

OPTIMAL RELAXATION FOR WITTEN LINDBLADIANS IN 1D

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ABSTRACT. We prove optimal trace-norm relaxation for the one-dimensional Lindbladian associated with the Witten differential $a = h\partial_x + V'$. That differential annihilates the classical Gibbs density $e^{V/h}$ and the Lindblad evolution is expected to settle at $|e^{-V/h}\rangle\langle e^{-V/h}|$. For a class of initial operators that happens at the exponential rate $\gamma_h = \lambda_1(h)/(2h)$ where $\lambda_1(h)$ is the first positive eigenvalue of the Witten Laplacian $H = a^*a$. The Schwartz kernels of initial data, after a Gibbs conjugation, satisfy L^2 estimates for the restriction to the diagonal and for the normal derivative. An appendix by Chat GPT 6 presents a stronger result specialised to positive initial data.

1. STATEMENT OF THE RESULT AND MOTIVATION

With motivation and references provided below, we study the evolution governed by the following superoperator:

$$\mathcal{L}T = \frac{1}{h}(aTa^* - \frac{1}{2}(a^*aT + Ta^*a)), \quad a := h\partial_x + V'(x). \quad (1.1)$$

Since the jump operator, a , is given by a Witten differential [Wi82] (and $H := a^*a$ is the scalar Witten Laplacian), we refer to $\mathcal{L}$ as the *Witten Lindbladian*. The assumptions on V are given in (1.8). By adding an irrelevant h -dependent constant we assume that

$$\int_{\mathbb{R}} e^{-2V(x)/h} dx = 1. \quad (1.2)$$

For initial conditions for the Lindblad evolution, we consider $T \in \mathcal{L}_1(L^2(\mathbb{R}))$, the space of trace-class operators on $L^2(\mathbb{R})$,

$$Tu(x) = \int_{\mathbb{R}} e^{-(V(x)+V(y))/h} r(x, y) u(y) dy, \quad r \in C^1,$$

where, with $d\mu(y) := e^{-2V(y)/h} dy$, r satisfies

$$r|_{\{(x, x)\}} \in L^2(d\mu), \quad |\partial_y r(x, y)| \leq G_0(y) \in L^2(d\mu). \quad (1.3)$$

Theorem 1. *There is a constant $C_{h,V} > 0$ such that for T above with $\text{tr } T = 1$, we have for $t \geq 0$ and $\Pi := |e^{-V/h}\rangle\langle e^{-V/h}|$,*

$$\|e^{t\mathcal{L}}T - \Pi\|_{\mathcal{L}_1} \leq C_{h,V} e^{-\gamma_h t} (\|T\|_{\mathcal{L}_1} + \|r|_{\{(x, x)\}} - 1\|_{L^2(\mu)} + \|G_0\|_{L^2(\mu)}), \quad (1.4)$$

where

$$\gamma_h := \lambda_1(h)/(2h) > 0, \quad \lambda_1(h) := \inf(\text{Spec}(H) \setminus \{0\}), \quad (1.5)$$

and this exponent is optimal.

Remarks. 1. The regularity assumptions on r are chosen to keep the proof elementary. The pointwise differentiability in y is likely to be replaceable by an appropriate absolute-continuity condition. We do not pursue this endpoint question here, but some restrictions on T are always needed, as $\mathcal{L}$ does not have a spectral gap on $\mathcal{L}_1$ (easily seen in the case of $V(x) = x^2$). The more substantial problem is the extension to higher dimensions: our methods are very much one dimensional.

2. Tracing the constants in the proof, one may take

$$C_{h,V} \leq \max(2e^{\gamma_h}, 1, 2A_{h,V}(1 + \gamma_h)^{1/2}\mathcal{V}(V)),$$

where (with S defined in (2.3))

$$A_{h,V} = \|\langle x \rangle (1 - S)^{-1/2}\|_{L^2(\mu) \rightarrow L^2(\mu)},$$

and $\mathcal{V}(V)$ was defined in (3.2). For $0 < h \leq h_0$, the assumptions on V give $A_{h,V} = \mathcal{O}_V(1)$ and $\gamma_h = \mathcal{O}_V(1)$. Thus the h -dependence provided by the proof is contained in $\mathcal{V}(V)$. It is bounded as $h \rightarrow 0$ in the one-well case, while for a nondegenerate double well of barrier height S_0 the proof gives

$$C_{h,V} = \mathcal{O}_V(\sqrt{h} e^{S_0/h}) = \mathcal{O}_V(\sqrt{h/\gamma_h}).$$

We do not claim that this is the optimal common prefactor in the estimate.

3. When, in the theorem, $T \geq 0$, we can replace the hypotheses by $r|_{\{(x,x)\}} \in L^\infty$ and the assumption that $\text{tr} HT = \|aT^{\frac{1}{2}}\|_2^2 < \infty$. The result then follows from the following inequality proved at the of this introduction:

$$|\partial_y r(x, y)| \leq G_0(y), \quad \|G_0\|_{L^2(\mu)} \leq h^{-1} \|r|_{\{(x,x)\}}\|_{L^\infty}^{\frac{1}{2}} \|aT^{\frac{1}{2}}\|_{\mathcal{L}_2}. \quad (1.6)$$

At our request, Chat GPT 6 upgraded this by weakening the hypothesis to $r|_{\{(x,x)\}} \in L^q(d\mu)$, $q > 1$. That result, Theorem 2, is presented in the appendix (heavily edited by us).

When $V(x) = x^2/2$, (1.1) is, after the standard normalization of the annihilation operator, the one-mode pure-loss channel, with transmissivity e^{-2t} ; see [HoWe01, We*12, De*17, De*18]. The general potential replaces the harmonic-oscillator annihilation operator by the Witten differential $h\partial_x + V'$. Related first-order factorizations for anharmonic oscillators appear in [Ga99, NeSa00, Ne02].

Lindblad generators [GKS76, Li76] provide the standard Markovian framework for dissipative quantum dynamics. Recently, they have also appeared in quantum Gibbs sampling; see [CKG23, DiLiLi25] for finite-dimensional constructions and [GaZw26, BRS26] for related infinite-dimensional work. A model based on the simpler equation considered here appeared in Leng–Ding–Chen–Lin [Le*26], where Witten differentials were used as Lindblad jump operators. The stationary state Π is obtained from the square root $e^{-V/h}$ of the Gibbs density. The trace norm is natural here since

$$|\text{tr}((T - \Pi)O)| \leq \|T - \Pi\|_{\mathcal{L}_1} \|O\|$$

for every bounded observable O .

The diagonal evolution is governed by the Witten Laplacian H . The theorem says that the full Lindblad evolution, including the off-diagonal part of the kernel, has the same optimal relaxation rate γ_h . For a self-contained one-dimensional account of the semiclassical asymptotics of $\lambda_1(h)$, and references, see [MiZw18]. If V has exactly one critical point, a unique nondegenerate minimum x_0 , harmonic approximation gives

$$\gamma_h = V''(x_0) + \mathcal{O}(h). \quad (1.7)$$

For a multiwell potential, γ_h is exponentially small and is governed by the Eyring–Kramers law. Thus the result includes the metastable regime. Related semiclassical Lindblad–Fokker–Planck constructions are discussed in [HRR25, GaZw25, Li25, Sm26].

We now impose, in addition to the normalization (1.2),

$$V^{(k)}(x) = \mathcal{O}(\langle x \rangle), \quad k \geq 1, \quad V''(x) \geq -C_0, \quad xV'(x) \geq x^2/C_1 - C_1. \quad (1.8)$$

These assumptions are chosen for a short proof rather than for maximal generality. Since $-H/(2h)$ generates a contraction semigroup on $L^2(\mathbb{R})$, the Davies construction gives the trace-class contraction semigroup used below; see [GaZw25, §1, Remark]. The remaining bounds are used in the maximum-principle, weighted, and tail estimates. We write

$$H := a^*a = -h^2\partial_x^2 + |V'|^2 - hV'', \quad \nu := e^{-V/h}, \quad \Pi := |\nu\rangle\langle\nu|. \quad (1.9)$$

Then, $a\nu = 0$, $\ker H = \mathbb{C}\nu$, and, from (1.2), $\|\nu\|_{L^2} = 1$. The assumptions (1.8) imply that H has compact resolvent and hence that $\lambda_1(h) > 0$.

Proof of (1.6). By the spectral theorem, $T = \sum_{j \geq 0} \lambda_j |\varphi_j\rangle\langle\varphi_j|$, $\lambda_j \geq 0$, $\sum_{j \geq 0} \lambda_j = 1$, where the φ_j are orthonormal in $L^2(\mathbb{R})$. Writing $\varphi_j = \nu f_j$ we introduce the ℓ^2 -valued function $F(x) := \{\sqrt{\lambda_j} f_j(x)\}_{j \in \mathbb{N}} \in L^2(\mathbb{R}; \ell^2(\mathbb{N}), d\mu)$. It follows that

$$r(x, y) = \sum_{j \geq 0} \lambda_j f_j(x) \overline{f_j(y)} = \langle F(x), F(y) \rangle_{\ell^2}, \quad r(x, x) = \|F(x)\|_{\ell^2}^2. \quad (1.10)$$

The definition (1.9), the spectral decomposition above, and the fact that $a(\nu f) = h\nu f'$ give

$$\begin{aligned} \operatorname{tr} HT &= \|aT^{\frac{1}{2}}\|_{\mathcal{L}_2}^2 = \sum_{j=0}^{\infty} \lambda_j \|a\varphi_j\|_{L^2(\mathbb{R})}^2 \\ &= \sum_{j=0}^{\infty} \lambda_j \|h\nu f'_j\|_{L^2(\mathbb{R})}^2 = h^2 \int_{\mathbb{R}} \|F'(x)\|_{\ell^2(\mathbb{N})}^2 d\mu(x). \end{aligned} \quad (1.11)$$

Differentiating (1.10) in the weak sense gives $\partial_y r(x, y) = \langle F(x), F'(y) \rangle_{\ell^2}$. Hence, by the Cauchy–Schwarz inequality and (1.10),

$$|\partial_y r(x, y)|^2 \leq r(x, x) \|F'(y)\|_{\ell^2}^2 \leq \|r\|_{\{(x,x)\}} \|F'(y)\|_{\ell^2}^2. \quad (1.12)$$

This and (1.11) proves (1.6). $\square$

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2. THE CONJUGATED KERNEL AND THE MAXIMUM PRINCIPLE

In the language of the theorem in §1, let $T_0 = T$ and $T_t = e^{t\mathcal{L}}T_0$. Using (1.9), write

$$K_t(x, y) = \nu(x)\nu(y)r_t(x, y). \quad (2.1)$$

The identities

$$a(\nu f) = h\nu f', \quad a^*a(\nu f) = h\nu(2V'f' - hf'')$$

give the following simple

Lemma 1. *The conjugated kernel satisfies*

$$\partial_t r_t = M r_t, \quad M = \frac{1}{2}h(\partial_x + \partial_y)^2 - V'(x)\partial_x - V'(y)\partial_y. \quad (2.2)$$

If $Rr(x) := r(x, x)$ and

$$Q = \frac{1}{2}h\partial_x^2 - V'\partial_x, \quad S = Q - V'', \quad (2.3)$$

then

$$RM = QR, \quad \partial_y M = (M - V''(y))\partial_y. \quad (2.4)$$

One striking consequence of (2.4) is

$$r_t(x, x) = e^{tQ}r_0(x, x). \quad (2.5)$$

Multiplication by ν gives the unitary equivalences of (unbounded) operators on $L^2(d\mu)$ (on the left) with operators on $L^2(dx)$ (on the right):

$$Q \simeq -\frac{a^*a}{2h}, \quad S \simeq -\frac{aa^*}{2h}. \quad (2.6)$$

The operator aa^* has no zero mode, and its spectrum agrees with the nonzero spectrum of $H = a^*a$. Hence

$$\|e^{tQ}f - \mu(f)\|_{L^2(\mu)} \leq e^{-\gamma_h t} \|f - \mu(f)\|_{L^2(\mu)}, \quad (2.7)$$

$$\|e^{tS}g\|_{L^2(\mu)} \leq e^{-\gamma_h t} \|g\|_{L^2(\mu)}. \quad (2.8)$$

Since $\int r_0(x, x)d\mu(x) = \text{tr } T_0 = 1$, this gives

$$\|r_t(x, x) - 1\|_{L^2(\mu)} \leq e^{-\gamma_h t} \|r_0(x, x) - 1\|_{L^2(\mu)}. \quad (2.9)$$

For the whole-space comparison we will use the second condition in (1.3) and the maximum principle for degenerate heat equations. Thus we start with

Lemma 2. *Let $w \in C^2([0, T] \times \mathbb{R}^2; \mathbb{R})$ have at most polynomial growth, and satisfy*

$$(\partial_t - M + V''(y))w \geq 0, \quad w(0, x, y) \geq 0. \quad (2.10)$$

Then $w(t, x, y) \geq 0$ for $(t, x, y) \in [0, T] \times \mathbb{R}^2$.

Proof. For C_0 in (1.8) put

$$z = e^{-C_0 t} w, \quad q(y) = -V''(y) - C_0 \leq 0, \quad P = \partial_t - M - q(y).$$

Then (2.10) gives $Pz \geq 0$. Let $\rho(x, y) = 1 + x^2 + y^2$. For every integer $k \geq 1$, (1.8) and direct differentiation give $M\rho^k \leq C_k \rho^k$ (see (2.2) for the definition of M).

Choose m so that $|z(t, x, y)| \leq C_T \rho(x, y)^m$ on $[0, T] \times \mathbb{R}^2$ and put

$$\varphi(t, x, y) = e^{At} \rho(x, y)^{m+1}.$$

Since $q \leq 0$, choosing $A > C_{m+1}$ gives $P\varphi > 0$.

For $\varepsilon > 0$, set $v = z + \varepsilon\varphi$. It is positive at $t = 0$, and it tends to $+\infty$ as $x^2 + y^2 \rightarrow \infty$, uniformly for $0 \leq t \leq T$. If it first vanished at (t_0, x_0, y_0) , then

$$\partial_t v \leq 0, \quad \partial_x v = \partial_y v = 0, \quad \nabla_{x,y}^2 v \geq 0.$$

In particular, $(\partial_x + \partial_y)^2 v \geq 0$. At the contact point the drift and zeroth-order terms vanish, and therefore

$$Pv = \partial_t v - \frac{1}{2}h(\partial_x + \partial_y)^2 v \leq 0.$$

But $Pv = Pz + \varepsilon P\varphi > 0$, a contradiction. Thus $z + \varepsilon\varphi > 0$ and letting $\varepsilon \rightarrow 0+$ proves the claim. $\square$

The next proposition propagates the estimate (1.3) on the initial condition:

Proposition 3. *For all $t \geq 0$, and for S defined in (2.3),*

$$|\partial_y r_t(x, y)| \leq (e^{tS} G_0)(y), \quad x, y \in \mathbb{R}. \quad (2.11)$$

Proof. Put $u_t = \partial_y r_t$. By (2.4),

$$\partial_t u_t = (M - V''(y))u_t.$$

Let $g_t = e^{tS} G_0$ and regard $g_t(y)$ as a function of (x, y) . Since M reduces to Q on functions of y alone, it satisfies the same equation. For bounded smooth data, Lemma 2, applied for every $\theta \in \mathbb{R}$ to

$$g_t(y) - \operatorname{Re}(e^{-i\theta} u_t(x, y)),$$

gives (2.11). The same lemma shows that the closed semigroup generated by $M - V''(y)$ is positivity preserving. Since u_0 and $G_0(y)$ belong to $L^2(\mathbb{R}^2; d\mu(x)d\mu(y))$, the order inequalities

$$-G_0(y) \leq \operatorname{Re}(e^{-i\theta} u_0(x, y)) \leq G_0(y)$$

pass to the L^2 closure. This proves the result for the stated $G_0 \in L^2(\mu)$. $\square$

For quadratic confinement the Volterra estimate below requires one spatial weight. It is obtained using smoothing for positive times:

Lemma 4. *There is $C_{h,V} > 0$ such that, for $g \in L^2(\mu)$ and $t > 0$,*

$$\|\langle x \rangle e^{tS} g\|_{L^2(\mu)} \leq C_{h,V} (1 + t^{-\frac{1}{2}}) e^{-\gamma_h t} \|g\|_{L^2(\mu)}. \quad (2.12)$$

Proof. As stated in (2.6), $1 - S$ is unitarily equivalent to $1 + aa^*/(2h)$. The assumptions (1.8) imply $|V'(x)|^2 \geq cx^2 - C$, while

$$aa^* = -h^2 \partial_x^2 + |V'|^2 + hV''.$$

Hence its quadratic form gives

$$\|\langle x \rangle f\|_{L^2(\mu)} \leq C_{h,V} \|(1 - S)^{1/2} f\|_{L^2(\mu)}. \quad (2.13)$$

The spectral theorem and the fact that $\text{Spec}(S) \subset (-\infty, -\gamma_h]$ give

$$\|(1 - S)^{1/2} e^{tS}\|_{L^2(\mu) \rightarrow L^2(\mu)} \leq C_{h,V} (1 + t^{-1/2}) e^{-\gamma_h t}.$$

Combining the two estimates proves the lemma. $\square$

3. THE VOLTERRA FACTORISATION AND PROOF OF THEOREM 1

Define

$$F_-(s) = \int_{-\infty}^s e^{-2V(x)/h} dx, \quad F_+(s) = \int_s^{\infty} e^{-2V(x)/h} dx, \quad (3.1)$$

and

$$\mathcal{V}(V)^2 = \int_{\mathbb{R}} F_-(s) F_+(s) e^{2V(s)/h} \langle s \rangle^{-2} ds. \quad (3.2)$$

Lemma 5. *For V satisfying (1.8), $\mathcal{V}(V) < \infty$.*

Proof. For s sufficiently large, (1.8) gives $V'(x) \geq cx$ for $x \geq s$. Therefore

$$F_+(s) \leq C_{h,V} s^{-1} e^{-2V(s)/h}.$$

Since $F_-(s) \leq 1$, the integrand in (3.2) is $\mathcal{O}(s^{-3})$ as $s \rightarrow +\infty$. The argument for $s \rightarrow -\infty$ is identical. $\square$

The next lemma provides the crucial Volterra trace factorisation inspired by [GK70]:

Lemma 6. *Suppose that $|u(x, s)| \leq v(s)$ and $\langle s \rangle v \in L^2(\mu)$. Then the operator with kernel*

$$R(x, y) := \nu(x) \nu(y) \int_x^y u(x, s) ds \quad (3.3)$$

is of trace class and

$$\|R\|_{\mathcal{L}_1} \leq 2\mathcal{V}(V) \|\langle s \rangle v\|_{L^2(\mu)}. \quad (3.4)$$

Proof. We write

$$R = R_+ + R_-, \quad R_{\pm}(x, y) := R(x, y) \mathbf{1}_{\pm x < \pm y},$$

and for the factorisation of R_+ , we define

$$w_+(s) = F_-(s) e^{2V(s)/h} \langle s \rangle^{-2} > 0.$$

We then put

$$A_+(x, s) := \nu(x)u(x, s)\mathbf{1}_{\{x < s\}}w_+(s)^{-1/2}, \quad B_+(s, y) := w_+(s)^{1/2}\nu(y)\mathbf{1}_{\{s < y\}},$$

so that (identifying integral kernels with operators), $R_+ = A_+B_+$. We now have

$$\|A_+\|_{\mathcal{L}_2}^2 \leq \int w_+(s)^{-1}v(s)^2F_-(s)ds = \|\langle s \rangle v\|_{L^2(\mu)}^2, \quad (3.5)$$

$$\|B_+\|_{\mathcal{L}_2}^2 = \int w_+(s)F_+(s)ds = \mathcal{V}(V)^2. \quad (3.6)$$

This gives $\|R_+\|_{\mathcal{L}_1} \leq \mathcal{V}(V)\|\langle s \rangle v\|_{L^2(\mu)}$. For R_- , put $w_-(s) = F_+(s)e^{2V(s)/h}\langle s \rangle^{-2}$ and

$$A_-(x, s) := \nu(x)u(x, s)\mathbf{1}_{\{s < x\}}w_-(s)^{-1/2}, \quad B_-(s, y) := w_-(s)^{1/2}\nu(y)\mathbf{1}_{\{y < s\}}.$$

Then $R_- = -A_-B_-$, and the same estimates give $\|R_-\|_{\mathcal{L}_1} \leq \mathcal{V}(V)\|\langle s \rangle v\|_{L^2(\mu)}$. Together with the estimate for R_+ this gives (3.4). $\square$

Proof of the Theorem 1. Put $B_t = T_t - \Pi$. For $t > 0$, split its kernel as

$$\begin{aligned} B_t(x, y) &= \nu(x)\nu(y)(r_t(x, x) - 1) + \nu(x)\nu(y)(r_t(x, y) - r_t(x, x)) \\ &=: D_t(x, y) + R_t(x, y). \end{aligned} \quad (3.7)$$

The first term has rank one, and (2.9) gives

$$\|D_t\|_{\mathcal{L}_1} \leq e^{-\gamma h t} \|r_0(x, x) - 1\|_{L^2(\mu)}. \quad (3.8)$$

For the second term,

$$r_t(x, y) - r_t(x, x) = \int_x^y \partial_y r_t(x, s) ds.$$

By Proposition 3, the integrand is dominated by $v_t = e^{tS}G_0$. For $t \geq 1$, Lemmas 4 and 6 give

$$\|R_t\|_{\mathcal{L}_1} \leq C_{h,V} e^{-\gamma h t} \|G_0\|_{L^2(\mu)}. \quad (3.9)$$

This proves (1.4) for $t \geq 1$. For $0 \leq t \leq 1$, stationarity of Π and trace-norm contractivity give

$$\|B_t\|_{\mathcal{L}_1} \leq \|T_0 - \Pi\|_{\mathcal{L}_1} \leq \|T_0\|_{\mathcal{L}_1} + 1,$$

and the short interval is absorbed by enlarging $C_{h,V}$.

For optimality, let $H\varphi_1 = \lambda_1(h)\varphi_1$, with $\varphi_1 \perp \nu$, and write $\varphi_1 = \nu f_1$. Then $f_1 \in L^2(\mu)$ and $f_1' \in L^2(\mu)$. To see the second claim we note that

$$h^2 \|f_1'\|_{L^2(dx)}^2 = \|a(\nu f_1)\|_{L^2(dx)}^2 = \langle H\varphi_1, \varphi_1 \rangle_{L^2(dx)} = \lambda_1(h) \|\varphi_1\|_{L^2(dx)}^2 < \infty.$$

This shows that operator

$$T_0 = \Pi + |\nu\rangle\langle\varphi_1|, \quad \text{tr } T_0 = 1 + \langle\nu, \varphi_1\rangle_{L^2(\mathbb{R})} = 1,$$

has conjugated kernel $r_0(x, y) = 1 + \overline{f_1(y)}$, so that $r_0(x, x) - 1 = \overline{f_1(x)} \in L^2(\mu)$ and $|\partial_y r_0(x, y)| = |f_1'(y)|$. Hence it satisfies the hypotheses of the theorem, and

$$\mathcal{L}(|\nu\rangle\langle\varphi_1|) = -\gamma_h |\nu\rangle\langle\varphi_1|.$$

Thus no larger uniform exponent is possible. $\square$

APPENDIX BY CHATGPT 6

The following strengthens Remark 3 in §1, under the same assumptions (1.2) and (1.8). Inner products are linear in the first factor.

Theorem 2. *Let $T \geq 0$, $\text{tr } T = 1$, and let $d(x) = r(x, x)$ be its conjugated diagonal, as in (1.10). If*

$$E(T) := \text{tr } HT = \|aT^{\frac{1}{2}}\|_{\mathcal{L}_2}^2 < \infty \quad \text{and} \quad d \in L^q(d\mu), \quad 1 < q < \infty,$$

then

$$\|e^{t\mathcal{L}}T - \Pi\|_{\mathcal{L}_1} \leq C_{h,V,q} e^{-\gamma_h t} \left(\frac{\|d\|_{L^q(d\mu)} E(T)}{\lambda_1(h)} \right)^{1/2}, \quad t \geq 0. \quad (\text{A.1})$$

For $d \in L^\infty(d\mu)$, replace $\|d\|_{L^q(d\mu)}$ by $\|d\|_{L^\infty(d\mu)}$; the estimate then holds with an absolute constant.

Proof. Let $F(x) = \{\sqrt{\lambda_j} f_j(x)\}_j$ be the ℓ^2 -valued function introduced in the proof of (1.6). Equations (1.10)–(1.11) give

$$r(x, y) = \langle F(x), F(y) \rangle, \quad d(x) = \|F(x)\|^2, \quad h^2 \|F'\|_{L^2(d\mu)}^2 = E(T).$$

We first suppose that $F \in C_c^\infty(\mathbb{R}; \mathbb{C}^N)$ and $\|F\|_{L^2(d\mu)} = 1$; the approximation is given below.

For any positive trace-one operator A , the operator $A - \Pi$ has at most one negative eigenvalue, since its quadratic form is nonnegative on $\nu^\perp$. Its trace is zero, so writing $\lambda \leq 0$ for its lowest eigenvalue gives

$$\|A - \Pi\|_{\mathcal{L}_1} = 2|\lambda| \leq 2\|A - \Pi\|_{\mathcal{L}_2}. \quad (\text{A.2})$$

(See [CCC19] for a similar argument). It therefore suffices to estimate the Hilbert–Schmidt norm. To this end, put

$$b(x) = \int_0^x 2\|F(s)\| \|F'(s)\| ds, \quad k(x, y) = r(x, y) - \frac{1}{2}(d(x) + d(y)). \quad (\text{A.3})$$

Then b is bounded and nondecreasing, and

$$|k(x, y)| \leq |b(x) - b(y)|, \quad |d(x) - d(y)| \leq |b(x) - b(y)|. \quad (\text{A.4})$$

Indeed, for complex Hilbert-space vectors u, v , with $\alpha = \|u\|$, $\beta = \|v\|$,

$$|\langle u, v \rangle - \frac{1}{2}(\alpha^2 + \beta^2)| \leq \frac{1}{2}(\alpha + \beta)\|u - v\| \leq \|\alpha u - \beta v\|.$$

The first inequality follows by Cauchy–Schwarz, the second by expanding squared norms. Apply them to $u = F(x)$, $v = F(y)$ and use

$$\|(\|F(x)\|F(x))'\| \leq 2\|F(x)\|\|F'(x)\| = b'(x), \quad |d'(x)| \leq b'(x).$$

For $p = 2q/(q+1) \in (1, 2)$, Hölder also gives

$$\|b'\|_{L^p(d\mu)} \leq 2\|F\|_{L^{2q}(d\mu)}\|F'\|_{L^2(d\mu)} = \frac{2}{h}\sqrt{\|d\|_{L^q(d\mu)}E(T)}. \quad (\text{A.5})$$

To propagate (A.4), use M, Q, S from (2.2)–(2.3) and put (see (2.5))

$$P_t = e^{tQ}, \quad d_t = P_t d, \quad b_t = P_t b, \quad k_t = r_t - \frac{1}{2}(d_t(x) + d_t(y)), \quad d_t(x) = r_t(x, x).$$

On $x < y$, both k_t and $b_t(y) - b_t(x)$ solve $\partial_t u = Mu$ and vanish on $x = y$. The barrier proof of Lemma 2, with the zeroth-order term omitted, applied to $b_t(y) - b_t(x) - \text{Re}(e^{i\theta} k_t)$ for every θ , propagates the first bound. Apply the same argument to $d_t(y) - d_t(x)$ for the second bound, and reflect for $x > y$. Smooth majorants converging uniformly to b' justify comparison when b' is not smooth.

By (2.6), $-Q$ has kernel the constants and first positive eigenvalue $\gamma_h = \lambda_1(h)/(2h)$. Since $-\langle Qf, f \rangle_{L^2(d\mu)} = (h/2)\|f'\|_{L^2(d\mu)}^2$, we obtain the Poincaré inequality

$$\|g - \mu(g)\|_{L^2(d\mu)} \leq \frac{h}{\sqrt{\lambda_1(h)}}\|g'\|_{L^2(d\mu)}, \quad \mu(g) := \int g(x) d\mu(x). \quad (\text{A.6})$$

Now set $\sigma_t = \|b_t - \mu(b_t)\|_{L^2(d\mu)}$. Since $\mu(d_t) = 1$, the identity

$$2\|f - \mu(f)\|_{L^2(d\mu)}^2 = \iint |f(x) - f(y)|^2 d\mu(x)d\mu(y)$$

gives

$$\|d_t - 1\|_{L^2(d\mu)} \leq \sigma_t, \quad \|k_t\|_{L^2(d\mu(x)d\mu(y))} \leq \sqrt{2}\sigma_t.$$

The $L^2(d\mu(x)d\mu(y))$ norm of $\frac{1}{2}(d_t(x) + d_t(y)) - 1$ is $\|d_t - 1\|_{L^2(d\mu)}/\sqrt{2}$. Since $\partial_x Q = S\partial_x$, we have $b'_t = e^{tS}b'$. Thus (A.2) and the Poincaré inequality (A.6) give

$$\|e^{t\mathcal{L}}T - \Pi\|_{\mathcal{L}_1} \leq 3\sqrt{2}\sigma_t \leq \frac{3\sqrt{2}h}{\sqrt{\lambda_1(h)}}\|e^{tS}b'\|_{L^2(d\mu)}. \quad (\text{A.7})$$

The confinement assumptions (1.8) give $c|x| \leq |V'(x)| \leq C|x|$, $xV'(x) > 0$, near infinity. Writing $\rho = e^{-2V/h}$, with $F_\pm$ as in (3.1), and letting m be the median of μ (so $\int_m^\infty \rho(s) ds = \frac{1}{2}$), these bounds give for large $x > 0$,

$$F_+(x) \asymp_{h,V} \frac{\rho(x)}{x}, \quad \int_m^x \frac{ds}{\rho(s)} \lesssim_{h,V} \frac{1}{x\rho(x)}, \quad \log \frac{1}{F_+(x)} \lesssim_{h,V} x^2.$$

Hence $F_+(x) \log(1/F_+(x)) \int_m^x \rho(s)^{-1} ds$ is bounded; the estimate with F_- follows by reflection. Thus μ satisfies a logarithmic Sobolev inequality [BR03, Theorem 1], and [Gr75, Theorem 6] gives

$$\|P_{t_p}\|_{L^p(d\mu) \rightarrow L^2(d\mu)} \leq 1, \quad \text{for some } t_p < \infty, t_2 := 0.$$

Since $V'' \geq -C_0$, comparison of the propagation by Q and S (see (2.3)) gives

$$0 \leq e^{tS}G \leq e^{C_0 t}P_t G, \quad \text{for } G \geq 0.$$

The spectral bound for S in (2.8) therefore yields

$$\|e^{tS}b'\|_{L^2(d\mu)} \leq e^{-\gamma_h(t-t_p)}e^{C_0t_p}\|b'\|_{L^p(d\mu)}, \quad t \geq t_p. \quad (\text{A.8})$$

Equations (A.5)–(A.8) prove (A.1) for $t \geq t_p$. The interval $0 \leq t \leq t_p$ is absorbed by enlarging $C_{h,V,q}$, using contractivity, $\|d\|_{L^q(d\mu)} \geq 1$, and $\|T - \Pi\|_{\mathcal{L}_1}^2 \leq 8E(T)/\lambda_1(h)$ (by (A.2), $\text{tr}(T^2) \leq 1$, and $H \geq \lambda_1(h)(I - \Pi)$). For $q = \infty$, take $p = 2$, $t_p = 0$; the constant is $6\sqrt{2}$.

Finally, (1.11) and $d \in L^q(d\mu)$ show that $F \in H^1(d\mu; \ell^2) \cap L^{2q}(d\mu; \ell^2)$. For finite q , component and spatial cutoffs, then mollification and normalization, give smooth finite-dimensional F_n converging in both spaces, with $\|F_n\|_{L^2(d\mu)} = 1$. Thus $\|T_n - T\|_{\mathcal{L}_1} \leq 2\|F_n - F\|_{L^2(d\mu)} \rightarrow 0$, and the energies and diagonal norms converge. Contractivity passes (A.1) to T . For $q = \infty$, the same approximation gives $\|F_n\|_{L^\infty(d\mu)} \leq (1 + o(1))\|F\|_{L^\infty(d\mu)}$. $\square$

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