A Remark on Isopolar Potentials

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1. In the approach to scattering by compactly supported perturbations introduced by Lax and Phillips [3] the scattering poles, or resonances, might be considered as the natural analogue of the discrete spectrum of operators on compact manifolds. One could then ask to what extent do the resonances determine the scatterer and consider the 'isopolar' rather than the isospectral problem. I would like to make a simple remark concerning the Schrödinger operator on the line.

Different situations could also be considered. In hyperbolic scattering, using a modification of Sunada's construction, Zelditch [7] and Bérard [1] obtained examples of 'isopolar manifolds'. We recall that in the finite volume case, hyperbolic scattering can be considered as a type of scattering by a compactly supported perturbation of a half-line – see for instance Example 1.1 in [5] and references given there.

2. We consider the Schrödinger operator

$$-\frac{d^2}{dx^2}+p(x), \quad x\in\mathbb{R},$$

we shall follow the notation from [4]. Our conclusion is a simple consequence of standard facts about inverse scattering. We shall be concerned with real even potential p in $L^1_{\mathrm{comp}}(\mathbb{R})$ which is equivalent to considering scattering on a half line, or the lowest angular momentum scattering by a radially symmetric potential in $\mathbb{R}^3$. The scattering matrix has the form

$$S_p(\lambda)=\begin{bmatrix} i\lambda/\widehat{X}(\lambda) & \widehat{Y}(\lambda)/\widehat{X}(\lambda) \\ \\ \widehat{Y}(-\lambda)/\widehat{X}(\lambda) & i\lambda/\widehat{X}(\lambda)\end{bmatrix}$$

The potential p is said to have a half bound state at 0 if $\widehat{X}(0)=0$. The scattering poles, or resonances, are given by the zeros of $\widehat{X}$ in $\Im\lambda<0$ but we will also include in that set the square roots of the negative eigenvalues of $-d^2/dx^2+p(x)$, which correspond to the zeros of $\widehat{X}$ in $\Im\lambda>0$. Other than at $\lambda=0$, $\widehat{X}$ cannot vanish on the real axis.

Proposition *Let $p\in L^1_{\mathrm{comp}}(\mathbb{R})$ be even. Then, if p does not have a half-bound state at 0, the scattering poles determine p uniquely. If p has a half-bound state, then there exists precisely one $p_1\in L^1_{\mathrm{comp}}(\mathbb{R})$, $p_1\neq p$, with the same set of scattering poles.*

Proof: Since the scattering matrix determines a compactly supported potential uniquely (see for instance Proposition 8, [8]) we only need to show that the poles determine $S_p(\lambda)$, i.e. $\widehat{X}$ and $\widehat{Y}$. Let us first assume that $\widehat{X}(0)\neq 0$. We use the facts about the properties of zeros of Fourier transforms from [6], extended to Fourier transforms of distributions as in Lemma 1 of [8]. If the support of $X\in\mathcal{E}'$ is $[-4a,0]$ (where in fact $2a$ is the diameter of the support of p) then the Carleman formula and the bound $\mathcal{O}(r)$ on the number of poles give, as in Theorem 6 of [6]:

$$\widehat{X}(\lambda)=e^{2a\lambda}\widehat{X}(0)\prod_{\text{poles}}\left(1-\frac{\lambda}{\lambda_j}\right), \quad |\lambda_1|\leq\cdots|\lambda_j|\leq\cdots.$$

The density of the poles is given by $4a/\pi$ so the exponential factor is fully determined. Since $X - \delta' - C\delta \in L^1$, the factor $\widehat{X}(0)$ is determined uniquely – multiplication of $\widehat{X}(0)$ by a constant would destroy the property $\widehat{X}(\lambda) - i\lambda - C \to 0$, $\lambda \to \pm\infty$. It remains to construct $\widehat{Y}(\lambda)$. For that we recall the unitarity relation

$$\widehat{X}(\lambda)\widehat{X}(-\lambda) = \lambda^2 + \widehat{Y}(\lambda)^2, \tag{1}$$

where $\widehat{Y}(\lambda) = \widehat{Y}(-\lambda)$, as p is even. Thus $\widehat{Y}(\lambda) = \pm(-\lambda^2 + \widehat{X}(\lambda)\widehat{X}(-\lambda))^{\frac{1}{2}}$. To determine the branch we recall that the definition on $\widehat{X}$ and $\widehat{Y}$ (see [4] or [8]) implies $\widehat{Y}(0) = -\widehat{X}(0) \neq 0$.

When $\widehat{X}(0) = 0$, the unitarity relation (1) shows that the zero is of order one. Thus we can apply the above argument to $\widehat{X}(\lambda)/\lambda$. Now, however, there is no obstruction to the choice of either branch in the formula for $\widehat{Y}$ and we obtain two different scattering matrices. $\square$

We conclude with a few remarks. The restriction to compactly supported potentials is natural if one wants to define resonances globally. However, if the potential decays faster than any exponential then $\widehat{X}$ is still holomorphic in $\mathbb{C}$ and again the resonances are globally defined as its zeros. In fact, if $p \in C^\infty(\mathbb{R})$ and $p(x) = \mathcal{O}(\exp(-A|x|^{1+a})$ then the number of resonances of p in a disc of radius r, $N_p(r)$, satisfies

$$N_p(r) = \mathcal{O}(r^{\frac{1+a}{a}}).$$

This can be seen from the methods of [4] by showing that $X(y) = \delta'(y) - \frac{1}{2}(\int p(z)dz)\delta(y) + \mathcal{O}(\exp(-2^{-2(1+a)}A|y|^{1+a}))$. But, unlike in the compactly supported case (see [8]), the existence of asymptotics and the structure of the Hadamard factorization of $\widehat{X}$ are not clear and one misses up to $[(1+a)/a]+1$ constants needed for the recovery procedure in the proof of the proposition.

The various methods of constructing potentials with the same scattering matrix (see for instance [2]) certainly yield isopolar potentials if one starts with, say, a compactly supported potential (so that the scattering matrix is globally meromorphic) but they immediately destroy the compact support property or in fact the super-exponential decay (see Proposition 8 of [8]).

Remark 1. I wrote this paper in 1988 and revised it in 1992. I decided to publish it now in view of renewed interest in one dimensional resonance problems [9],[11],[12],[13].

For other relevant developments in the theory of resonances I refer to the bibliography of [14] and to [10].

Remark 2. Determining potentials on a half line from their resonances was recently studied by Kargaev and Korotyaev [12]. Some results on that can also be obtained by the method above: $\hat{X}(\xi)\hat{X}(-\xi) = (-i\xi + \hat{Y}(\xi))(i\xi + \hat{Y}(\xi))$ and the poles for the Neumann and Dirichlet problems are given by zeros of the two factors on the right, respectively. Proceeding as in the proof of the proposition shows that we can determine $\hat{Y}$ from either set, and hence the zeros of $\hat{X}$, and hence $\hat{X}$ itself. The additional information of distinguishing Dirichlet and Neumann spectra eliminates now the ambiguity in the case of a zero resonance. Amusingly, when a zero resonance is present, then the two potentials in Proposition above have their Dirichlet and Neumann resonances interchanged.

Kargaev and Korotyaev also pointed out a mistake in [8, Proposition 8]: the compactness of the support of a potential cannot be concluded from the transmission coefficient alone. That can however be done when the potential is even and that is all that is relevant in this note.

References

[1] P. Bérard. *Transplantation et isospecralité I.* Math. Ann. **292** (1992), 547-559.

[2] P.A. Deift. *Application of a commutation formula.* Duke Math. J. **45**(2)(1978), 267-310.

[3] P. Lax and R. Phillips. *Scattering theory*, Academic Press, 1967.

[4] A. Melin. *Operator methods for inverse scattering on the real line.* Comm. P.D.E. **10**(1985), 677-766.

[5] J. Sjöstrand and M. Zworski. *Distribution of scatering ples near the real axis.* Comm. P.D.E. **17**(5&6), 1021-1035 (1992).

[6] E.C. Titchmarsh. *The zeros of certain integral functions* Proc. London Math. Soc. **25**(1926), 283-302.

[7] S. Zelditch. *Kuznecov formulæ and Szego limit formulæ on manifolds.* Comm. P.D.E. **17**(1992), 221-260.

[8] M. Zworski. *Distribution of poles for scattering on the real line.* J. of Funct. Anal. **73**(2)(1987), 277-296.

[9] R. Froese. *Asymptotic distribution of resonances in one dimension,* J. Diff. Eqns. **137**(1997), 251-272.

[10] A. Hassell and M. Zworski. *Resonant rigidity of S^2.* J. of Funct. Anal. **169**(1999), 604-609.

[11] M. Hitrik. *Bounds on scattering poles in one dimension,* Comm. Math. Phys. **208**(1999), 381-411.

[12] P. Kargaev and E. Korotyaev. *Inverse resonance scattering and conformal mappings for Schrödinger operators,* preprint, 2000.

[13] B. Simon. *Resonances in one dimension and Fredholm determinants,* preprint, 2000.

[14] M. Zworski. *Resonances in physics and geometry.* Notices Amer. Math. Soc. **46**(1999), 319-328.

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