Quantum Monodromy revisited

Johannes Sjöstrand and Maciej Zworski

We present a few simplifications of the presentation of the *quantum monodromy* operator in [1] and [2].

We first repeat the comment made in [1, §4]: the trace formula of [2] is formulated in terms of a general Hamiltonian, $P(z)$ (for instance an effective Hamiltonian with a non-linear dependence on z). However, the proof can be reduced to the case of $P - z$. In fact, the assumptions of [2, Theorem 2], the implicit function theorem, and the usual symbolic iteration, imply that

$$P(z) = A(z)^*(P - z)A(z)\,,$$

with $A(z) \in \Psi_h^{0,k/2}(X)$ elliptic near $\gamma(0)$, and $P \in \Psi_h^{0,0}(X)$ self-adjoint. Replacing $P(z)$ by $P-z$ in [2, Theorem 2] changes the trace by $\mathcal{O}(h^\infty)$.

In the special case of $P(z) = P - z$ the monodromy operator can be written quite simply (though we still believe that it is interesting to consider $M(z)$ for the non-linear $P(z)$ as done in [2]). Let us recall that at a point on an integral curve of $P - z$, $\gamma(z)$, $m_0(z) \in \gamma$, we can define the microlocal kernel of $P - z$ at $m_0(z)$, to be the set of families $u(h)$, such that $u(h)$ are microlocally defined near m_0 and

$$(P - z)u(h) = \mathcal{O}(h^\infty) \text{ near } m_0.$$

We denote it by $\ker_{m_0(z)}(P - z)$. Since microlocally, near a given point, the operator $P - z$ can be reduced to hD_{x_1} any solution can be continued microlocally along $\gamma(z)$ and we denote the corresponding forward and backward continuations by $I_\pm(z)$. We can also define the propagator $\exp(-it(P - z)/h)$ and we see that

$$\exp(-it(P - z)/h) \;:\; \ker_{m_0(z)}(P - z) \;\longrightarrow\; \ker_{\exp(tH_p)m_0(z)}(P - z)\,.$$

This follows from the fact that $(P-z)\exp(-it(P-z)/h) = \exp(-it(P-z)/h)(P-z)$, and propagation of semi-classical wave fronts: $WF_h(\exp(-it(P-z)/h)u)$ is contained in a neighbourhood of $\exp(tH_p)m_0(z)$ if $WF_h(u)$ is contained in a neighbourhood of $m_0(z)$. Hence we have

$$I_\pm(z) = \exp(\mp it(P - z)/h) \tag{1}$$

microlocally near $(m_0(z), \exp(tH_p)(m_0(z)))$.

To define the quantum monodromy we take $m_1(z) \neq m_0(z)$ be another point on $\gamma(z)$ and put

$$\begin{gathered} I_-(z)\mathcal{M}(z) = I_+(z)\,, \quad \text{near } m_1, \\ \mathcal{M}(z) \;:\; \ker_{m_0(z)}(P - z) \;\longrightarrow\; \ker_{m_0(z)}(P - z)\,. \end{gathered} \tag{2}$$

In view of (1) we now have

$$\mathcal{M}(z) = \exp(-iT(z)(P - z)/h) \;:\; \ker_{m_0(z)}(P - z) \;\longrightarrow\; \ker_{m_0(z)}(P - z)\,, \tag{3}$$

where $T(z)$ is the period of $\gamma(z)$ but for z small we can replace it by a fixed period, $T(0)$.

The operator $P(z)$ is assumed to be self-adjoint with respect to some inner product $\langle \bullet, \bullet \rangle$, and we define the *quantum flux* norm on $\ker_{m_0(z)}(P - z)$ as follows: let χ be a microlocal cut-off function supported near γ and equal to one near the part of γ between m_0 and m_1, in the positive direction determined by H_p. We denote by $[P, \chi]_+$ the part of the commutator supported near m_0, or more generally, near the left end point (using the orientation determined by H_p) of the support of $\chi|_\gamma$. We then put

$$\langle u, v\rangle_{\rm QF} \stackrel{\rm def}{=} \langle [(i/h)P, \chi]_+ u, v\rangle\,, \quad u, v \in \ker_{m_0(z)}(P - z)\,.$$

It is easy to check that this norm is independent of the choice of χ: if $\tilde{\chi}$ agrees with χ near m_1 we see that $[P, \tilde{\chi} - \chi]_+ = [P, \tilde{\chi} - \chi]$ and clearly $\langle [P, \tilde{\chi} - \chi] u, v \rangle = 0$ (see [2, Lemma 4.4] for more details). This independence leads to the unitarity of $\mathcal{M}(z)$:

$$\begin{aligned} \langle \mathcal{M}(z)u, \mathcal{M}(z)u \rangle_{\mathrm{QF}} &= \langle (i/h)[P,\chi]_+ e^{-iT(z)(P-z)/h} u, e^{-iT(z)(P-z)/h} v \rangle \\ &= \langle (i/h)[P, e^{iT(z)(P-z)/h} \chi e^{-iT(z)(P-z)/h}]_+ u, v \rangle \\ &= \langle (i/h)[P,\chi]_+ u, v \rangle = \langle u, v \rangle_{\mathrm{QF}} \end{aligned} \tag{4}$$

As already recalled above the operator $P - z$ can be reduced to hD_{x_1} we can identify $\ker_{m_0(z)}(P - z)$ with $\mathcal{D}'(\mathbb{R}^n)$. This is done microlocally near $(0, 0)$, and we can choose the identification, $K(z)$, so that

$$K^*(z)(i/h)[P,\chi]_+ K(z) = Id\,.$$

This guarantees that the corresponding monodromy operator,

$$M(z) \stackrel{\mathrm{def}}{=} K(z)^{-1} \mathcal{M}(z) K(z)^{-1} \; : \; \mathcal{D}'(\mathbb{R}^n) \; \longrightarrow \; \mathcal{D}'(\mathbb{R}^n)\,,$$

microlocally defined near $(0, 0)$, is unitary (microlocally near $(0, 0)$. Here $(0, 0)$ corresponds to the closed orbit intersecting a transversal identified with $T^*\mathbb{R}^n$. We easily see that $M(z)$ is a semi-classical Fourier integral operator which quantizes the Poincaré map of $\gamma(z)$.

Using (3),

$$M(z) = K(z)^{-1} \circ \exp(-iT(0)(P - z)/h) \circ K(z)\,. \tag{5}$$

This expression trivializes the proof of [2, Lemma 6.2] in the case $P(z) = P - z$. For $P = hD_{x_1}$, $K(z)u(x) = e^{izx_1/h}u(x')$, $x = (x_1, x')$, $x' \in \mathbb{R}^n$, and hence the complexification of z in $K(z)$ produces growth of size $\mathcal{O}(e^{\epsilon|\operatorname{Im} z|/h})$. Then (5) shows that (for z close to 0)

$$\begin{aligned} \|M(z)\| &\leq e^{-(T(0)-\epsilon)\operatorname{Im} z/h}\,, \quad 0 < \operatorname{Im} z\,, \\ \|M(z)^{-1}\| &\leq e^{(T(0)-\epsilon)\operatorname{Im} z/h}\,, \quad \operatorname{Im} z < 0\,. \end{aligned} \tag{6}$$

The rather subtle [2, Lemma 6.1] is altogether unnecessary (unless we want the results for the general $P(z)$; they are however not needed for the trace formula). The estimates (6) also give a slight improvement in [2, Theorem 1]: we can make the conditions on the support of $\hat{f}$ there optimal: $\hat{f} \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$, supp $\hat{f} \subset (-NT, NT) \setminus \{0\}$,

For a discussion of the quantum monodromy operator in a concrete setting and a relation of the quantum flux norm to the more standard objects, see the appendix to [1].

References

[1] A. Iantchenko, J. Sjöstrand, and M. Zworski, *Birkhoff normal forms in semi-classical inverse problems,* to appear in Math. Res. Lett. `http://arXiv.org/abs/math.SP/0201190`

[2] J. Sjöstrand and M. Zworski, *Quantum monodromy and semi-classical trace formulæ*, J. Math. Pure Appl. **81**(2002), 1-33.