RESONANT RIGIDITY OF S^2

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1. Introduction

In this note we show that in obstacle scattering in $\mathbb{R}^3$, the sphere is uniquely determined by its resonances. The result is a simple consequence of the Poisson formula for resonances and of the extremal properties of the Willmore functional. It is somewhat disappointing that at the moment the argument works only for connected obstacles and in three dimensions. We discuss the connectedness assumption further at the end of Sect. 3.

The singularity at zero in the Poisson formula was first used for obstacle problems by Sjöstrand-Zworski [19]. Our argument is also related to Melrose's proof of existence of resonances in three dimensional potential scattering [10] and to the subsequent generalization to all odd dimensions by Sá Barreto-Zworski [17]. One could say that those results showed *resonant rigidity* of the zero potential among real super-exponentially decaying potentials. The analogous result in even dimensional ($n \geq 4$) potential scattering was recently proved by Sá Barreto-Tang [16].

Let $\mathcal{O}$ be a *connected* open precompact subset of $\mathbb{R}^n$ such that $\partial\mathcal{O}$ is smooth and $\mathbb{R}^n \setminus \mathcal{O}$ is connected. Let $-\Delta_{\mathbb{R}^n\setminus\mathcal{O}}$ be the Dirichlet Laplacian on $\mathbb{R}^n \setminus \mathcal{O}$ and let $R_{\mathcal{O}}(\lambda) = (-\Delta_{\mathbb{R}^n\setminus\mathcal{O}} - \lambda^2)^{-1}$ be its resolvent with the convention that $R_{\mathcal{O}}(\lambda) : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to H^2 \cap H^1_0(\mathbb{R}^n \setminus \mathcal{O})$ for $\operatorname{Im}\lambda > 0$. It is classical that $R_{\mathcal{O}}(\lambda)$ continues meromorphically when we restrict the domain and extend the range:

$$R_{\mathcal{O}}(\lambda) \; : \; L^2_{\rm comp}(\mathbb{R}^n \setminus \mathcal{O}) \longrightarrow H^2_{\rm loc} \cap H^1_{0,{\rm loc}}(\mathbb{R}^n \setminus \mathcal{O}) \,, \quad \lambda \in \left\{ \begin{array}{ll} \mathbb{C} & n \text{ odd} \\ \Lambda & n \text{ even} \end{array} \right. , \tag{1.1}$$

where Λ is the logarithmic plane.

The multiplicity of the pole at λ can be defined in many equivalent ways and we can for instance take

$$m_{\mathcal{O}}(\lambda) = \operatorname{rank} \oint_{|\zeta-\lambda|=\epsilon} R_{\mathcal{O}}(\zeta) d\zeta \,, \quad 0 < \epsilon \ll 1 \,. \tag{1.2}$$

The poles of $R_{\mathcal{O}}(\lambda)$ are called *resonances* or *scattering poles* and they constitute a natural replacement of discrete spectral data for problems on exterior domains. That point of view was emphasized by Lax-Phillips [5] – see the appendix to [5] as well as [10],[18] and [24] for an overview of recent results.

We denote by $\mathbb{B}(0,R)$ an open ball $\{x \; : \; |x| < R\}$ and we prove the following simple

Theorem. *If* $\mathcal{O} \subset \mathbb{R}^3$ *then*

$$\forall \, \lambda \in \mathbb{C} \;\, m_{\mathcal{O}}(\lambda) = m_{\mathbb{B}(0,R)}(\lambda) \implies \mathcal{O} = \mathbb{B}(0,R) \,.$$

Hence we can say that we showed the *resonant rigidity* of the sphere in three dimensional space. The poles of $R_{\mathbb{B}(0,1)}$ are well understood. They arise naturally in the study of diffraction by the sphere and are given by the zeros of the Hankel function, [20],[11]. They asymptotic location was given very precisely by Olver [12] and it is shown schematically in Fig.1. The spherical symmetry produces of course high multiplicities.

Another natural object in scattering theory, which is the exact analogue of the counting function of eigenvalues, is the scattering phase, $\sigma_{\mathcal{O}}(\lambda)$ – see for instance [10]. In a heuristic sense it measures the

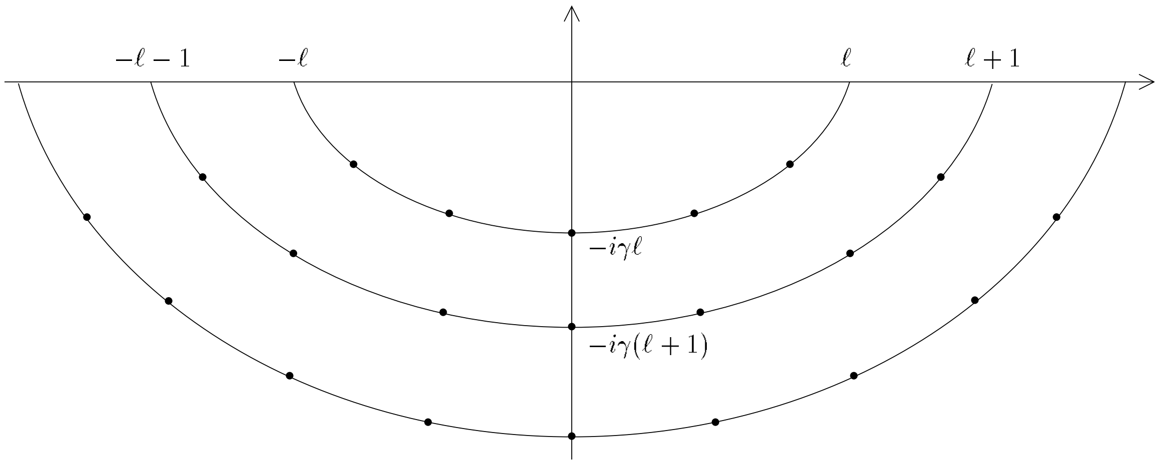

FIGURE 1. Resonances for the sphere

averaged phase shift of waves scattered by the obstacle. The asymptotics of $\sigma_{\mathcal{O}}(\lambda)$ are analogous to the asymptotics of the counting function of eigenvalues:

$$\sigma_{\mathcal{O}}(\lambda) = c_n \mathrm{vol}\,(\mathcal{O})\lambda^n + \mathcal{O}(\lambda^{n-1})\,,$$

and when the measure of the set of closed transversally reflected trajectories of the broken geodesic flow is zero in $T^*\mathbb{R}^n \setminus \mathcal{O}$ then

$$\sigma_{\mathcal{O}}(\lambda) = c_n \mathrm{vol}\,(\mathcal{O})\lambda^n + c_n' \mathrm{vol}\,(\partial\mathcal{O})\lambda^{n-1} + o(\lambda^{n-1})\,.$$

This was shown in odd dimensions by Melrose [9] and in greater generality which included all dimensions by Robert [14]. The proofs were based on the work of Ivrii [4] but had to resolve serious new difficulties due to non-compactness of the domain. Since the dynamical assumption is expected to hold for all obstacles in $\mathbb{R}^n$ it is clear from the isoperimetric inequality that

$$\sigma_{\mathcal{O}}(\lambda) - \sigma_{\mathbb{B}(0,R)}(\lambda) = o(\lambda^{n-1}) \implies \mathcal{O} = \mathbb{B}(0,R)\,. \tag{1.3}$$

If we assume the equality of the scattering phases then the Krein-Birman formula (see for instance [1]) and the heat or wave asymptotics immediately give the same conclusion without any conditions.

On the other hand it was pointed out in [22] that in odd dimensions $\sigma_{\mathcal{O}}$ is determined by $\{m_{\mathcal{O}}(\lambda)\}_{\lambda\in\mathbb{C}}$ up to an odd polynomial of degree n, that is up to $(n+1)/2$ numbers. In fact, the behaviour of the scattering matrix at zero (see [23] and references given there) shows that only two highest order coefficients are non-zero [15]. Hence the theorem above says that we do not need these two numbers to determine the sphere in three dimensions. However, the resonances are more natural to measure than the scattering phase, as they describe the long time behaviour of solutions of the wave equation and that makes the inverse problem for them quite reasonable. Indeed, Lax-Phillips [5] showed that if $\mathcal{O}$ is non-trapping and n is odd then

$$\left.\begin{array}{l} (\partial_t^2 - \Delta_{\mathbb{R}^n\setminus\mathcal{O}})u(t,x) = 0 \\[1ex] u\restriction_{t=0} = f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n\setminus\overline{\mathcal{O}}) \\[1ex] \frac{1}{i}\partial_t u\restriction_{t=0} = g \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n\setminus\overline{\mathcal{O}}) \end{array}\right\} \Longrightarrow u(t,x) \sim_{t\to\infty} \sum_{\lambda\in\mathbb{C}} m_{\mathcal{O}}(\lambda) \sum_{j=1}^{m_{\mathcal{O}}(\lambda)} w_{\lambda,j}(x) e^{-it\lambda} t^{j-1}\,,\ x \in K\,, \tag{1.4}$$

$K \subset \mathbb{R}^n \setminus \mathcal{O}$ compact.

2. Preliminaries

One of the basic tools for proving the existence of resonances is the Poisson formula. In odd dimensions it is the exact analogue of the trace formula for eigenvalues which is immediate from the spectral theorem:

$$\operatorname{tr}\,(U_{\mathcal{O}}(t) - U_0(t)) = \sum_{\lambda \in \mathbb{C}} m_{\mathcal{O}}(\lambda) e^{-i\lambda|t|}\,, \quad t \neq 0\,, \quad \mathcal{O} \subset \mathbb{R}^n\,, \quad n \text{ odd}\,, \tag{2.1}$$

where the equality is meant in the sense of distributions on $\mathbb{R} \setminus \{0\}$. Here $U_{\mathcal{O}}(t)$ denotes the wave group: $U_{\mathcal{O}}(t)^t(f,g) = {}^t(u(t), -i\partial_t u(t))$, where $u(t)$ is given by (1.4) and $U_0(t)$ is the free wave group ($\mathcal{O} = \emptyset$). The finite speed of propagation implies that $U_{\mathcal{O}}(t) - U_0(t)$ is of trace class as a distribution on $\mathbb{R}$. Formula (2.1) was proved by Melrose [6],[7] – see [1] for earlier work and [19],[18],[22] and [23] for further developments.

From the work of Ivrii [4] we know the behaviour of the trace at $t = 0$:

$$\begin{aligned} \operatorname{tr}\,(U_{\mathcal{O}}(t) - U_0(t)) = & a_0(\mathcal{O})|D_t|^{n-1}\delta_0(t) + a_1(\mathcal{O})|D_t|^{n-2}\delta_0(t) + \cdots \\ & + a_{n-1}(\mathcal{O})\delta_0(t) + a_n(\mathcal{O}) + a_{n+1}(\mathcal{O})|t| + \cdots . \end{aligned} \tag{2.2}$$

The coefficients $a_j(\mathcal{O})$ are given by non-zero multiples of the heat coefficients and can be found in [2]. Specializing to dimension three we obtain

$$\begin{aligned} a_0(\mathcal{O}) &= \alpha_0 \operatorname{vol}\,(\mathcal{O}) \\ a_1(\mathcal{O}) &= \alpha_1 \operatorname{vol}\,(\partial\mathcal{O}) \\ a_2(\mathcal{O}) &= \alpha_2 \int_{\partial\mathcal{O}} H_{\mathcal{O}} \\ a_3(\mathcal{O}) &= \alpha_3 \int_{\partial\mathcal{O}} (-3H_{\mathcal{O}}^2 + 20K_{\mathcal{O}}) \end{aligned} \tag{2.3}$$

where $H_{\mathcal{O}}$ is the mean curvature of $\partial\mathcal{O}$, $K_{\mathcal{O}}$ the Gaussian curvature and $\alpha_j \neq 0$.

3. Proof of Theorem

Assume that $\mathcal{O}$ has the same resonances as the ball of radius R, that is, $m_{\mathcal{O}}(\lambda) = m_{\mathbb{B}(0,R)}(\lambda)$ for all $\lambda \in \mathbb{C}$. The Poisson formula (2.1) immediately implies that

$$\operatorname{tr}\,(U_{\mathcal{O}}(t) - U_0(t)) = \operatorname{tr}\,(U_{\mathbb{B}(0,R)}(t) - U_0(t))\,, \quad t \neq 0\,. \tag{3.1}$$

If we knew this for all t then the result would follow trivially from the comparison of the first two coefficients in (2.2). We can detect the coefficients of $|D_t|^{2k+1}\delta_0(t)$ since they have the whole real line as supports, so their coefficients are determined by the asymptotics of the wave trace as $t \to 0$. However, we cannot see the coefficients of derivatives of δ_0 which are supported at zero. Thus we can only conclude that

$$\operatorname{vol}\,(\partial\mathcal{O}) = \operatorname{vol}\,(\mathbb{B}(0,R))\,, \quad \int_{\partial\mathcal{O}} (-3H_{\mathcal{O}}^2 + 20K_{\mathcal{O}}) = \int_{\partial\mathbb{B}(0,R)} (-3H_{\mathbb{B}(0,R)}^2 + 20K_{\mathbb{B}(0,R)})\,, \tag{3.2}$$

Using the Gauss-Bonnet formula and the fact that $\mathcal{O}$ is assumed to have the same number of connected components as $\mathbb{B}(0,R)$, we conclude that

$$\int_{\partial\mathcal{O}} H_{\mathcal{O}}^2 = \int_{\partial\mathbb{B}(0,R)} H_{\mathbb{B}(0,R)}^2 - \frac{80\pi}{3} g\,, \ g = \text{genus of } \partial\mathcal{O}\ .$$

It is well known however that the *Willmore functional*, $\int_S H^2$, is uniquely minimized by spheres with the standard metric – see [21]. The result is now immediate as we can use a_1 or a scaling argument to determine the radius.

Remark. The following heuristic argument, based on (3.1), strongly suggests that an obstacle with more than one component cannot have the resonances of the sphere. Since $\mathbb{R}^3 \setminus \mathbb{B}(0,R)$ has no trapped rays there are no singularities in the wave trace for $t > 0$. However every disconnected obstacle has trapped rays, for example the shortest line segment between two components, which 'should' produce wave-trace singularities.

This is known to be true generically (see [3] for the singularities of the wave trace and [13], Chapter 3 for generic properties of reflected rays) but no proof is known which holds in general.

Remark. For the Neumann boundary condition, the coefficient a_3 is given by the integral of $15H^2 - 4K$ over the boundary [2] and the same argument goes through.

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