GEOMETRIC STRUCTURE OF NLS EVOLUTION

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ABSTRACT. We compare the Hamiltonian and Lagrangian approaches to nonlinear evolution equations presenting the nonlinear Schrödinger equation on the line as a simple concrete example. In particular we explain the least action principle and the Noether theorem in this context. The specific point of view, adapted from Souriau's book [10], has not been applied to nonlinear evolution equations and it offers an elegant simple presentation of these topics.

1. INTRODUCTION

The purpose of this note is to clarify the relation between the Hamiltonian and Lagrangian approaches to nonlinear evolution equations. In particular we explain the least action principle and the Noether theorem in this context. These topics are well known and are discussed in many texts – see for instance [9, Chapter 2] and [11, §1.4]. As pointed out in [6, Section 2] the presentation in the case of symmetries which mix time and space, such as the Galilean invariance (3.8), has never been clear. In [6] a presentation based on a Poisson algebra structure was given.

In this note we adapt the approach of Souriau [10] which according to him is close the orginal approach of Lagrange. A "Lagrangian one-form" (3.1), the integral of which over paths defines actions, and whose differential has one dimensional kernel, lies at the center of that approach. Although not as general as the study of currents, this approach reduces the proof of Noether's theorem and the least action principle to the "Cartan's magic formula" (3.7) and is immediately applicable to non-linear evolution equations which are Hamiltonian. The conceptual and non-computational aspect of that proof is particularly appealing.

As a simple example we consider the nonlinear Schrödinger equation (NLS) on the line, with the quintic case particularly interesting due to its large group of symmetries. The application to other equations which are Hamiltonian with respect to some symplectic structure is done along the same lines.

We comments that in the mathematics literature, as in [1],[4],[5], and in references given there, the Hamiltonian point of view is prevalent. In the physics literature, see for instance

[3],[8], the Lagrangian point of view rules with the symplectic structure largely neglected. In §5 we present one possible mathematical reason for that.

2. The Hamiltonian structure

In this section we recall well known facts about the Hamiltonian structure of the nonlinear Schrödinger equation. The same point of view applies to other evolution equations, see for instance [4] and references given there.

For simplicity we will consider the case of dimension one, and

$$V \stackrel{\text{def}}{=} H^1(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C}),$$

viewed as a *real* Hilbert space. The inner product and the symplectic form are given by

$$\langle u, v\rangle \stackrel{\text{def}}{=} \operatorname{Re} \int u\bar{v}, \quad \omega(u, v) \stackrel{\text{def}}{=} \langle u, iv\rangle = \operatorname{Im} \int u\bar{v}, \tag{2.1}$$

Let $H : V \to \mathbb{R}$ be a function, a Hamiltonian. The associated Hamiltonian vector field is a map $\Xi_H : V \to TV$, which means that for a particular point $u \in V$, we have $(\Xi_H)_u \in T_uV$. The vector field Ξ_H is defined by the relation

$$\omega(v, (\Xi_H)_u) = d_uH(v), \tag{2.2}$$

where $v \in T_uV$, and $d_uH : T_uV \to \mathbb{R}$ is defined by

$$d_uH(v) = \frac{d}{ds}\Big|_{s=0} H(u + sv).$$

In the notation above

$$dH_u(v) = \langle dH_u, v\rangle, \quad (\Xi_H)_u = \frac{1}{i} dH_u. \tag{2.3}$$

If we take $V = H^1(\mathbb{R}, \mathbb{C})$ with the symplectic form (2.1), and

$$H(u) = \int \frac{1}{4}|\partial_x u|^2 - \frac{1}{p+1}|u|^{p+1}$$

then we can compute

$$\begin{aligned} d_uH(v) &= \operatorname{Re} \int ((1/2)\partial_x u \partial_x \bar{v} - |u|^{p-1}u\bar{v}) \\ &= \operatorname{Re} \int (-(1/2)\partial_x^2 u - |u|^{p-1}u)\bar{v}. \end{aligned}$$

Thus, in view of (2.3) and (2.2),

$$(\Xi_H)_u = \frac{1}{i}\left(-\frac{1}{2}\partial_x^2 u - |u|^{p-1}u\right)$$

The flow associated to this vector field (Hamiltonian flow) is

$$\dot{u} = (\Xi_H)_u = \frac{1}{i}\left(-\frac{1}{2}\partial_x^2 u - |u|^{p-1}u \right) . \tag{2.4}$$

3. The Lagrangian point of view and the Noether theorem

According to [10] the following point of view towards dynamics goes back to Lagrange. We consider

$$\widetilde{V} = V \times \mathbb{R} = H^1(\mathbb{R}, \mathbb{C}) \times \mathbb{R} ,$$

and the following one form on $\widetilde{V}$ (we are rather informal here about dual spaces etc):

$$\alpha_{(u,t)}(v, T) \stackrel{\text{def}}{=} \frac{1}{2}\omega(u, v) - H(u)T , \quad (v, T) \in T_{(u,t)}\widetilde{V} , \quad (u, t) \in \widetilde{V} . \tag{3.1}$$

Remark. The presence of the factor 1/2 in front of ω in the definition of α is best understood using the finite dimensional analogy: if $z = x + i\xi$, $x, \xi \in \mathbb{R}^n$, then

$$\alpha = \frac{1}{2}\operatorname{Im} z d\bar{z} - H(x, \xi)dt = \frac{1}{2}(\xi dx - x d\xi) - H(x, \xi)dt . \tag{3.2}$$

We then define the differential of α:

$$\widetilde{\omega} \stackrel{\text{def}}{=} d\alpha ,$$

that is

$$\widetilde{\omega}_{(u,t)}((v_1, T_1), (v_2, T_2)) = \omega(v_1, v_2) - dH_u(v_1)T_2 + dH_u(v_2)T_2 ,$$

where we used the notation of §2. This calculation is easily understood using the analogy with (3.2):

$$d(\xi dx - x d\xi)/2 = d\xi \wedge dx .$$

Having $\widetilde{\omega}$ makes $\widetilde{V}$ a presymplectic space in the sense that $\widetilde{\omega}$ has a kernel of dimension one. Here, the kernel is

$$\ker \widetilde{\omega}_{(u,t)} \stackrel{\text{def}}{=} \left\{ (v, T) \in T_{(u,t)}\widetilde{V} \; ; \; \forall \, (v', T') \in T_{(u,t)}\widetilde{V} , \; \widetilde{\omega}_{(u,t)}((v, T), (v', T')) = 0 \right\} .$$

The following proposition replaces (2.4) with a condition related to $\widetilde{\omega} = d\alpha$:

Proposition 1. *The curve $t \mapsto u(t) \in V$ is a solution to*

$$i u_t = -\frac{1}{2}\partial_x^2 u - |u|^{p-1}u , \tag{3.3}$$

if and only if

$$(\dot{u}(t), 1) \in \ker \widetilde{\omega}_{u(t)} . \tag{3.4}$$

In other words,

$$\ker \widetilde{\omega}_u = \mathbb{R}(\Xi_u, 1)\,.$$

Proof. We already know that (3.3) is equivalent to (2.4). We then check that

$$\begin{aligned}\widetilde{\omega}(((\Xi_H)_u, 1), (v, T)) &= \omega((\Xi_H)_u, v) - \langle dH_u, T(\Xi_H)_u - v\rangle \\ &= -\langle dH_u, v\rangle - \langle dH_u, T(\Xi_H)_u - v\rangle \\ &= T\langle dH_u, (\Xi_H)_u\rangle = T\langle dH_u, (1/i)dH_u\rangle = 0\,.\end{aligned}$$

□

A special case of Noether's Theorem (see [10, (11.12)] for a more general version using the moment map) is now nicely given using this point of view:

Proposition 2. *Suppose that*

$$A(s) \;:\; (s, U) \mapsto U(s)\,, \quad s \in \mathbb{R}\,, \quad U \in \widetilde{V}\,,$$

is a one parameter group acting on $\widetilde{V}$ and preserving α:

$$A(s)^*\alpha = \alpha\,, \tag{3.5}$$

(here the pullback is given by $f^*\alpha_{(u,t)}(v, T) \stackrel{\text{def}}{=} \alpha_{f(u,t)}(f_*(v, T))$). Then

$$F(u, t) \stackrel{\text{def}}{=} \alpha_{(u,t)}\left(\frac{d}{ds}A(s)(u, t)|_{s=0}\right)\,,\ (u, t) \in \widetilde{V} \tag{3.6}$$

is conserved by the flow (3.3)*.*

Proof. In the finite dimensional case we use Cartan's formula: if $(d/ds)f_s|_{s=0} = X$ (here $f_s : \widetilde{V} \to \widetilde{V}$), then at $s = 0$,

$$\frac{d}{ds}f_s^*\alpha = d(\alpha(X)) + (d\alpha)(X, \bullet)\,. \tag{3.7}$$

If we take $f_s = A(s)$ then the left hand side is 0 and $X = (d/ds)A(s)(u, t)|_{s=0}$. The invariance of F is then equivalent to

$$d(\alpha(X))(\dot{u}, 1) = 0$$

but since

$$d(\alpha(X)) = -\widetilde{\omega}(X, \bullet)\,,$$

this follows from Proposition 1. The same argument applies formally in the case of evolution equations and can be easily verified. □

3.1. **Standard group actions.**

The basic group action to consider are

$$(u,t) \mapsto (e^{-is}u,t)\,, \quad (u,t) \mapsto (u(\bullet - s),t)\,, \quad (u,t) \mapsto (u,t-s)\,,$$

and in each case we quickly see that $A(s)^*\alpha = \alpha$. In the three cases we have

$$(d/ds)A(s)(u,t)|_{s=0} = (-iu,0)\,, \quad (d/ds)A(s)(u,t)|_{s=0} = (-u_x,0)\,,$$
$$(d/ds)A(s)(u,t)|_{s=0} = (0,-1)\,,$$

respectively, and the conserved quantities obtained using the formula (3.6) are easily seen to be

$$\int |u|^2 dx\,, \quad \mathrm{Im}\int u_x\bar{u}dx\,, \quad H(u)\,.$$

A more interesting example is given by considering the Galilean invariance:

$$A(s)(u,t) = (A_0(s,t)u,t)\,, \quad A_0(s,t)u \stackrel{\mathrm{def}}{=} e^{-its^2/2+i\bullet s}u(\bullet - st)\,. \tag{3.8}$$

We first check that (3.5) holds. In fact,

$$[A(s)_*]_{(u,t)}(v,T) = (A_0(s,t)v + \partial_t(A_0(s,t)u)T,T) = (A_0(s,t)(v-(is^2u/2+su_x)T),T)\,,$$

and hence

$$\begin{aligned}(A(s)^*\alpha)_{(u,t)}(v,T) &= \alpha_{A(s)(u,t)}([A(s)_*]_{(u,t)}(v,T)) \\ &= \omega(A_0(s,t)u, A_0(s,t)(v-(is^2u/2+su_x)T)) - 2H(A_0(s,t)u)T \\ &= \alpha_{(u,t)}(v,T)\,,\end{aligned}$$

since

$$\omega(A_0(s,t)u, A_0(s,t)v) = \omega(u,v)\,,$$

and

$$\begin{aligned}H(A_0(s,t)u) &= H(u) + s^2\langle u,u\rangle/4 + s\langle iu,u_x\rangle/2 \\ &= H(u) + s^2\omega(iu,u)/4 + s\omega(u,u_x)/2\,.\end{aligned}$$

We also see that

$$\frac{d}{ds}A(s)(u,t)|_{s=0} = (ixu - tu_x,0)\,,$$

formula (3.6) gives

$$F(u,t) = t\,\mathrm{Im}\int u_x\bar{u}dx - \int x|u|^2dx = F(u,0) = -\int x|u|^2dx = 0\,,$$

which of course corresponds to $p = mq/t$ where

$$p = \mathrm{Im}\int u_x\bar{u}dx\,, \quad q = \frac{1}{m}\int x|u|^2dx\,, \quad m = \int |u|^2dx\,,$$

are the momentum, position, and mass, respectively.

3.2. **Scaling.** Let us now consider another group action preserving solutions of (3.3) ($p > 1$):

$$(s, u, t) \longmapsto (A_0(s)u, s^{-2}t)\,, \quad A_0(s)u(\bullet) \stackrel{\text{def}}{=} s^{\frac{2}{p-1}} u(s\bullet)\,. \tag{3.9}$$

Then

$$[A(s)_*]_{(u,t)}(v, T) = A(s)(v, T)\,,$$

and

$$(A(s)^*\alpha)_{(u,t)}(v, T) = \tfrac{1}{2}\omega(A_0(s)u, A_0(s)v) - H(A_0(s)u)s^{-2}T = s^{\frac{5-p}{p-1}}(\tfrac{1}{2}\omega(u, v) - H(u)T)\,.$$

That means that the form is preserved for $p = 5$. For $p \neq 5$ we still preserve the kernel of $\widetilde{\omega} = d\alpha$ which is consistent with (3.9) preserving the solutions.

To see the invariant quantity given by Noether's theorem (formula (3.6)) for $p = 5$ we compute

$$\frac{d}{ds} A(s)(u, t)|_{s=1} = (u/2 + xu_x, -2t)\,,$$

and the conserved quantity is

$$F(u, t) = \tfrac{1}{2}\omega(u, u/2 + xu_x) + 2tH(u) = -\tfrac{1}{2}\,\mathrm{Im} \int xu_x\bar{u} + 2tH(u)\,, \tag{3.10}$$

which is a version of the virial identity, typically written

$$\partial_t \left(\mathrm{Im} \int xu_x\bar{u} \right) = 4H(u)\,. \tag{3.11}$$

3.3. **Case of $p = 5$.** Here the scaling symmetry is part of a more general scaling property:

$$u(t, x) \longmapsto (ct + d)^{-1/2} e^{\frac{icx^2}{2(ct+d)}} u\left(\frac{at + b}{ct + d}, \frac{x}{ct + d}\right)\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})\,,$$

see [7] for this and a recent study of the quintic NLS.

Motivated by this, for $g \in SL_2(\mathbb{R})$ we define the standard action on $\overline{\mathbb{R}}$:

$$g(t) = \frac{at + b}{ct + d}\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})\,.$$

Then

$$A(g) \,:\, \widetilde{V} \to \widetilde{V}\,,$$

is given as follows

$$A(g)(u, t) = (A_0(g)u, g^{-1}(t))\,, \quad A_0(g)u = (g'(t))^{-\frac{1}{4}} e^{-ig''(t)x^2/(4g'(t))} u\left((g'(t))^{\frac{1}{2}} x\right)\,.$$

Since $g'(t) = (ct + d)^{-2}$, $(g'(t))^{\frac{1}{2}} = (ct + d)^{-1}$ is well defined.

The cases of

$$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

correspond to scaling and translation with the invariant quantities already discussed.

For

$$g(s) \stackrel{\text{def}}{=} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix},$$

we obtain

$$\frac{d}{ds}(A(g(s))(u,t)|_{s=0} = ((-t/2 + ix^2/2)u - txu_x, 1 + t^2),$$

so that the conserved quantity is

$$F(u,t) = -\frac{1}{4}\int x^2|u|^2 dx + \frac{1}{2}t\,\mathrm{Im}\int xu_x\bar{u} - H(u)t^2 - H(u).$$

Since $H(u)$ is conserved and we also have (3.10) we conclude that

$$\int x^2|u(x,t)|^2 dx = \int x^2|u(x,0)|^2 dx + 2t\,\mathrm{Im}\int xu_x\bar{u} - 4H(u)t^2 \tag{3.12}$$

which is again a version of the virial identity. This version of the virial identity is usually written

$$\partial_t^2 \int x^2|u(x,t)|^2\,dx = 8H(u)$$

Two time integrations, then substituting the identity

$$\partial_t \int x^2|u|^2\,dx = 2\,\mathrm{Im}\int x\bar{u}\partial_x u\,dx$$

evaluated at $t = 0$, give

$$\int x^2|u(t)|^2\,dx = \int x^2|u|^2\,dx\Big|_{t=0} + 2\,\mathrm{Im}\int x\bar{u}\partial_x u\,dx\Big|_{t=0} t + 4H(u)t^2$$

Integrating (3.11) from 0 to t gives an expression for $\mathrm{Im}\int x\bar{u}\partial_x u\,dx\Big|_{t=0}$, which substituted here gives (3.12).

4. The least action principle

To formulate the least action principle we need to define the Lagrangian. In the last section, although we took the Lagrangian point of view, we used the form α given by (3.1). The Lagrangian,

$$\mathcal{L} : T\widetilde{V} \longrightarrow \mathbb{R},$$

is defined as follows:

$$\mathcal{L}(u,t,X,T) \stackrel{\text{def}}{=} \alpha_{(u,t)}(X,T), \quad X \in T_uV.$$

If $t \mapsto u$ is a curve in V we use a simplified notation

$$\mathcal{L}(u) \stackrel{\text{def}}{=} \alpha_{(u,t)}(\dot{u}, 1)\,. \tag{4.1}$$

For the equation (3.3) we obtain

$$\mathcal{L}(u) = \frac{1}{2}\omega(u, \dot{u}) - H(u) = -\frac{1}{2}\operatorname{Im}\int u_t \bar{u} - \frac{1}{4}\int |u_x|^2 + \frac{1}{p+1}\int |u|^{p+1}\,. \tag{4.2}$$

Action is more natural than considering Lagrangian. Let γ be a curve in $\widetilde{V}$. Then the action on γ is defined as

$$\mathcal{S}_\gamma \stackrel{\text{def}}{=} \int_\gamma \alpha\,. \tag{4.3}$$

When the curve is given by $t \mapsto (u(t), t)$ we get, in the notation of (4.1),

$$\mathcal{S}_\gamma \stackrel{\text{def}}{=} \int \mathcal{L}(u) dt\,.$$

The least action principle can be formulated as follows:

Proposition 3. *The curve $\gamma : s \mapsto (u(s), t(s))$ is critical for $\mathcal{S}_\gamma$ if and only if $\dot{\gamma}(s) \in \ker \widetilde{\omega}_{\gamma(s)}$. In other words*

$$\delta \mathcal{S}_\gamma = 0 \iff \dot{\gamma}(s) \in \ker \widetilde{\omega}_{\gamma(s)}\,. \tag{4.4}$$

Proof. We first give the proof in finite dimensions. Let γ_r be a smooth family of curves such that $\gamma_0 = 0$, and γ_r is equal to γ outside of a compact subset, disjoint from $\partial\gamma$. Being stationary means that for any such family,

$$\frac{d}{dr}\int_{\gamma_r} \alpha \;\Big|_{r=0} = 0\,.$$

Let F_r be a smooth family of diffeomorphism such that, for r small, $\gamma_r = F_r(\gamma)$, and let $X = (d/dr)F_r|_{r=0}$ be a vector field defined on γ. Then, as in the proof of Proposition 2, we use Cartan's formula:

$$\begin{aligned} \frac{d}{dr}\int_{\gamma_r} \alpha \;\Big|_{r=0} &= \frac{d}{dr}\int_\gamma F_r^* \alpha \;\Big|_{r=0} = \int_\gamma (d\alpha(X, \bullet) + d(\alpha(X))) \\ &= \int_\gamma \widetilde{\omega}(X, \bullet) + \alpha(X)|_{\partial\gamma} = \int_\gamma \widetilde{\omega}(X, \bullet)\,, \end{aligned}$$

since by the assumptions on γ_r, $X \equiv 0$ near $\partial\gamma$. This means that

$$\frac{d}{dr}\int_{\gamma_r} \alpha \;\Big|_{r=0} = 0 \implies \widetilde{\omega}_{\gamma(s)}(X_{\gamma(s)}, \dot{\gamma}(s)) = 0 \;\forall\, X \;\forall\, s\,,$$

which proves the proposition in finite dimensions.

The same formal argument applies to evolution equation and in our case we check it by a sandard direct computation:

$$\mathcal{S}(u+\delta u, t+\delta t) = \int (\omega(u+\delta u, \dot u + \delta \dot u)/2 - H(u+\delta u)(\dot t + \delta \dot t))ds\,.$$

Integrating by parts and neglecting higher order terms we obtain the first variation of $\mathcal{S}$:

$$\begin{aligned}\delta\mathcal{S} &= \int (\omega(\delta u, \dot u) - \dot t d_u H(\delta u) + d_u H(\dot u)\delta t)ds \\ &= \int \widetilde{\omega}_{(u,t)}((\delta u, \delta t), (\dot u, \dot t))ds\,,\end{aligned}$$

and this vanishes for all δu and δt if and only if $(\dot u, \dot t) \in \ker \widetilde{\omega}_{(u,t)}$. □

5. Effective dynamics

Suppose that $\widetilde{M} \subset \widetilde{V}$ is a submanifold which is *presymplectic* in the sense that

(5.1) $$\dim \ker \widetilde{\omega}|_{\widetilde{M}} = k\,, \quad \text{where } k \text{ is constant on } \widetilde{M}.$$

Then $\ker \widetilde{\omega}|_{\widetilde{M}}$ defines a foliation of $\widetilde{M}$ with leaves of dimension k. We note that the fact that $d\widetilde{\omega} = 0$ and the formula for $d\rho(X,Y,Z)$,

$$X\rho(Y,Z) - Y\rho(X,Z) + Z\rho(X,Y) - \rho([Y,Z],X) + \rho([X,Z],Y) - \rho([X,Y],Z)\,,$$

show that the $\ker \widetilde{\omega}$ satisfies the Frobenius integrability condition.

The method of collective coordinates for motion close to $\widetilde{M}$ is based on the following principle:

> Suppose that γ is critical for $\mathcal{S}$ (for instance $t \mapsto u(t)$ which satisfies (3.4) or, equivalently, (2.4)). Suppose also that γ is *close* to $\widetilde{M}$. Then it is close to a fixed leaf of the above foliation.

Here is a trivial example to illustrate this. Let $V = T^*\mathbb{R}$ and $H(x,\xi) = \xi^2/2$. Then suppose that $\widetilde{M} = \{\xi = 0\}$. In that case $\dim \ker \widetilde{\omega}|_{\widetilde{M}}$ is 3. If

$$\gamma(t) = ((x + t\epsilon, \epsilon), t)\,,$$

then it is close $\widetilde{M}$, which the only leaf of the foliation.

What one normally wants is (see [8] for examples from the physics literature and [2] for an implicit application of this principle in the mathematics literature):

> Let $\widetilde{M}$ satisfy (5.1) with $k = 1$. Suppose that γ is critical for $\mathcal{S}$. Suppose also that γ is *close* to $\widetilde{M}$. Then γ is close to a $\gamma_{\widetilde{M}} \subset \widetilde{M}$ which is critical for $\mathcal{S}_{\gamma_0}$, $\gamma_0 \subset \widetilde{M}$. In other words, we restrict the Lagrangian to the submanifold and compute the action there.

The simplest case is given by $\widetilde{M} = M \times \mathbb{R}$ with M symplectic, that is M for which $\omega|_M$ is nondegenerate. In that case the foliation is given by

$$s \longrightarrow (\exp(s\Xi_{H\,|_M}), s)\,,$$

where Ξ_p is the Hamilton vector field of a Hamiltonian p. This is very clear in finite dimensions since then, locally,

$$M = (x, \xi) : x'' = \xi'' = 0\,, \quad x = (x', x'')\,, \quad \xi = (\xi', \xi'')\,,$$

and

$$\widetilde{\omega}|_{\widetilde{M}} = (d\xi' + H_{x'}(x', 0, \xi', 0)dt) \wedge (dx' - H_{\xi'}(x', 0, \xi', 0)dt)\,.$$

However we may have situations in which $\omega|_M$ is degenerate yet $\ker \widetilde{\omega}|_{M\times\mathbb{R}}$ keeps fixed rank 1. That means that the Hamiltonian formalism is not applicable but the Lagrangian one is. Here is a simple example:

$$\begin{gathered} M = \{(x_1, 0, \xi_2^2, \xi_2)\} \subset V = T^*\mathbb{R}^2\,, \quad H(x, \xi) = x_1\,, \\ \omega|_M = 2\xi_2 d\xi_2 \wedge dx_1\,, \quad \dim \ker \omega|_{M\cap\{\xi_2=0\}} = 2 = \dim M\,, \\ \widetilde{\omega}|_{\widetilde{M}} = (2\xi_2 d\xi_2 + dt) \wedge dx_1 \quad \dim \ker \widetilde{\omega}|_{\widetilde{M}} = 1\,. \end{gathered}$$

A complicated example from the physics literature comes from [3, §5]. If one considers $M \subset V = H^1(\mathbb{R}, \mathbb{C})$ given by all sums of time-modulated solitons and time modulated nonlinear ground states (see [3, (5.1)]),

$$\begin{gathered} u(x) = u_S(x; \eta, Z, V, \phi) + u_D(x; a, \phi, \psi)\,, \\ u_S(x; \eta, Z, V, \phi) \stackrel{\text{def}}{=} \eta \operatorname{sech}(\eta x - Z) e^{iVx - i\phi}\,, \\ u_D(x; a, \phi, \psi) \stackrel{\text{def}}{=} a \operatorname{sech}\left(ax + \tanh^{-1}\left(\frac{\gamma}{a}\right)\right) e^{-i(\phi+\psi)}\,, \end{gathered} \tag{5.2}$$

then in the reduced six dimensional space described by $(\eta, Z, V, \phi, a, \psi)$ is *not* symplectic with respect to ω given by (2.1). This can be checked by computing the determinant of a matrix corresponding to $\omega|_M$ – see Fig.1. Despite that the method of collective coordinates is used by the authors in constructing an effective Lagrangian [3, (5.4)] and it is then used to obtained approximate equations of motion [3, (5.9)]. It is numerically shown to give a good agreement with the solution of the equation.

Finally we comment on the effective dynamics of solitons interacting with slowly varying potentials. In [5] we followed [1] and used a symplectic approach improving the results of [1] and [2] (same method apply to in that setting) by obtaining equations of motion without

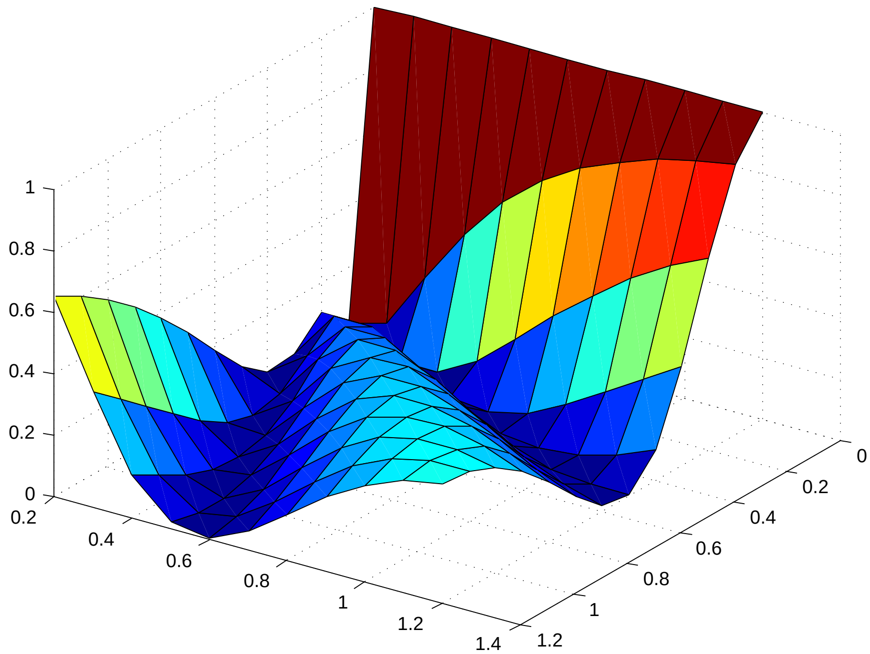

FIGURE 1. The plot of $\det(\omega|_{M\cap\{V=Z=0\}})$ for $\gamma = 0.1$ (in the notation of [3]), $\phi = 0$, $\psi = \pi/4$, $0 < \eta < 1.2$ and $0.11 < a < 1.2$. The two lines along which $\omega|_{M\cap\{V=Z=0\}}$ is degenerate are clearly visible. The restriction to $Z = V = 0$ is not essential since these two variables are essentially conjugate.

errors and obtaining a better accuracy of approximation by a moving soliton ($h \to h^2$, where h is slowness parameter of the potential). When attempting to reproduce these results using the Lagrangian formalism we could obtain the same equations of motions (we later learned that they were implicit in [8]), but could not obtain the $h \to h^2$ improvement.

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