(x)$ independent of h for $|\alpha| = 2$, $a_\alpha(x;h) \in C_b^\infty(\mathbb{R}^n)$ uniformly bounded with respect to h (here $C_b^\infty(\mathbb{R}^n)$ denotes the space of C^∞ functions on $\mathbb{R}^n$ with bounded derivatives of all orders), and

$$\begin{aligned} &\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq (1/c)|\xi|^2,\ \ \forall \xi \in \mathbb{R}^n\,,\ \text{for some constant } c > 0,\\ &\sum_{|\alpha|\leq 2} a_\alpha(x;h)\xi^\alpha \longrightarrow \xi^2 - 1 \text{ uniformly with respect to } h \text{ as } |x| \to \infty. \end{aligned} \tag{1.6}$$

We also need the following analyticity assumption in a neighbourhood of infinity: there exist $\theta \in [0,\pi)$, $\epsilon > 0$ and $R \geq R_0$ such that the coefficients $a_\alpha(x;h)$ of $Q(h)$ extend

holomorphically in x to

$$\{r\omega : \omega \in \mathbb{C}^n\,,\ \ \mathrm{dist}(\omega, \mathbf{S}^n) < \epsilon\,,\ r \in \mathbb{C}\,,\ |r| > R\,,\ \arg r \in [-\epsilon, \theta_0 + \epsilon)\}\,,$$

with (1.6) valid also in this larger set of x's. We remark that in [13],[30] the operators were required to be globally real analytic but the conditions at infinity were much more general. A particularly nice feature of the theory developed in [13] is allowing arbitrary homogeneous polynomials as potentials (see [30, (c.31)-(c.33)]).

The first theorem we present fits naturally in the methodology of this paper. It is a slight generalization of a result of Martinez [21] which in turn was a $\mathcal{C}^\infty$ version of a similar result in [30], already implicit in [13]. As discussed at the end of §4 it is essentially optimal.

Theorem 1. *Suppose that X is non compact and euclidean outside of a compact set, and that the operator $P \in \Psi^{0,2}(X)$ satisfies the assumptions below. Suppose in addition that no orbit of H_p on $p^{-1}(0)$ is trapped:*

$$\forall\ K \Subset p^{-1}(0)\,,\ \exists\ T_K\,,\ \ (x,\xi) \in K \implies \exp(tH_p(x,\xi)) \notin K\,,\ t > T_K\,. \tag{1.7}$$

Then, for any $M > 0$ there exists $h(M)$ such that for $0 < h < h(M)$,

$$|\,\mathrm{Res}(P) \cap D(0, Mh\log(1/h))| = \emptyset\,. \tag{1.8}$$

Here $\mathrm{Res}(P)$ *denotes the set of resonances of P defined in some h-independent neighbourhood of* 0.

Before stating the main result which requires hyperbolicity of the flow on the energy surface we first give the general upper bound which generalizes slightly[†] the results of Bony [2] which in turn generalized earlier results of Petkov and the second author [27],[28].

Theorem 2. *Suppose that X is Euclidean outside a compact set, and that the operator $P \in \Psi^{2,0}(X)$ satisfies the general assumptions below. Then, as $h \to 0$,*

$$|\,\mathrm{Res}(P) \cap D(0, Ch)| = \mathcal{O}(h^{-n+1})\,. \tag{1.9}$$

The estimate (1.9) is also optimal in the same way that the analogous estimate for eigenvalues of a self-adjoint operator with a compact and smooth energy surface. That follows for instance from applying [25, Corollary, §5].

The basic hyperbolicity assumption at an energy E can be stated as follows: for $\rho \in K_E$ we have,

$$\begin{aligned}
&T_\rho(p^{-1}(E)) = \mathbb{R}H_p(\rho) \oplus E_+(\rho) \oplus E_-(\rho)\ \ ,\ \ \dim E_\pm(\rho) = n-1\,,\\
&p^{-1}(E) \ni \rho \longmapsto E_\pm(\rho) \subset T_\rho(p^{-1}(E)) \ \text{ is continuous,}\\
&d(\exp tH_p)_\rho(E_\pm(\rho)) = E_\pm(\exp tH_p(\rho))\,,\\
&\exists\ \lambda > 0 \quad \|d(\exp tH_p)_\rho(X)\| \le Ce^{\pm\lambda t}\|X\|\,,\ \ \text{for all } X \in E_\pm(\rho),\ \mp t \ge 0.
\end{aligned} \tag{1.10}$$

[†]Here we consider an arbitrary manifold in the compact part. It is clear that the methods of [2] easily allow this type of generalization and the point is that our method is different and more robust.

An example of a potential satisfying this assumption at a range of non-zero energies is given in Fig.1 – see [24] and [30, Appendix c]. Following [30] we will formulate a weaker dynamical hypothesis in §7.

The main result of this paper is

Theorem 3. *Suppose that $P(h)$ satisfies our general assumptions* (1.5)*, the flow of H_p near zero energy is hyperbolic in the sense of* (1.10)*, or the weaker sense given in* §*7.1, and that the trapped set at zero energy has Minkowski dimension $m_0 = 2\nu_0 + 1$. Then for any $\nu > \nu_0$, and $C_0 > 0$ there exists C_1 such that*

$$|\operatorname{Res}(P(h)) \cap D(0, C_0 h)| \leq C_1 h^{-\nu}\,. \tag{1.11}$$

When the trapped is set is of pure dimension*, ν can be replaced by ν_0.*

A sharp rigorous lower bound is known when K is an isolated hyperbolic trajectory. A very precise asymptotic description of resonances in that case is given in [11] and it implies (1.11) with $\nu = 0$. In spite of the convincing numerical evidence cited above no rigorous examples with non-integral values ν_0 are known. A recent indication of the delicate nature of lower bounds for resonances was given in [7] where a class of *complex* compactly supported potentials in $\mathbb{R}^3$ with *no* resonances at all. We have no reasonable hope of obtaining any analogue of (1.2) at the present moment.

The methods of this paper apply also to a simpler problem of operators with complex absorbing barriers. Let $V \in \mathcal{C}^\infty_{\rm c}(B(0,R_0);\mathbb{R})$ be a potential for which H_p, $p = \xi^2 + V(x) - 1$, has hyperbolic flow on $p = 0$, for instance a "three bump" potential [24]. Now let $W \in \mathcal{C}^\infty(\mathbb{R}^n)$ satisfy

$$W(x) \geq 0\,, \quad \operatorname{supp} W \cap B(0,R_0) = \emptyset\,, \quad W(x) \geq 2C_0 > 0 \ \text{ for } |x| > 2R_0.$$

Consider then (see [35] and references given there)

$$\widetilde{P}(h) = -h^2\Delta + V(x) - iW(x)\,.$$

The spectrum of this non-selfadjoint operator lies in $\overline{\mathbb{C}}_-$ and we have the exact analogue of (1.3):

$$|\operatorname{Spec} \widetilde{P}(h) \cap D(0, Ch)| = \mathcal{O}(h^{-\nu})\,.$$

Acknowledgements. The second author would like to thank Jean-Yves Chemin and Jean-Michel Bony for useful discussions, the National Science Foundation for partial support under the grant DMS-0200732, and École Polytechnique for its generous hospitality in Fall 2004.

2. Outline of the proof

To prove the main result on fractal upper bounds (Theorem 3) we first develop methods for proving the natural results on the absence of resonances (Theorem 1) and on general

upper bounds at non-degenerate energies (Theorem 2). In this section we present the general ideas. All of them have origins in other works and pointers to the literature will be given in corresponding sections.

The absence of resonances for operators with $\mathcal{C}^\infty$ coefficients in domains of size $h\log(1/h)$ around an energy level hold under a nontrapping condition:

$$\exists\, \epsilon_0 > 0\,,\ \forall\, K \Subset p^{-1}(0)\,,\ \exists\, T_K\,,\quad (x,\xi)\in K \implies \exp(tH_p(x,\xi)) \notin K\,,\ t > T_K\,.$$

This implies the existence of an escape function in a neighbourhood of $p^{-1}(0)$:

$$\exists\, G_1 \in \mathcal{C}^\infty(T^*X)\,,\quad H_pG_1(x,\xi) \geq c_0 > 0\,,\ \text{for } |p(x,\xi)| < \epsilon_0.$$

The resonances of P are given by the eigenvalues of the deformed operator P_θ. In the case of $P = -h^2\Delta + V(x)$ with V analytic in a conic neighbourhood of $\mathbb{R}^n$,

$$V(x) + 1 \longrightarrow 0,\quad |x| \longrightarrow \infty\,,$$

the scaled operator is simply

$$P_\theta = -h^2 e^{-2i\theta}\Delta + V(e^{i\theta}x)\,,$$

and it behaves as $-h^2e^{-2i\theta}\Delta - 1$ near infinity. For $\theta > 0$ that last operator is clearly invertible.

We can introduce a modified $G = \chi G_1$, $\chi \in \mathcal{C}^\infty_{\rm c}(X)$, so that for

$$\theta \sim \epsilon \sim Mh\log(1/h)\,,$$

we have

$$|\operatorname{Re} p_\theta| < \delta \implies -\operatorname{Im} p_\theta + \epsilon H_pG \geq c_0\epsilon\,,\quad p_\theta = \sigma(P_\theta)\,.$$

The operators $\exp(\pm\epsilon G^w(x,hD)/h)$ are now pseudodifferential operators in a mildly exotic class and we consider

$$P_{\theta,\epsilon} = e^{-\epsilon G^w/h} P_\theta e^{\epsilon G^w/h}\,.$$

The spectrum of P_θ in $D(0, Mh\log(1/h))$ is the same as that of $P_{\theta,\epsilon}$ but the properties of G imply that

$$\|P_{\theta,\epsilon}^{-1}\| \leq C/\epsilon$$

showing that in fact there is no spectrum in $D(0, M'h\log(1/h))$. This approach allows us to obtain the absence of resonances very directly.

In Theorem 2 we show that if 0 is a non-critical energy level then

$$|\operatorname{Res} P \cap D(0,Ch)| = \mathcal{O}(h^{-n+1})\,.$$

The proof follows from a "robust" proof of the same estimate for an operator with a compact resolvent (for instance, an elliptic operator on a compact manifold). Let P be such an operator, say, $P = -h^2\Delta_g - 1$, on a compact Riemannian manifold. We would like to consider a modified operator

$$\widetilde{P}(h) \stackrel{\rm def}{=} P(h) - iMh\psi(MP(h)/h)\,,\quad \psi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})\,,$$

whose "symbol", $p - iMh\psi(Mp/h)$, has the absolute value bounded from below by $Mh/2$ everywhere. That does not make sense at first since

$$\psi(MP(h)/h)$$

is *not* an h-pseudodifferential operator. To remedy this we construct a second microlocal calculus with a new Planck constant $\tilde{h} \sim 1/M$. The new operator $\widetilde{P}(h)$ becomes elliptic in this calculus and for $\tilde{h}$ small enough, independent of h, it is invertible. We then have

$$(P(h)-z)^{-1} = (I+K(z))^{-1}(\widetilde{P}(h)-z)^{-1}\,, \quad K(z) \stackrel{\rm def}{=} i(\widetilde{P}(h)-z)^{-1}Mh\psi(P(h)/h)$$

and the eigenvalues of $P(h)$ near 0 coincide with the zeros of $\det(I+K(z))$. The zeros of this determinant are the same as the zeros of a determinant $\det(I+R(z))$ where $R(z)$ is a finite rank operator with the rank proportional to h^{-n} times the volume of the support of $\psi(Mp/h)$. That gives estimates on the determinant which imply (1.1). A slight modification of this argument is needed to obtain Theorem 2.

We now assume that the flow of H_p is hyperbolic and introduce the sets

$$\Gamma_\pm \stackrel{\rm def}{=} \{(x,\xi)\in T^*X \,:\, p(x,\xi)=0\,,\quad \exp(tH_p)(x,\xi)\not\to\infty\,,\ t\to\mp\infty\}\,. \tag{2.1}$$

The trapped set at zero energy is

$$K = \Gamma_+\cap\Gamma_-\,. \tag{2.2}$$

If we assume that $K\subset\overline{\Gamma_\pm\setminus K}$, that is K has no component isolated from infinity, then K is a set of Liouville measure 0.

To prove an upper bound involving the dimension of K we combine the methods used to prove Theorems 1 and 2. There exist functions $\varphi_\pm\in\mathcal{C}^{1,1}(T^*X)$ such that, uniformly on compact sets,

$$H_p\varphi_\pm \sim \mp\varphi_\pm\,,\quad \varphi_\pm\sim d(\Gamma_\pm,\bullet)^2\,,\quad \varphi_+ + \varphi_- \sim d(K,\bullet)^2\,.$$

A local model for the simplest case of one trajectory is given by $p=\xi_1+x_2\xi_2$, $(x,\xi)\in T^*\mathbb{R}^2$, so that

$$H_p = \partial_{x_1}+x_2\partial_{x_2}-\xi_2\partial_{\xi_2}\,,\quad \varphi_+=\xi_2^2\,,\quad \varphi_-=x_2^2\,,\quad K=\{(t,0;0,0)\,:\,t\in\mathbb{R}\}\,. \tag{2.3}$$

A new escape function is given by

$$G\stackrel{\rm def}{=}\left(\log(C\epsilon+\widehat{\varphi}_-)-\log(C\epsilon+\widehat{\varphi}_+)\right)\,,\quad \epsilon\sim Mh\,,\quad M\gg 1\,, \tag{2.4}$$

where $\widehat{\varphi}_\pm$ are suitable h-dependent regularizations of $\varphi_\pm$.

The logarithmic flattening of the more straightforward escape function $\varphi_--\varphi_+$ is forced by the requirement that $G=\mathcal{O}(\log(1/h))$ so that the conjugation used in the proof of Theorem 1 can be applied. However, even for uniformly smooth $\widehat{\varphi}_\pm$ the regularization of

G is essentially in the symbolic class $S_{\frac{1}{2}}$ and the situation becomes more complicated in general. Nevertheless we obtain the following estimates:

$$\partial^\alpha_{(x,\xi)} H_p^k G = \mathcal{O}(\epsilon^{-\frac{|\alpha|}{2}}), \quad \text{for } |\alpha| + k \geq 1, \text{ uniformly on compact sets,}$$

and

$$d((x,\xi),K)^2 \geq C\epsilon \implies H_p G \geq 1/C\,.$$

As in the proof of Theorem 1 (but with very different parameters and escape functions) we introduce a conjugated operator,

$$P_{\theta,t}(h) \stackrel{\text{def}}{=} e^{-tG^w} P_\theta(h) e^{tG^w}\,,$$

which now is in an exotic $\frac{1}{2}$-class, with the second Planck constant $\tilde{h} \sim 1/M$ playing the rôle of the asymptotic parameter. The escape function used here, G, has compact support.

We now build a second microlocal calculus which combines this exotic class with the one used in the proof of Theorem 2. The first allows us the use of the irregular escape function and the second allows a localization to an h-neighbourhood of the energy surface. In the new calculus the operator

$$\widetilde{P}_{\theta,t} = P_{\theta,t} - iMh\widetilde{\mathrm{Op}^w_h}(a)\,, \quad a(x,\xi) \stackrel{\text{def}}{=} \chi\left(\frac{p(x,\xi)}{K_1 h}\right)\chi\left(\frac{C_0 H_p G(x,\xi)}{K_1 h}\right)\,,$$

is globally elliptic (here $\widetilde{\mathrm{Op}^w_h}$ describes a second microlocal quantization operator). As in the proof of Theorem 2 the number of eigenvalues of $P_{\theta,t}$, and hence P_θ, near 0, is estimated by h^{-n} times the volume of the support of a. That volume is bounded by $h^{1+(2n-2-2\nu)/2}$, where $\nu > \nu_0$, $2\nu_0 + 1$ is the dimension of K. That gives (1.3).

3. Preliminaries

3.1. **Review of semiclassical pseudodifferential calculus.** Let X be a $\mathcal{C}^\infty$ manifold which agrees with $\mathbb{R}^n$ outside a compact set, or more generally

$$X = X_0 \sqcup (\mathbb{R}^n \setminus B(0,R_0)) \sqcup \cdots \sqcup (\mathbb{R}^n \setminus B(0,R_0))\,, \quad X_0 \Subset X\,.$$

We introduce the usual class of semiclassical symbols on X:

$$S^{m,k}(T^*X) = \{a \in \mathcal{C}^\infty(T^*X \times (0,1]) : |\partial^\alpha_x \partial^\beta_\xi a(x,\xi;h)| \leq C_{\alpha,\beta} h^{-m} \langle\xi\rangle^{k-|\beta|}\}\,,$$

where outside X_0 we take the usual $\mathbb{R}^n$ coordinates in this definition. The corresponding class of pseudodifferential operators is denoted by $\Psi^{m,k}_h(X)$, and we have the quantization and symbol maps:

$$\begin{aligned} \mathrm{Op}^w_h &: S^{m,k}(T^*X) \longrightarrow \Psi^{m,k}_h(X) \\ \sigma_h &: \Psi^{m,k}_h(X) \longrightarrow S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X)\,, \end{aligned}$$

with both maps surjective, and the usual properties

$$\begin{gathered}\sigma_h(A\circ B)=\sigma_h(A)\sigma_h(B)\,,\\ 0\to\Psi^{m-1,k-1}(X)\hookrightarrow\Psi^{m,k}(X)\overset{\sigma_h}{\to}S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X)\to 0\,,\end{gathered}\tag{3.1}$$

a short exact sequence, and

$$\sigma_h\circ \mathrm{Op}_h^w : S^{m,k}(T^*X)\ \longrightarrow S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X)\,,$$

the natural projection map. The class of operators and the quantization map are defined locally using the definition on $\mathbb{R}^n$:

$$\mathrm{Op}_h^w(a)u(x)=\frac{1}{(2\pi h)^n}\int\int a\left(\frac{x+y}{2},\xi\right)e^{i\langle x-y,\xi\rangle/h}u(y)dyd\xi\,,\tag{3.2}$$

and we refer to [10] for a detailed discussion. We remark only that when we consider the operators acting on half-densities we can define the symbol map, σ_h, onto

$$S^{m,k}(T^*X)/S^{m-2,k-2}(T^*X)\,,$$

see [34, Appendix]. We keep this in mind but for notational simplicity we suppress the half-density notation.

For $a\in S^{m,k}(T^*X)$ we define

$$\text{ess-supp}_h\, a\subset T^*X\sqcup S^*X\,,\quad S^*X\overset{\text{def}}{=}(T^*X\setminus 0)/\mathbb{R}_+\,,$$

where the usual $\mathbb{R}_+$ action is given by multiplication on the fibers: $(x,\xi)\mapsto(x,t\xi)$, as

$$\begin{aligned}\text{ess-supp}_h\, a=\complement\{(x,\xi)\in T^*X\ :\ \exists\,\epsilon>0\,,\ \partial_x^\alpha\partial_\xi^\beta a(x',\xi')=\mathcal{O}(h^\infty)\,,\ d(x,x')+|\xi-\xi'|<\epsilon\}\\ \sqcup\ \complement\{(x,\xi)\in S^*X:\exists\,\epsilon>0\,,\ \partial_x^\alpha\partial_\xi^\beta a(x',\xi')=\mathcal{O}(h^\infty\langle\xi'\rangle^{-\infty})\,,\\ d(x,x')+1/|\xi'|+|\xi-\xi'/|\xi'||<\epsilon\}/\mathbb{R}_+\,,\end{aligned}$$

where the second complement is in S^*X. For $A\in\Psi_h^{m,k}(X)$, then define

$$\mathrm{WF}_h(A)=\text{ess-supp}_h\, a\,,\quad A=\mathrm{Op}_h^w(a)\,,$$

noting that, as usual, the definition does not depend on the choice of Op_h^w, and

$$\mathrm{Char}(A)=\complement\left\{(x,\xi)\ :\ \exists\,B\in\Psi_h^{-m,-k}(X)\,,(x,\xi)\notin\mathrm{WF}_h(AB-I)\right\}\,,$$

where the complement is taken in $T^*X\sqcup S^*X$. For

$$u\in\mathcal{C}^\infty((0,1]_h;\mathcal{C}^\infty(X))\,,\quad\forall\,K\Subset X\,,N\in\mathbb{N}\,\exists\,P\,,h_0\,,\quad\|u\|_{C^N(K)}\le h^{-P}\,,\ h<h_0\,.$$

we define

$$\mathrm{WF}_h(u)=\bigcap\left\{\mathrm{Char}(A)\ :\ A\in\Psi^{0,0}(X)\ :\ \ Au\in h^\infty\mathcal{C}^\infty((0,1]_h;\mathcal{C}^\infty(X))\right\}\,.$$

Here we will be concerned with a purely semiclassical theory and deal only with *compact* subsets of T^*X.

To illustrate the h-pseudodifferential calculus at work we prove two simple lemmas which will be used later. We say that $A \in \Psi^{m,k}(X)$ is elliptic on $K \Subset T^*X$ if $|\sigma(A)\restriction_K| > h^{-m}/C$. This is equivalent to saying

Lemma 3.1. *Suppose $Q \in \Psi^{0,m}(X)$ is elliptic at (x_0,ξ_0), $\|u\|_{L^2} = 1$, and $\mathrm{WF}_h(u)$ is contained in a sufficiently small neighbourhood of (x_0,ξ_0). Then for h small enough,*

$$\|Qu\|_{L^2} \geq 1/C\,.$$

Lemma 3.2. *Suppose that $\psi_j \in \mathcal{C}^\infty_{\mathrm{b}}(T^*X)$, $\psi_1^2+\psi_2^2 = 1$, $\operatorname{supp}\psi_1 \subset \{(x,\xi)\ :\ |\xi|\leq C\}$. Then, there exist $\Psi_1 \in \Psi^{0,-\infty}(X)$ and $\Psi_2 \in \Psi^{0,0}(X)$, with principal symbols ψ_1 and ψ_2 respectively, such that*

$$\Psi_1^2+\Psi_1^2 = I+R\,,\quad R\in\Psi^{-\infty,-\infty}(X)\,,\quad \Psi_j^* = \Psi_j\,.$$

Proof. Symbolic calculus gives

$$(\psi_1^w)^2+(\psi_2^w)^2 = I+r_1^w\,,\quad r_1\in S^{-1,-\infty}(T^*X)\,,$$

in particular $r = \mathcal{O}(h) : H^{-M}(X)\to H^M(X)$. If h is small enough we put

$$\Psi_j^1 = (1+r_1^w)^{-\frac14}\psi_j^w(1+r_1^w)^{-\frac14}\,,$$

so that

$$(\Psi_1^1)^2+(\Psi_2^1)^2 = I+r_2^w\,,\quad r_2\in S^{-2,-\infty}(T^*X)\,,\quad (\Psi_j^1)^* = \Psi_j^1\,.$$

and we can then proceed by iteration. □

The semiclassical Sobolev spaces, $H^s_h(X)$ are defined by choosing a globally elliptic, self-adjoint operator, $A\in\Psi^{0,1}(X)$ (that is an operator satisfying $\sigma(A)\geq\langle\xi\rangle/C$ everywhere) and putting

$$\|u\|_{H^s_h} = \|A^s u\|_{L^2(X)}\,.$$

When $X=\mathbb{R}^n$,

$$\|u\|^2_{H^s_h}\sim\int_{\mathbb{R}^n}\langle h\xi\rangle^{2s}|\mathcal{F}u(\xi)|^2d\xi\,,\quad \mathcal{F}u(\xi)\stackrel{\mathrm{def}}{=}\int_{\mathbb{R}^n}u(x)e^{-i\langle x,\xi\rangle}dx\,.$$

The following lemma will also be useful:

Lemma 3.3. *Suppose that P_t, $t\in(0,\infty)$, is a family of operators such that*

$$P_t\ :\ H^s_h(X)\ \longrightarrow\ H^{s-m}_h(X)\,,$$
$$\forall\, A\in\Psi^{0,-\infty}(X)\,,\quad \mathrm{ad}_{P_t}A = \mathcal{O}(h)\ :\ L^2(X)\ \longrightarrow\ L^2(X)\,,\quad 0<h<h_0(t)\,,$$

with the bound depending on A but not on t. Let Ψ_j be as in Lemma 3.2 and suppose that

$$\|P_t\Psi_j u\|\geq th\|\Psi_j u\|-\mathcal{O}(h)\|u\|\,,\quad j=1,2\,,\quad u\in\mathcal{C}^\infty_{\mathrm{c}}(X)\,.$$

Here the constants in $\mathcal{O}$ are independent of h and t. Then for $t>t_0\gg 1$ and $0<h<h_0(t)$,

$$\|P_t u\|\geq th\|u\|/2\,.$$

Proof. We recall from Lemma 3.2 that

$$\|\Psi_1 v\|^2 + \|\Psi_2 v\|^2 = \|v\|^2 + \langle Rv, v\rangle = \|v\|^2 + \mathcal{O}(h^\infty)\|v\|_{H_h^{-N}}\,, \tag{3.3}$$

and hence with $v = P_t u$,

$$\begin{aligned}
\|P_t u\|^2 &= \|\Psi_1 P_t u\|^2 + \|\Psi_2 P_t u\|^2 - \mathcal{O}(h^\infty)\|u\|^2 \\
&\geq \|P_t\Psi_1 u\|^2 + \|P_t\Psi_2 u\|^2 - \|[\Psi_2, P_t]u\|^2 - \|[\Psi_2, P_t]u\|^2 \\
&\quad - \ 2\left(\|\Psi_1 P_t u\|\|[\Psi_1, P_t]u\|^2 + \|\Psi_2 P_t u\|\|[\Psi_2, P_t]u\|^2\right)^{\frac{1}{2}} - \mathcal{O}(h^\infty)\|u\|^2 \\
&\geq \|P_t\Psi_1 u\|^2 + \|P_t\Psi_2 u\|^2 \\
&\quad - 2C(\|[\Psi_1, P_t]u\|^2 + \|[\Psi_2, P_t]u\|^2) - \|P_t u\|^2/C - \mathcal{O}(h^\infty)\|u\|^2 \\
&\geq \|P_t\Psi_1 u\|^2 + \|P_t\Psi_2 u\|^2 - C'h^2\|u\|^2 - \|P_t u\|^2/C\,.
\end{aligned}$$

We now use the hypothesis of the lemma and (3.3) with $v = u$ to obtain

$$\begin{aligned}
\|P_t u\|^2 &\geq t^2h^2(\|\Psi_1 u\|^2 + \|\Psi_2 u\|^2) - C'h^2\|u\|^2 - \|P_t u\|^2/C \\
&\geq t^2h^2\|u\|^2 - C''h^2\|u\|^2 - \|P_t u\|^2/C
\end{aligned}$$

and the lemma follows. □

3.2. **Semiclassical Fourier integral operators.** We now follow [34] and review some aspects of the theory of semiclassical Fourier Integral Operators. We take a point of view which will be used in showing invariance of the second microlocal calculus developed below in §5.

Thus let $A(t)$ be a smooth family of pseudodifferential operators,

$$A(t) = \mathrm{Op}_h^w(a(t))\,,\quad a(t) \in \mathcal{C}^\infty([-1,1]_t; S^{0,-\infty}(T^*X))\,,$$

such that for all t, $\mathrm{WF}_h(A(t)) \Subset T^*X$. We then define a family of operators

$$\begin{gathered}
U(t)\ :\ L^2(X) \to L^2(X)\,, \\
hD_t U(t) + U(t)A(t) = 0\,,\quad U(0) = U_0 \in \Psi_h^{0,0}(X)\,.
\end{gathered} \tag{3.4}$$

This is an example of a family of h-Fourier Integral Operators, $U(t)$, associated to canonical transformations $\kappa(t)$, generated by the Hamilton vector fields $H_{a_0(t)}$, where the real valued $a_0(t)$ is the h-principal symbol of $A(t)$,

$$\frac{d}{dt}\kappa(t)(x,\xi) = (\kappa(t))_*(H_{a_0(t)}(x,\xi))\,,\quad \kappa(0)(x,\xi) = (x,\xi)\,,\quad (x,\xi) \in T^*X\,.$$

We will often need the *Egorov theorem* which can be proved directly from this definition: when U_0 in (3.4) is elliptic (that is $|\sigma(U_0)| > c > 0$) on T^*X, then for $B \in \Psi_h^{m,k}(X)$

$$\sigma(V(t)BU(t)) = (\kappa(t))^*\sigma(B)\,, \tag{3.5}$$

where $V(t)$ is an approximate inverse to $U(t)$,

$$V(t)U(t) - I\,,\, U(t)V(t) - I \in \Psi_h^{-\infty,-\infty}(T^*X)\,.$$

The approximate inverse is constructed by taking

$$hD_tV(t) - A(t)V(t) = 0\,,\quad V(0) = V_0\,,\quad V_0U_0 - I\,,\, U_0V_0 - I \in \Psi_h^{-\infty,-\infty}(T^*X)\,,$$

the existence of V_0 being guaranteed by the ellipticity of U_0. The proof of (3.5) follows from writing $B(t) = V(t)BU(t)$, so that, in view of the properties of $V(t)$,

$$hD_tB(t) \equiv [A(t), B(t)] \mod \Psi_h^{-\infty,-\infty}\,,\quad B(0) = B_0\,.$$

Since the symbol of the commutator is given by $(h/i)H_{a_0(t)}\sigma(B(t))$, (3.5) follows directly from the definition of $\kappa(t)$.

If $U = U(1)$, say, and the graph of $\kappa(1)$ is denoted by C, we conform to the usual notation and write

$$U \in I_h^0(X \times X; C')\,,\quad C' = \{(x,\xi;y,-\eta)\ :\ (x,\xi) = \kappa(y,\eta)\}\,,$$

which means that U is an h-Fourier Integral Operator associated to the canonical graph C. Locally all h-Fourier Integral Operators associated to canonical graphs are of the form $U(1)$ since each local canonical transformation with a fixed point can be deformed to identity, see [34, Lemma 3.2] and the proof of Lemma 5.8 below.

Our definitions of pseudo-differential operators and of (the special class of) h-Fourier Integral Operators were global. It is useful and natural to consider the operators and their properties microlocally. We consider classes of *tempered* operators:

$$T\ :\ \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)\,,$$

and for any semi-norm $\|\bullet\|_1$ on $\mathcal{C}^\infty(X)$ there exist a seminorm $\|\bullet\|_2$ on $\mathcal{C}^\infty(X)$ and a constant M_0 such that

$$\|Tu\|_1 = \mathcal{O}(h^{-M_0})\|u\|_2\,.$$

We remark that since we deal with compact subsets of T^*X here, we could consider operators $T\ :\ \mathcal{D}'(X) \to \mathcal{C}^\infty(X)$ in which case we can ask for existence of M_0 for any *two* seminorms $\|\bullet\|_j$, $j = 1, 2$.

For open sets, $V \subset T^*X$, $U \subset T^*X$, the operators *defined microlocally* near $V \times U$ are given by equivalence classes of tempered operators given by the relation

$$T \sim T' \iff A(T - T')B = \mathcal{O}(h^\infty)\ :\ \mathcal{D}'(X)\ \longrightarrow \mathcal{C}^\infty(X)\,,$$

for any $A, B \in \Psi_h^{0,0}(X)$ such that

$$\begin{gathered}\mathrm{WF}_h(A) \subset \widetilde{V}\,,\quad \mathrm{WF}_h(B) \subset \widetilde{U}\,,\\ \overline{V} \Subset \widetilde{V} \Subset T^*X\,,\quad \overline{U} \Subset \widetilde{U} \Subset T^*X\,,\quad \widetilde{U}\,,\widetilde{V}\quad \text{open}\,.\end{gathered} \tag{3.6}$$

The equivalence class T is an h-Fourier Integral Operator associated to a local canonical graph C if for any A and B with the properties above

$$ATB \in I^0(X \times X; \widetilde{C}')\,,$$

where C needs to be defined only near $U \times V$.

We say that $P = Q$ *microlocally* near $U \times V$ if $APB - AQB = \mathcal{O}_{L^2 \to L^2}(h^\infty)$, where because of the assumed pre-compactness of U and V the L^2 norms can be replaced by any other norms. For operator identities this will be the meaning of equality of operators in this paper, with U, V specified (or clear from the context). Similarly, we say that $B = T^{-1}$ microlocally near $U \times V$, if $BT = I$ microlocally near $U \times U$, and $TB = I$ microlocally near $V \times V$. More generally, we could say that $P = Q$ microlocally on $W \subset T^*X \times T^*X$ (or, say, P is microlocally defined there), if for any U, V, $U \times V \subset W$, $P = Q$ microlocally in $U \times V$.

If the open sets U or V in (3.6) are small enough, so that they can be identified with neighbourhoods of points in $T^*\mathbb{R}^n$, we can use that identification to state that T is microlocally defined near, say, $(m, (0,0))$, $m \in T^*X$, $(0,0) \in T^*\mathbb{R}^n$.

To give a useful example of this formalism we state a semiclassical version of Egorov's theorem.

Proposition 3.4. *Suppose that F is an h-Fourier integral operator, microlocally defined near $U \times V \subset T^*X \times T^*X$, and associated to a locally defined canonical transformation $\kappa : V \to U$, and elliptic near $(\kappa(\rho), \rho) \in U \times V$. Let F^{-1} be the microlocal inverse of F near $(\kappa(\rho), \rho)$. Then for any $A \in \Psi^{m,k}(X)$,*

$$F^{-1} \circ A \circ F = B \in \Psi^{m,k}(X)\,, \quad \sigma_h(B) = \sigma_h(A) \circ \kappa\,, \tag{3.7}$$

microlocally near $\rho \in T^*X$.

The proof is a localized adaptation of the argument giving (3.5) and the observation recalled above that any canonical transformation with a fixed point (as we can assume that $\kappa(m) = m$) can be deformed to the identity.

3.3. $S_{\frac12}$ **spaces with two parameters.** We define the following symbol class:

$$a \in S^{m,\widetilde{m},k}_{\frac12}(T^*\mathbb{R}^n) \iff |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} h^{-m} \tilde{h}^{-\widetilde{m}} \left(\frac{\tilde{h}}{h}\right)^{\frac12(|\alpha|+|\beta|)} \langle \xi \rangle^{k-|\beta|}\,, \tag{3.8}$$

where in the notation we suppress the dependence of a on h and $\tilde{h}$. We define the Weyl quantization of a in the usual way

$$a^w(x, hD_x)u = \frac{1}{(2\pi h)^n} \int a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle} u(y) dy d\xi\,,$$

and the standard results (see [10]) show that if $a \in S^{m,\widetilde{m},k}_{\frac12}(T^*\mathbb{R}^n)$ and $b \in S^{m',\widetilde{m}',k'}_{\frac12}(T^*\mathbb{R}^n)$ then

$$a(x, hD_x) \circ b(x, hD_x) = c(x, hD_x) \ \text{ with } \ c \in S^{m+m',\widetilde{m}+\widetilde{m}',k+k'}_{\frac12}(T^*\mathbb{R}^n)\,.$$

The presence of the additional parameter $\tilde{h}$ allows us to conclude that

$$c \equiv \sum_{|\alpha|<M} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \mod S^{m+m',\widetilde{m}+\widetilde{m}'-M,k+k'-M}_{\frac{1}{2}}(T^*\mathbb{R}^n)\,,$$

that is, we have a symbolic expansion in powers of $\tilde{h}$. We could also consider an expansion in the Weyl quantization – see (3.11).

We denote our class of operators by $\Psi^{m,\widetilde{m},k}_{\frac{1}{2}}(T^*\mathbb{R}^n)$. For simplicity we will only state the characterization à la Beals for a simpler class of symbols:

Lemma 3.5. *Suppose that $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Then $A = \mathrm{Op}^w_h(a)$ with*

$$\partial_x^\alpha \partial_\xi^\beta a = \mathcal{O}(h^{-m}\tilde{h}^{-\widetilde{m}}) \left(\frac{\tilde{h}}{h}\right)^{\frac{1}{2}(|\alpha|+|\beta|)}\,, \tag{3.9}$$

if and only if for any sequence $\{\ell_j\}_{j=1}^N$ of linear functions on $T^*\mathbb{R}^n$ we have

$$\| \operatorname{ad}_{\mathrm{Op}^w_h(\ell_1)} \circ \cdots \operatorname{ad}_{\mathrm{Op}^w_h(\ell_N)} A u\|_{L^2(\mathbb{R}^n)} \leq C h^{-m+N/2}\tilde{h}^{-\widetilde{m}+N/2}\|u\|_{L^2(\mathbb{R}^n)}\,,$$

for any $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. We can assume that $m = \widetilde{m} = 0$. The statement follows from the proof in [10, Chapter 8] and a rescaling:

$$(\tilde{x},\tilde{\xi}) = (\tilde{h}/h)^{\frac{1}{2}}(x,\xi)\,.$$

In fact, we define the following unitary operator on $L^2(\mathbb{R}^n)$:

$$U_{h,\tilde{h}} u(\tilde{x}) = (\tilde{h}/h)^{\frac{n}{4}} u((h/\tilde{h})^{\frac{1}{2}}\tilde{x})\,,$$

for which we can check that

$$a(x,hD_x) = U^{-1}_{h,\tilde{h}} a_{h,\tilde{h}}(\tilde{x},\tilde{h}D_{\tilde{x}}) U_{h,\tilde{h}}\,,\quad a_{h,\tilde{h}}(\tilde{x},\tilde{\xi}) = a((h/\tilde{h})^{\frac{1}{2}}(\tilde{x},\tilde{\xi}))\,.$$

Clearly a satisfies (3.9) if and only if $a_{h,\tilde{h}} \in \mathcal{C}^\infty_{\mathrm{b}}(T^*\mathbb{R}^n)$. The Beals condition for $\tilde{h}$-pseudodifferential operators is

$$\| \operatorname{ad}_{\tilde{\ell}_1(\tilde{x},\tilde{h}D_{\tilde{x}})} \circ \cdots \circ \operatorname{ad}_{\tilde{\ell}_N(\tilde{x},\tilde{h}D_{\tilde{x}})} a_{h,\tilde{h}}(\tilde{x},\tilde{h}D_{\tilde{x}}) u\|_{L^2} \leq C\tilde{h}^N \|u\|_{L^2}\,.$$

But this is the condition in the lemma since we should take

$$\tilde{\ell}_j = (\ell_j)_{h,\tilde{h}} = (\tilde{h}/h)^{\frac{1}{2}}\ell_j\,,$$

and this completes the proof. □

We remark that the proof given in [14],[10, Chapter 8] is recalled, in a more complicated setting, in the proof of Proposition 5.2 below. It is based on showing that

$$\forall \alpha \in \mathbb{N}^{2n}\,,\ \|(\partial^\alpha a)(x,D)u\|_{L^2} \leq C_\alpha \|u\|_{L^2} \iff \forall \beta \in \mathbb{N}^{2n}\,,\ \sup|\partial^\beta a| \leq C'_\beta\,,$$

which in the setting presented in Lemma 3.5 becomes

$$\begin{gathered}\forall \alpha\in\mathbb{N}^{2n}\,,\ \|(\partial^\alpha a)(x,hD)u\|_{L^2}\leq C_\alpha(\tilde h/h)^{|\alpha|/2}\|u\|_{L^2}\\ \Longleftrightarrow\\ \forall\beta\in\mathbb{N}^{2n}\,,\ \sup|\partial^\beta a|\leq C'_\beta(\tilde h/h)^{|\beta|/2}\,.\end{gathered}\tag{3.10}$$

We will also need the following application of the semi-classical calculus:

Lemma 3.6. *Suppose that $\partial^\alpha a$, $\partial^\alpha b=\mathcal{O}_\alpha((\tilde h/h)^{|\alpha|/2})$, and that $c^w(x,hD)=a^w(x,hD)\circ b^w(x,hD)$. Then*

$$c(x,\xi)=\sum_{k=0}^N\frac{1}{k!}\left(\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)\right)^k a(x,\xi)b(y,\eta)\restriction_{x=y,\xi=\eta}+e_N(x,\xi)\,,\tag{3.11}$$

where for some M

$$\begin{gathered}|\partial^\alpha e_N|\leq C_N h^{N+1}\times\\ \sum_{\alpha_1+\alpha_2=\alpha}\sup_{\substack{(x,\xi)\in T^*\mathbb{R}^n\\(y,\eta)\in T^*\mathbb{R}^n}}\sup_{|\beta|\leq M\,,\beta\in\mathbb{N}^{2n}}\left|(h^{\frac12}\partial_{(x,\xi;y,\eta)})^\beta(i\sigma(D)/2)^{N+1}\partial^{\alpha_1}a(x,\xi)\partial^{\alpha_2}b(y,\eta)\right|\,,\end{gathered}\tag{3.12}$$

where $\sigma(D)=\sigma(D_x,D_\xi;D_y,D_\eta)$.

Proof. This follows from from the standard estimates of symbolic calculus (see [10, Proposition 7.6]): suppose that $A(D)$ is a non-degenerate real quadratic form. Then there exists M such that

$$|\partial^\alpha\exp(iA(D))a(x,\xi)|\leq C\sum_{|\beta|\leq M}\sup_{(x,\xi)\in T^*\mathbb{R}^n}|\partial^{\alpha+\beta}a(x,\xi)|\,.$$

We observe that a rescaling $\tilde x=x/\sqrt{s}$, $s>0$, shows that

$$|\partial^\alpha\exp(isA(D))a(x,\xi)|\leq C\sum_{|\beta|\leq M}\sup_{(x,\xi)\in T^*\mathbb{R}^n}|\partial^\alpha(\sqrt{s}\partial)^\beta a(x,\xi)|\,.$$

To obtain an expansion we use the Taylor expansion:

$$\exp(ihA(D))=\sum_{k=0}^N\frac{(ihA(D))^k}{k!}+\frac{1}{N!}\int_0^1(1-t)^N\exp(ithA(D))(ihA(D))^{N+1}dt\,.$$

In the notation of the lemma and with $A(D)=\sigma(D_x,D_\xi;D_y,D_\eta)/2$,

$$c(x,\xi)=\exp(ihA(D))a(x,\xi)b(y,\eta)\restriction_{x=y,\eta=\xi}\,,$$

and the lemma follows. □

As a particular consequence we notice that if $a \in S^{0,0,-\infty}_{\frac{1}{2}}(T^*\mathbb{R}^n)$ and $b \in S^{0,-\infty}(T^*\mathbb{R}^n)$ then

$$a^w(x,hD)\circ b^w(x,hD) = c^w(x,hD)\,,\quad c(x,\xi) =$$
$$\sum_{k=0}^{N}\frac{1}{k!}\left(ih\sigma(D_x,D_\xi;D_y,D_\eta)\right)^k a(x,\xi)b(y,\eta)\restriction_{x=y,\xi=\eta} + \mathcal{O}(h^{\frac{N+1}{2}}\tilde{h}^{\frac{N+1}{2}})\,,$$

and the usual pseudodifferential calculus allows a remainder improvement to

$$\mathcal{O}(h^{\frac{N+1}{2}}\tilde{h}^{\frac{N+1}{2}}\langle\xi\rangle^{-\infty})\,.$$

3.4. **One parameter groups of elliptic operators.** We recall a special case of a result of Bony and Chemin [4, Théoreme 6.4]. Let $m(x,\xi)$ be an order function in the sense of [10]:

$$m(x,\xi) \le Cm(y,\eta)\langle(x-y,\xi-\eta)\rangle^N\,. \tag{3.13}$$

The class of symbols, $S(m)$, corresponding to m is defined as

$$a \in S(m) \iff |\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}m(x,\xi)\,.$$

If m_1 and m_2 are order functions in the sense of (3.13), and $a_j \in S(m_j)$ then (we put $h=1$ here),

$$a_1^w(x,D)a_2^w(x,D) = b^w(x,D)\,,\quad b \in S(m_1m_2)\,,$$

with b given by the usual formula,

$$\begin{aligned} b(x,\xi) &= a_1 \# a_2(x,\xi)\\ &\overset{\text{def}}{=} \exp(i\sigma(D_{x^1},D_{\xi^1};D_{x^2},D_{\xi^2})/2)a_1(x^1,\xi^1)a_2(x^2,\xi^2)\restriction_{x^1=x^2=x,\xi^1=\xi^2=\xi}\,. \end{aligned} \tag{3.14}$$

A special case of [4, Théoreme 6.4] gives

Proposition 3.7. *Let m be an order function in the sense of* (3.13) *and suppose that $G \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n;\mathbb{R})$ satisfies*

$$G(x,\xi) - \log m(x,\xi) = \mathcal{O}(1)\,,\quad \partial_x^\alpha\partial_\xi^\beta G(x,\xi) = \mathcal{O}(1)\,,\quad |\alpha|+|\beta|\ge 1\,. \tag{3.15}$$

Then

$$\exp(tG^w(x,D)) = B_t^w(x,D)\,,\quad B_t \in S(m^t)\,. \tag{3.16}$$

Here $\exp(tG^w(x,D))$ *is constructed using spectral theory of bounded self-adjoint operators. The estimates on $B_t \in S(m^t)$ depend* only *on the constants in* (3.15) *and in* (3.13)*. In particular they are independent of the support of G.*

Since m^t is the order function $\exp(t\log m(x,\xi))$, we can say that on the level of order functions "quantization commutes with exponentiation".

In Appendix at the end of the paper we give a simple direct proof of this proposition. We should stress that the main difficulties in [4] came from considering general Weyl calculi

of pseudodifferential opearators. Here we need only the case of the simplest metric $g = dx^2 + d\xi^2$.

3.5. **Review of complex scaling.** We very briefly recall the procedure described in [31]. It follows the long tradition of the complex scaling method – see [32] for the presentation for compactly supported perturbations and references to earlier work.

Let $\Gamma_\theta \subset \mathbb{C}^n$ be a totally real contour with the following properties:

$$\begin{gathered} \Gamma_\theta \cap B_{\mathbb{C}^n}(0,R_0) = B_{\mathbb{R}^n}(0,R_0)\,, \\ \Gamma_\theta \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0,2R_0) = e^{i\theta}\mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0,2R_0)\,, \\ \Gamma_\theta = \{x + if_\theta(x) \,:\, x \in \mathbb{R}^n\}\,,\quad \partial_x^\alpha f_\theta(x) = \mathcal{O}_\alpha(\theta)\,. \end{gathered} \tag{3.17}$$

The contour can be considered as a deformation of the manifold X as nothing is being done in the compact region. The operator P defines a dilated operator:

$$P_\theta \stackrel{\text{def}}{=} P\restriction_{\Gamma_\theta}\,,\quad P_\theta u = \widetilde{P}(\tilde{u})\restriction_{\Gamma_\theta}\,,$$

where $\widetilde{P}$ is the holomorphic continuation of the operator P, and $\tilde{u}$ is an almost analytic extension of $u \in \mathcal{C}^\infty_{\rm c}(\Gamma_\theta)$ (here we are only concerned with $\Gamma_\theta \cap B_{\mathbb{C}^n}(0,R_0)$).

For θ fixed, the scaled operator, P_θ, is uniformly elliptic in $\Psi^{0,2}(X)$ outside a compact set (see (4.9) below) and hence the resolvent, $(P_\theta - z)^{-1}$, is meromorphic for $z \in D(0,1/C)$. We can also take θ to be h dependent and the same statement holds for $z \in D(0,\theta/C)$. The spectrum of P_θ in $z \in D(0,\theta/C)$ is independent of θ and consists of quantum resonances of P which are defined as the poles of the meromorphic continuation of

$$(P-z)^{-1} \,:\, \mathcal{C}^\infty_{\rm c}(X) \,\longrightarrow\, \mathcal{C}^\infty(X)\,.$$

In fact, that is one of the ways of defining resonances, and in this paper we will be estimating the number of eigenvalues of P_θ.

4. Resonance free regions under the non-trapping assumption

4.1. **Estimates using weight functions.** We follow the presentation given in [9, §4.2] and inspired by many previous works, including [21].

Let us suppose that $P_\theta \in \Psi^{0,2}(X)$ (we identify X and X_θ here) is a complex scaled operator with $\theta = M_1 h\log(1/h)$. We choose ϵ

$$M_3 h \le \epsilon \le M_2 h \log\frac{1}{h}\,, \tag{4.1}$$

where $M_2 > M_1$ and M_3 are large constants to be fixed later.

Let $G \in \mathcal{C}^\infty_{\rm c}(T^*X)$ and define

$$P_{\epsilon,\theta} \stackrel{\text{def}}{=} e^{-\epsilon G/h} P_\theta e^{\epsilon G/h} = e^{-\frac{\epsilon}{h}{\rm ad}_G} P_\theta \sim \sum_0^\infty \frac{\epsilon^k}{k!}(-\frac{1}{h}{\rm ad}_G)^k(P_\theta)\,,\ G = G^w(x,hD)\,.$$

We note that the assumption on ϵ and the boundedness of ad_G/h show that the expansion makes sense. The operators $\exp(\epsilon G/h)$ are pseudo-differential in an exotic class $S^{C_2}_\delta$ for any $\delta > 0$ (see [10]) but that is not relevant here.

Using the same letters for operators and and the corresponding symbols, we see that

$$P_{\epsilon,\theta} = P_\theta - i\epsilon\{P_\theta, G\} + \mathcal{O}(\epsilon^2) = p_\theta - i\epsilon\{p_\theta, G\} + \mathcal{O}(h+\epsilon^2),$$

so that

$$\begin{aligned}
\operatorname{Re} P_{\epsilon,\theta} &\stackrel{\mathrm{def}}{=} (P_{\epsilon,\theta} + P^*_{\epsilon,\theta})/2 = \operatorname{Re} p_\theta + \epsilon\{\operatorname{Im} p_\theta, G\} + \mathcal{O}(h+\epsilon^2) = \operatorname{Re} p_\theta + \mathcal{O}(h+\theta\epsilon+\epsilon^2)\,,\\
\operatorname{Im} P_{\epsilon,\theta} &\stackrel{\mathrm{def}}{=} (P_{\epsilon,\theta} - P^*_{\epsilon,\theta})/(2i) = \operatorname{Im} p_\theta - \epsilon\{\operatorname{Re} p_\theta, G\} + \mathcal{O}(h+\epsilon^2).
\end{aligned}$$

We now make the following assumption: for a fixed $\delta > 0$

$$|\operatorname{Re} p_\theta| < \delta \implies -\operatorname{Im} p_\theta + \epsilon H_p G \geq c_0\epsilon\,. \tag{4.2}$$

Now let $\psi_1, \psi_2 \in \mathcal{C}^\infty_{\mathrm{b}}(T^*X)$ be two functions satisfying

$$\psi_1^2 + \psi_2^2 = 1\,, \quad \psi_1\restriction_{|\operatorname{Re} p_\theta|<\delta/2} \equiv 1\,, \quad \operatorname{supp}\psi_1 \subset \{|\operatorname{Re} p_\theta| < \delta\}\,.$$

Lemma 3.2 gives two selfadjoint operators Ψ_1 and Ψ_2 with principal symbols ψ_1 and ψ_2 respectively, such that

$$\Psi_1^2 + \Psi_1^2 = I + R\,, \quad R = \mathcal{O}(h^\infty) \,:\, H^{-M}(X) \to H^M\,.$$

We then write $P_{\epsilon,\theta} = A_{\epsilon,\theta} + iB_{\epsilon,\theta}$, where

$$A_{\epsilon,\theta} = \frac{1}{2}(P_{\epsilon,\theta} + P^*_{\epsilon,\theta})\,, \quad B_{\epsilon,\theta} = \frac{1}{2i}(P_{\epsilon,\theta} - P^*_{\epsilon,\theta})\,.$$

The principal symbol of $B_{\epsilon,\theta}$ is given by $\operatorname{Im} p_\theta - \epsilon H_p G$ and on the essential support of Ψ_1 it is bounded below by $c_0\epsilon \gg h$. Hence the sharp Gårding inequality (see [10, Theorem 7.12]) implies that for h small enough

$$\begin{aligned}
\|P_{\epsilon,\theta}\Psi_1 u\|\|\Psi_1 u\| &\geq |\langle P_{\epsilon,\theta}\Psi_1 u, \Psi_1 u\rangle| \geq |\operatorname{Im}\langle P_{\epsilon,\theta}\Psi_1 u, \Psi_1 u\rangle|\\
&= -\langle B_{\epsilon,\theta}\Psi_1 u, \Psi_1 u\rangle \geq \frac{\epsilon}{C}\|\Psi_1 u\|^2\,,
\end{aligned}$$

and hence

$$\|P_{\epsilon,\theta}\Psi_1 u\| \geq \frac{\epsilon}{C}\|\Psi_1 u\|\,.$$

On the support of ψ_2 the operator $A_{\epsilon,\theta}$ is elliptic and by Lemma 3.1,

$$\|P_{\epsilon,\theta}\Psi_2 u\| \geq \frac{1}{C}\|\Psi_2 u\| - \mathcal{O}(h^\infty)\|u\|\,.$$

Applying Lemma 3.3 with $t = \epsilon/h \gg 1$ we conclude that

$$\|P_{\epsilon,\theta} u\| \geq \frac{\epsilon}{C}\|u\|\,.$$

This shows that the conjugated operator has no spectrum in $D(0, \epsilon/(2C))$.

4.2. **Construction of an escape function.** Using the results of §4.1 all we need to do is to construct G so that (4.2) holds. For that we modify a standard argument with the presentation borrowed in part from [38, Sect.4].

We recall that (1.7) implies the same condition with $p^{-1}([-\epsilon_0,\epsilon_0])$ for some $\epsilon_0>0$. That follows from the compactness of the trapped set in $p^{-1}([-\delta,\delta])$ – see [11, Appendix] for a detailed discussion.

Let us now fix R a large parameter. We will define $G_\rho\in\mathcal{C}^\infty_{\rm c}(T^*X)$, a local escape function supported in a neighbourhood of the bicharacteristic segment

$$I_\rho=\{\exp(tH_p)(\rho):\ t\in[-T,T]\}\,,$$

and which satisfies $H_pG_\rho\geq 1$ on the part of I_ρ lying over

$$K'=\{\rho'\in T^*X\ :\ |x(\rho')|\leq R\} \tag{4.3}$$

For that, let Γ be a hypersurface through ρ which is transversal to H_p. Then there is a neighbourhood U_ρ of ρ, such that

$$V_\rho=\{\exp(t(U_\rho\cap\Gamma)):\ t\in(-T-1,T+1)\}\subset p^{-1}([-\epsilon_0/2,\epsilon_0/2])\,,$$

is a neighbourhood of I_ρ. That neighbourhood can be identified with a product,

$$V_\rho\simeq(-T-1,T+1)\times(U_\rho\cap\Gamma)\,,$$

and, in this identification, we will choose T and $0<\alpha<1$ so that

$$(((-T-1,-\alpha T)\cup(\alpha T,T+1))\times(U_\rho\cap\Gamma)))\cap K'=\emptyset\,.$$

We now need the following elementary

Lemma 4.1. *For any $0<\alpha<1/2$ and $T>0$ there exist as function $\chi=\chi_{T,\alpha}\in\mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ such that*

$$\chi(t)=\begin{cases}0 & |t|>T\\ t & |t|<\alpha T\end{cases}\quad,\quad \chi'(t)\geq-2\alpha\,.$$

Proof. The piecewise linear function

$$\chi_\#(t)=\begin{cases}0 & |t|>T\\ t & |t|<\alpha T\\ \pm\alpha(T-t)/(1-\alpha) & \alpha T\leq\pm t\leq T\end{cases}$$

satisfies $\chi_\#'\geq-\alpha/(1-\alpha)>-2\alpha$ wherever the derivative is defined. A regularization of this function gives $\chi_{T,\alpha}$. □

Now let $\phi_\rho\in\mathcal{C}^\infty_{\rm c}(U_\rho\cap\Gamma)$ be identically 1 near ρ, and let χ_T be given by the lemma. Using the product coordinates, we can think of ϕ_ρ, t, and hence $\chi(t)$, as functions on T^*X. The functions ϕ_ρ and $\chi_T(t)$ have compact support in V_ρ. Let

$$\psi\in\mathcal{C}^\infty_{\rm c}((-\epsilon_0,\epsilon_0))\,,\quad \psi\restriction_{[-\epsilon_0/2,\epsilon_0/2]}\equiv 1\,,$$

and put

$$G_\rho = \chi_T(t)\phi_\rho\psi(p)\,,\quad G_\rho \in \mathcal{C}^\infty_{\rm c}(V_\rho)\,. \tag{4.4}$$

so that

$$H_pG_\rho = \chi_T'\phi_\rho\psi(p), \tag{4.5}$$

satisfies

$$H_pG_\rho = 1 \text{ on } V_\rho\cap\{|x|<R\} \text{ and } H_pG_\rho \geq -2\alpha \text{ everywhere.}$$

Now let $K \Subset T^*X$ be the compact set

$$K = \{\rho\in p^{-1}([-\epsilon_0/3,\epsilon_0/3])\ :\ |x(\rho)|\leq R/2\}. \tag{4.6}$$

Since K is compact, applying the previous argument for every $\rho\in K$ gives a U_ρ, and a $U'_\rho\subset U_\rho$ on which $\phi_\rho = 1$. Since $\{U'_\rho:\ \rho\in K\}$ covers K, the compactness of K shows that we can pass to a finite subcover, $\{U'_{\rho_j}:\ j=1,\ldots,N\}$. We let

$$G = \sum_{j=1}^N G_{\rho_j}. \tag{4.7}$$

The construction of G_{ρ_j}'s now shows that by choosing α small enough (depending on the maximal number of support overlaps we obtain

$$H_pG(\rho)\geq 1\,,\ \ \rho\in p^{-1}((-\epsilon_0/2,\epsilon_0/2))\cap\{|x(\rho)|<R\} \text{ and } H_pG(\rho)\geq -\delta\,,\ \ \rho\in T^*X\,. \tag{4.8}$$

4.3. **Resonance free region.** We now want to choose the scaling so that (4.2) holds with G satisfying (4.8). Once that is done the results of §4.1 will give Theorem 3.

For that we choose the complex scaling so that

$$\begin{gathered} -\operatorname{Im} p_\theta(x,\xi)\geq\theta \ \text{ when } |p(x,\xi)|\leq\epsilon_0 \text{ and } |x|\geq R,\\ \operatorname{Im} p_\theta < C_1\theta \quad \text{when } |p(x,\xi)|\leq\epsilon_0,\end{gathered} \tag{4.9}$$

where R is independent of θ. With $\epsilon = M_2h\log(1/h)$ we now choose $\theta = M_1h\log(1/h)$ such that

$$M_1 < M_2/C_1\,,\quad \delta M_2 < M_1\,,$$

where C_1 comes from (4.9) and δ comes from (4.8). Since we can choose δ as small as we want this can certainly be arranged leading to (4.2).

For completeness we include a quantitative corollary of Theorem 1 from [25]:

Theorem 1′. *Suppose that the assumptions of Theorem 1 are satisfied and that $(P_\theta - z)^{-1}$ is the scaled resolvent defined for $0<\theta<2Mh\log h$, $M\gg 1$. Then for $0<h<h_0(M)$ we have*

$$\|(P_\theta - z)^{-1}\|_{L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)} = C\frac{\exp(C|\operatorname{Im} z|/h)}{h}\,,\quad z\in D(0,Mh\log(1/h))\,.$$

Theorem 1 is essentially optimal as shown by the well known one dimensional result going back to Regge (see [40] for a proof and references): if $V \in \mathcal{C}^N([a,b])$ is extended by 0 to a potential on $\mathbb{R}$, and

$$V(x) \sim \begin{cases} (x-a)^p & x \simeq a+ \\ (x-b)^q & x \simeq b- \end{cases}, \quad p,q < N\,,$$

then the scattering poles for $-\Delta + V(x)$ are given at high energies by the sequence

$$\lambda_k = \frac{\pi k}{b-a} - i\alpha \log|k| + \mathcal{O}(1)\,, \quad k \in \mathbb{Z}\,, \quad \alpha = \frac{p+q+4}{2(b-a)}\,.$$

The semiclassical resonances, $z_k(h)$, of $-h^2\Delta + h^2 V(x)$ are related to these scattering poles by the formula $z_k(h) = h^2\lambda_k^2$. Hence

$$\operatorname{Re} z_k(h) \sim 1 \implies \operatorname{Im} z_k(h) \sim h\log(1/h)\,.$$

5. Second microlocal calculus associated to a hypersurface

To obtain Theorem 2 we need to localize to an h-size neighbourhood of the energy surface

$$\Sigma \stackrel{\text{def}}{=} \{(x,\xi) \in T^*X \;:\; p(x,\xi) = 0\}\,.$$

That means that we have to work with functions of the form

$$a(x,\xi;h) = \psi(p(x,\xi)/h)\,.$$

The usual quantization procedure (the passage from symbols to pseudodifferential operators) is prohibited as

$$\partial^\alpha_{x,\xi} a \simeq h^{-|\alpha|}\,.$$

The troublesome symbols have a special form and we can construct a calculus which includes them by straightening Σ locally by means of canonical transformations. That means moving Σ to

$$\Sigma_0 = \{\xi_1 = 0\}\,.$$

We may then localize to rectangles in the (x_1,ξ_1)-space of length ~ 1 in x_1 and of length $\sim h$ in ξ_1. This amounts to a form of semiclassical second microlocalization. The presentation here is essentially self-contained and we refer to [33] for pointers to the literature.

For Theorem 2 we need even more singular calculus related to the $\Psi_{\frac{1}{2}}$ calculus described in §3.3.

We assume that Σ is a compact $\mathcal{C}^\infty$ hypersurface in T^*X. Since the delicate constructions will only be used in a compact set and since we are working in the $\mathcal{C}^\infty$ category this creates no restrictions.

5.1. **Basic properties.** To construct the calculus, let $\Sigma \Subset T^*X$ be a $\mathcal{C}^\infty$ compact hypersurface. We consider a class of symbols associated to Σ, a multiindex

$$\mathfrak{p} \stackrel{\text{def}}{=} (m, \widetilde{m}, k_1, k_2)\,,$$

and depending on *two* small parameters,

$$0 < h < \tilde{h}\,.$$

$$a \in S^{\mathfrak{p}}_{\Sigma,\delta}(T^*X) \Longleftrightarrow \begin{cases} \underline{\text{near } \Sigma} \quad V_1 \cdots V_{l_1} W_1 \cdots W_{l_2} H_q^k a = \\ \mathcal{O}\left(h^{-m-\delta l_1 - l_2} \tilde{h}^{-\widetilde{m}+\delta l_1 + l_2} \langle (\tilde{h}/h) q\rangle^{k_2+\delta k-(1-\delta)l_2}\right)\,, \\ \text{where } V_1, \cdots, V_{l_1} \text{ are vector fields tangent to } \Sigma \text{ and} \\ W_1, \cdots W_{l_2} \text{ are any vectorfields} \\ \\ \underline{\text{away from } \Sigma} \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) = \\ \mathcal{O}(h^{-m-k_2-\delta(|\alpha|+|\beta|)} \tilde{h}^{-\widetilde{m}+k_2+\delta(|\alpha|+|\beta|)} \langle \xi\rangle^{k_1-|\beta|})\,. \end{cases} \tag{5.1}$$

Here q is any defining function of Σ, that is a function which vanishes simply on Σ. To assure invariance we need

Lemma 5.1. *Definition* (5.1) *is independent of the choice of q and the vector fields applied to a can be taken in any order.*

Proof. The independence of the order of vector fields follows from the fact that $[V_{\ell_j}, H_q]$ is a vector field tangent to Σ. To see independence of q we note that $H_{uq} = uH_q + qH_u$. The vector field H_u can be considered as an arbitrary vector field and qH_u is tangent to Σ. Hence the application of the second term is estimated by

$$|qH_u a| \le C \min(|q|h^{-1}\langle qh^{-1}\rangle^{-(1-\delta)}, h^{-\delta}) \le C'\langle h^{-1}q\rangle^{\delta}\,,$$

and this estimate can be iterated. Hence we have the same estimates for q replaced by uq, $u \neq 0$ near Σ. □

The symbol classes in (5.1) are best understood in the simple case when $q = \xi_1$, say. The condition for $|\xi| \le C$, means that, with $\xi = (\xi_1, \xi')$,

$$\partial_{x_1}^k \partial_{x'}^\alpha \partial_{\xi'}^\beta (\xi_1 \partial_{\xi_1})^{\ell_1} \partial_{\xi_1}^{\ell_2} a = \mathcal{O}\left(h^{-m} \tilde{h}^{-\widetilde{m}} (\tilde{h}/h)^{\delta(|\alpha|+|\beta|)+\delta\ell_1+\ell_2} \langle (\tilde{h}/h)\xi_1\rangle^{k_2+\delta k-(1-\delta)\ell_2)}\right)\,, \tag{5.2}$$

or, if we eliminate the vector fields vanishing at $\xi_1 = 0$,

$$\partial_{x_1}^k \partial_{x'}^\alpha \partial_{\xi'}^\beta ((h/\tilde{h})\partial_{\xi_1})^p a = \mathcal{O}\left(h^{-m} \tilde{h}^{-\widetilde{m}} (\tilde{h}/h)^{\delta(|\alpha|+|\beta|)} \langle (\tilde{h}/h)\xi_1\rangle^{k_2-(1-\delta)p+\delta k}\right)\,. \tag{5.3}$$

We used the fact that if $|\xi_1| \le C$ then

$$\left((\tilde{h}/h)\xi_1\right)^{\delta-1} \le \left((\tilde{h}/h)\xi_1\right)^{-1} (\tilde{h}/h)^{\delta}\,,$$

to eliminate the need for the $\xi_1\partial_{\xi_1}$. The advantage of the formulation in (5.1) (and (5.2)) is the geometric invariance. In explicit coordinates (5.3) is however more transparent.

Since it is sufficient for our purposes, and for simplicity of presentation we will consider the case of $\delta = 1/2$ only.

The second microlocalization associates to the space of symbols $S^{\mathfrak{p}}_{\Sigma,\frac12}$ a space of operators, $\Psi^{\mathfrak{p}}_{\Sigma,\frac12}$, defined in §5.4 below. The basic properties of these spaces are described in the following

Theorem 4. *Let us define two multiindices,*

$$\mathfrak{p} = (m, \widetilde{m}, k_1, k_2)\,, \quad \mathfrak{p}' = (m, \widetilde{m}-1, k_1-1, k_2)\,.$$

With the definitions of $S^{\mathfrak{p}}_{\Sigma,\delta}$ above and $\Psi^{\mathfrak{p}}_{\Sigma,\delta}$ in §5.4 below, there exist maps

$$\mathrm{Op}_{\Sigma,h,\tilde h} \ : \ S^{\mathfrak{p}}_{\Sigma,\frac12}(T^*X) \longrightarrow \Psi^{\mathfrak{p}}_{\Sigma,\frac12}(X)$$

$$\sigma_{\Sigma,h,\tilde h} \ : \ \Psi^{\mathfrak{p}}_{\Sigma,\frac12}(X) \longrightarrow S^{\mathfrak{p}}_{\Sigma,\frac12}(T^*X)/S^{\mathfrak{p}'}_{\Sigma,\frac12}(T^*X)$$

such that

$$\sigma_{\Sigma,h,\tilde h}(A\circ B) = \sigma_{\Sigma,h,\tilde h}(A)\sigma_{\Sigma,h,\tilde h}(B) \tag{5.4}$$

$$0 \longrightarrow \Psi^{\mathfrak{p}'}_{\Sigma,\frac12}(X) \longrightarrow \Psi^{\mathfrak{p}}_{\Sigma,\frac12}(X) \overset{\sigma_{\Sigma,h,\tilde h}}{\longrightarrow} S^{\mathfrak{p}}_{\Sigma,\frac12}(T^*X)/S^{\mathfrak{p}'}_{\Sigma,\frac12}(T^*X) \longrightarrow 0$$

is a short exact sequence and

$$\sigma_{\Sigma,h,\tilde h}\circ \mathrm{Op}_{\Sigma,h,\tilde h} \ : \ S^{\mathfrak{p}}_{\Sigma,\frac12}(T^*X) \longrightarrow S^{\mathfrak{p}}_{\Sigma,\frac12}(T^*X)/S^{\mathfrak{p}'}_{\Sigma,\frac12}(T^*X)$$

is the natural projection map. If

$$a \in S^{\mathfrak{p}}_{\Sigma,\frac12}(T^*X)\,, \quad d(\mathrm{supp}\, a, \Sigma) \geq 1/C$$

then

$$a \in S^{m+k_2,\widetilde{m}-k_2,k_1}_{\frac12}(T^*X)\,, \quad \mathrm{Op}_{\Sigma,h,\tilde h}(a) = \mathrm{Op}^w_h(a) \in \Psi^{m+k_2,\widetilde{m}-k_2,k_1}_{\frac12}(X)\,,$$

where $\Psi^{\bullet,\bullet,\bullet}_{\frac12}(X)$ is the class of pseudodifferential operators defined in §3.3.

5.2. **Calculus in the model case.** To define $\Psi^{\mathfrak{p}}_{\Sigma,\frac12}(X)$ we proceed locally and put Σ into a normal form $\Sigma_0 = \{\xi_1 = 0\}$ (locally).

The model case is obtained by taking symbols satisfying (5.3) globally and defining

$$a = a(x,\xi,\lambda,h,\tilde h)\,, \quad \lambda = \tilde h\xi_1/h\,,$$

satisfying

$$\partial^k_{x_1}\partial^\alpha_{x'}\partial^\beta_\xi\partial^p_\lambda a(x,\xi,\lambda;h) = \mathcal{O}(h^{-m}\tilde h^{-\widetilde{m}})(\tilde h/h)^{(|\alpha|+|\beta|)/2}\langle\lambda\rangle^{k_2+k/2-p/2}\,, \tag{5.5}$$

which is the same as (5.1). We will write (5.5) as

$$a \in \widetilde{\mathcal{O}}_{\frac12}\left(h^{-m}\tilde h^{-\tilde m}\langle\lambda\rangle^{k_2}\right)\,.$$

and define an exact quantization in the usual way,

$$\begin{aligned}&\widetilde{\mathrm{Op}}_{h,\tilde h}(a)u(x)=\\ &\frac{1}{2\pi\tilde h}\frac{1}{(2\pi h)^{n-1}}\int a(x,\xi',(h/\tilde h)\lambda,\lambda;h)e^{i\langle x'-y',\xi'\rangle/h+i\langle x_1-y_1,\lambda\rangle/\tilde h}u(y)dyd\xi'd\lambda\,,\end{aligned}\tag{5.6}$$

$n=\dim X$, and where

$$\lambda=(\tilde h/h)\xi_1\,.$$

For $a\in\widetilde{\mathcal{O}}_{\frac12}(\langle\lambda\rangle^{k_2})$ and $b\in\widetilde{\mathcal{O}}_{\frac12}(\langle\lambda\rangle^{k_2'})$ we have

$$\begin{aligned}&\widetilde{\mathrm{Op}}_{h,\tilde h}(a)\circ\widetilde{\mathrm{Op}}_{h,\tilde h}(b)=\widetilde{\mathrm{Op}}_{h,\tilde h}(a\sharp_{h,\tilde h}b)\,,\quad a\,\sharp_{h,\tilde h}\,b=\widetilde{\mathcal{O}}_{\frac12}\left(\langle\lambda\rangle^{k_2+k_2'}\right)\,,\\ &a\,\sharp_{h,\tilde h}\,b(x,\xi,\lambda;h)\sim\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}(h\partial_{\xi'})^{\alpha'}(h\partial_{\xi_1}+\tilde h\partial_\lambda)^{\alpha_1}a\,D_x^\alpha b\,,\end{aligned}\tag{5.7}$$

where the asymptotic sum is defined up to terms in

$$\widetilde{\mathcal{O}}_{\frac12}(\tilde h^\infty\langle\lambda\rangle^{k_2+k_2'})\,.$$

To see this we write $\widetilde{\mathrm{Op}}_{h,\tilde h}(a)$ as a quantization of an operator valued symbol, $\mathrm{Op}_h(a)(x_1,\lambda)$,

$$\mathrm{Op}_h(a)(x_1,\lambda)=\frac{1}{(2\pi h)^{n-1}}\int a(x,\xi',(h/\tilde h)\lambda,\lambda;h)e^{i\langle x'-y',\xi'\rangle/h}d\xi'\,,$$

so that

$$\widetilde{\mathrm{Op}}_{h,\tilde h}(a)=\mathrm{Op}_h(a)(x_1,\tilde hD_{x_1})\,.\tag{5.8}$$

In view of (3.10) we have

$$a\in\widetilde{\mathcal{O}}_{\frac12}(\langle\lambda\rangle^{k_2})\iff$$

$$\|\partial_{x_1}^k\partial_\lambda^p\mathrm{Op}_h(\partial_{x,\xi}^\alpha a)(x_1,\lambda)\|_{L^2(\mathbb{R}^{n-1})\to L^2(\mathbb{R}^{n-1})}\le C_{\alpha,p,k}(\tilde h/h)^{|\alpha|/2}\langle\lambda\rangle^{k_2+k/2-p/2}\,.$$

Observing that

$$\begin{aligned}&\widetilde{\mathrm{Op}}_{h,\tilde h}(a)\circ\widetilde{\mathrm{Op}}_{h,\tilde h}(b)=\\ &\left(\exp(i\tilde h\langle D_\lambda,D_{y_1}\rangle)\mathrm{Op}_h a(x_1,\lambda)\circ\mathrm{Op}_h b(y_1,\lambda')\restriction_{x_1=y_1,\lambda=\lambda'}\right)(x_1,\tilde hD_{x_1})\,,\end{aligned}$$

proves (5.7).

For the operator $\widetilde{\mathrm{Op}}_{h,\tilde h}(a)$, with a supported in $|\xi_1|\le C$, we define its principal symbol as the equivalence class of a in

$$\widetilde{\mathcal{O}}_{\frac12}(\langle\lambda\rangle^{k_2})/\widetilde{\mathcal{O}}_{\frac12}(\tilde h\langle\lambda\rangle^{k_2})\,.$$

In view of (5.7) this symbol map is a homomorphism onto the quotient of symbol spaces:

$$\widetilde{\mathrm{Op}}_{h,\tilde h}(a)\mapsto[a]\in\widetilde{\mathcal{O}}_{\frac12}(\langle\lambda\rangle^{k_2})/\widetilde{\mathcal{O}}_{\frac12}(\tilde h\langle\lambda\rangle^{k_2})\tag{5.9}$$

We note that in the local model we are not concerned here with the behaviour as $|\xi|\to\infty$.

It will be useful to have the analogue of Beals's characterization of pseudodifferential operators by stability under taking commutators. It follows from the proof of the semiclassical analogue of Beals's result [14] and its adaptations in [10, Chapter 8] and [33, Lemma 4.2].

Proposition 5.2. *Let $A = A_h : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ and put $x' = (x_2, \cdots, x_n)$. For $p \in \mathbb{R}$, we define the norms*

$$\|u\|_{(p)} = \|\langle \tilde{h} D_{x_1}\rangle^p u\|_{L^2}\,.$$

Then

$$A = \widetilde{\mathrm{Op}}_{h,\tilde{h}}(a) \ \text{ for } \ a = \widetilde{\mathcal{O}}_{\frac{1}{2}}(\langle\lambda\rangle^k), \quad a = a(x,\xi',\lambda;h,\tilde{h})\,,$$

if and only if for all $N, p, q \geq 0$ and every sequence $\ell_1(x',\xi'), \cdots, \ell_N(x',\xi')$ of linear forms on $\mathbb{R}^{2(n-1)}$ there exists $C > 0$ for which

$$\begin{aligned} &\|\operatorname{ad}_{\ell_1(x',hD_{x'})} \circ \cdots \circ \operatorname{ad}_{\ell_N(x',hD_{x'})} \circ (\operatorname{ad}_{\tilde{h}D_{x_1}})^p \circ (\operatorname{ad}_{x_1})^q A u\|_{(q/2-\min(k,0))} \\ &\qquad \leq C h^{N/2} \tilde{h}^{N/2+p+q} \|u\|_{p/2+(\max(k,0))}\,. \end{aligned} \tag{5.10}$$

Proof. We first observe that for $A = \widetilde{\mathrm{Op}}_{h,\tilde{h}}(a)$ for $a = \widetilde{\mathcal{O}}_{\frac{1}{2}}(\langle\lambda\rangle^k)$, (5.10) follows from the calculus.

Let

$$V_h u(x_1, \tilde{x}') \stackrel{\text{def}}{=} h^{-\frac{n-1}{4}} u(x_1, h^{\frac{1}{2}}\tilde{x}')$$

so that

$$A = V_h^{-1} \widetilde{A} V_h\,,$$

where, as in the proof of Proposition 3.5, we have

$$\begin{aligned} &\|\operatorname{ad}_{\ell_1(x',D_{x'})} \circ \cdots \circ \operatorname{ad}_{\ell_N(x',D_{x'})} \circ (\operatorname{ad}_{D_{x_1}})^p \circ (\operatorname{ad}_{x_1})^q \widetilde{A} u\|_{(q/2-\min(k,0))} \\ &\qquad \leq C \tilde{h}^{N/2+q} \|u\|_{p/2+(\max(k,0))}\,. \end{aligned} \tag{5.11}$$

We can write $\widetilde{A} = a_{h,\tilde{h}}(x, D_x)$, so that, $A = \widetilde{\mathrm{Op}}_{h,\tilde{h}}(a)$, where

$$a(x,\lambda,\xi';h) = a_{h,\tilde{h}}(x_1, h^{-1/2}x'; \tilde{h}^{-1}\lambda, h^{-1/2}\xi')\,.$$

The required estimate on a becomes

$$\partial_{x_1}^p \partial_{\xi_1}^q \partial_{\tilde{x}'}^{\alpha'} \partial_{\tilde{\xi}'}^{\beta'} a_{h,h'} = \mathcal{O}(1) \tilde{h}^{(|\alpha'|+|\beta'|)/2+q} \langle \tilde{h}\xi_1\rangle^{k-q/2+p/2} \tag{5.12}$$

We have

$$\langle \widetilde{A}\psi, \phi\rangle = \frac{1}{(2\pi)^n} \int\int e^{i\langle x,\xi\rangle} a_{h,\tilde{h}}(x,\xi) \widehat{\psi}(\xi) \overline{\phi(x)} dx d\xi\,, \tag{5.13}$$

with

$$\widehat{\psi}(\xi) = (\mathcal{F}\psi)(\xi) = \int e^{-i\langle x,\xi\rangle} \psi(x) dx\,,$$

and for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Let us fix $(x_0, \xi_0), (y_0, \eta_0) \in T^*\mathbb{R}^n$, and $\lambda \gg 1$. With $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ we put

$$\psi_{x_0,\xi_0}(x) = \lambda^{\frac{1}{4}} \psi\left(\lambda^{\frac{1}{2}}(x_1 - x_{0,1}), x' - x_0'\right) e^{i\langle x, \xi_0\rangle},$$
$$\phi_{y_0,\eta_0}(x) = \lambda^{\frac{1}{4}} \phi\left(\lambda^{\frac{1}{2}}(x_1 - y_{0,1}), x' - y_0'\right) e^{i\langle x - y_0, \eta_0\rangle}.$$

We see that

$$\widehat{\psi}_{x_0,\xi_0}(\xi) = \lambda^{-\frac{1}{4}} \hat{\psi}\left(\lambda^{-\frac{1}{2}}(\xi_1 - \xi_{0,1}), \xi' - \xi_0'\right) e^{-i\langle \xi - \xi_0, x_0\rangle},$$
$$\widehat{\phi}_{y_0,\eta_0}(\xi) = \lambda^{-\frac{1}{4}} \hat{\phi}\left(\lambda^{-\frac{1}{2}}(\xi_1 - \eta_{0,1}), \xi' - \eta_0'\right) e^{-i\langle \xi, y_0\rangle}.$$

We have

$$B \stackrel{\text{def}}{=} \mathrm{ad}_{x'}^{\alpha'} \mathrm{ad}_{D_{x'}}^{\beta'} \mathrm{ad}_{x_1}^{q} \mathrm{ad}_{D_{x_1}}^{p} \widetilde{A} = (-i)^{|\alpha'|+|\beta'|+q+p} b_{h,\tilde{h}}(x, D),$$
$$b_{h,\tilde{h}}(x, \xi) = (-\partial_{\xi'})^{\alpha'} \partial_{x'}^{\beta'} (-\partial_{\xi_1})^{q} \partial_{x_1}^{p} a_{h,\tilde{h}}(x, \xi).$$

Let us now assume we have the commutator estimate in the proposition with $k \geq 0$. Since

$$\|u\|_{(-q/2)} = \|\langle \tilde{h} D_{x_1}\rangle^{-q/2} u\|_{L^2}$$

is the norm dual to $\| \bullet \|_{(q/2)}$, we get from (5.11)

$$|\langle B\psi_{x_0,\xi_0}, \phi_{y_0,\eta_0}\rangle| \leq \tilde{h}^{(|\alpha'|+|\beta'|)/2+q} \|\psi_{x_0,\xi_0}\|_{(p/2+k)} \|\phi_{y_0,\eta_0}\|_{(-q/2)}. \tag{5.14}$$

Let $\bullet_{0,1}$ denote the first component of $\bullet_0 \in \mathbb{R}^n$. For fixed $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{gathered} \|\psi_{x_0,\xi_0}\|^2_{(p/2+k)} \leq C_N I^N_{p/2+k}(\lambda, \xi_{0,1}), \quad \|\phi_{y_0,\eta_0}\|^2_{(-q/2)} \leq C_N I^N_{-q/2}(\lambda, \eta_{0,1}) \\ I^N_r(\lambda, \tau) \stackrel{\text{def}}{=} \lambda^{-\frac{1}{2}} \int_{\mathbb{R}} \langle \tilde{h}\rho\rangle^{2r} \langle \lambda^{-\frac{1}{2}}(\rho - \tau)\rangle^{-N} d\rho \end{gathered} \tag{5.15}$$

Using (5.13) we rewrite the left hand side of (5.14) as

$$\frac{1}{(2\pi)^n} \left| \int\int e^{i\langle x,\xi\rangle} b_{h,\tilde{h}}(x, \xi) \widehat{\psi}_{x_0,\xi_0}(\xi) \bar{\phi}_{y_0,\eta}(x) dx d\xi \right|.$$

Decomposing the first exponent in the integral as

$$\langle x, \xi\rangle = \langle y_0, \xi_0\rangle + \langle x - y_0, \xi_0\rangle + \langle \xi - \xi_0, y_0\rangle + \langle x - y_0, \xi - \xi_0\rangle$$

and using the formulæ for $\widehat{\psi}_{x_0,\xi_0}$, ϕ_{y_0,η_0}, we rewrite it further as

$$\frac{1}{(2\pi)^n} \Big| \int\int b_{h,\tilde{h}}(x, \xi) \exp(i\langle x - y_0, \xi - \xi_0) \widehat{\psi}(\lambda^{-\frac{1}{2}}(\xi_1 - \xi_{0,1}), \xi' - \xi_0')$$
$$\bar{\phi}(\lambda^{\frac{1}{2}}(x_1 - y_{0,1}), x' - y_0') \exp(i(\langle \xi - \xi_0, y_0 - x_0\rangle + \langle x - y_0, \xi_0 - \eta_0\rangle)) dx d\xi \Big|$$

Summing up, we get

$$\begin{aligned}\big|\mathcal{F}\big(\chi b_{h,\tilde h}\big)(\eta_0-\xi_0,x_0-y_0)\big| &= \mathcal{O}(1)\tilde h^{(|\alpha'|+|\beta'|)/2+q}\|\psi_{x_0,\xi_0}\|_{p/2+k}\|\phi_{y_0,\eta_0}\|_{-q/2}\,,\\ \chi(x,\xi) &= e^{i\langle x-y_0,\xi-\xi_0\rangle}\widehat{\psi}(\lambda^{-\frac12}(\xi_1-\xi_{0,1}),\xi'-\xi_0')\bar\phi(\lambda^{\frac12}(x_1-y_{0,1}),x'-y_0')\,.\end{aligned}\tag{5.16}$$

Writing

$$\zeta_1 \overset{\text{def}}{=} \frac{\eta_{0,1}-\xi_{0,1}}{\lambda^{\frac12}}\,,\quad z_1\overset{\text{def}}{=}\lambda^{\frac12}(x_{0,1}-y_{0,1})\,,\quad \zeta'=\eta_0'-\xi_0'\,,\quad z'=x_0'-y_0'\,,$$

we therefore get

$$\mathcal{F}\left(\widetilde{\chi}\widetilde{b}_{h,\tilde h}\right)(\zeta,z)=\mathcal{O}(1)\tilde h^{(|\alpha'|+|\beta'|)/2+q}\|\psi_{x_0,\xi_0}\|_{(p/2+k)}\|\phi_{y_0,\eta_0}\|_{(-q/2)}\,,$$

where

$$\widetilde{b}_{h,\tilde h}(X,\Xi)\overset{\text{def}}{=} b_{h,\tilde h}(y_0+\lambda^{-\frac12}X,\xi_0+\lambda^{\frac12}\Xi)\,,$$

and

$$\widetilde{\chi}(X,\Xi)\overset{\text{def}}{=} e^{i\langle X,\Xi\rangle}\widehat{\psi}(\Xi)\bar\phi(X)\,.$$

We then have

$$\begin{aligned}&\mathcal{F}\left(\partial_{\Xi'}^{\tilde\alpha'}\partial_{\Xi_1}^{\tilde q}\partial_{X'}^{\tilde\beta'}\partial_{X_1}^{\tilde p}\widetilde{\chi}\widetilde{b}_{h,\tilde h}\right)(\zeta,z)=\\ &\quad\mathcal{O}_N(1)\tilde h^{(|\alpha'|+|\beta'|)/2+q}I^N_{(p/2+k+\tilde p/2)}(\lambda,\xi_{0,1})I_{(-q/2-\tilde q/2)}(\lambda,\eta_{0,1})\lambda^{-\tilde p/2+\tilde q/2}\,,\end{aligned}$$

which putting

$$\tilde\alpha=(\tilde q,\tilde\alpha')\,,\quad \tilde\beta=(\tilde p,\tilde\beta')\,,$$

we rewrite as

$$\begin{aligned}&z^{\tilde\alpha}\zeta^{\tilde\beta}\mathcal{F}\left(\widetilde{\chi}\widetilde{b}_{h,\tilde h}\right)(\zeta,z)=\\ &\quad\mathcal{O}(1)\tilde h^{(|\alpha'|+|\beta'|)/2+q}I^N_{(p/2+k+\tilde p/2)}(\lambda,\xi_{0,1})I_{(-q/2-\tilde q/2)}(\lambda,\eta_{0,1})\lambda^{-\tilde p/2+\tilde q/2}\,.\end{aligned}$$

We now go back to (5.15) and choose $\lambda=\langle\tilde h\xi_{0,1}\rangle$. Then

$$\begin{aligned}I^N_{(p/2+\tilde p/2+k)}(\langle\tilde h\xi_{0,1}\rangle,\xi_{0,1}) &= \left(\int_{\mathbb{R}}\langle\tilde h(\xi_{0,1}+\langle\tilde h\xi_{0,1}\rangle^{\frac12}r)\rangle^{p+\tilde p+2k}\langle r\rangle^{-N}dr\right)^{\frac12}\\ &= C_N\left(\int_{\mathbb{R}}\langle R+\langle R\rangle^{\frac12}\tilde h r)\rangle^{p+\tilde p+2k}\langle r\rangle^{-N}dr\right)^{\frac12}\restriction_{R=\tilde h\xi_{0,1}}\\ &\sim \langle R\rangle^{(p+\tilde p+2k)/2}\restriction_{R=\tilde h\xi_{0,1}}=\langle\tilde h\xi_{0,1}\rangle^{p/2+\tilde p/2+k}\,.\end{aligned}$$

It follows that for any $\tilde\alpha$ and N we have

$$|z^{\tilde\alpha}\mathcal{F}\left(\widetilde{\chi}\widetilde{b}_{h,\tilde h}\right)(\zeta,z)|=\mathcal{O}_N(1)\tilde h^{(|\alpha'|+|\beta'|)/2+q}I^N_{(-q/2-\tilde q/2)}\lambda^{\tilde q/2+p/2+k}\langle\zeta\rangle^{-N}\,,\quad\lambda=\langle\tilde h\xi_{0,1}\rangle\,.\tag{5.17}$$

We now integrate the left hand side in $\zeta_1 = (\eta_{0,1} - \xi_{0,1})/\lambda^{\frac{1}{2}}$:

$$\begin{aligned}
&\int_{\mathbb{R}} I^N_{-q/2-\tilde{q}/2} \left\langle \frac{\eta_{0,1}-\xi_{0,1}}{\langle \tilde{h}\xi_{0,1}\rangle^{\frac{1}{2}}} \right\rangle^{-N} \langle \tilde{h}\xi_{0,1}\rangle^{-\frac{1}{2}} d\eta_{0,1} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle \tilde{h}(\eta_{0,1} + \langle \tilde{h}\xi_{0,1}\rangle^{\frac{1}{2}} r)\rangle^{-q-\tilde{q}} \langle r\rangle^{-N} dr \right)^{\frac{1}{2}} \left\langle \frac{\eta_{0,1}-\xi_{0,1}}{\langle \tilde{h}\xi_{0,1}\rangle^{-N}} \right\rangle^{\frac{1}{2}} \langle \tilde{h}\xi_{0,1}\rangle^{-\frac{1}{2}} d\eta_{0,1} \\
&\leq C'_N \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle \tilde{h}(\xi_{0,1} + \langle \tilde{h}\xi_{0,1}\rangle^{\frac{1}{2}}(r+\zeta_1))\rangle^{-q/2-\tilde{q}/2} \langle r\rangle^{-N} dr \right)^{\frac{1}{2}} \langle \zeta_1\rangle^{-N} d\zeta_1 \\
&\leq \langle \tilde{h}\xi_{0,1}\rangle^{-q/2-\tilde{q}/2} .
\end{aligned}$$

Returning to (5.17) we see that

$$\int_{\mathbb{R}^n} |\mathcal{F}\left(\widetilde{\chi} \widetilde{b}_{h,\tilde{h}}\right)(\zeta, z)| d\zeta = \mathcal{O}(1)\tilde{h}^{(|\alpha'|+|\beta'|)/2+q} \langle \tilde{h}\xi_{0,1}\rangle^{p/2+k-q/2} \langle z\rangle^{-N}$$

and consequently

$$\widetilde{b}_{h,\tilde{h}}\widetilde{\chi}(0,0) = \mathcal{O}(1)\tilde{h}^{(|\alpha'|+|\beta'|)/2+q} \langle \tilde{h}\xi_{0,1}\rangle^{k+p/2-q/2} .$$

Combining this with the definition of $b_{h,\tilde{h}}$ and $\widetilde{b}_{h,\tilde{h}}$ gives

$$\partial^{\alpha'}_{\xi'} \partial^q_{\xi_1} \partial^{\beta'}_{x'} \partial^p_{x_1} a_{h,\tilde{h}}(y_0, \xi_0) = \mathcal{O}(1)\tilde{h}^{(|\alpha'|+|\beta'|)/2+q} \langle \tilde{h}\xi_{0,1}\rangle^{k+p/2-q/2} ,$$

which is (5.12) for $k \geq 0$. When $k < 0$ we check that the assumption is satisfied for $\langle \tilde{h}D_{x_1}\rangle^{-k} A$ with k replaced by 0. The composition formula then gives the result. □

5.3. **Modified Sobolev spaces.** The norm used in Proposition 5.2 can be defined globally. We generalize it to include standard Sobolev spaces by adding additional information at a smooth compact hypersurface $\Sigma \subset T^*X$.

Let $Q \in \Psi^{0,0}_h$ be an operator, $Q = Q^*$, with the principal symbol q satisfying

$$\Sigma = \{(x,\xi) \,:\, q(x,\xi) = 0\} , \quad dq|_\Sigma \neq 0 , \quad |q(x,\xi)| \geq 1 \ \text{ for } |\xi| \geq C \quad . \tag{5.18}$$

For $s, m \in \mathbb{R}$ we define

$$H^{s,m}_\Sigma(X) = \{u \in \mathcal{S}'(X) \,:\, \langle (\tilde{h}/h)Q\rangle^m u \in H^s_h(X)\} , \quad \|u\|_{H^{s,m}_\Sigma(X)} \stackrel{\text{def}}{=} \|\langle (\tilde{h}/h)Q\rangle^m u\|_{H^s_h(X)} . \tag{5.19}$$

Here $H^s_h(X)$ denotes the usual semiclassical Sobolev spaces defined in §3.1. The spaces are (complex) interpolation spaces, in m, and in s.

When $m \in \mathbb{Z}$ the definition is equivalent to

$$H^{s,m}_\Sigma(X) = \begin{cases} \{u \,:\, (\tilde{h}/h)^k Q^k u \in H^s_h(X) , \ 0 \leq k \leq m\} & m \geq 0 \\ \{u \,:\, u = \sum_{k=0}^{|m|} (\tilde{h}/h)^k Q^k u_k , \ u_k \in H^s_h(X)\} & m \leq 0 \end{cases}$$

with the norms equivalent to the same natural way as for Sobolev spaces:

$$\|u\|_{H^{s,m}_\Sigma(X)} \simeq \begin{cases} \sum_{k=0}^m \|(\tilde h/h)^k Q^k u\|_{H^s_{\tilde h}(X)}\,, & m \geq 0 \\ \inf\{\sum_{k=0}^{|m|} \|u_k\|_{H^s_{\tilde h}(X)} \ :\ u = \sum_{k=0}^{|m|} (\tilde h/h)^k Q^k u_k\}\,, & m \leq 0 \end{cases}$$

This can be seen using a spectral decomposition of Q which is assumed to be bounded and self-adjoint. We use the following simplified notation

$$H^m_\Sigma(X) \overset{\text{def}}{=} H^{0,m}_\Sigma(X)\,,\quad m \in \mathbb{Z}\,.$$

The spaces $H^{s,m}_\Sigma(X)$ have the following basic invariance property:

Lemma 5.3. *The definition* (5.19) *does not depend on the choice of* $Q \in \Psi^{0,0}_h(X)$ *satisfying* (5.18). *If* $A \in \Psi^{0,0}(X)$ *has the property that* $d(\mathrm{WF}_h(A), \Sigma) > 1/C$ *then for* $M \geq 0$

$$\begin{aligned} \|Au\|_{H^{-M}_\Sigma(X)} &\leq (h/\tilde h)^M \|u\|_{L^2(X)}\,, \\ \|Au\|_{L^2(X)} &\leq (h/\tilde h)^M \|u\|_{H^M_\Sigma(X)}\,, \end{aligned} \tag{5.20}$$

for $u \in \mathcal{C}^\infty_{\mathrm{c}}(X)$.

Also, suppose that F *is a* 0*th order* h*-Fourier Integral operator associated to a canonical transformation which maps* Σ *to another hypersurface* Σ' *satisfying our hypothesis. Then*

$$F\ :\ H^{s,m}_\Sigma(X)\ \longrightarrow H^{s,m}_{\Sigma'}(X)\,.$$

Proof. Let Q' be another operator satisfying (5.18). Then

$$Q' = AQ + E = QA + E'\,,$$

where $A \in \Psi^{0,0}(X)$ is uniformly elliptic, and $E, E' \in \Psi^{-1,-1}(X)$. Because of the interpolation property of $H^{s,m}_\Sigma$ we only need to check the independence for $k \in \mathbb{Z}$. Then, by induction on k,

$$\|(\tilde h/h)^k (Q')^k u\|_{H^s(X)} \leq C_k \sum_{\ell=0}^k \tilde h^{k-\ell} \|(\tilde h/h)^\ell Q^\ell u\|_{H^s(X)}\,,$$

where C_k depends only on A and E. This shows that the definition for $m \geq 0$ is independent of the choice of Q. The case of $m < 0$ is similar. To see the first inequality in (5.20) we recall that

$$\|Au\|_{H^{-M}_\Sigma(X)} = \inf\left\{ \sum_{k=0}^M \|u_k\|_{L^2(X)}\ :\ Au = \sum_{k=0}^M (\tilde h/h)^k Q^k u_k \right\}\,.$$

Because of the ellipticity of Q on $\mathrm{WF}_h(A)$ we can find B such that

$$Au = Q^k BAu + v\,,\quad \|v\|_{L^2} = \mathcal{O}(h^\infty)\|u\|_{L^2}\,.$$

Hence we can take $u_0 = v$ and $u_M = (h/\tilde{h})^M BAu$, $u_k = 0$ for all other k's, so that

$$\|Au\|_{H_\Sigma^{-M}(X)} \leq C(h/\tilde{h})^M \|u\|_{L^2(X)} + \mathcal{O}(h^\infty)\|u\|_{L^2(X)}$$

proving the first estimate in (5.20) ($h < \tilde{h}$ everywhere here). Since $H_\Sigma^M(X)$ is the dual of H_Σ^{-M} this is equivalent to

$$\forall\, v \in \mathcal{C}^\infty_{\rm c}(X) \quad \langle Au, v\rangle_{L^2(X)} \leq C(h/\tilde{h})^M \|u\|_{L^2(X)} \|v\|_{H_\Sigma^M(X)}\,,$$

which in turn proves the second estimate with A replaced by A^*.

The Egorov theorem (Proposition 3.4 above) shows that QF is equal to $FQ' + E$ where Q' satisfies (5.18) with Σ replaced by Σ'. The mapping property follows from the boundedness of F on $H^s(X)$ and an argument similar to that above. □

5.4. **Globally defined class of operators.** For the global definition we cannot use the classes

$$\widetilde{\rm Op}_{h,\tilde{h}}\left(\widetilde{\mathcal{O}}_{\frac{1}{2}}(\langle\lambda\rangle^m)\right)$$

due to the presence of propagation in the h-sense. To see the problem let us consider the operator P introduced in §1 and define

$$A = h^{-m}\chi\left(\frac{\tilde{h}P}{h}\right)\,,\quad \chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})\,.$$

In the model theory of §5.2 this operator is obtained from the symbol $h^{-m}\chi(\lambda)$. However, when $\tilde{h}$ is fixed, the operator A is an h-Fourier integral operator:

$$A = \frac{h^{-m}}{2\pi\tilde{h}}\int_{\mathbb{R}} \widehat{\chi}(t/\tilde{h})\exp(itP/h)dt\,,$$

associated to the relation

$$\{(\rho,\rho')\ :\ \exists\, t\,,\ \exp(tH_p)(\rho) = \rho'\,,\ p(\rho) = p(\rho') = 0\}\,.$$

Hence the presence of almost closed orbits of the flow prevents a definition which would be purely local in the $\mathcal{O}(h^\infty)$ sense. Since $\widehat{\chi}(t/\tilde{h})\restriction_{|t|>\epsilon} = \mathcal{O}(\tilde{h}^\infty)$, the non-local contributions in A are of the order $\mathcal{O}(h^{-m}\tilde{h}^\infty)$.

These global concerns suggest the following definition of the residual class. First we introduce a useful cut-off operator. Let V_1, V_2 be two open neighbourhoods of Σ, satisfying

$$\overline{V_1} \Subset V_2\,.$$

Then we choose

$$\gamma_\Sigma \in \Psi^{0,0}(X)\,,\quad {\rm WF}_h(\gamma_\Sigma) \subset V_2\,,\quad {\rm WF}_h(I-\gamma_\Sigma) \subset \complement V_1\,. \tag{5.21}$$

We now define the spaces of operators. Let

$$\mathfrak{p} = (m, \tilde{m}, k_1, k_2)\,,\quad \mathfrak{p}_\infty = (m, -\infty, -\infty, k_2)\,.$$

As before we start with the definition of a residual class:

Definition 1. We say that $A \in \Psi^{\mathfrak{p}_\infty}_{\Sigma,\frac12}(X)$ if

$$A : \mathcal{C}^\infty_{\rm c}(X) \to \mathcal{C}^\infty(X)\,,\quad (I-\gamma_\Sigma)A\,,\ A(I-\gamma_\Sigma) \in \Psi^{m+k_2,-\infty,-\infty}_{\frac12}(X)\,,$$

and for any $u \in \mathcal{C}^\infty_{\rm c}(X)$, any sequence $\{b_j\}_{j=1}^N \subset S^0(T^*X)$, and any k, p, and M, we have

$$\| \operatorname{ad}_{\operatorname{Op}^w_h(b_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}^w_h(b_N)} \operatorname{ad}^k_Q Au\|_{H^{p+N/2-k_2}_\Sigma(X)} \le Ch^{-m+k}\tilde h^M \|u\|_{H^{p+k/2}_\Sigma}\,,$$

where Q is the operator in (5.19).

Definition 2. We say that $A \in \Psi^{\mathfrak{p}}_{\Sigma,\frac12}(X)$ if

$$A : \mathcal{C}^\infty_{\rm c}(X) \to \mathcal{C}^\infty(X)\,,\quad (I-\gamma_\Sigma)A\,,\ A(I-\gamma_\Sigma) \in \Psi^{m+k_2,\tilde m-k_2,k_1}_{\frac12}(X)\,,$$

for some cut-off operator γ_Σ satisfying (5.21).

- For any $\chi \in \mathcal{C}^\infty_{\rm c}(T^*X)$,
$$A = A^\sharp_\chi + A^\flat_\chi\,,\quad A^\flat_\chi \in \Psi^{\mathfrak{p}_\infty}_{\Sigma,\frac12}(X)\,,$$
so that for any p, and any sequences
$$\{b_j\}_{j=1}^N\,,\quad \{a_j\}_{j=1}^M \subset S^0(T^*X)\,,\ H_p a_j\restriction_{\operatorname{supp}\chi} \equiv 0\,,$$
$$\begin{aligned}\| \operatorname{ad}_{\operatorname{Op}^w_h(b_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}^w_h(b_N)} \circ \operatorname{ad}_{\operatorname{Op}^w_h(a_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}^w_h(a_M)} \circ \operatorname{ad}^k_Q \chi^w\gamma_\Sigma A^\sharp_\chi \chi^w u\|_{H^{p+N/2-k_2}_\Sigma(X)}\\ \le\ Ch^{M/2+k-m}\tilde h^{M/2+N-\tilde m}\|u\|_{H^{p+k/2}_\Sigma(X)}\,,\quad \chi^w = \operatorname{Op}^w_h(\chi)\,,\end{aligned}$$
for any $u \in \mathcal{C}^\infty_{\rm c}(X)$.
- For any $\psi_1,\psi_2 \in \mathcal{C}^\infty_{\rm b}(T^*X)$, $\operatorname{supp}\psi_1 \cap \operatorname{supp}\psi_2 = \emptyset$,
$$\operatorname{Op}^w_h(\psi_1) A \operatorname{Op}^w_h(\psi_2) \in \Psi^{\mathfrak{p}_\infty}_{\Sigma,\frac12}(X)\,.$$

It is important to record

Lemma 5.4. *Definitions 1 and 2 are independent of the choice of the cut-off operator satisfying* (5.21).

Proof. Suppose that γ'_Σ is another cut-off operator satisfying (5.21). We need to show that

$$(\gamma_\Sigma - \gamma'_\Sigma)A\,,\quad A(\gamma_\Sigma-\gamma'_\Sigma) \in \Psi^{m+k_2,\tilde m-k_2,-\infty}_{\frac12}(X)\,, \tag{5.22}$$

and for that we will use only the commutators involving b_j's in Definitions 1 and 2. The first inclusion is, in view of Lemma 3.5 and the fact that $\gamma_\Sigma - \gamma'_\Sigma$ is smoothing, equivalent to

$$\begin{aligned}\| \operatorname{ad}_{\operatorname{Op}^w_h(b_1)} \circ \cdots \circ \operatorname{ad}_{\operatorname{Op}^w_h(b_N)}(\gamma_\Sigma-\gamma'_\Sigma)Au\|_{L^2(X)}\\ \le Ch^{-m-k_2+N/2}\tilde h^{-\tilde m+k_2+N/2}\|u\|_{L^2(X)}\,.\end{aligned} \tag{5.23}$$

To see this we first use (5.20) to see that if $E \in \Psi^{0,0}(X)$ and $d(\mathrm{WF}_h(E), \Sigma) > 1/C$ then for any subsequence $\{i_j\}_{j=1}^J$ of $1, \cdots, N$,

$$\begin{aligned} &\|E \operatorname{ad}_{\mathrm{Op}_h^w(b_{i_1})} \circ \cdots \circ \operatorname{ad}_{\mathrm{Op}_h^w(b_{i_J})} Au\|_{L^2(X)} \\ &\qquad \leq C(h/\tilde{h})^{J/2-k_2} \| \operatorname{ad}_{\mathrm{Op}_h^w(b_{i_1})} \circ \cdots \circ \operatorname{ad}_{\mathrm{Op}_h^w(b_{i_J})} Au\|_{H_\Sigma^{J/2-k_2}(X)} \\ &\qquad \leq Ch^{-m}(h/\tilde{h})^{J/2-k_2}\tilde{h}^{J-\tilde{m}}\|u\|_{L^2(X)} = Ch^{-m-k_2+J/2}\tilde{h}^{-\tilde{m}+k_2+J/2}\|u\|_{L^2(X)}\,. \end{aligned} \tag{5.24}$$

Using the derivation property of $\operatorname{ad}_{\mathrm{Op}_h^w(b_j)}$ and the fact that

$$\operatorname{ad}_{\mathrm{Op}_h^w(b_j)}(\gamma_\Sigma - \gamma'_\Sigma) = hE_j\,, \quad E_j \in \Psi^{0,0}(X)\,, \quad d(\mathrm{WF}_h(E_j), \Sigma) > 1/C\,,$$

we can estimate the left hand side of (5.23) by a linear combination of terms of the form

$$h^{N-J}\|E \operatorname{ad}_{\mathrm{Op}_h^w(b_{i_1})} \circ \cdots \circ \operatorname{ad}_{\mathrm{Op}_h^w(b_{i_J})} Au\|_{L^2(X)}\,, \quad E \in \Psi^{0,0}(X)\,, \quad d(\mathrm{WF}_h(E), \Sigma) > 1/C\,.$$

Consequently (5.23) follows from (5.24).

Since, by the calculus of §3.3

$$B \in \Psi^{m+k_2,-\infty,-\infty}_{\frac{1}{2}}(X) \iff B^* \in \Psi^{m+k_2,-\infty,-\infty}_{\frac{1}{2}}(X)\,,$$

and since

$$\operatorname{ad}_{\mathrm{Op}_h^w(b_1)} \circ \cdots \circ \operatorname{ad}_{\mathrm{Op}_h^w(b_N)} B^* = (-1)^N \left(\operatorname{ad}_{\mathrm{Op}_h^w(\bar{b}_1)} \circ \cdots \circ \operatorname{ad}_{\mathrm{Op}_h^w(\bar{b}_N)} B\right)^*\,,$$

the second inclusion in (5.22) follows from the first one. □

We now have a natural mapping and invariance properties:

Proposition 5.5. *The operators $\Psi^{0,0,0,0}_{\Sigma,\frac{1}{2}}(X)$ form an algebra and $\Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac{1}{2}}(X)$ is an ideal in that algebra. If $A \in \Psi^{\mathfrak{p}_\infty}_{\Sigma,\frac{1}{2}}(X)$ then*

$$A = \mathcal{O}(h^{-m}\tilde{h}^M)\ :\ H^{-M,r}_\Sigma(X) \to H^{M,r-k_2}_\Sigma(X)\,,$$

for any m, M. If $A \in \Psi^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(X)$ then

$$A = \mathcal{O}(h^{-m}\tilde{h}^{-\widetilde{m}})\ :\ H^{s+k_1,p+k_2}_\Sigma(X) \longrightarrow H^{s,p}_\Sigma(X)\,,$$

for any $s \in \mathbb{R}$ and $p \in \mathbb{Z}$.

Also, suppose that F, G are 0th order h-Fourier Integral operators, F associated to a canonical transformation which maps Σ to another hypersurface Σ' satisfying our hypothesis, and G to its inverse (both transformation need to be defined only locally). Then

$$A \in \Psi^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(X) \implies F \circ A \circ G \in \Psi^{\mathfrak{p}}_{\Sigma',\frac{1}{2}}(X)\,. \tag{5.25}$$

Proof. We first show that $\Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac12}(X)$ is an algebra. In fact, in the notation of Definition 1,

$$\| \operatorname{ad}_{\operatorname{Op}^w_h(b_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}^w_h(b_N)} \operatorname{ad}^k_Q ABu\|_{H^{p+N/2}_\Sigma(X)}\,, \quad A, B \in \Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac12}(X)\,,$$

is bounded by a linear combination of

$$\| \operatorname{ad}_{\operatorname{Op}^w_h(b_{i_1})} \circ \cdots \operatorname{ad}_{\operatorname{Op}^w_h(b_{i_J})} \operatorname{ad}^l_Q A \operatorname{ad}_{\operatorname{Op}^w_h(b_{k_1})} \circ \cdots \operatorname{ad}_{\operatorname{Op}^w_h(b_{i_{N-J}})} \operatorname{ad}^{k-l}_Q Bu\|_{H^{p+N/2}_\Sigma(X)}\,,$$
$$\{i_j\}_{j=1}^J \cup \{k_j\}_{j=1}^{N-J} = \{1, \cdots, N\}\,.$$

These terms are estimated by

$$\begin{aligned} Ch^l\tilde h^{M_1} \| \operatorname{ad}_{\operatorname{Op}^w_h(b_{k_1})} \circ \cdots \operatorname{ad}_{\operatorname{Op}^w_h(b_{i_{N-J}})} \operatorname{ad}^{k-l}_Q Bu\|_{H^{p+(N-J)/2+l/2}_\Sigma(X)} \\ \leq C' h^k \tilde h^{M_1+M_2} \|u\|_{p+N/2+k}\,, \end{aligned}$$

where we used Definition 1 with A and with B. Here M_1 and M_2 are any large integers. Checking that

$$(I-\gamma_\Sigma)AB\,,\ AB(I-\gamma_\Sigma) \in \Psi^{0,-\infty,-\infty}_{\frac12}(X)\,,$$

is the same as in the proof of Lemma 5.4.

We now check that for $B \in \Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac12}(X)$ and $A \in \Psi^{0,0,0,0}_{\Sigma,\frac12}(X)$ we have

$$BA \in \Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac12}(X)\,.$$

In the notation of Definition 2, we can write

$$B = \gamma_\Sigma B_1 \gamma_\Sigma + (1-\gamma_\Sigma)B_1 + \gamma_\Sigma B_1(1-\gamma_\Sigma) + B_2\,,$$

and it suffices to check that

$$A\gamma_\Sigma B_1 \gamma_\Sigma \in \Psi^{0,-\infty,-\infty}_{\frac12}(X)\,. \tag{5.26}$$

Since Σ is compact (and hence we can take $\chi^w = \gamma_\Sigma$) and Definition 2 is independent of the choice of γ_Σ we have, as its special case,

$$\| \operatorname{ad}_{\operatorname{Op}^w_h(b_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}^w_h(b_N)} \circ \operatorname{ad}^k_Q \gamma_\Sigma B_1 \gamma_\Sigma u\|_{H^{p+N/2}_\Sigma(X)} \ \leq\ Ch^k\tilde h^N \|u\|_{H^{p+k/2}_\Sigma(X)}\,,$$

and the verification of (5.26) follows the proof of the algebra property of $\Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac12}(X)$.

To conclude the algebraic part of the proof we show that $\Psi^{0,0,0,0}_{\Sigma,\frac12}(X)$ is closed under composition of operators. Let $B_1, B_2 \in \Psi^{0,0,0,0}_{\Sigma,\frac12}(X)$ and let $\chi \in \mathcal{C}^\infty_{\rm c}(T^*X)$. Since we already established that composition with elements of $\Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac12}(X)$ produces an operator in that space we only have to show that

$$B_1B_2 = A_1 + A_2\,, \quad A_2 \in \Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac12}(X)\,,$$

so that for any p, and any sequences

$$\{b_j\}_{j=1}^N\,, \quad \{a_j\}_{j=1}^M \subset S^0(T^*X)\,,\ H_p a_j \restriction_{\operatorname{supp}\chi} \equiv 0\,,$$

and

$$\begin{aligned}\| \operatorname{ad}_{\operatorname{Op}_h^w(b_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}_h^w(b_N)} \circ \operatorname{ad}_{\operatorname{Op}_h^w(a_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}_h^w(a_M)} \circ \operatorname{ad}_Q^k \chi^w \gamma_\Sigma A_1 \chi^w u \|_{H_\Sigma^{p+N/2-k_2}(X)} \\ \leq\ C h^{M/2+k-m} \tilde{h}^{M/2+N-\widetilde{m}} \|u\|_{H_\Sigma^{p+k/2}(X)}\end{aligned}$$

To find the decomposition of $B_1 B_2$ we introduce $\chi_j \in \mathcal{C}_{\mathrm{c}}^\infty(T^*X)$ such that $\chi_1 \equiv 1$ on $\operatorname{supp}\chi_0$, and $\chi_0 \equiv 1$ on $\operatorname{supp}\chi$. We choose $\operatorname{supp}\chi_1$ sufficiently close to the support of χ so that the functions a_j satisfy $H_p a_j \restriction_{\operatorname{supp}\chi_1} \equiv 0$. We then put

$$A_2 = \chi_0^w (B_1)^\sharp_{\chi_1} (1 - \chi_1^w)(B_2)^\sharp_{\chi_1} + B_1 (B_2)^\flat_{\chi_1} + (B_1)^\flat_{\chi_1} B_2 \,,$$

which is in $\Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac{1}{2}}(X)$ since $\operatorname{supp}\chi_0 \cap \operatorname{supp}(1-\chi_1^w) = \emptyset$, and we can use the second part of Definition 2.

We have

$$A_1 = (1-\chi_0^w)(B_1)^\sharp_{\chi_1}(B_2)^\sharp_{\chi_1} + \chi_0^w (B_1)^\sharp_{\chi_1} \chi_1^w (B_1)^\sharp_{\chi_1} \,.$$

Up to negligible, $\mathcal{O}(h^\infty)$, errors

$$\chi^w (1-\chi_0^w) \equiv 0 \,, \quad \chi^w \chi_0^w \equiv \chi^w \,.$$

Hence we need to check that

$$\begin{aligned}\operatorname{ad}_{\operatorname{Op}_h^w(b_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}_h^w(b_N)} \circ \operatorname{ad}_{\operatorname{Op}_h^w(a_1)} \circ \cdots \operatorname{ad}_{\operatorname{Op}_h^w(a_M)} \circ \operatorname{ad}_Q^k \chi^w (B_1)^\sharp_{\chi_1} \chi_0^w (B_2)^\sharp_{\chi_1} \chi^w \\ =\ \mathcal{O}(h^{M/2+k-m} \tilde{h}^{M/2+N-\widetilde{m}}) \ :\ H_\Sigma^{p+N/2-k_2}(X) \ \longrightarrow\ H_\Sigma^{p+k/2}(X) \,,\end{aligned}$$

and this follows from the Leibnitz rule for ad_{b_j} and assumptions on $(B_j)^\sharp_{\chi_1}$.

The mapping properties are immediate from the definitions: we apply them with no commutators, and in the case of Definition 2, with $\chi^w = \gamma_\Sigma$.

Lemma 5.3 shows that the spaces $H_\Sigma^{\pm M}$ transform correctly under F and hence Proposition 3.4 (Egorov's Theorem) shows that for

$$A \in \Psi^{\mathfrak{p}_\infty}_{\Sigma,\frac{1}{2}}(X) \implies F \circ A \circ G \in \Psi^{\mathfrak{p}_\infty}_{\Sigma',\frac{1}{2}}(X) \,,$$

that is (5.25) holds for the residual class. Since the conditions on a_j, b_j's in Definition 2 are symplectically invariant, Egorov's theorem (Proposition 3.4) again shows that $F \circ A \circ G \in \Psi^{\mathfrak{p}}_{\Sigma'}(X)$. □

5.5. **The symbol map.** To define the symbol map we will use the invariance given by Proposition 5.5 and the symbol map in the model case. We start with

Lemma 5.6. *In the notation of Definition 2, suppose that $X = T^*\mathbb{R}^n$ and that*

$$\Sigma \cap V = \{\xi_1 = 0\} \cap V \,, \quad V \subset T^*\mathbb{R}^n \,, \quad \textit{is open, and } \operatorname{supp}\chi \Subset V \,.$$

Then

$$\operatorname{Op}_h^w(\chi) A_1 \operatorname{Op}_h^w(\chi) = \widetilde{\operatorname{Op}}_{h,\tilde{h}}(a_\chi) \,, \quad a_\chi = \widetilde{\mathcal{O}}_{\frac{1}{2}}(h^{-m} \tilde{h}^{-\widetilde{m}} \langle \lambda \rangle^{k_2}) \,.$$

Proof. This is a consequence of Proposition 5.2 and the properties of the term A_1 in Definition 2. □

To construct a symbol map, that is a homomorphism

$$\sigma_{\Sigma,h,\tilde{h}} \ : \ \Psi^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(X) \longrightarrow S^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(T^*X)/S^{\mathfrak{p}'}_{\Sigma,\frac{1}{2}}(T^*X)\,,$$
$$\mathfrak{p} = (m, \tilde{m}, k_1, k_2)\,, \quad \mathfrak{p}' = (m, \tilde{m}-1, k_1-1, k_2)\,,$$

such that the sequence

$$0 \longrightarrow \Psi^{\mathfrak{p}'}_{\Sigma,\frac{1}{2}}(X) \longrightarrow \Psi^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(X) \stackrel{\sigma_{\Sigma,h,\tilde{h}}}{\longrightarrow} S^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(T^*X)/S^{\mathfrak{p}'}_{\Sigma,\frac{1}{2}}(T^*X) \longrightarrow 0$$

is exact, for an arbitrary Σ we will use Lemma 5.6. That requires putting Σ locally to the model hypersurface $\xi_1 =$ which on the quantum level is done using h-Fourier integral operators. Hence we need a local invariance statement given in the next

Proposition 5.7. *Let U be an h-Fourier Integral Operator, elliptic in $V \times V$, where V is a neighbourhood of $(0,0) \in T^*\mathbb{R}^n$, the compact hypersurface Σ satisfies,*

$$\Sigma \cap V = \{\xi_1 = 0\} \cap V\,.$$

Assume also that the canonical transformation associated to U, κ, satisfies:

$$\kappa(0,0) = (0,0)\,, \quad \kappa(\{\xi_1 = 0\} \cap V) \subset \{\xi_1 = 0\}\,. \tag{5.27}$$

Let $A = \widetilde{\mathrm{Op}}_{h,\tilde{h}}(a)$ where $a = \widetilde{\mathcal{O}}_{\frac{1}{2}}(h^{-m}\tilde{h}^{-\tilde{m}}\langle\lambda\rangle^k)\,, \quad a(x,\xi,\lambda) \equiv 0$ for $(x,\xi) \notin V$.

If U^{-1} is the microlocal inverse of U near $V \times V$, then

$$U^{-1} \circ A \circ U = \widetilde{\mathrm{Op}}_{h,\tilde{h}}(b) + E\,, \quad b = a \circ K\,,$$
$$E \in \Psi^{\mathfrak{p}'}_{\Sigma}(\mathbb{R}^n)\,, \quad \mathfrak{p}' = (m, \widetilde{m}-1, -\infty, k-1)\,,$$

where K is the natural lift of κ to the (x,ξ,λ) variables:

$$K(y,\eta,\mu) = (x,\xi,\lambda) \iff (x,\xi) = \kappa(y,\eta)\,, \quad \lambda = (\xi_1/\eta_1)\mu\,. \tag{5.28}$$

Proof. We start by observing that the proposition holds in the special cases $\kappa(x,\xi) = (x,\xi)$ and $\kappa(x,\xi) = (-x,-\xi)$. The first special case concerns conjugation with elliptic classical h-pseudodifferential operators and it follows from the discussion after (5.7). The second special case follows from the first one and the fact that the proposition is easily checked for $Uu(x) = u(-x)$. As a consequence we can asssume that κ preserves the sign of ξ_1:

$$\kappa(y,\eta) = (x,\xi) \implies \xi_1\eta_1 \geq 0\,. \tag{5.29}$$

We will prove the proposition by a deformation method inspired by the "Heisenberg picture of quantum mechanics" and for that we need the following geometric

Lemma 5.8. *Let κ be a smooth canonical transformation satisfying* (5.27) *and* (5.29). *Then we can find a piecewise smooth family of canonical transformations $[0,1] \ni t \mapsto \kappa_t$ satisfying* (5.27), (5.29) *and such that $\kappa_0 = id$ and $\kappa_1 = \kappa$.*

Proof. Let us denote by Σ the hypersurface given by $\xi_1 = 0$. We first observe that if κ is a linear symplectic transformation preserving Σ then we can find a family of linear symplectic transformations, κ_t, satisfying the conclusions of the lemma: the subgroup of elements of $Sp(n, \mathbb{R})$ preserving a half space bounded by Σ is connected.

Hence we can assume that $d\kappa(0,0) = Id$. Now introduce

$$\kappa_t(x,\xi) = t^{-1}\kappa\left(tx, t\xi\right) ,$$

a smooth family of symplectic transformations, preserving a half space bounded by Σ, with $\kappa_1 = \kappa$ and $\kappa_0 = Id$. □

We now return to the proof of Proposition 5.7. For simplicity we can assume that $m = \tilde{m} = 0$. Let κ_t be a piecewise smooth family with $\kappa_1 = \kappa$, $\kappa_0 = id$. Using (3.4) we construct a piecewise smooth family of classical elliptic h-Fourier integral operators, U_t, defined microlocally near $(0,0)$ and associated to κ_t. If we demand that $U_1 = U$ then U_0 is a pseudodifferential operator elliptic at $(0,0)$. For notational convenience we assume that our deformation is smooth in t – the piecewise smooth case follows from the same argument applied in several steps and that $U_0 = Id$ (the last condition can be arranged by composing U with a elliptic pseudodifferential operator). Thus we have

$$hD_tU_t + U_tQ_t = 0\,, \tag{5.30}$$

where Q_t is a smooth family of classical h-pseudodifferential operators of order 0 with the leading symbol q_t satisfying

$$\frac{d}{dt}\kappa_t(x,\xi) = (\kappa_t)_*\left(H_{q_t}(x,\xi)\right) , \tag{5.31}$$

in a neighbourhood of V. It follows from (5.31) that H_{q_t} is tangent to Σ and hence

$$\partial_{x_1} q_t(x,\xi) = \xi_1 r_t(x,\xi)\,. \tag{5.32}$$

We extend q_t to a globally defined function in $S(T^*\mathbb{R}^n, \langle x,\xi\rangle)$, keeping the property (5.32). This defines a family of global canonical transformation which coincide with κ_t near V.

Let V_t satisfy

$$hD_tV_t = Q_tV_t\,, \quad V_0 = Id\,. \tag{5.33}$$

It follows that $V_t = U_t^{-1}$, and if we take Q_t to be self-adjoint, $V_t^* = U_t$.

We will now use (5.30) and (5.31) to prove Egorov's theorem (3.5) for the new class. Thus we consider $A_t = U_t^{-1}AU_t$, so that A_1 is the operator we want to study and $A_0 = A$ is the given operator. From Proposition 5.5 we already know that $A_t \in \Psi^{0,0,0,k}_{\Sigma,\frac{1}{2}}(\mathbb{R}^n)$. Using Lemma 5.6 we can write

$$A_t \equiv \widetilde{\mathrm{Op}}_{h,\tilde{h}}(a_t) \mod \Psi^{0,-\infty,-\infty,k}_{\Sigma,\frac{1}{2}}(\mathbb{R}^n)\,,$$

microlocally near V. From (5.30) and (5.33), we get

$$\left\{ \begin{array}{l} hD_t A_t = [Q_t, A_t]\,, \\ A_0 = A\,, \end{array} \right. \tag{5.34}$$

To compute the commutator on the symbolic level we need:

Lemma 5.9. *Suppose that $a \in \widetilde{\mathcal{O}}_{\frac{1}{2}}(\langle\lambda\rangle^k)$ and that $b \in S^{0,-\infty}(T^*\mathbb{R}^n)$ (that is, b is a symbol in the sense of §3.1), does not depend on λ. In addition let us assume that*

$$\partial_{x_1} b = \xi_1 r(x,\xi)\,.$$

Then

$$\frac{i}{h}[\widetilde{\mathrm{Op}}_{h,\tilde h}(b), \widetilde{\mathrm{Op}}_{h,\tilde h}(a)] = \widetilde{\mathrm{Op}}_{h,\tilde h}(c)\,, \quad c = (H_b - r\lambda\partial_\lambda)a + \widetilde{\mathcal{O}}_{\frac{1}{2}}(\tilde h\langle\lambda\rangle^k)\,. \tag{5.35}$$

Proof. We will use (5.8) to compute $a\sharp_{h,\tilde h} b$ (and $b\sharp_{h,\tilde h} a$), noting that

$$\mathrm{Op}_h a(x_1, \tilde h D_{x_1}) \circ \mathrm{Op}_h b(x_1, \tilde h D_{x_1}) = (\mathrm{Op}_h(a)\sharp_{\tilde h}\mathrm{Op}_h(b))\,(x_1, \tilde h D_{x_1})\,.$$

Since $b \in S^{0,-\infty}$, Lemma 3.6 shows that the composition formula in the (x',ξ') is given by an asymptotic series in $(h\tilde h)^{\frac{1}{2}}$, and a $\mathcal{O}(h^\infty)$ error. The only subtlety lies in the dependence on (x_1,λ) and to explain it we suppress the other variables. We have

$$\partial^\ell_{x_1}\partial^p_\lambda b(x_1,\lambda) = \left\{ \begin{array}{ll} \mathcal{O}((h/\tilde h)^p\langle(h/\tilde h)\lambda\rangle^{-\infty}) & \ell = 0 \\ \mathcal{O}((h/\tilde h)^{p+1}\lambda\langle(h/\tilde h)\lambda\rangle^{-\infty}) & \ell > 0\,, \end{array} \right.$$

and $\partial^p_{x_1}\partial^\ell_\lambda a = \mathcal{O}(\langle\lambda\rangle^{k-\ell/2+p/2})$. Consequently the terms in the expansions of $a\sharp_{\tilde h} b$ and $b\sharp_{\tilde h} a$ are bounded by

$$C_p h\tilde h^{p-1}\langle\lambda\rangle^{k-p/2+1}\langle(h/\tilde h)\lambda\rangle^{-\infty}\,, \quad C_p(h/\tilde h)^p\tilde h^p\langle\lambda\rangle^{k+p/2}\langle(h/\tilde h)\lambda\rangle^{-\infty}\,, \quad p>0\,,$$

respectively. Hence we have expansions such that for $p > 1$ the terms are bounded by $h\tilde h^{p-1}\langle\lambda\rangle^k$, and the errors are $\mathcal{O}(h\tilde h^\infty\langle\lambda\rangle^k)$. This argument shows that

$$a\ \sharp_h\ b = ab + h\left(\sum_{i=1}^n \partial_{\xi_i} a\, D_{x_i} b + r\lambda\partial_\lambda a\right) + \widetilde{\mathcal{O}}_{\frac{1}{2}}\left(h\tilde h\langle\lambda\rangle^k\right)\,,$$

and

$$b\ \sharp_h\ a = ab + h\sum_{i=1}^n \partial_{\xi_i} b\, D_{x_i} a + \widetilde{\mathcal{O}}_{\frac{1}{2}}\left(h\tilde h\langle\lambda\rangle^k\right)\,,$$

from which the lemma follows. □

Since q_t satisfies (5.32), Lemma 5.9 gives

$$\frac{i}{h}[Q_t, \widetilde{\mathrm{Op}}_{h,\tilde h}(a_t)] = \widetilde{\mathrm{Op}}_{h,\tilde h}\left((H_{q_t} - r_t\lambda\partial_\lambda)a_t + \widetilde{\mathcal{O}}_{\frac{1}{2}}(h\tilde h\langle\lambda\rangle^k)\right)\,.$$

If we write $a_t = a_t^0 + \mathcal{O}_{\frac{1}{2}}(\tilde{h}\langle\lambda\rangle^k)$ then

$$\frac{\partial}{\partial t}a_t^0 = \left(H_{q_t} - r_t(x,\xi)\lambda\frac{\partial}{\partial\lambda}\right)a_t^0\,,\quad a_0^0 \equiv a \mod \mathcal{O}_{\frac{1}{2}}(\tilde{h}\langle\lambda\rangle^k)\,. \tag{5.36}$$

We now note that

$$\left(H_{q_t} - r_t\lambda\frac{\partial}{\partial\lambda}\right)a_t^0 = H_{q_t}(a_t^0\restriction_{\lambda=(\tilde{h}/h)\xi_1})\,.$$

Hence, if K_t is the transformation in (x,ξ,λ)-space corresponding to κ_t as in the statement of the proposition, it follows that

$$a_t^0 = a\circ K_t\,,$$

that the principal symbol of A_t is $a\circ K_t$ and the proposition follows. □

We can now define the symbol map,

$$\Psi^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(X) \ni A \longmapsto \sigma_{\Sigma,h,\tilde{h}}(A) \in S^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(T^*X)/S^{\mathfrak{p}'}_{\Sigma,\frac{1}{2}}(T^*X)\,.$$

We recall from §3.1 and §3.3 that we already have the symbol map for the $S_{\frac{1}{2}}$ calculus:

$$\Psi^{m,\widetilde{m},k}_{\frac{1}{2}}(X) \ni B \longmapsto \sigma_h(A) \in S^{m,\tilde{m},k}_{\frac{1}{2}}(T^*X)/S^{m,\widetilde{m}-1,k-1}_{\frac{1}{2}}(T^*X)\,.$$

The definition of the symbol classes (5.1) shows that

$$\begin{aligned}(S^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(T^*X)\cap\mathcal{C}^\infty(T^*X\setminus U_\Sigma))/S^{\mathfrak{p}'}_{\Sigma,\frac{1}{2}}(T^*X) = \\ (S^{m+k_2,\widetilde{m}-k_2,k_1}(T^*X)\cap\mathcal{C}^\infty(T^*X\setminus U_\Sigma))/S^{m+k_2,\widetilde{m}-k_2-1,k-1}(T^*X)\,,\end{aligned}$$

for any open (h-independent) neighbourhood of Σ, U_Σ. Hence we define

$$\sigma_{\Sigma,h,\tilde{h}}((I-\gamma_\Sigma)A) \stackrel{\text{def}}{=} \sigma_h((I-\gamma_\Sigma)A)\,,$$

and as in the proof of Lemma 5.4 we see that this definition is independent of the choice of γ_Σ.

To define $\sigma_{\Sigma,h,\tilde{h}}(\gamma_\Sigma A)$ we use Lemma 5.6. We choose a partition of the cut-off operator, γ_Σ,

$$\sum_{j=1}^J \gamma_j^2 = \sigma_h(\gamma_\sigma)\,,\quad \gamma_j \in \mathcal{C}^\infty_{\rm c}(T^*X)\,,$$

such that for each j, $\operatorname{supp}\gamma_j\cap\Sigma$ can be put into the normal form $\Omega\cap\{\xi_1=0\}$ by a local canonical transformation,

$$\begin{gathered}\kappa_j : \Omega \longrightarrow \Omega_j\,,\quad (0,0)\in\Omega\subset T^*\mathbb{R}^n\,,\\ \kappa_j(\{\xi_1=0\}\cap\Omega) = \Sigma\cap\Omega_j\,.\end{gathered} \tag{5.37}$$

We then *choose* elliptic h-Fourier Integral Operators, U_j, microlocally defined in neighbourhoods of $\Omega\times\Omega_j$ and associated to κ_j's. By Proposition 5.5

$$U_j^{-1}\mathrm{Op}^w_h(\chi_j)A\mathrm{Op}^w_h(\chi_j)U_j = \mathrm{Op}^w_h(\tilde{\chi}_j)\widetilde{A}\mathrm{Op}^w_h(\tilde{\chi}_j) \in \Psi^{\mathfrak{p}}_{\Sigma',\frac{1}{2}}(\mathbb{R}^n)\,,\quad \Sigma'\cap\Omega = \{\xi_1=0\}\,.$$

In the notation of Lemma 5.6 we then define

$$\sigma_{\Sigma,h,\tilde{h}}(\gamma_\Sigma A) = \sum_{j=1}^{J} (\kappa_j^{-1})^* a_{\tilde{\chi}_j} \gamma_j^2 \,. \tag{5.38}$$

Proposition 5.7 shows that this definition is independent of the choices made here.

5.6. **Global quantization map.** From the local quantization given in §5.2 we can define a global map $\mathrm{Op}_{\Sigma,h,\tilde{h}}$. Thus let $a \in S^{\mathfrak{p}}_{\Sigma}(T^*X)$ be a symbol in the class defined in (5.1).

Let γ_Σ (where we will use the same letter for the symbol and the operator) and V_j's be as in (5.21) By shrinking V_2 if necessary we can find a finite open cover

$$V_2 \subset \bigcup_{j=1}^{J} \Omega_j$$

such that for each j there exists a canonical transformation κ_j satisfying (5.37).

Let ϕ_j be a partition of unity on V_2 subordinate to the cover by Ω_j's. Let a_j be the unique symbol of the form

$$a_j = a_j(x, \xi_2, \cdots, \xi_n, \lambda; h)$$

such that

$$(a_j)_{\lambda=(\tilde{h}/h)\xi_1} = (\gamma_\Sigma \phi_j a) \circ \kappa_j \,.$$

Using the h-Fourier integral operators defined after (5.37) we put

$$\mathrm{Op}_{\Sigma,h,\tilde{h}}(a) \overset{\mathrm{def}}{=} \mathrm{Op}^w_h\left((1-\gamma_\Sigma)a\right) + \sum_j U_j \widetilde{\mathrm{Op}}_{h,\tilde{h}}\left(a_j\right) U_j^{-1} \,. \tag{5.39}$$

In view of Proposition 5.5 we have $\mathrm{Op}_{\Sigma,h,\tilde{h}}(a) \in \Psi^{\mathfrak{p}}_{\Sigma,\frac{1}{2}}(X)$.

The construction of the symbol map in §5.5 shows that

$$\sigma_{\Sigma,h,\tilde{h}}(\mathrm{Op}_{\Sigma,h,\tilde{h}}(a)) \equiv a \bmod S^{\mathfrak{p}'}_{\Sigma}(X)\,, \quad \mathfrak{p}' = (m, \widetilde{m}-1, k_1-1, k_2)\,.$$

We recall from (5.1) that *away from* Σ,

$$S^{\mathfrak{p}}_{\Sigma,\frac{1}{2}} \ \text{ becomes } \ S^{m+k_2,\widetilde{m}-k_2,k_1}_{\frac{1}{2}}\,,$$

and

$$S^{\mathfrak{p}'}_{\Sigma,\frac{1}{2}} \ \text{ becomes } \ S^{m+k_2,\widetilde{m}-k_2-1,k_1-1}_{\frac{1}{2}}\,.$$

If $\delta < 1/2$ in (5.1) then we have the usual filtration of the h-pseudodifferential calculus: near Σ we only gain in $\tilde{h}$ and away from Σ, in h. In the case of $\delta = 1/2$ considered here in detail we only gain in $\tilde{h}$, near and away Σ.

This completes the proof of Theorem 4 and provides an explicit quantization $\mathrm{Op}_{\Sigma,h,\tilde{h}}$.

5.7. **Approximation by finite rank operators.** To estimate the number of resonances we will need to use approximation by finite rank operators.

For $a \in S^{m,\widetilde{m},-\infty,k_2}_{\Sigma,\frac{1}{2}}(T^*X)$ we need a notion of essential support. Unlike the essential support defined in §3.1 it now has to depend on $h, \tilde{h}$. As in [33], rather than introduce an equivalence class of families of sets, we will say that for an $(h,\tilde{h})$-dependent family of sets $W_{h,\tilde{h}} \subset T^*X$

$$\begin{gathered}\operatorname{ess\,supp} a \subset W_{h,\tilde{h}} \iff \\ \exists\, a' \in S^{m,\widetilde{m},-\infty,k_2}_{\Sigma,\frac{1}{2}}(T^*X)\,,\ \operatorname{supp} a' \subset W_{h,\tilde{h}}\,,\quad a - a' \in S^{m,-\infty,-\infty,k_2}_{\Sigma,\frac{1}{2}}(T^*X)\,.\end{gathered}$$

We notice that

$$\operatorname{ess\,supp} a \subset V^j_{h,\tilde{h}}\,,\ j = 1, \cdots, N \implies \operatorname{ess\,supp} a \subset V^1_{h,\tilde{h}} \cap \cdots \cap V^N_{h,\tilde{h}}\,,$$

so that this *formal notion* of essential support behaves correctly under finite products and sums. We can now state

Proposition 5.10. *Suppose that* $a \in S^{0,0,-\infty,-\infty}_{\Sigma,\frac{1}{2}}(T^*X)$ *and*

$$\operatorname{ess\,supp} a \subset W_{h,\tilde{h}}\,,$$

where $W_{h,\tilde{h}}$ *satisfies*

$$\begin{gathered}W_{h,\tilde{h}} \subset \{(x,\xi)\ :\ d((x,\xi), \Sigma \cap W_{h,\tilde{h}}) \leq C_1 h/\tilde{h}\}\,, \\ W_{h,\tilde{h}} \subset \bigcup_{k=1}^{K(h)} \exp([-1,1]H_q)(B_k)\,,\quad \operatorname{diam} B_k \leq C_1 (h/\tilde{h})^{\frac{1}{2}}\,,\end{gathered}$$

for some $K(h) \in \mathbb{N}$. *Then for* $0 < h < h_0$, *there exists a finite rank operator* $R(h)$ *such that for*

$$\operatorname{Op}_{\Sigma,h,\tilde{h}}(a) - R(h) \in \Psi^{0,-\infty,-\infty,-\infty}_{\Sigma,\frac{1}{2}}(X)\,,\quad \operatorname{rank} R(h) = C_2 \tilde{h}^{-n} K(h)\,.$$

Proof. We take an open covering of $W_{h,\tilde{h}}$,

$$\begin{gathered}W_{h,\tilde{h}} \subset \bigcup_{k=1}^{K'(h)} U_k\,,\quad K'(h) \leq C' K(h)\,,\quad U_k = \exp([-1/C, 1/C]H_q)V_k \\ \operatorname{diam}(V_k) \leq C(h/\tilde{h})^{\frac{1}{2}}\,,\quad \sup_{(x,\xi)\in U_k} d((x,\xi), U_k \cap \Sigma) \leq Ch/\tilde{h}\,,\end{gathered}$$

with a partition of unity on $W_{h,\tilde{h}}$,

$$\sum_{k=1}^{K'(h)} \chi_k = 1 \quad \text{on } W_{h,\tilde{h}},\quad \operatorname{supp}\chi_k \subset U_k\,,\quad \chi_k \in S^{0,0,-\infty,-\infty}_{\Sigma,\frac{1}{2}}(T^*X)\,.$$

If $\psi = 1 - \sum_k \chi_k \in S^{0,0,-\infty,-\infty}_{\Sigma,\frac{1}{2}}$ then the condition on the support of a shows that

$$\forall\ \alpha,\beta \in \mathbb{N}^n\,,\quad \partial^\alpha a\,\partial^\beta \psi \equiv 0\,.$$

Consequently the calculus of Theorem 4 gives

$$\mathrm{Op}_{\Sigma,h,\tilde h}(\psi) A \in \Psi^{0,-\infty,-\infty,-\infty}_{\Sigma,\frac{1}{2}}(X)\,.$$

Hence it suffices to show that for each k there exists an operator R_k such that

$$\mathrm{Op}_{\Sigma,h,\tilde h}(\chi_k) A - R_k \in \Psi^{0,-\infty,-\infty,-\infty}_{\Sigma,\frac{1}{2}}(X)\,,\quad \mathrm{rank}(R_k) \leq C\tilde h^{-n}\,,$$

with C independent of k. By taking a finer partition (with a number of elements $K''(h) \leq C''K(h)$) we can assume that

$$\mathrm{Op}_{\Sigma,h,\tilde h}(\chi_k) A = U_k A_k V_k$$

where U_k, V_k are h-semiclassical Fourier Integral Operators of the form used in the construction of $\mathrm{Op}_{\Sigma,h,\tilde h}$, and

$$A_k = \widetilde{\mathrm{Op}}_{h,\tilde h}(a_k)$$
$$\mathrm{supp}\, a_k \subset \{(x,\xi',\lambda)\ :\ |\lambda| \leq C\,,\quad |x'| + |\xi'| \leq C(h/\tilde h)^{\frac{1}{2}}\,, |x_1| < 1/C\}$$

Consider commuting operators

$$Q = \left(\tilde h D_{x_1}\right)^2 + x_1^2 + (hD_{x_2})^2 + x_2^2 + \cdots (hD_{x_n})^2 + x_n^2\,,$$
$$Q = \widetilde{\mathrm{Op}}_{h,\tilde h}(q)\,,\quad q = \lambda^2 + x_1^2 + \cdots \xi_n^2 + x_n^2\,,$$
$$Q' = (hD_{x_2})^2 + x_2^2 + \cdots (hD_{x_n})^2 + x_n^2\,.$$

If $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$, $\chi(t) = 1$ for $t \leq \widetilde{C}$, $\chi(t) = 0$ for $t > \widetilde{2}C$, then

$$\chi(\tilde h Q'/h) \in \Psi^{0,0,0,0}_{\Sigma_0,\frac{1}{2}}(\mathbb{R}^n)\,,$$

$$\chi(\tilde h Q'/h)\chi(Q) A_k - A_k \in \Psi^{0,-\infty,-\infty,-\infty}_{\Sigma_0,\frac{1}{2}}\,.$$

The standard analysis of the spectrum of harmonic oscillators shows that $\chi(\tilde h Q'/h)\chi(Q)$ is a finite rank operator and its rank is bounded by $C'\tilde h^{-n}$. Hence we can take $R_k = \chi(Q)\chi(\tilde h Q'/h) A_k$. □

6. General upper bounds in regions of size h

6.1. **Bound for the number of eigenvalues of a self-adjoint operator.** With the calculus developed in §5 we can follow the standard procedure of modifying the operator near the energy surface, now at the limiting scale. For that we introduce the second small parameter $\tilde h$ which eventually will be fixed, as $h \to 0$.

Let $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R};[0,1])$ be equal to one near 0. We then define

$$a(x,\xi,h) \stackrel{\rm def}{=} \chi\left(\frac{\tilde h p(x,\xi)}{h}\right). \tag{6.1}$$

Then, in terms of Definition 5.1, $a \in S^{0,0,-\infty,-\infty}_{\Sigma,0}(T^*X)$, $\Sigma = p^{-1}(0)$. Although this class of symbols corresponds to a class $\Psi^\star_{\Sigma,0}$ we will use the larger class $\Psi^\star_{\Sigma,\frac12}$ since it was presented in detail in §5. We stress that this is done for convenience only and an examination of §5 shows how the simpler calculus is constructed without the $S_{\frac12}$ complications.

The operator

$$\widetilde P \stackrel{\rm def}{=} P + i(h/\tilde h)A - z\,,\quad A \stackrel{\rm def}{=} \widetilde{\rm Op}_{h,\tilde h}(a)\,,$$

is elliptic in $\Psi^{0,0,2,1}_\Sigma(X)$, in the sense that

$$|\sigma_{\Sigma,h,\tilde h}(\widetilde P)| > C\left(d(\bullet,\Sigma) + h/\tilde h\right).$$

This is most clearly seen locally when $p = \xi_1$ and $\lambda = (h/\tilde h)^{-1}\xi_1$:

$$\tilde p = \frac{h}{\tilde h}\left(\lambda + i\chi\left(\lambda\right)\right),\quad |\tilde p| > C\frac{h}{\tilde h}\langle\lambda\rangle\,.$$

Using Theorem 4 we can construct a parametrix for $\widetilde P - z$, $|z| \le Ch$, uniformly in h, which for $\tilde h$ small enough (keeping $\tilde h M$ large and constant) gives an exact inverse:

$$(\widetilde P - z)^{-1} = \mathcal{O}\left(\frac{\tilde h}{h}\right) \,:\, L^2(X) \,\longrightarrow L^2(X)\,,\quad |z| \le Ch\,.$$

Proposition 5.10 gives a finite rank operator R such that

$$A = R + E\,,\quad \operatorname{rank}(R) \le C\tilde h^{-n}h^{-n+1}\,,\quad E \in \Psi^{0,-\infty,-\infty,-\infty}_{\Sigma,\frac12}\,,$$

so that in particular, by Proposition 5.5,

$$\|E\|_{L^2\to L^2} = \mathcal{O}(\tilde h^\infty)\,.$$

Now we write

$$\begin{aligned} P - z &= (\widetilde P - z)(I - i(h/\tilde h)(\widetilde P - z)^{-1}(R+E)) \\ &= (\widetilde P - z)(I - i(h/\tilde h)(\widetilde P - z)^{-1}E)(I + K(z))\,, \end{aligned}$$

where

$$\begin{gathered} K(z) = -i(h/\tilde h)(I + i(h/\tilde h)(\widetilde P - z)^{-1}E)^{-1}(\widetilde P - z)^{-1}R\,, \\ \operatorname{rank}(K(z)) \le Mh^{-n+1}\,,\quad K(z) = \mathcal{O}(1) \,:\, L^2(X) \,\longrightarrow\, L^2(X)\,. \end{gathered}$$

This implies that

$$h(z) \stackrel{\rm def}{=} \det(I + K(z)) = \mathcal{O}(\exp(CMh^{-n+1}))\,,\quad |z| \le C_0 h\,,$$

and that the zeros of $h(z)$ are the eigenvalues of P in $|z| \leq Ch$.

The bound on the number of eigenvalues will follow standard estimates [‡]

once we show that

$$|h(z_0)| > \exp(-CMh^{-n+1}) \text{ at some } z_0,\ |z_0| \leq C_1 h,\ C_1 < C_0.$$

For that we take z_0 with $|\operatorname{Im} z_0| > C_1 h$ so that, by self-adjointness,

$$(P - z_0)^{-1} = \mathcal{O}((C_1 h)^{-1}) \,:\, L^2(X) \longrightarrow L^2(X)\,.$$

We then see that

$$(I + K(z_0))^{-1} = I + L(z_0)\,,$$

where

$$I + L(z) = (I + i(h/\tilde{h})(P - z)^{-1}(R + E))(I + i(h/\tilde{h})(\widetilde{P} - z)^{-1}E)\,, \quad |\operatorname{Im} z| > C_1 h\,.$$

The operator $L(z_0)$ is of trace class and using the rank of R we have the estimate

$$h(z_0)^{-1} = \det(I + L(z_0)) = \mathcal{O}(\exp(CMh^{-n+1}))\,,$$

which shows that

$$|\text{Spec } P \cap D(0,h)| = \mathcal{O}(h^{-n+1})\,. \tag{6.2}$$

6.2. **Proof of Theorem 2.** To establish the estimate (1.9) we proceed as in the case of eigenvalues but using the scaled operator P_θ instead – see §3.5. In this section, following [2], we take

$$\theta = C_0 h$$

for C_0, a large and fixed constant. We recall from §3.1 that

$$X = X_0 \sqcup (\mathbb{R}^n \setminus B(0, R_0))\,,$$

We can also have more neighbourhoods of infinity (see §3.1) but for simplicity of notation we restrict ourselves to the case above.

The operator P_θ can be written as $P_\theta = P_1 + iP_2$, where P_j are formally self-adjoint on $L^2(X)$. We consider the *Weyl symbols*, p_1 and p_2 (defined, we recall, modulo $\mathcal{O}(h^2)$) of these two operators. In view of (3.17) we have

$$\begin{gathered} p_1 \restriction_{T^*_{X_0 \cup \mathbb{R}^n \setminus B(0,2R_0)} X} = p \restriction_{T^*_{X_0 \cup \mathbb{R}^n \setminus B(0,2R_0)} X}\,, \quad |p_1 - p| \leq c_1 h \langle \xi \rangle^2\,, \\ p_2 \restriction_{X_0} = 0\,, \quad -3C_1 h|\xi|^2 \leq p_2 \restriction_{T^*_{\mathbb{R}^n \setminus B(0,2R_0)} X} \leq -C_1 h|\xi|^2\,, \quad |p_2| \leq c_1 h \langle \xi \rangle^2\,. \end{gathered} \tag{6.3}$$

We also note that our assumptions give

$$|p_1| \leq \delta \implies \langle \xi \rangle \sim |\xi|\,,$$

[‡]The estimate we need is this: if $\log|h(z)| \leq K$ in R_1, where R_1 is a rectangle, and R_2 is another rectangle strictly inside R_1, $\log|h(z_0)| > -K$, for some $z_0 \in R_2$, then the number of zeros of $h(z)$ in R_2 is bounded by $C(R_1, R_2)K$, with a dilation invariant constant, $C(tR_1, tR_2) = C(R_1, R_2)$. This follows from Jensen's formula and its standard consequences for zero counting, see for instance [37, §3.61, §8.21].

with some small fixed δ. Now let $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}, [0,1])$, satisfy

$$\chi(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & |t| \geq 2 \end{cases}$$

With this χ we define

$$a(x,\xi,h) \stackrel{\rm def}{=} \chi\left(\frac{\tilde{h}p(x,\xi)}{h}\right)\chi\left(\frac{|x|}{3R_0}\right),$$

where $\tilde{h}$ is small,

$$\tilde{h}^{-1} \sim C_1 \sim c_1 \sim C_0\,,$$

in (6.3) but eventually fixed.

We now choose a *compact* $\mathcal{C}^\infty$ hypersurface Σ so that

$$\Sigma \cap T^*_{X_0 \sqcup B(0,6R_0)\setminus B(0,R_0)} X = \{(x,\xi)\ :\ p(x,\xi) = 0\}\,.$$

It follows that

$$a(x,\xi,h) \in S^{0,0,-\infty,0}_\Sigma(T^*X)\,,$$

and continuing in the same spirit as in §6.1, we put

$$A \stackrel{\rm def}{=} \widetilde{\rm Op}_{h,\tilde{h}}(a)\,.$$

On the level of symbols we have

$$|p_1| \leq (h/\tilde{h}) \implies -p_2(x,\xi) + (h/\tilde{h})a(x,\xi,h) \geq (h/\tilde{h})/C\,. \tag{6.4}$$

Hence if we put

$$\widetilde{P} = P_\theta - i(h/\tilde{h})A\,,$$

then for $|\operatorname{Re} z| \leq Ch$, $\operatorname{Im} z \geq -Ch$,

$$(\tilde{h}/h)(\widetilde{P} - z) \in \Psi^{0,0,2,1}_\Sigma(X)\,.$$

We claim that $\widetilde{P} - z$ is invertible for $z \in D(0,Ch)$ and

$$\|(\widetilde{P} - z)^{-1}\| = \mathcal{O}(\tilde{h}/h)\,. \tag{6.5}$$

In fact, let $W \subset T^*X$ be set in which $(\tilde{h}/h)(\tilde{p} - z)$ is elliptic in $S^{0,0,2,1}_\Sigma(T^*X)$. The estimate (6.4) shows that $\operatorname{Im}(\tilde{p} - z) \leq -C(h/\tilde{h})$ on a neighbourhood of $\complement W \cap \{|\operatorname{Re}(\tilde{p} - z)| < \delta\}$. Now, let Ψ_1 and Ψ_2 be as in Lemma 3.2, with $\operatorname{ess\,supp}\Psi_1 \subset W$. Then proceeding as in §4.1 we obtain

$$\|(\widetilde{P} - z)\Psi_2 u\| \geq (h/\tilde{h})\|\Psi_2 u\|/C - \mathcal{O}(h^\infty)\|u\|\,,\quad u \in \mathcal{C}^\infty_{\rm c}(X)\,.$$

Ellipticity of $(\tilde{h}/h)(\widetilde{P} - z)$ in W shows that

$$\|(\tilde{h}/h)(\widetilde{P} - z)\Psi_1 u\| \geq \|\Psi_1 u\|/C - \mathcal{O}(\tilde{h}^\infty)\,,$$

that is

$$\|(\widetilde{P} - z)\Psi_1 u\| \geq (h/\tilde{h})\|\Psi_1 u\|/C - \mathcal{O}(h\tilde{h}^\infty)\,.$$

Lemma 3.3 then gives (6.5).

We can now proceed as in §6.1 and obtain the bound on the number of resonances (1.9). The complex analytic argument outlined in the footnote is used in a rectangle

$$[-Ch, Ch] + i[C_2 h, -Ch]$$

where C_2 is large enough to guarantee a lower bound for $-\operatorname{Im}(P_\theta - z)$ when $\operatorname{Im} z \sim C_2 h$. We can still take C_2 proportional to the other large constants. □

7. The escape function for hyperbolic flows and its h dependent regularizations

In this section we modify [30, Sect.5] and construct a regularized escape function depending on a small parameter, essentially $h/\tilde{h}$. We recall that we assume that $p \in \mathcal{C}^\infty(T^*X; \mathbb{R})$ satisfies

$$\begin{gathered} p = 0 \implies dp \neq 0 \\ |x| \geq R\,, \quad |p(x,\xi)| < 2\delta \implies \exp tH_p(x,\xi) \to \infty \ \text{ for either } t \to \infty \text{ or } t \to -\infty. \end{gathered} \tag{7.1}$$

We also recall the result of [11, Appendix]:

Proposition 7.1. *Suppose that* (7.1) *holds and that* $\widehat{K}$ *is the trapped set,*

$$\widehat{K} \stackrel{\text{def}}{=} \{\rho \in T^*X \,:\, \exp(tH_p)(\rho) \not\to \infty\,,\ t \to \pm\infty\,, |p(\rho)| \leq \delta\} \Subset T^*X\,. \tag{7.2}$$

Then for any two neighbourhoods, U, V*, of* $\widehat{K}$*,* $\overline{U} \subset V$ *there exists* $G_0 \in \mathcal{C}^\infty(T^*X)$ *such that*

$$\begin{gathered} \operatorname{supp} G_0 \subset T^*X \setminus U\,, \quad H_p G_0 \geq 0\,, \quad H_p G_0 \restriction_{p^{-1}([2\delta,2\delta])} \leq C\,, \\ H_p G_0 \restriction_{p^{-1}([-\delta,\delta])\setminus V} \geq 1\,. \end{gathered} \tag{7.3}$$

7.1. **Dynamical assumptions.** We start with the hyperbolicity assumptions [30, §5] weaker than the more standard assumptions in §1. Let $\widehat{K}$ be the compact trapped set near zero energy given by (7.2). The trapped set at zero energy is given by $K = \widehat{K} \cap p^{-1}(0)$. We also have $\widehat{K} = \widehat{\Gamma}_+ \cap \widehat{\Gamma}_-$, where

$$\widehat{\Gamma}_\pm \stackrel{\text{def}}{=} \{(x,\xi) \in T^*X \,:\, |p(x,\xi)| \leq \delta\,,\ \exp(tH_p)(x,\xi) \not\to \infty\,,\ t \to \mp\infty\}\,, \tag{7.4}$$

and the sets $\widehat{K}$, $\widehat{\Gamma}_\pm$ are clearly invariant under the flow,

$$\exp(tH_p)(\widehat{K}) \subset \widehat{K}\,, \quad \exp(tH_p)(\widehat{\Gamma}_\pm) \subset \widehat{\Gamma}_\pm\,. \tag{7.5}$$

We can now state the dynamical hypothesis.

- In a neighbourhood, Ω_{ρ_0} of any $\rho_0 \in K$,

$$\begin{gathered} \widehat{\Gamma}_\pm = \bigcup_{\rho \in \Omega_{\rho_0} \cap \widehat{\Gamma}_\pm} \widehat{\Gamma}_{\pm,\rho}\,, \quad \rho \in \widehat{\Gamma}_{\pm,\rho}\,, \\ \widehat{\Gamma}_{\pm,\rho} \cap \widehat{\Gamma}_{\pm,\rho'} = \emptyset\,, \quad \text{or} \quad \widehat{\Gamma}_{\pm,\rho} = \widehat{\Gamma}_{\pm,\rho'}\,. \end{gathered}$$

- Each $\widehat{\Gamma}_{\pm,\rho}$ is a closed $\mathcal{C}^1$ manifold of dimension $n+d$, with $d \geq 0$ fixed, and the dependence
$$\Omega_{\rho_0} \cap \widehat{\Gamma}_\pm \ni \rho \longmapsto T_\rho \widehat{\Gamma}_{\pm,\rho}$$
is continuous.
- If $E^\pm_\rho \stackrel{\text{def}}{=} T_\rho \widehat{\Gamma}_{\pm,\rho}$, then $E^+_\rho + E^-_\rho = T_\rho p^{-1}(p(\rho)) \subset T_\rho(T^*X)$, $\mathbb{R}H_p(\rho) \in E^\pm_\rho$, and

$$\|d(\exp tH_p)_\rho(X)\| \leq Ce^{\pm\lambda t}\|X\|\,, \quad \rho \in K\,, \quad \text{for all } X \in T_\rho(T^*X)/E^\mp_\rho,\ \mp t \geq 0. \tag{7.6}$$

The above definition makes sense since by (7.5) $d(\exp tH_p)_\rho(E^\pm_\rho) = E^\pm_{\exp tH_p(\rho)}$, $\rho \in \widehat{\Gamma}_\pm$, we have
$$d(\exp tH_p)_\rho\, T_\rho(T^*X)/E^\mp_\rho \longrightarrow T_{\exp tH_p(\rho)}(T^*X)/E^\mp_{\exp tH_p(\rho)}\,, \quad \rho \in K\,,$$
and we choose continuously dependent norms in the last estimate in (7.6). We also note that $X \in T_\rho(T^*X)/E^\mp_\rho$ implies that X can be identified with a vector tangent to $p^{-1}(p(\rho))$.

In [30, §5] it is shown that there exist two functions, $\varphi_\pm \in \mathcal{C}^{1,1}(T^*X)$, $\varphi_\pm \geq 0$, $H^k_p\varphi_\pm \in \mathcal{C}^{1,1}(T^*X)$, $k \in \mathbb{N}$, such that for ρ in a small neighbourhood of K,
$$\mp H_p\varphi_\pm(\rho) \sim \varphi_\pm(\rho)\,, \quad H^k_p\varphi_\pm(\rho) = \mathcal{O}(\varphi_\pm(\rho))\,, \quad k \in \mathbb{N}\,,\,,$$
$$\varphi_\pm(\rho) \sim d(\rho, \widehat{\Gamma}_\pm)\,, \quad \varphi_+(\rho) + \varphi_-(\rho) \sim d(\rho, \widehat{K})^2\,,$$
and where $d(\bullet, \Gamma)$ is the distance to a closed set Γ. The notation $f \sim g$, means that there exists a constant $C > 0$ such
$$0 \leq g/C \leq f \leq Cg\,.$$
The simple model (2.3) is given in §2. Here we modify the construction to obtain suitably regularized functions $\widehat{\varphi}_\pm$

7.2. **Regularization of $\varphi_\pm$.** We start with two general lemmas:

Lemma 7.2. *Suppose $\Gamma \subset \mathbb{R}^m$ is a closed set. For any $\epsilon > 0$ there exists $\varphi_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^m)$ such that*
$$\varphi_\epsilon \geq \epsilon\,, \quad \varphi_\epsilon \sim d(\bullet, \Gamma)^2 + \epsilon\,, \quad \partial^\alpha\varphi_\epsilon = \mathcal{O}(\varphi_\epsilon^{1-|\alpha|/2})\,,$$
uniformly on compact sets.

Proof. Using a Whitney convering argument (see for instance [15, Lemma 1.4.9, Lemma 2.3.7]) we can find a sequence $x_j \in \mathbb{R}^m$ such that
$$\bigcup_j B(x_j, d(x_j, \Gamma)/8) = \mathbb{R}^m \setminus \Gamma\,,$$
every $x \in Q \setminus \Gamma$, $Q \Subset \mathbb{R}^m$, is in at most $N_0 = N_0(Q)$ balls $B(x_j, d(x_j, \Gamma)/2)$.

Let $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^m; [0,1])$ be supported in $B(0, 1/4)$, and be identically one in $B(0, 1/8)$. We define
$$\varphi_\epsilon(x) \stackrel{\text{def}}{=} \epsilon + \sum_{d(x_j,\Gamma) > \sqrt{\epsilon}} d(x_j, \Gamma)^2 \chi\left(\frac{x - x_j}{d(x_j, \Gamma) + \sqrt{\epsilon}}\right)$$

We first note that the number non-zero terms in the sum is uniformly bounded by N_0. In fact, $d(x_j,\Gamma)+\sqrt{\epsilon}<2d(x_j,\Gamma)$, and hence if $\chi((x-x_j)/(d(x_j,\Gamma)+\sqrt{\epsilon}))\neq 0$ then

$$1/4\geq |x-x_j|/(d(x_j,\Gamma)+\sqrt{\epsilon})\geq (1/2)|x-x_j|/d(x_j,\Gamma)\,,$$

and $x\in B(x_j,d(x_j,d(x_j,\Gamma))/2)$. This shows that $\varphi_\epsilon(x)\leq 2N_0(\epsilon+d(x,\Gamma)^2)$, and

$$\partial^\alpha\varphi_\epsilon(x)=\mathcal{O}((d(x,\Gamma)^2+\epsilon)^{1-|\alpha|/2})\,,$$

uniformly on compact sets.

To see the lower bound on φ_ϵ we first consider the case when $d(x,\Gamma)\leq C\sqrt{\epsilon}$.

$$\varphi_\epsilon(x)\geq\epsilon\geq(\epsilon+d(x,\Gamma)^2)/C'\,.$$

If $d(x,\Gamma)>C\sqrt{\epsilon}$ then for at least one j, $\chi((x-x_j)/(d(x_j,\Gamma)+\sqrt{\epsilon}))=1$ (since the balls $B(x_j,d(x_j,\Gamma)/8)$ cover the complement of Γ, and $\chi(t)=1$ if $|t|\leq 1/8$). Thus

$$\varphi_\epsilon(x)\geq\epsilon+d(x_j,\Gamma)^2\geq(\epsilon+d(x,\Gamma)^2)/C\,,$$

which concludes the proof. □

For future use we also record the following

Lemma 7.3. *Suppose $\varphi\in\mathcal{C}^{1,1}(\mathbb{R}^m)$, $\varphi\geq 0$, and for a vectorfield $V\in\mathcal{C}^\infty(\mathbb{R}^m;\mathbb{R}^m)$, $V^k\varphi=\mathcal{O}(\varphi)$, $V^k\phi\in\mathcal{C}^{1,1}(\mathbb{R}^m)$, $k\in\mathbb{N}$. Then, uniformly on compact sets,*

$$dV^k\varphi=\mathcal{O}(\varphi^{\frac{1}{2}})\,,\quad k\in\mathbb{N}\,.$$

Proof. For some $C>0$ the $\mathcal{C}^{1,1}$ function $C\varphi-V^k\varphi$ is non-negative. Hence using the standard estimate based on Taylor's formula,

$$|d\varphi|^2=\mathcal{O}(\varphi)\,,\quad |d(C\varphi-V^k\varphi)|^2=\mathcal{O}(C\varphi-V^k\varphi)=\mathcal{O}(\varphi)\,.$$

The lemma follows. □

We now have

Proposition 7.4. *Let $\widehat{\Gamma}_\pm$ be given by (7.4). For any small $\epsilon>0$ there exist functions $\widehat{\varphi}_\pm\in\mathcal{C}^\infty(T^*X;[0,\infty))$ such that in a neighbourhood of $\widehat{K}$,*

$$\begin{aligned}\widehat{\varphi}_\pm(\rho)&\sim d(\rho,\widehat{\Gamma}_\pm)^2+C\epsilon\,,\\ \mp H_p\widehat{\varphi}_\pm(\rho)+C\epsilon&\sim\widehat{\varphi}_\pm(\rho)\,,\\ \partial^\alpha H_p^k\widehat{\varphi}_\pm(\rho)&=\mathcal{O}(\widehat{\varphi}_\pm(\rho)^{1-|\alpha|/2})\,,\quad k\in\mathbb{N}\,,\\ \widehat{\varphi}_+(\rho)+\widehat{\varphi}_-(\rho)&\sim d(\rho,\widehat{K})^2+C\epsilon\,.\end{aligned}\tag{7.7}$$

Proof. We modify the arguments of [30, §5], roughly speaking, adding an $\mathcal{O}(\epsilon)$ error to all the estimates. Let $\varphi_\pm$ be the functions obtained using Lemma 7.2 with $\Gamma = \Gamma_\pm$. We now put

$$\widehat{\varphi}_\pm(\rho) \stackrel{\text{def}}{=} \int_{\mathbb{R}} g_T(t)\varphi_\pm(\exp tH_p(\rho))dt\,,$$
$$g_T \in \mathcal{C}^\infty_{\rm c}((-1,T+1))\,,\quad \operatorname{supp} g_T' \subset [-1,1]\cup[T-1,T+1]\,,$$
$$g_T'\restriction_{[-1,1]} \geq 0\,,\quad g_T'\restriction_{[T-1,T+1]} \leq 0\,,\quad g_T'(0)=1\,,\quad g_T'(T)=-1\,.$$

To check (7.7) we note that, by definition, $\varphi_\pm(\rho) \sim d(\rho,\widehat{\Gamma}_\pm)^2 + C\epsilon$. The assumptions (7.6) imply (see [30, Lemma 5.2]) that

$$\exists\, C\,,\ \forall\, T\geq 0\,,\ \exists\, \Omega_T \supset K\,,\text{an open set},\quad d(\exp(\pm TH_p)(\rho),\widehat{\Gamma}_\pm) \leq Ce^{-T/C}d(\rho,\widehat{\Gamma}_\pm)\,.$$

Hence, with constants depending on T,

$$\widehat{\varphi}_+(\rho) \sim \varphi_+(\exp(TH_p)(\rho)) \sim \varphi_+(\rho) \sim d(\rho,\Gamma_+)^2 + C\epsilon\,,$$
$$\widehat{\varphi}_-(\rho) \sim \varphi_-(\rho) \sim d(\rho,\Gamma_-)^2 + C\epsilon\,.$$

This shows the first statement in (7.7).

The assumptions on g_T also show that

$$H_p\widehat{\varphi}_\pm(\rho) \sim \varphi_\pm(\exp TH_p(\rho)) - \varphi_\pm(\rho) \sim d(\exp TH_p(\rho),\widehat{\Gamma}_\pm)^2 - d(\rho,\widehat{\Gamma}_\pm)^2 + \mathcal{O}(\epsilon)\,.$$

so that for T large enough and for ρ in a small neighbourhood of K, (again with T depenendent constants)

$$\mp H_p\widehat{\varphi}_\pm(\rho) + C\epsilon \sim d(\rho,\widehat{\Gamma}_\pm)^2 + C'\epsilon \sim \widehat{\varphi}_\pm(\rho)\,.$$

This proves the second part of (7.7). The third part is proved using Lemma 7.3 for $|\alpha|=1$ and the estimates on $\varphi_\pm$ in general.

To prove the last statement in (7.7) we first see that the transversality, $E^+_{\rho_0} + E^-_{\rho_0} = T_{\rho_0}(T^*X)$, and the continuity, $\rho \mapsto E^\pm_\rho$, assumed in (7.6) imply that for $\rho\,,\rho_1\,,\rho_2\,,$ near a point $\rho_0 \in K$,

$$d(\rho,\widehat{\Gamma}_{+,\rho_1}\cap\widehat{\Gamma}_{-,\rho_2}) \sim d(\rho,\widehat{\Gamma}_{+,\rho_1}) + d(\rho,\widehat{\Gamma}_{-,\rho_2})\,.$$

Hence

$$\begin{aligned}\widehat{\varphi}_+(\rho)+\widehat{\varphi}_-(\rho)+\mathcal{O}(\epsilon) &\sim d(\rho,\widehat{\Gamma}_+)^2 + d(\rho,\widehat{\Gamma}_-)^2 + C\epsilon\\ &\leq d(\rho,\widehat{\Gamma}_{+,\rho'})^2 + d(\rho,\widehat{\Gamma}_{-,\rho'})^2 + C\epsilon\\ &\sim d(\rho,\widehat{\Gamma}_{+,\rho'}\cap\widehat{\Gamma}_{-,\rho'})^2 + C\epsilon\,.\end{aligned}$$

If we choose $\rho' \in K$ so that $d(\rho,\widehat{K}) = d(\rho,\rho')$ then

$$d(\rho,\widehat{\Gamma}_{+,\rho'}\cap\widehat{\Gamma}_{-,\rho'})^2 \leq d(\rho,\rho')^2 = d(\rho,\widehat{K})^2\,,$$

proving that

$$\widehat{\varphi}_+(\rho)+\widehat{\varphi}_-(\rho) \leq d(\rho,\widehat{K})^2 + \mathcal{O}(\epsilon)\,.$$

The opposite inequality is obtained by choosing $\rho_\pm \in \widehat{\Gamma}_\pm$ such that $d(\rho,\rho_\pm) = d(\rho,\widehat{\Gamma}_\pm)$. Then using the transversality of $\widehat{\Gamma}_+$, $\widehat{\Gamma}_-$

$$\begin{aligned} d(\rho,\widehat{K})^2 &\leq d(\rho,\widehat{\Gamma}_{+,\rho_+}\cap\widehat{\Gamma}_{-,\rho_-})^2 \sim d(\rho,\widehat{\Gamma}_{+,\rho_+})^2 + d(\rho,\widehat{\Gamma}_{-,\rho_-})^2 \\ &\leq d(\rho,\rho_+)^2 + d(\rho,\rho_-)^2 = d(\rho,\widehat{\Gamma}_+)^2 + d(\rho,\widehat{\Gamma}_-)^2 \\ &\leq \widehat{\varphi}_+(\rho) + \widehat{\varphi}_-(\rho) + \mathcal{O}(\epsilon)\,. \end{aligned}$$

□

7.3. **Regularized escape function.** We now use the functions constructed in Proposition 7.4 to obtain an escape function near K. We first need the following

Lemma 7.5. *Then for $|\alpha|+k\geq 1$ we have*

$$\partial_\rho^\alpha H_p^k \log(\widehat{\varphi}_\pm) = \mathcal{O}(\widehat{\varphi}_\pm^{-\frac{|\alpha|}{2}})$$

Proof. Let $f(t)=\log(t)$. Then

$$f^{(k)}(\widehat{\varphi}_\pm) = \mathcal{O}\left(\frac{1}{\widehat{\varphi}_\pm^k}\right)\,,\quad k\geq 1\,,$$

and for $|\alpha|+k\geq 1$, $\partial_\rho^\alpha H_p^k f(\widehat{\varphi}_\pm)$ is a finite linear combination of terms

$$f^{(l)}(\widehat{\varphi}_\pm)\left(\partial_\rho^{\alpha_1}H_p^{k_1}\widehat{\varphi}_\pm\right)\cdots\left(\partial_\rho^{\alpha_\ell}H_p^{k_\ell}\widehat{\varphi}_\pm\right) = \mathcal{O}(1)\prod_{j=1}^{\ell}\frac{\partial_\rho^{\alpha_j}H_p^{k_j}\widehat{\varphi}_\pm}{\widehat{\varphi}_\pm}\,,$$

with

$$|\alpha_j|+k_j\geq 1\,,\quad \alpha_1+\cdots+\alpha_\ell=\alpha\,,\quad k_1+\cdots+k_\ell=k\,.$$

The estimates in (7.7) show that $\partial_\rho^{\alpha_j}H_p^{k_j}\widehat{\varphi}_\pm/\widehat{\varphi}_\pm = \mathcal{O}(\widehat{\varphi}_\pm^{-|\alpha_j|/2})$, and hence

$$\partial_\rho^\alpha H_p^k f(\widehat{\varphi}_\pm) = \mathcal{O}(\widehat{\varphi}_\pm^{-\frac{|\alpha|}{2}})\,,$$

proving the lemma. □

We are now ready for the main results of this section.

Lemma 7.6. *Let $\widehat{\varphi}_\pm$ be given in Proposition 7.4 and*

$$\widehat{G} \overset{\text{def}}{=} \left(\log(M\epsilon+\widehat{\varphi}_-) - \log(M\epsilon+\widehat{\varphi}_+)\right)\,. \tag{7.8}$$

Then in a neighbourhood of K we have

$$\begin{gathered} \partial_\rho^\alpha H_p^k\widehat{G} = \mathcal{O}_M(\min(\widehat{\varphi}_+,\widehat{\varphi}_-)^{-\frac{|\alpha|}{2}}) = \mathcal{O}_M(\epsilon^{-\frac{|\alpha|}{2}})\,,\quad |\alpha|+k\geq 1\,,\\ d(\rho,\widehat{K})^2\geq C\epsilon \Longrightarrow\ H_p\widehat{G}\geq 1/C\,, \end{gathered} \tag{7.9}$$

where, for the second estimate, M has to be chosen large enough, independently of ϵ, and C is a large constant.

Proof. We observe that, with constants depending on M, $\widehat{\varphi}_\pm + M\epsilon$ has the same properties as $\widehat{\varphi}_\pm$. Hence the estimates on $\partial_\rho^\alpha H_p^k \widehat{G}$ follow directly from the definition (7.8) and from Lemma 7.5. To check the second part of (7.9) we compute, using Proposition 7.4,

$$H_p \widehat{G} = \left(\frac{H_p \widehat{\varphi}_-}{\widehat{\varphi}_- + M\epsilon} - \frac{H_p \widehat{\varphi}_+}{\widehat{\varphi}_+ + M\epsilon} \right) \geq \frac{1}{C_1} \left(\frac{\widehat{\varphi}_- - C_2\epsilon}{\widehat{\varphi}_- + M\epsilon} + \frac{\widehat{\varphi}_+ - C_2\epsilon}{\widehat{\varphi}_+ + M\epsilon} \right) .$$

From (7.7) we also have

$$d(\rho, \widehat{K})^2 \geq C\epsilon \implies \max(\widehat{\varphi}_+, \widehat{\varphi}_-) \geq (C/2 - \mathcal{O}(1))\epsilon > C_3\epsilon \,,$$

where C_3 can be as large as we like depending on the choice of C. Hence, since $x \mapsto (x - C_2)/(x + M)$ is increasing,

$$H_p \widehat{G} \geq \frac{1}{C_1} \left(\frac{C_3 - C_2}{C_3 + M} - \frac{C_2}{M} \right) \geq \frac{1}{C} \,,$$

if we choose $C_3 \gg M \gg C_2$. □

We now modify $\widehat{G}$ using G_0 given in Proposition 7.1:

Proposition 7.7. *Let us fix $\delta_0 > 0$. Then there exist $\widehat{\chi}, \chi_0 \in \mathcal{C}^\infty_{\rm c}(T^*X)$, $C_0 > 0$, and a neighbourhoood V of K, such that*

$$G \stackrel{\rm def}{=} \widehat{\chi}\widehat{G} + C_0 \left(\log \frac{1}{\epsilon} \right) \chi_0 G_0 \,,$$

satisfies

$$\partial^\alpha H_p^k G = \begin{cases} \mathcal{O}(\log(1/\epsilon)) & \alpha = 0 \\ \mathcal{O}(\epsilon^{-|\alpha|/2}) & \text{otherwise} \end{cases} ,$$

$$\begin{gathered} d(\rho, \widehat{K})^2 \geq C\epsilon \,,\ \rho \in V \implies H_p G(\rho) \geq 1/C \,, \\ \rho \in p^{-1}([-\delta, \delta]) \setminus V \,, \quad |x(\rho)| \leq 3R_0 \implies H_p G(\rho) \geq \log(1/\epsilon) \,, \\ H_p G(\rho) \geq -\delta_0 \log(1/\epsilon) \,, \quad \rho \in T^*X \,. \end{gathered} \tag{7.10}$$

In addition we have

$$\frac{\exp G(\rho)}{\exp G(\mu)} \leq C_0 \left\langle \frac{\rho - \mu}{\sqrt{\epsilon}} \right\rangle^{N_0} , \tag{7.11}$$

for some constants C_0 and N_0.

Proof. We obtain G_0 from Proposition 7.1 taking for V a neighbourhood of $\widehat{K}$ in which the estimates of Lemma 7.6 hold. We have $\partial^\alpha H_p^k G_0 = \mathcal{O}_{k,|\alpha|}(1)$, and consequently for any $\chi_0 \in \mathcal{C}^\infty_{\rm c}(T^*X)$,

$$\partial^\alpha H_p^k \left(\log(1/\epsilon) \chi_0 G_0 \right) = \mathcal{O}_{k,|\alpha|}(\log(1/\epsilon)) = \begin{cases} \mathcal{O}(\log(1/\epsilon)) & \alpha = 0 \\ \mathcal{O}(\epsilon^{-|\alpha|/2}) & \text{otherwise} \end{cases} .$$

From Lemma 7.6 we obtain, again for any $\widehat{\chi} \in \mathcal{C}^\infty_{\rm c}(T^*X)$,

$$\partial^\alpha H_p^k(\widehat{\chi}\widehat{G}) = \begin{cases} \mathcal{O}(\log(1/\epsilon)) & \alpha = 0 \\ \mathcal{O}(\epsilon^{-|\alpha|/2}) & \text{otherwise} \end{cases} .$$

The loss compared to (7.9) is due to the presence of the cut-off function.

We take $\chi_0 \in \mathcal{C}^\infty_{\rm c}(T^*X;[0,1])$ to be identically equal to 1 in

$$p^{-1}([-\delta,\delta]) \cap \{(x,\xi) \,:\, |x| \leq 3R_0\} .$$

For $\widehat{\chi} \in \mathcal{C}^\infty_{\rm c}(T^*X)$ we take a function which is supported in a neighbhourhood of $\widehat{K}$ where (7.9) holds, and identically 1 in V. Hence for $\rho \in p^{-1}([-\delta,\delta]) \setminus V$, $|x(\rho)| \leq 3R_0$,

$$H_pG(\rho) = C_0 \log(1/\epsilon) H_pG_0(\rho) + H_p(\widehat{\chi}\widehat{G})(\rho) \geq C_0 \log(1/\epsilon) - \mathcal{O}(1)\log(1/\epsilon) \geq \log(1/\epsilon) ,$$

if C_0 is taken large enough. For $\rho \in V$, $\widehat{\chi}(\rho) = 1$, and

$$H_pG(\rho) = C_0 \log(1/\epsilon) H_pG_0(\rho) + H_p\widehat{G}(\rho) \geq H_p\widehat{G}(\rho) ,$$

and if $d(\rho,\widehat{K}) \geq C\epsilon$, $H_pG(\rho) \geq 1/C$. To complete the proof of (7.10) we need to define χ_0 for $|x| \geq R_0$. That is essentially done as in §4.2 where it was based on Lemma 4.1. Let T and R be large positive constants to be fixed later, $\chi(t)$ be given by Lemma 4.1, and let $\psi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R};[0,1])$ be equal to 1 for $|t| \leq 1$, and to 0 for $|t| \geq 2$. We define

$$\chi_0(\rho) \stackrel{\rm def}{=} \frac{\chi(G_0(\rho))}{G_0(\rho)} \psi\left(\frac{p(\rho)}{\delta}\right) \psi\left(\frac{|x(\rho)|}{R}\right) .$$

Then

$$\begin{aligned} H_p(\chi_0 G_0)(\rho) = {} & \chi'(G_0(\rho)) H_pG_0(\rho) \psi\left(\frac{p(\rho)}{\delta}\right) \psi\left(\frac{|x(\rho)|}{R}\right) \\ & + \frac{1}{R}\chi(G_0(\rho))\psi\left(\frac{p(\rho)}{\delta}\right)\psi'\left(\frac{|x(\rho)|}{R}\right) H_p(|x|)(\rho) , \end{aligned}$$

and

$$H_p(\chi_0 G_0)(\rho) \geq -C_1\left(\alpha + \frac{T}{R}\right) ,$$

where C_1 is independent of T and R: we note that (7.3) guarantees the boundedness of H_pG_0, and the assumptions on p imply that $H_p(|x|)$ is uniformly bounded for $|p| \leq 2\delta$. For any $\alpha > 0$ we can choose $T = T(\alpha)$ such that $|G_0(\rho)| \leq \alpha T$ for $|x(\rho)| \leq 3R_0$, $|p(\rho)| \leq 2\delta$. We then choose α and R so that

$$C_0C_1(\alpha + T(\alpha)/R) < \delta_0 .$$

Hence for $|x(\rho)| \geq R_0$

$$H_pG = C_0 \log(1/\epsilon) H_p(\chi_0 G_0) \geq -\delta_0 \log(1/\epsilon) ,$$

which is the last statement in (7.10).

It remains to show (7.11) and for simplicity of presentation we replace T^*X with $\mathbb{R}^{2n}$. We first prove that

$$\frac{\widehat{\varphi}_\pm(\rho)+M\epsilon}{\widehat{\varphi}_\pm(\mu)+M\epsilon}\leq C_1\left\langle\frac{\rho-\mu}{\sqrt{\epsilon}}\right\rangle^2\,,\quad M\geq 0\,, \tag{7.12}$$

with constants depending on M. We can replace $\widehat{\varphi}_\pm+M\epsilon$ with $\widehat{\varphi}_\pm$, as $\widehat{\varphi}_\pm+M\epsilon\sim_M\widehat{\varphi}_\pm$. Thus we claim that,

$$\frac{\widehat{\varphi}_\pm(\rho)}{\widehat{\varphi}_\pm(\mu)}\leq C_1\left\langle\frac{\rho-\mu}{\sqrt{\epsilon}}\right\rangle^2\,.$$

Since $\widehat{\varphi}_\pm\sim d(\bullet,\Gamma_\pm)^2+\epsilon$, $\widehat{\varphi}_\pm\geq\epsilon$, we have

$$\begin{aligned}\widehat{\varphi}_\pm(\rho)&\leq C(d(\rho,\Gamma_\pm)^2+\epsilon)\leq C(d(\mu,\Gamma_\pm)^2+|\mu-\rho|^2+\epsilon)\\&\leq C'(\widehat{\varphi}_\pm(\mu)+|\mu-\rho|^2)=C'(\widehat{\varphi}_\pm(\mu)+\epsilon\langle(\rho-\mu)/\sqrt{\epsilon}\rangle^2)\\&\leq 2C'\widehat{\varphi}_\pm(\mu)\langle(\rho-\mu)/\sqrt{\epsilon}\rangle^2\,.\end{aligned}$$

In the notation of Lemma 7.6, (7.12) gives

$$|\widehat{G}(\rho)-\widehat{G}(\mu)|\leq C+2\log\langle(\rho-\mu)/\sqrt{\epsilon}\rangle\,,$$

and with $\widehat{\chi}\in\mathcal{C}^\infty_{\rm c}$,

$$|\widehat{\chi}(\rho)\widehat{G}(\rho)-\widehat{\chi}(\mu)\widehat{G}(\mu)|\leq C|\rho-\mu|\log(1/\epsilon)+C\log\langle(\rho-\mu)/\sqrt{\epsilon}\rangle\,.$$

Clearly,

$$|\chi_0(\rho)G_0(\rho)-\chi_0(\mu)G_0(\mu)|\leq C|\rho-\mu|\log(1/\epsilon)\,,$$

and hence to obtain (7.11) we need

$$|\rho-\mu|\log(1/\epsilon)\leq C\log\langle(\rho-\mu)/\sqrt{\epsilon}\rangle+C\,,\quad\rho,\mu\in Q\Subset\mathbb{R}^{2n}\,.$$

If we put $\delta=\sqrt{\epsilon}$, $t=|\rho-\mu|/(C\delta)$ this becomes

$$\delta\log\frac{1}{\delta}\leq\frac{\log\langle t\rangle+1}{t}\,,\quad 0\leq t\leq\frac{1}{\delta}\,,$$

and that is clear as $t\mapsto(\log\langle t\rangle+1)/t$ is decreasing. □

8. Proof of the main result

Let G be the escape function given in Proposition 7.7, $\epsilon=h/\tilde{h}$. and let G^w be its Weyl quantization,

$$G^w=\mathcal{O}(\log(\tilde{h}/h))\ :\ L^2(X)\to L^2(X)\,.$$

We define a family of conjugated operators:

$$P_{\theta,t}\stackrel{\rm def}{=}e^{-tG^w}P_\theta e^{tG^w}\,,\quad\theta=C_0h\log(1/h)\,. \tag{8.1}$$

It is easy to see that, in the notation of §3.3,

$$\exp(tG^w)\in\Psi^{|t|C,0,0}_{\frac{1}{2}}(X)\,, \tag{8.2}$$

that is $\exp(tG^w) = B_t^w$, $\partial^\alpha B_t = \mathcal{O}(h^{-|t|C-|\alpha|/2}\tilde{h}^{|\alpha|/2})$. Finer estimates are however possible thanks to the results of Bony-Chemin [4]. The first of these is given in

Lemma 8.1. *Suppose that $Q \in \Psi^{0,0,0}_{\frac{1}{2}}(\mathbb{R}^n)$. Then*

$$\exp(-tG^w)Q\exp(tG^w) \in \Psi^{0,0,0}_{\frac{1}{2}}(\mathbb{R}^n)\,. \tag{8.3}$$

Proof. We follow §3.3 and change to the variables

$$\begin{gathered}(\tilde{x},\tilde{\xi}) = (\tilde{h}/h)^{\frac{1}{2}}(x,\xi)\,,\\ \widetilde{G}(\tilde{x},\tilde{\xi}) = G(x,\xi)\,,\quad \widetilde{Q}_t(\tilde{x},\tilde{\xi}) = Q_t(x,\xi)\,,\\ U^{-1}G^w(x,hD)U = \widetilde{G}^w(\tilde{x},\tilde{h}D_{\tilde{x}})\,,\quad U^{-1}Q_t^w(x,hD)U = \widetilde{Q}_t^w(\tilde{x},\tilde{h}D_{\tilde{x}})\,,\\ Uv(\tilde{x}) = (\tilde{h}/h)^{\frac{n}{4}}v((h/\tilde{h})^{\frac{1}{2}}\tilde{x})\,.\end{gathered}$$

We also note that

$$R \in \Psi^{0,0,0}_{\frac{1}{2}}(\mathbb{R}^n) \iff U^{-1}RU \in \Psi^{0,0}(\mathbb{R}^n)\,,$$

where on the right, $\tilde{h}$ is the small parameter – see the proof of Lemma 3.5. The estimate (7.11) shows that, in $(\tilde{x},\tilde{\xi})$ coordinates, $\widetilde{G}$ satisfies the hypothesis of Proposition 3.7 and that proves (8.3). □

The basic properties of $P_{t,\theta}$ are given in

Proposition 8.2. *Let $P_{\theta,t}$ be given by* (8.1) *and let $\Sigma \Subset T^*X$ be a compact surface coinciding with $p^{-1}(0)$ in a neighbourhood of the support of G. Then for $|t| \leq C$,*

$$\begin{gathered}P_{\theta,t} = P_\theta - ith\mathrm{Op}_h^w(H_pG) + E_t\,,\quad P_\theta - ith\mathrm{Op}_h^w(H_pG) \in \Psi^{0,0,2,0}_{\Sigma,\frac{1}{2}}(X)\cap\Psi^{0,0,2}_{\frac{1}{2}}(X)\,,\\ E_t \in \Psi^{-1,-1,0}_{\frac{1}{2}}(X)\,,\quad E_t = \mathcal{O}(h\tilde{h})\ :\ L^2(X)\longrightarrow L^2(X)\,,\end{gathered} \tag{8.4}$$

uniformly in h and $\tilde{h}$.

Proof. Let V_1, V_2 be open neighbourhoods of $\operatorname{supp} G$,

$$\operatorname{supp} G \subset V_1 \Subset \overline{V}_2 \Subset T^*X\,.$$

We first observe that if $\Psi \in \Psi^{0,-\infty}(X)$ satisfies

$$\mathrm{WF}_h(\Psi) \subset V_2\,,\quad \mathrm{WF}_h(I-\Psi) \subset \complement V_1\,,$$

then

$$[\exp(tG^w),\Psi] \in \Psi^{-\infty,-\infty}(X)\,,\quad (I-\Psi)(\exp(tG^w)-I) \in \Psi^{-\infty,-\infty}(X)\,,\quad |t|\leq 1\,. \tag{8.5}$$

In fact, using the calculus in §3.3 we see that $[G^w,\Psi] \in \Psi^{-\infty,-\infty}(X)$, Hence, using (8.2)

$$\begin{aligned}\frac{d}{dt}[\exp(tG^w),\Psi] &= G^w[\exp(tG^w),\Psi] + [G^w,\Psi]\exp(tG^w)\\ &= G^w[\exp(tG^w),\Psi] + A_t\,,\quad A_t \in \Psi^{-\infty,-\infty}(X)\,.\end{aligned}$$

Thus

$$[\exp(tG^w),\Psi]=\int_0^t\exp((t-s)G^w)A_s ds\in\Psi^{-\infty,-\infty}(X)\,,$$

which is the first statement in (8.5). We also compute

$$\frac{d}{dt}(I-\Psi)(\exp(tG^w)-I)=(I-\Psi)G^w\exp(tG^w)\in\Psi^{-\infty,-\infty}(X)\,,$$

and the second statement in (8.5) follows. Treating the equivalence of $(I-\Psi)P_\theta e^{tG^w}$ and $(I-\Psi)P_\theta$ similarly we conclude that

$$P_{\theta,t}-e^{-tG^w}\Psi P_\theta e^{tG^w}-(I-\Psi)P_\theta\in\Psi^{-\infty,-\infty}(X)\,.$$

We now put

$$Q_\theta\stackrel{\rm def}{=}\Psi P_\theta\in\Psi^{0,0}(X)\,,\quad Q_{\theta,t}\stackrel{\rm def}{=}e^{-tG^w}Q_\theta e^{tG^w}\,,$$

and we only need to prove (8.4) with $P_\bullet$ replaced by $Q_\bullet$. By a localization argument similar to the one used to construct $Q_\bullet$ we can assume that $X=\mathbb{R}^n$ when applying Lemma 8.1 and that shows that

$$Q_{\theta,t}\in\Psi^{0,0,0}_{\frac12}(X)\,.$$

We now establish the expansion in (8.4). Lemma 3.6 implies that

$$[Q_\theta,G^w]=(h/i){\rm Op}^w_h(H_{p_\theta}G)+R\,,$$

where $R\in\Psi^{-3/2,-3/2,0}_{\frac12}(X)\subset\Psi^{-1,-1,0}_{\frac12}(X)$. It also shows that

$$[[Q_\theta,G^w],G^w]=(h/i)[{\rm Op}^w_h(H_{p_\theta}G),G^w]+[R,G^w]\in\Psi^{-1,-1,0}_{\frac12}(X)\,.$$

Here we used the special structure of G,

$$G=\widehat{\chi}\widehat{G}+C_0\log(1/h)\chi_0 G_0\,,$$

where $\widehat{\chi},\chi_0$ and G_0 are uniformly smooth. When derivatives fall on these terms in error estimates (3.12) the gain in h compensates for the logarithmic growth, while for $|\alpha|>0$, $\partial^\alpha\widehat{G}\in S^{|\alpha|/2,-|\alpha|/2}_{\frac12}$.

This gives,

$$\frac{d}{dt}E_t=[Q_{\theta,t},G^w]-(h/i){\rm Op}^w_h(H_{p_\theta}G)+(h/i){\rm Op}^w_h(H_{p_\theta-p}G)=[Q_{\theta,t}-Q_\theta,G^w]+R_t\,,$$

with

$$E_0=(h/i){\rm Op}^w_h(H_{p_\theta-p}G)\in(h\log(1/h))^2\Psi^{0,0,0}_{\frac12}(X)\subset\Psi^{-1,-1,0}_{\frac12}(X)\,,$$

and $R_t\in\Psi^{-1,-1,0}_{\frac12}(X)$. We also have

$$\frac{d}{dt}[(Q_{\theta,t}-Q_\theta),G^w]=e^{-tG^w}[[Q_\theta,G^w],G^w]e^{tG^w}\in\Psi^{-1,-1,0}_{\frac12}(X)\,,\quad Q_{\theta,0}-Q_\theta=0\,.$$

Hence $[Q_{\theta,t}-Q_\theta,G^w]\in\Psi^{-1,-1,0}_{\frac12}(X)$, and consequently $E_t\in\Psi^{-1,-1,0}_{\frac12}$.

To show that

$$Q^0_{\theta,t} \overset{\text{def}}{=} Q_\theta - thi\mathrm{Op}^w_h(H_pG) \in \Psi^{0,0,0,0}_{\Sigma,\frac{1}{2}}(X)\,,$$

it suffices to show, in view of Definition 2, that for any sequence

$$\{a_j\}_{j=1}^M \subset S^0(T^*X)\,,$$

we have

$$\|\operatorname{ad}_{\mathrm{Op}^w_h(a_1)} \circ \cdots \operatorname{ad}_{\mathrm{Op}^w_h(a_M)} \circ \operatorname{ad}^k_P Q^0_{\theta,t} u\|_{L^2(X)} \;\leq\; Ch^{M/2+k}\tilde{h}^{M/2}\|u\|_{L^2(X)}\,.$$

This will follow if we show that

$$\operatorname{ad}^k_P Q^0_{\theta,t} \in \Psi^{-k,0,0}_{\frac{1}{2}}(X)\,,$$

and since that is clear for Q_θ we only need to check that

$$h\operatorname{ad}^k_P \mathrm{Op}^w_h(H_pG) \in \Psi^{-k,0,0}_{\frac{1}{2}}(X)\,. \tag{8.6}$$

This follows from the following stronger result

Lemma 8.3. *Let G and P be as above. Then for any $\epsilon > 0$*

$$\operatorname{ad}^\ell_P(H_pG)^w \in \Psi^{-\ell+\epsilon,0,0}_{\frac{1}{2}}(X)\,,\quad \ell \in \mathbb{N}\,,\quad \ell \geq 1\,.$$

Proof. We start by proving that

$$\operatorname{ad}^\ell_P(H_pG_1)^w \in \Psi^{-\ell+\epsilon,0,0}_{\frac{1}{2}}(X)\,,\quad \ell \geq 1\,, \tag{8.7}$$

where $G_1 = \widehat{G}\widehat{\chi}$ is given in Proposition 7.7. We claim that $\operatorname{ad}^\ell_P(H_pG_1)^w = E^w_\ell$, where

$$E_\ell \sim \left(\frac{h}{i}\right)^\ell \left(H_p^{\ell+1}G_1 + \sum_{s=0}^{\ell-1}\sum_{r=2(\ell-s)}^{\infty} h^r V^\ell_{rs}(H_p^{s+1}G_1)\right)\,,$$

with V^ℓ_{rs} a differential operator of order less than or equal to $r+\ell-s$, and where in view of good symbolic properties of P the error is $\mathcal{O}(h^\infty)$. In fact, Lemma 3.6 gives for any $a \in S^{0,0,0}_{\frac{1}{2}}$, $\operatorname{ad}_P a^w = a^w_1$,

$$a_1 \sim (h/i)\left(H_pa + \sum_{r=2}^{\infty} h^r V_r(a)\right)\,,$$

where V_r is a differential operator of order $r+1$. The expansion for E_ℓ comes from iterating this and observing that for any differential operator B of order q

$$H_p^m B H_p^k G_1 = \sum_{s=0}^{m} B_s H_p^{k+s} G_1\,, \tag{8.8}$$

where each B_s is a differential operator of order q. Using (7.9) we now see that for $r \geq 2(\ell - s)$,

$$h^\ell h^r V_{rs}^\ell (H_p^{s+1} G_1) \in h^{\ell+r} S^{(r+\ell-s)/2+\epsilon,-(r+\ell-s)/2,0}_{\frac{1}{2}} \subset S^{-\ell-r/2+\epsilon+(\ell-s)/2,0,0}_{\frac{1}{2}} \subset S^{-\ell-1/2+\epsilon,0,0}_{\frac{1}{2}},$$

where the $\epsilon > 0$ correction in the order comes from the term $(H_p^{s+1}\widehat{\chi})\widehat{G} \in S^{\epsilon,0,0}_{\frac{1}{2}}$. Hence

$$E_\ell = (h/i)^\ell (H_p^{\ell+1} G_1)^w + \mathcal{O}(h^{\ell+\frac{1}{2}})\Psi^{\epsilon,0,0}_{\frac{1}{2}}(X) \in \Psi^{-\ell+\epsilon,0,0}_{\frac{1}{2}}(X)\,,$$

which gives (8.7). On the other hand, again in the notation of Proposition 7.7,

$$\mathrm{ad}_P^\ell (H_p G_0)^w \in \Psi^{-\ell,0}(X)\,,$$

and consequently

$$\mathrm{ad}_P^\ell (\log(1/h) H_p G_0)^w \in \log(1/h)\Psi^{-\ell,0,0}_{\frac{1}{2}}(X) \subset \Psi^{-\ell+\epsilon,0,0}_{\frac{1}{2}}(X)\,.$$

Since $G = G_1 + \log(\tilde{h}/h)\chi_0 G_0$ the lemma follows. □

The lemma immediately gives (8.6) completing the proof of Proposition 8.2. □

The next lemma follows from Proposition 7.7:

Lemma 8.4. *Let $\widehat{G}$ be given in Lemma 7.6, $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$, and let $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(T^*X)$ be one in a fixed small neighbourhood of K and zero outside of another sufficiently small neighbourhood of K. Then*

$$\psi(x,\xi)\chi(H_p\widehat{G}(x,\xi)) \in S^{0,0,-\infty,0}_{\Sigma,\frac{1}{2}}(T^*X)\,.$$

Proof. Lemma 7.6 gives a stronger condition

$$H_p^\ell \partial_\rho^\alpha H_p^k (\psi(\rho)\chi(H_p G(\rho)) = \mathcal{O}((\tilde{h}/h)^{|\alpha|/2})\,,$$

as can be verified using (8.8) □

As in §6.2 we modify our operator to obtain global invertibility. Thus we define $a \in S^{0,0,-\infty,-\infty}_{\Sigma,\frac{1}{2}}(T^*X)$ as follows

$$a(x,\xi) \stackrel{\mathrm{def}}{=} \chi\left(\frac{\tilde{h}}{h} p(x,\xi)\right)\chi(H_p G(x,\xi))\psi(x,\xi)\,,$$
$$\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R};[0,1])\,,\quad \chi(t) \equiv 1\,,\quad |t| \leq 1\,,$$

and ψ is as in Lemma 8.4. In particular by taking its support close to K we can replace G by $\widehat{G}$ in the definition of a.

We then put

$$\widetilde{P}_{\theta,t} = P_{\theta,t} - i(h/\tilde{h})\widetilde{\mathrm{Op}}_{h,\tilde{h}}(a) \in \Psi^{0,0,2,1}_{\Sigma,\frac{1}{2}}(X)\,, \tag{8.9}$$

and first treat the region away from the trapped set:

Lemma 8.5. *Suppose that $P_{\theta,t}$ is given by* (8.9) *and $\Psi_0 \in \Psi^{0,0}(T^*X)$ satisfies*

$$\mathrm{WF}_h(\Psi_0) \cap K = \emptyset\,.$$

Then for $u \in \mathcal{C}^\infty_{\mathrm{c}}(X)$, $z \in D(0,Ch)$ we have

$$\|(\widetilde{P}_{\theta,t} - z)\Psi_0 u\|_{L^2} \geq th\|\Psi_0 u\|_{L^2(X)}/C - \mathcal{O}(h^\infty)\|u\|_{L^2(X)}\,,$$
$$0 < h \leq h_0(\tilde{h})\,, \quad 0 < \tilde{h} \leq \tilde{h}_0(t)\,.$$

Proof. Let us assume that $\|u\| = 1$. Once h is small enough, $a \equiv 0$ in a neighbourhood of $\mathrm{WF}_h(\Psi_0)$ and Theorem 4 gives

$$\|(\widetilde{P}_{\theta,t} - z)\Psi_0 u\|_{L^2} = \|(P_{\theta,t} - z)\Psi_0 u\|_{L^2} + \mathcal{O}(h\tilde{h}^\infty)\,.$$

We then observe that $P_{\theta,t} \in \Psi^{0,0,2}_{1/2}(X)$, and consequently we can use the simpler calculus of §3.3. Microlocally near $\mathrm{WF}_h(\Psi_0)$, for $z \in D(0,Ch)$, and for t sufficiently large, Proposition 7.7 and the choice of the angle of scaling give,

$$P_{\theta,t} - z = \mathrm{Op}^w_h(\operatorname{Re} p_\theta - \operatorname{Re} z) + i\mathrm{Op}^w_h(\operatorname{Im} p_\theta - iht H_p G - \operatorname{Im} z) + \mathcal{O}_t(h\tilde{h} + h^2\log(1/h))\,,$$
$$|\operatorname{Re} p_\theta - \operatorname{Re} z| < \delta \implies -\operatorname{Im} p_\theta + ht H_p G + \operatorname{Im} z \geq th/C\,.$$

(This is the analogue of (4.2) in the non-trapping case of §4.) Lemma 3.3 applied with Ψ_j's such that $|\operatorname{Re} p_\theta - \operatorname{Re} z| > \delta$ on $\mathrm{WF}_h(\Psi_1)$ (with Ψ_j's constructed using Lemma 3.2) completes the proof. □

Near the trapped set we use the second microlocal calculus to obtain

Lemma 8.6. *Suppose that $P_{\theta,t}$ is given by* (8.9) *and let $z \in D(0,Ch)$. For $u \in \mathcal{C}^\infty_{\mathrm{c}}(X)$, $\|u\| = 1$, with $\mathrm{WF}_h(u)$ in a fixed small neighbourhood of K we have*

$$\|(\widetilde{P}_{\theta,t} - z)u\|_{L^2(X)} \geq th\|u\|_{L^2(X)}/C\,, \quad 0 < h \leq h_0(\tilde{h})\,, 0 < \tilde{h} \leq \tilde{h}_0(t)\,. \tag{8.10}$$

provided that t is large enough.

Proof. In a small neighbourhood of K the operator is microlocally equal to

$$P_t^\flat \stackrel{\mathrm{def}}{=} P - ith\widetilde{\mathrm{Op}}_{h,\tilde{h}}(H_p\widehat{G}) - i(h/\tilde{h})\widetilde{\mathrm{Op}}_{h,\tilde{h}}(a) + \mathcal{O}_{L^2\to L^2}(h\tilde{h})\,,$$

that is,

$$\|(\widetilde{P}_{\theta,t} - z)u\|_{L^2(X)} = \|(P_t^\flat - z)u\|_{L^2(X)} + \mathcal{O}(h^\infty)\,, \quad \|u\|_{L^2(X)} = 1\,,$$

for u with $\mathrm{WF}_h(u)$ near K. For $z \in D(0,Ch)$,

$$P_t^\sharp - Z \stackrel{\mathrm{def}}{=} (\tilde{h}/h)(P_t^\flat - z) \in \Psi^{0,0,2,1}_{\Sigma,\frac{1}{2}}(X)\,, \quad Z \stackrel{\mathrm{def}}{=} (\tilde{h}/h)z\,,$$

has the symbol given by

$$p^\sharp - Z = \lambda - Z - it\tilde{h}H_pG - i\chi(\lambda)\chi(H_pG) + \mathcal{O}(\tilde{h}^2)\,, \quad Z \in D(0,C\tilde{h})\,, \quad \lambda = (\tilde{h}/h)p\,.$$

Now let $\psi_0, \psi_1 \in \mathcal{C}^\infty_{\rm b}(\mathbb{R})$ satisfy

$$\psi_0(t)^2 + \psi_1^2(t) = 1\,, \quad \operatorname{supp}\psi_0 \subset \{t \,:\, \chi(t) = 1\}\,, \quad \psi_1(t) \equiv 0\,, \quad |t| \leq 1/2\,.$$

As in Lemma 3.2 we can now find two operators $\Psi^\sharp_j \in \Psi^{0,0,0,0}_{\Sigma,\frac{1}{2}}(T^*X)$ such that

$$\sigma_{h,\tilde h}(\Psi^\sharp_j) = \psi_j(H_pG)\,, \quad (\Psi^\sharp_0)^2 + (\Psi^\sharp_1)^2 = Id + \mathcal{O}_{L^2\to L^2}(\tilde h^\infty)\,.$$

In a neighbourhoood of the support of $\psi_1(H_pG)$ the operator $P^\sharp_\theta - Z$ is elliptic in $\Psi^{0,0,2,1}_{\Sigma,\frac{1}{2}}$:

$$|\lambda - Z - i\tilde h H_pG - i\chi(\lambda)\chi(H_pG)| \geq (|\lambda - Z| + \chi(\lambda)\chi(H_pG))/2 \geq \langle\lambda\rangle/C\,,$$

when $Z \in D(0, C\tilde h)$ and $\chi(H_pG) > 1/2$, say. This implies (see Lemma 3.1) that for u with $\mathrm{WF}_h(u)$ near K, $\|u\|_{L^2(X)} = 1$,

$$\|(P^\sharp_t - Z)\Psi^\sharp_0 u\|_{L^2(X)} \geq \|\Psi^\sharp_0 u\|_{L^2(X)}/C - \mathcal{O}(\tilde h^\infty)\,,\ 0 < \tilde h \leq \tilde h_0\,.$$

To estimate $\|(P^\sharp_t - Z)\Psi^\sharp_1 u\|$ from below we proceed as in §4.1. Let

$$B^\sharp_t = \frac{1}{2i}\left(P^\sharp_t - (P^\sharp_t)^*\right)\,,$$

so that the Weyl symbol (in the sense of $\Psi_{\Sigma,\frac{1}{2}}$) of $B^\sharp_t$ is equal to

$$-\chi(\lambda)\chi(H_pG) + \mathcal{O}(\tilde h) - t\tilde h H_pG + \mathcal{O}_t(\tilde h^2)\,,$$

where we indicated the dependence on t in the second bound. Since $H_pG \geq 1/C$ in a neighbourhood of the support of $\psi_1(H_p)$ we see that for u with $\mathrm{WF}_h(u)$ near K, $\|u\|_{L^2(X)}$,

$$-\langle B^\sharp_t \Psi^\sharp_1 u, \Psi^\sharp_1 u\rangle \geq (t\tilde h - \mathcal{O}(\tilde h) - \mathcal{O}_t(\tilde h^2))\|\Psi^\sharp_1 u\|^2 - \mathcal{O}(\tilde h^\infty)\,.$$

We now first take t large enough to dominate the first error term and then $\tilde h$ small enough to dominate the second one. Hence,

$$\begin{aligned}\|(P^\sharp_t - Z)\Psi^\sharp_1 u\|\|\Psi^\sharp_1 u\| &\geq |\langle (P^\sharp_t - Z)\Psi^\sharp_1 u, \Psi^\sharp_1 u\rangle| \geq |\operatorname{Im}\langle (P^\sharp_t - Z)\Psi^\sharp_1 u, \Psi^\sharp_1 u\rangle| \\ &= -\langle (B^\sharp_t - \operatorname{Im} Z)\Psi^\sharp_1 u, \Psi^\sharp_1 u\rangle \geq t\tilde h\|\Psi^\sharp_1 u\|^2/2 - \mathcal{O}(\tilde h^\infty)\,,\end{aligned}$$

provided that t was large enough, and then $\tilde h$ small enough. Lemma 3.3 (or rather its proof) gives

$$\|(P^\sharp_t - Z)u\| \geq t\tilde h\|u\|/C\,, t \geq t_0 \gg 1\,, \quad 0 < h < h_0(t)\,,$$

for u with $\mathrm{WF}_h(u)$ near K.

We complete the proof by writing

$$\begin{aligned}\|(\widetilde P_{\theta,t} - z)u\|_{L^2(X)} &= \|(P^\flat_t - z)u\|_{L^2(X)} + \mathcal{O}(h^\infty)\|u\|_{L^2(X)} \\ &= (h/\tilde h)\|(P^\sharp_t - Z)u\|_{L^2(X)} + \mathcal{O}(h^\infty)\|u\|_{L^2(X)} \\ &\geq th\|u\|_{L^2(X)}/C\,.\end{aligned}$$

□

The two lemmas are now combined using Lemma 3.3 which gives for large t, $0 < \tilde{h} \leq \tilde{h}_0(t)$, and $0 < h < h_0(t, \tilde{h})$, the invertibility of $\widetilde{P}_{\theta,t} - z$, $z \in D(0, Ch)$:

$$(\widetilde{P}_{\theta,t} - z)^{-1} = \mathcal{O}(1/h) \ : \ L^2(X) \ \longrightarrow \ L^2(X)\,.$$

As in §6.2, Theorem 3 is a consequence of writing

$$\widetilde{\mathrm{Op}}_{h,\tilde{h}}(a) = R + E\,, \ \ \mathrm{rank}(R) = \mathcal{O}(h^{-\nu})\,, \ \ E = \mathcal{O}(\tilde{h}^{\infty}) \ : \ L^2(X) \to L^2(X)\,, \tag{8.11}$$

$\nu > \nu(E)$, where $m(E) = 2\nu(E) + 1$ is the dimension of the trapped set at energy E, allowing $\nu = \nu(E)$ if the trapped set is of pure dimension.

The decomposition (8.11) follows from Proposition 5.10 and the definition of the Minkowski dimension:

$$m_0 = 2n - 1 - \sup\{d \ : \ \limsup_{\epsilon \to 0} \epsilon^{-d}\,\mathrm{vol}(\{\rho \in p^{-1}(0) \ : \ d(\rho, K) < \epsilon\}) < \infty\}\,,$$

with the set being of pure dimension if

$$\limsup_{\epsilon \to 0} \epsilon^{-2n+1+m_0}\,\mathrm{vol}(\{\rho \in p^{-1}(0) \ : \ d(\rho, K) < \epsilon\}) < \infty\,.$$

In other words, for ϵ small

$$\mathrm{vol}(\{\rho \in p^{-1}(0) \ : \ d(\rho, K) < \epsilon\}) \leq C\epsilon^{2n-1-m}\,, \ \ m > m_0\,,$$

and m replaceable by m_0 when K is of pure dimension. In particular,

$$\mathrm{vol}(\mathrm{supp}\, a \cap p^{-1}(0)) \leq C_{\tilde{h}} h^{(2n-1-m)/2} = C_{\tilde{h}} h^{n-\nu-1}\,, \ \ m = 2\nu + 1 > m_0\,,$$

with equality if K is of pure dimension. Since

$$\mathrm{supp}\, a \cap p^{-1}(0) \subset \bigcup_{\rho \in K} B_{\Sigma}(\rho, M(h/\tilde{h})^{\frac{1}{2}})\,,$$

where B_Σ are balls in Σ with respect to some fixed smooth metric, and since K is invariant under the flow, the standard covering arguments (see [30, Lemma 3.3]) show that the hypothesis of Proposition 5.10 are satisfied with

$$K(h) \leq C_{\tilde{h}} h^{-\nu}\,,$$

which completes the proof of Theorem 3.

Appendix

We present a direct proof of Proposition 3.7. The hypotheses on G in (3.15) are equivalent to the statement that $\exp(tG) \in S(m^t)$, for all $t \in \mathbb{R}$. We start with

Lemma A.1. *Let* $U(t) \stackrel{\mathrm{def}}{=} (\exp tG)^w(x, D) \ : \ \mathcal{S}(\mathbb{R}^n) \ \longrightarrow \ \mathcal{S}(\mathbb{R}^n)$. *For* $|t| < \epsilon_0(G)$, *the operator* $U(t)$ *is invertible, and*

$$U(t)^{-1} = B_t^w(x, D)\,, \ \ B_t \in S(m^{-t})\,.$$

Proof. We apply the composition formula (3.14) to obtain

$$U(-t)U(t) = Id + E_t^w(x,D)\,,\quad E_t \in S(1)\,.$$

More explicitely we write (see [10, Proposition 7.7] and Lemma 3.6 here)

$$\begin{aligned} E_t(x_1,\xi) &= \int_0^s e^{sA(D)}A(D)(e^{-tG(x_1,\xi_1)+tG(x_2,\xi_2)})\restriction_{x_2=x_1=x,\xi_2=\xi_1=\xi}\,ds \\ &= \int_0^s (it/2)e^{sA(D)}(D_{\xi_1}GD_{x_2}G - D_{x_1}GD_{\xi_2}G)e^{-tG(x_1,\xi_1)+tG(x_2,\xi_2)}\restriction_{x_2=x_1=x,\xi_2=\xi_1=\xi}\,ds\,, \end{aligned}$$

where $A(D) = i\sigma(D_{x_1},D_{\xi_1};D_{x_2},D_{\xi_2})/2$.

Hence $E_t = t\widetilde{E}_t$ where $\widetilde{E}_t \in S(1)$ uniformly, and thus

$$E_t^w(x,D) = \mathcal{O}(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,.$$

This shows that for $|t|$ small enough $Id + E_t^w(x,D)$ is invertible, and Beals's lemma (see for instance [10, Proposition 8.3]) gives

$$(Id + E_t^w(x,D))^{-1} = C_t^w(x,D)\,,\quad C_t \in S(1)\,.$$

Hence $B_t = C_t\#\exp(-tG(x,\xi)) \in S(m^{-t})$. □

We now observe that

$$\begin{aligned} &\frac{d}{dt}\left(U(-t)\exp(tG^w(x,D))\right) = V(t)\exp(tG^w(x,D))\,, \\ &V(t) = A_t^w(x,D)\,,\quad A_t \in S(m^{-t})\,. \end{aligned} \tag{A.1}$$

In fact, we see that

$$\frac{d}{dt}U(-t) = -(G\exp(-tG))^w(x,D)\,,\quad U(-t)G^w(x,D) = (\exp(-tG)\#G)^w(x,D)\,.$$

As before, the composition formula (3.14) gives

$$\exp(-tG)\#G - G\exp(-tG) =$$

$$\int_0^1 \exp(sA(D))A(D)\exp(-tG(x^1,\xi^1)G(x^2,\xi^2)\restriction_{x^1=x^2=x,\xi^1=\xi^2=\xi}\,,$$

$$A(D) = i\sigma(D_{x^1},D_{\xi^1};D_{x^2},D_{\xi^2})/2\,.$$

The hypothesis on G shows that $A(D)\exp(tG(x^1,\xi^1))G(x^2,\xi^2)$ is a sum of terms of the form $a(x^1,\xi^1)b(x^2,\xi^2)$ where $a \in S(m^{-t})$ and $b \in S(1)$. The continuity of $\exp(A(D))$ on the spaces of symbols (see [10, Proposition 7.6]) gives (A.1).

If we put

$$C(t) \overset{\text{def}}{=} -V(t)U(-t)^{-1}\,,$$

then by Lemma A.1, $C(t) = c_t^w$ where $c_t \in S(1)$. Symbolic calculus shows that c_t depends smoothly on t and

$$(\partial_t + C(t))(U(-t)\exp(tG^w(x,D))) = 0\,.$$

The proof of Proposition 3.7 is now reduced to showing

Lemma A.2. *Suppose that $C(t) = c_t^w(x,D)$, where $c_t \in S(1)$, depends continuously on $t \in (-\epsilon_0, \epsilon_0)$. Then the solution of*

$$(\partial_t + C(t))Q(t) = 0\,, \quad Q(0) = q^w(x,D)\,, \quad q \in S(1)\,, \tag{A.2}$$

is given by $Q(t) = q_t(x,D)$, where $q_t \in S(1)$ depends continuously on $t \in (-\epsilon_0, \epsilon_0)$.

Proof. The Picard existence theorem for ODEs shows that $Q(t)$ is bounded on L^2. If $\ell_j(x,\xi)$ are linear functions on $T^*\mathbb{R}^n$ then

$$\frac{d}{dt}\,\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)}\, Q(t) + \mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)}(C(t)Q(t)) = 0\,,$$
$$\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)}\, Q(0) \;:\; L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)\,.$$

If we show that for any choice of $\ell_j's$ and any N

$$\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)}\, Q(t) \;:\; L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)\,, \tag{A.3}$$

then Beals's lemma (see [10, Chapter 8]) concludes the proof. We proceed by induction on N:

$$\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)}(C(t)Q(t)) = C(t)\,\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)}\, Q(t) + R(t)\,,$$

where $R(t)$ is the sum of terms of the form

$$A_k(t)\,\mathrm{ad}_{\ell_1(x,D)} \circ \mathrm{ad}_{\ell_k(x,D)}\, Q(t)\,, \quad k < N\,, \quad A_k(t) = a_k(t)^w\,,$$

where $a_k(t) \in S(1)$ depend continuously on t (this statement can also be proved by induction using the derivation property of ad_ℓ: $\mathrm{ad}_\ell(CD) = (\mathrm{ad}_\ell\, C)D + C(\mathrm{ad}_\ell\, D)$). Hence by the induction hypothesis $R(t)$ is bounded on L^2, and depends continuously on t. Thus

$$\left(\frac{d}{dt} + C(t)\right)\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)}\, Q(t) = R(t) \;:\; L^2(\mathbb{R}^n) \;\longrightarrow\; L^2(\mathbb{R}^n)\,.$$

Since (A.3) is valid at $t = 0$ we obtain it for all $t \in (-\epsilon_0, \epsilon_0)$. □

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