DIMENSION OF THE LIMIT SET AND THE DENSITY OF RESONANCES FOR CONVEX CO-COMPACT HYPERBOLIC SURFACES.

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1. Introduction and statement of results

The purpose of this paper is to show how the methods of Sjöstrand for proving the geometric bounds for the density of resonances [28] apply to the case of convex co-compact hyperbolic surfaces. We prove that the exponent in the Weyl estimate for the number of resonances in subconic neighbourhoods of the continuous spectrum is related to the dimension of the limit set of the corresponding Kleinian group.

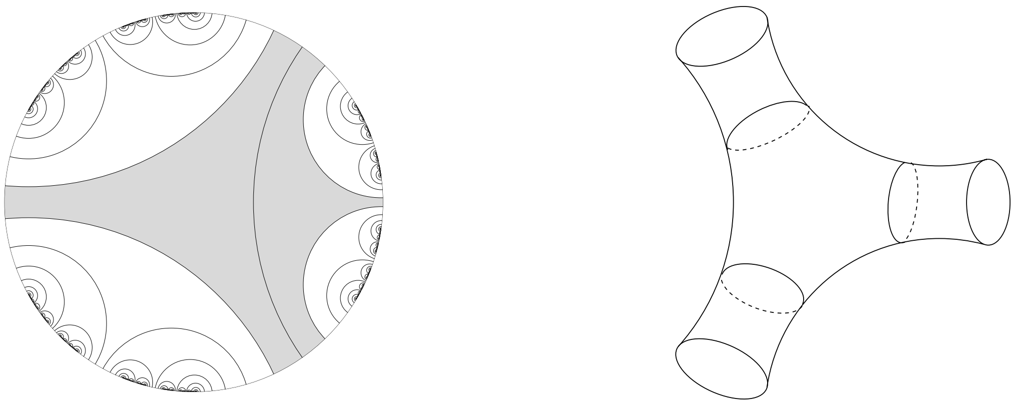

FIGURE 1. Tesselation by the Schottky group generated by inversions in three symmetrically placed circles each cutting the unit circle in an 110^o angle, with the fundamental domain of its subgroup of direct isometries and the associated Riemann surface. The dimension of the limit set is $\delta = 0.70055063\ldots$.

The convex co-compact surfaces can be defined as the infinite volume finite geometry quotients, $X = \Gamma\backslash\mathbb{H}^2$, for which the group Γ has only hyperbolic elements. The limit set of Γ, $\Lambda(\Gamma) \subset \partial\mathbb{H}^2$, is classically defined as the set of accumulation points of orbits of hyperbolic elements. As we will recall below it is naturally related to the dynamically defined *trapped set*, K, of X, $K \subset T^*X \setminus 0$, [23],[33]. The convex co-compact condition can also be formulated as saying that the projection of the convex hull of the limit set from $\mathbb{H}^2$ to $\Gamma\backslash\mathbb{H}^2$ is compact. Resonances of X are equivalently given as either the poles of the meromorphic continuation of the resolvent, $R_X(s) = (\Delta_X - s(1-s))^{-1}$, the scattering matrix of X, or of the Eisenstein series – see Sect.3 below and [24],[12]. They constitute the natural replacement of the discrete spectral data for problems on non-compact domains. Roughly speaking, if an eigenvalue gives the frequency of a state then the real of part of a resonances describes its frequency and the imaginary part its rate of decay. Hence the resonances close to the real axis are perhaps most interesting as they live the longest.

Theorem. *If $X = \Gamma\backslash\mathbb{H}^2$ is a finite geometry non-compact surface such that Γ has only hyperbolic elements and if $\mathcal{R}_X$ denotes the set of resonances of X included according to their multiplicities, then for any α,*

$0 \leq \alpha \leq 1$, and $a, b > 0$ we have

$$\sharp \left\{ s \in \mathcal{R}_X \ : \ \frac{1}{2} - \operatorname{Re} s < a |\operatorname{Im} s|^\alpha + b \,, \ |s| \leq r \right\} = \mathcal{O}\left(r^{1+\alpha+\delta(1-\alpha)} \right) \,, \quad \delta = \dim \Lambda(\Gamma) \tag{1.1}$$

where $\Lambda(\Gamma)$ is the limit set of Γ.

The most subtle part of the proof of this theorem comes almost directly from [28] (with some modifications from [30],[32],[29]). Interesting new difficulties are present however because we are working on a manifold and, more seriously, because of the more complicated structure of infinity which prevents an application of the general results of Helffer and Sjöstrand [15]. This second difficulty appear intractable at the moment for arbitrary conformally compact analytic manifolds (in the sense of Mazzeo and Melrose [20]). In the special case of dimension two a detailed analysis of a neighbourhood of infinity based on separation of variables and a delicate "gluing" argument allow nevertheless a solution to this problem.

The dimension, δ, of the limit set was related to the spectral properties of X in the work of Patterson [23] and then of Sullivan [33]. They proved that if $\delta > \frac{1}{2}$ then the ground state energy of X, that is the lowest eigenvalue of the Laplacian Δ_X, is equal to $\delta(1-\delta)$. More generally, they showed that the abscissa of convergence of the Poincaré series

$$\sum_{\gamma \in \Gamma} \exp(-s d_{\mathbb{H}^2}(z, \gamma z')) \quad z, z' \in \mathbb{H}^2$$

is given by δ. In the language of resonances we could restate it as saying that the meromorphic continuation of $R_X(s) = (\Delta_X - s(1-s))^{-1}$ is pole free *precisely* for $\operatorname{Re} s > \delta$.

As proposed by Sullivan, X can be viewed from the dynamical point of view as a hyperbolic manifold of dimension $1 + \delta$. If the resonances in a strip are considered as analogous to eigenvalues of compact surfaces then their Weyl estimate given by the Theorem above is $\mathcal{O}(r^{1+\delta})$. This is opposed to the usual estimate $\mathcal{O}(r^2) = \mathcal{O}(r^{\dim X})$ and is a quantum reflection of the dynamical point of view.

The number δ is also equal to the entropy of X which appears in the asymptotics of the counting function of the lengths of primitive closed geodesics on X, $\pi_X(l)$:

$$\pi_X(l) = \frac{e^{\delta l}}{\delta l} (1 + o(1)) \,, \quad l \longrightarrow \infty \tag{1.2}$$

see [8],[9],[18]. The dynamical zeta function of X is defined for $\operatorname{Re} s \gg \frac{1}{2}$ as

$$Z_X(s) = \prod_{\substack{\{\gamma\} \\ \gamma \in \Gamma}} \prod_{n=0}^{\infty} (1 - \exp(-(s+n) l(\gamma))) \tag{1.3}$$

where $\{\gamma\}$ denotes the conjugacy class of a promitive element $\gamma \in \Gamma$ and $l(\gamma)$ is the minimal displacement length of $\{\gamma\}$, that is the length of the primitive closed geodesic corresponding to $\{\gamma\}$. In view of the recent results of Patterson and Perry [25] combined with the identification of the divisors in [12], the Theorem above provides an upper bound on the number of zeros of the dynamical zeta function in a neighbourhood of $\operatorname{Re} s = \frac{1}{2}$.

It is perhaps possible that the Theorem above could follow directly from the asymptotics of $l(\gamma)$ given (1.2) and the analysis of the meromorphic continuation of Z_X given by (1.3). Considering the difficulties involved in proving the existence of this meromorphic continuation that seems rather unlikely. The relation between the density of zeros of the dynamical zeta function and the entropy (the exponent δ in (1.2)) could however be studied in greater generality.

Finally we would like to describe the existent lower bounds for the number of resonances of hyperbolic surfaces. For a more general class of two dimensional manifolds (any surface with finite or infinite volume hyperbolic ends), Guillopé and the author [10],[12] proved global upper and lower bounds with optimal exponents:

$$r^2 / C \leq \sharp \{ s \in \mathcal{R}_X \ : \ |s| < r + C \} \leq C r^2 \,. \tag{1.4}$$

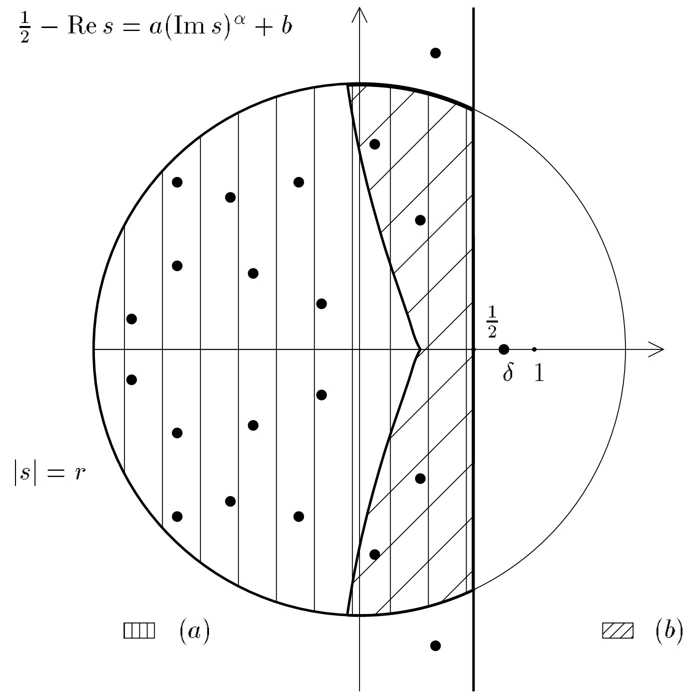

FIGURE 2. Density of resonances in different regions: (a) the number of resonances is $\geq r^2/C$; (b) the number of resonances is $\leq r^{1+\alpha+(1-\alpha)\delta}$.

Hence the density of resonances in discs is much higher ($\geq r^2/C$) than the density in subconic neighbourhoods of the continuous spectrum, $\mathrm{Re}\, s = \frac{1}{2}$ ($\leq Cr^{2-\epsilon}$, where by the Theorem above, $\epsilon > 0$ depends on the neighbourhood).

For the exact quotients of the type considered in this paper a basic bound was derived in [31]:

$$\sum_{\substack{|\,\mathrm{Im}\, s| < r \\ \mathrm{Re}(s-\frac{1}{2}) < \rho \log\langle s \rangle}} e^{-(d-\epsilon)(\frac{1}{2} - \mathrm{Re}\, s)} > \left(\frac{1}{2\pi \sinh(d/2)} \sum_{ml(\gamma)=d} l - o_\epsilon(1) \right) r \,. \tag{1.5}$$

This bound implies that the density of resonances in logarithmic neighbourhoods of $\mathrm{Re}\, s = \frac{1}{2}$ is at least r. A modification of the argument of Ikawa [16] shows also that there are infinitely many resonances in some strip $\frac{1}{2} > \mathrm{Re}\, s > \frac{1}{2} - \alpha$. A little more is true: for large enough strips the density of resonances cannot be less than $r^{1-\epsilon}$, $\epsilon > 0$ – see [13].

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2. **Outline of the proof**

The purpose of this section is to provide a broad outline of the proof and a general description of two methods which are the starting point of this paper: the method of complex scaling and Sjöstrand's geometric estimates on the number of resonances.

As stated in the Theorem in Sect.1 our goal is to obtain fine estimates for the number of resonances near the continuous spectrum. Unlike in previous works on hyperbolic scattering (see [24],[20],[9],[10],[5] and references given there) it is not enough to characterize resonances using the analytic Fredholm theory but we need an *effective meromorphic continuation.* In euclidean scattering it is provided by the method of complex scaling which has a long tradition following the original work of Aguilar-Combes [1] and Balslev-Combes [2] – see [14] and references given there. Roughly speaking it can be described as follows.

Let $V(r)$ be an analytic potential on $[0,\infty)$ (we will take, say, Dirichlet boundary conditions) and let us assume that V continues to a cone around the half-line and decays uniformly at infinity. We would like to define resonances as the poles of the meromorphic continuation of $(D_r^2+V(r)-z)^{-1}:\mathcal{C}_{\rm c}^\infty\to\mathcal{C}_{\rm loc}^\infty$ from $\operatorname{Im} z>0$ to $\operatorname{Im} z<0$ through the continuous spectrum. This of course requires establishing the existence of the meromorhic continuation. To achieve it we want to deform the operator $P=D_r^2+V(r)$ to an operator P_θ for which $P_\theta-z$ will be Fredholm when z is in a conic neighbourhood the continuous spectrum. The Fredholm property is guaranteed by local ellipticity (which of course suffices on a compact manifold) and by uniform ellipticity near infinity, that is by the non-vanishing of the symbol. In the case of the original operator, the symbol $\rho^2+V(r)-z$ can vanish when z is real so we cannot have a Fredholm property uniformly across the continuous spectrum. To gain ellipticity we deform the half-line to a contour, Γ_θ in $\mathbb{C}$ which coincides with $\mathbb{R}$ on a compact interval and eventually becomes a half line $a+e^{i\theta}\mathbb{R}$ – see Fig.5. We can think of P as the restriction of the holomorphic operator $D_w^2+V(w)$ to $[0,\infty)$. We then define P_θ as the restriction of $D_w^2+V(w)$ to Γ_θ.

The gist of the complex scaling method lies in two observations. First, the decay of V at infinity implies that the symbol of $P_\theta-z$ is essentially $e^{-2i\theta}\rho^2-z$ near infinity and hence it does not vanish for z in a cone. Thus $P_\theta-z$ is Fredholm and $(P_\theta-z)^{-1}$ is meromorphic. Second, if $f\in\mathcal{C}_{\rm c}^\infty$ is supported in the region where Γ_θ coincides with $\mathbb{R}$ and z is not a pole of $(P_\theta-z)^{-1}$ then to solve $(P-z)u=f$ we can solve $(P_\theta-z)u_\theta=f$ and take u to be the restriction of the holomorphic continuation of u_θ (which exists as u_θ solves a holomorphic equation, see Sect.4) to $[0,\infty)$. This provides the meromorphic continuation of $(P-z)^{-1}$ with the poles given by the eigenvalues of P_θ. The price one pays is the non-self-adjointness of P_θ but the compensation lies in being able to use microlocal techniques on P_θ directly.

It is convenient to work semi-classically, that is, to look at $-h^2\Delta_X-z$, where h is a small parameter. Hence we can work on a compact set in the spectral parameter z: letting $h\to0$ provides the asymptotic information. Also, the semi-classical *régime* allows us to localize all work in the phase-space to a neighbourhood of some fixed energy surface.

We would like to apply the method of complex scaling in the case of hyperbolic surfaces with infinite volume ends. Near infinity the Laplacian, $-h^2\Delta_X$, is roughly of the form $(hD_r)^2+e^{-2r}(hD_y)^2$ where D_y is the derivative on a circle (see Sect.3). The symbol is $\rho^2+e^{-2r}\eta^2$ and we easily see that the method described above cannot work uniformly in η due to oscillations produced by the exponential term when continued holomorphically. Hence we have to deform in a way which depends on η. When $e^{-2r}\eta^2$ is large we do not need to deform as that term provides ellipticity. So, we start moving into the complex plane at the turning point $r\sim\log\eta$ – see Fig.5. This could perhaps be done uniformly by considering deformation in phase space which was developed in the euclidean case by Helffer-Sjöstrand [15]. However as their method does not apply directly in our case, we take a more naïve appproach and define the deformed space using separation of variables. That is, we replace η by the eigenvalues of hD_y on the circle – see Sect.4 and 5. The effect of the modified complex scaling can be described as follows: when $|hD_y|<2A$ we perform the usual scaling in r independent of hD_y and we achieve uniform ellipticity for large r (Lemma 4.1) and if $A/2<|hD_y|<A$ for all r (Lemma 4.2). When $|hD_y|\geq 2A$ we perform the modified scaling from the

turning point only and get uniform ellipticity for all r (Lemma 4.3). Here A is some large fixed constant and the uniform ellipticity is meant in the semi-classical case: it implies invertibility for h small enough.

This provides an abstract space and an abstract non-self-adjoint operator whose eigenvalues are identified with resonances. Moreover, since we deformed only near infinity, both the space and the operator remain unchanged over the Fenchel-Nielsen region[1], the interesting, compact part of the surface. It is therefore natural to expect that the methods developed in the euclidean case should work there in a similar way. Some technical modifications are of course required since we are on a manifold.

Sjöstrand's method [28] for estimating resonances of semi-classical operators on $\mathbb{R}^n$ in terms of dimension of the trapped set of the flow of their symbols is based on his work with Helffer [15] and can be, very roughly, described as follows. In some sense it is a finer version of complex scaling. The operator is now deformed also in the compact region so that $p_t - z$, where p_t is the symbol of the deformed operator, P_t, is uniformly elliptic on as large a subset of the phase space as possible. Previously we were only concerned with Fredholm properties and for that uniform ellipticity was needed only near infinity. The deformation is performed microlocally and it can be thought of as a conjugation by a microlocal weight $\exp(tG(x,D)/h)$. The symbol of the conjugated operator is given by $p_t \sim p - itH_pG$ (see Lemma 8.3) and since we are interested in gaining some ellipticity of $p_t - z$ for $\operatorname{Im} z \geq -1/C$ we would like to make H_pG as large as possible. That is, we would like G to grow along the flow of H_p. Such a function is called an *escape function* and $H_pG > 0$ can be achieved away from the *trapped set*, K, of the flow – see Sect.3. When the flow is hyperbolic then one can get the optimal statement: $H_pG \sim d(\bullet, K)^2$ - see Sect.6.

For a self-adjoint semi-classical operator $p(x, hD)$ the number of its eigenvalues in $[a, b]$ is approximately given by the volume of the set $\{(x,\xi) : a \leq p(x,\xi) \leq b\}$ multiplied by h^{-n} (and a universal constant). For the non-self-adjoint operator $P_t = p_t(x, hD)$, we consequently expect that the number of eigenvalues in $[a,b] - i[0, Ch^{1-\alpha}]$ is bounded by

$$Ch^{-n}\mathrm{vol}\left(\{(x,\xi) \,:\, a \leq p(x,\xi) \leq b\,,\ tH_pG(x,\xi) \leq Ch^{1-\alpha}\}\right) \sim h^{-n}h^{\frac{1}{2}(1-\alpha)(2n-m)}\,,$$

where m is the Minkowski dimension of K (see Sect.3) since $H_pG((x,\xi)) \sim d((x,\xi), K)^2$. This intuition is at least partly justified using Weyl inequalities for characteristic values – see Sect.8 where we followed [28],[30],[32].

To implement this strategy in our case we need to connect the abstract complex scaling construction near infinity with the geometric construction of the operator P_t. We start by observing that when we work near a fixed energy surface, the bicharacteristics where the symbol of $(hD_y)^2$ can get large are far away from the cotangent bundle over the Fenchel-Nielsen region – see Fig.4. For $(hD_y)^2$ bounded by some large constant the scaling was independent of hD_y and can be interpreted geometrically as a deformation of the hyperbolic funnel inside its Grauert tube. More precisely, we can identify a part of the abstract deformed space (corresponding to $|hD_y| < A$) with a part of L^2 of a totally real submanifold of the Grauert tube of X – see the last part of Sect.5. Since outside that part[2] of the space and close to infinity we have uniform invertibility guaranteed by the scaling we only need to consider a compact subset of the totally real submanifold denoted by X_θ. It is there that we carry out the geometric construction. The escape function outside the Fenchel-Nielsen region needs to satisfy $H_pG > 1/C$ only for bounded r and η (symbol of hD_y) since otherwise the scaling provides the needed ellipticity. Hence we can cut it off rather crudely outside of that region. The deformed manifold X_θ is not real analytic but as it is close to the real analytic manifold X inside its Grauert tube, we can still define an FBI transformation with good analytic continuation properties. It allows a definition of spaces deformed using the cut-off escape function which are deformations of the abstract spaces obtained using complex scaling. This second deformation involves only an h-dependent change of norms and hence the eigenvalues remain unchanged. We can then apply a modification of Sjöstrand's method outlined above.

[1] That set is defined as the the projection of the convex hull of the limit set from $\mathbb{H}^2$ to $\Gamma\backslash\mathbb{H}^2$.

[2] This is admittedly rather vague: in Sect.5 and 8 we do not use subspaces and orthogonal projections but pseudo-differential cut-offs

3. **Preliminaries**

In the general discussion we will use the upper half plane model, $\mathbb{H}^2 = \{z = x + iy \; : \; x \in \mathbb{R}, y > 0\}$ with the hyperbolic metric $g_{\mathbb{H}^2} = y^{-2}(dx^2 + dy^2)$. We will denote by Γ a discrete subgroup of $SL(2, \mathbb{R})$ which is the group of hyperbolic isometries of $\mathbb{H}^2$. We will assume that Γ is of finite geometric type and that it contains no parabolic or elliptic elements (see [3],[23]). Hence each element of Γ can be conjugated to a dilation:

$$\gamma \in \Gamma \longrightarrow \exists \alpha \in SL(2, \mathbb{R}) \quad \alpha\gamma\alpha^{-1} \; : \; z \longmapsto e^{\ell(\gamma)} z \, .$$

A classical set of examples is provided by Schottky groups – see [19], Sect.X.H and Fig.1.

Under this assumptions the quotient $X = \Gamma \backslash \mathbb{H}^2$ with the metric g induced by $g_{\mathbb{H}^2}$ is a non-compact, finite geometry type Riemannian surface with infinite volume. We recall from Sect.1 that the set of accumulation points of the orbits of Γ is contained in $\partial \mathbb{H}^2$ (as is clear from the discreteness and the discontinuous action of Γ) and is called the limit set of Γ, $\Lambda(\Gamma)$. The surface X is convex co-compact in the sense that the projection of the convex hull of $\Lambda(\Gamma)$ from $\mathbb{H}^2$ to X is compact. That projection is called the Fenchel-Nielsen region and it will be denoted by N. The surface X can then be smoothly decomposed as follows

$$X = N \cup Y_1^0 \cup \cdots \cup Y_n^0 \tag{3.1}$$

where $\partial N = \partial Y_1^0 \cup \cdots \cup \partial Y_n^0$ and

$$Y_j^0 \simeq [0, \infty)_r \times (\mathbb{R}/\ell_j \mathbb{Z})_y \; , \; g\restriction_{Y_j^0} \simeq dr^2 + \cosh^2 r dy^2 \, . \tag{3.2}$$

We note that each boundary component of N is a geodesic on X.

To recall the basic properties of the geodesic flow on X we will also denote by $g \in \mathcal{C}^\infty(T^*X; \mathbb{R})$, the quadratic form dual to the metric. In particular, g is the symbol of the Laplace-Beltrami operator of (X, g), Δ_X. The geodesic flow lifted to the cotangent bundle is given by

$$\exp tH_g \; : \; T^*X \longrightarrow T^*X \, ,$$

where, as usual, H_g is the Hamilton vector field of g: $\omega(\bullet, H_g) = dg$ with ω, the canonical symplectic form on T^*X. The flow is hyperbolic (Anosov) in the following strong sense:

$$\begin{gathered} \exists C \; \forall m \in S^*X \; \exists E_m^\pm \;\; E_m^0 \subset T_m(S^*X) \, , \; \dim E_m^\pm = \dim E_m^0 = 1 \, , \; E_m^+ + E_m^- + E_m^0 = T_m(S^*X) \, , \\ d(\exp tH_g)_m(E_m^\bullet) = E^\bullet_{\exp tH_g(m)} \, , \; H_g(m) \in E_m^0 \, , \\ \|d(\exp tH_g)_m(v_\pm)\| \leq C \exp(\pm Ct) \|v_\pm\| \, , \; v_\pm \in E_m^\pm \, . \end{gathered} \tag{3.3}$$

Here $S^*X = \{m \in T^*X : g(m) = 1\}$ is the cosphere bundle.

We also note that if $\pi : T^*X \to X$ is the natural projection then

$$\pi(m) \notin \overline{N} \implies \pi(\exp tH_g(m)) \notin \overline{N} \; \text{ for either all } t > 0 \text{ or for all } t < 0 \, . \tag{3.4}$$

The trapped set, K, defined by

$$\begin{gathered} K = \Gamma_+ \cap \Gamma_- \, , \\ \Gamma_\pm = \{m \in T^*X \setminus 0 \; : \; \exists T(m) \; \forall \mp t > T(m) \quad \pi\left(\exp tH_g(m)\right) \in N\} \end{gathered} \tag{3.5}$$

is, in view of (3.4), contained in $T^*\overline{N} \setminus 0$. Using the Patterson-Sullivan measure on $\Lambda(\Gamma)$ (see [23, 33]) Sullivan constructed an invariant measure on K and his construction shows that

$$\dim K = 2 \dim \Lambda(\Gamma) + 2 \, , \tag{3.6}$$

where dim denotes the Hausdorff dimension. As shown in [33] the Hausdorff dimension coincides in this case with the Minkowski dimension used in [28]: for L a bounded set in $\mathbb{R}^m$ we put

$$\dim L = m - \sup \{d \; : \; \overline{\lim_{\epsilon \to 0}} \epsilon^{-d} (\text{vol} \, \{x \in \mathbb{R}^m \; : \; d(x, L) < \epsilon\}) < \infty\} \, . \tag{3.7}$$

Since K is homogeneous we have

$$\dim K = \dim \widetilde{K} + 1 \, , \; \widetilde{K} = K \cap S^*X \, ,$$

where $\widetilde{K}$ is compact. Hence, using local coordinates, the definition (3.7) is applicable to $\tilde{K}$. In addition, it is known that $\Lambda(\Gamma)$ and $\widetilde{K}$ are always of *pure dimension*, that is the supremum in the definition (3.7) is achieved or, in other words, that $\mathcal{H}^{\delta}(\Lambda(\Gamma)) < \infty$ – see [33].

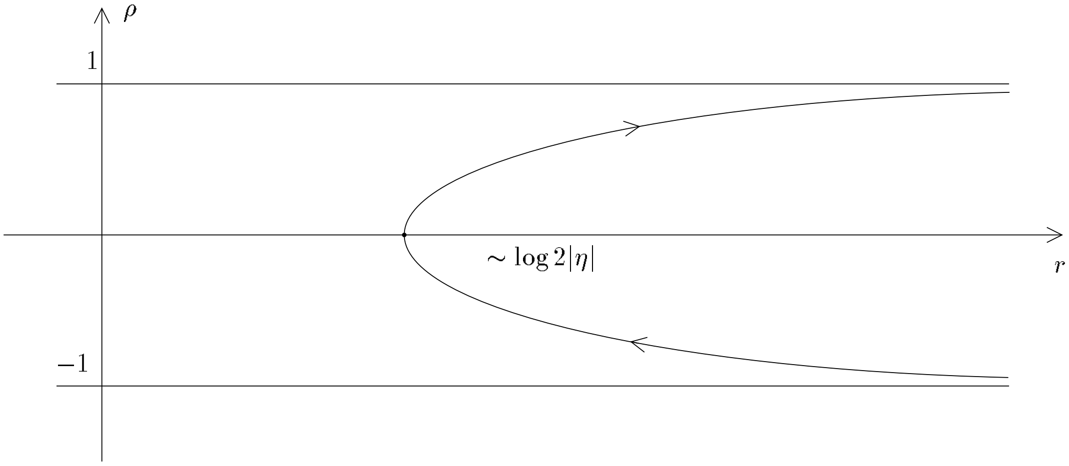

FIGURE 3. A geodesic on S^*Y^0 projected to the (r, ρ) coordinates.

In the funnels Y_j^0, the metric $g\restriction_{Y_j^0} \in \mathcal{C}^\infty(T^*Y_j^0; \mathbb{R})$ takes the explicit form $g\restriction_{Y_j^0} = \rho^2 + (\cosh r)^{-2}\eta^2$, where (ρ, η) are the dual variables to (r, y) in (3.2). A lifted geodesic on S^*X is shown in Fig.3 and it is clear that

$$r > 0\,,\ \rho^2 \leq 1\,,\ |\eta| > 1 \implies \forall t \in \mathbb{R} \ \ \pi(\exp tH_g(r, y; \rho, \eta)) \notin N\,. \tag{3.8}$$

The dynamical structure of X is shown schematically in Fig.4: the trajectories near infinity have fixed hyperbolic energy, that is the energy given by the Hamiltonian $g \in \mathcal{C}^\infty(T^*X; \mathbb{R})$ but they have large cylindrical energy, that is the energy given by the cylindrical metric on Y_j^0, $\rho^2 + \eta^2$. They do not affect the resonance phenomena.

We will now review briefly some basic scattering theory of X referring to [24],[9] and [12] for detailed presentations.

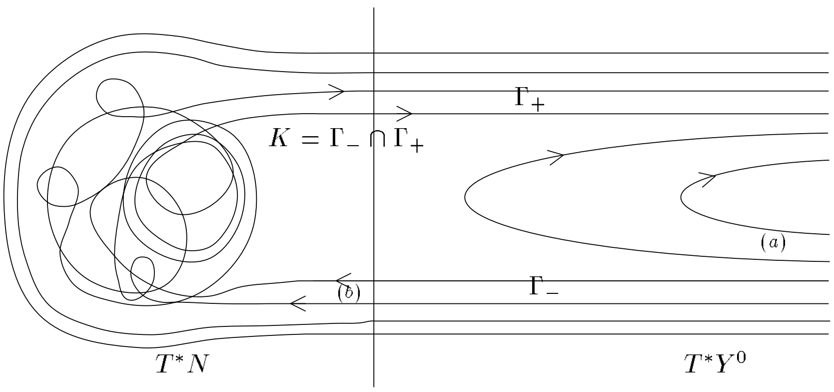

FIGURE 4. Schematic representation of the dynamics in T^*X: (a) trajectories near infinity with bounded hyperbolic energy but large cylindrical energy, (b) escaping trajectories close to the trapped set K: bounded hyperbolic and cylindrical energy.

Let Δ_X be the Laplace-Beltrami operator of (X, g) acting on functions. As a positive, unbounded self-adjoint operator on $L^2(X, d\mathrm{vol}_g)$ with the domain $H^2(X, d\mathrm{vol}_g)$, its spectrum, spec (Δ_X), is given by

$$\mathrm{spec}\,(\Delta_X) = \{E_1, \cdots, E_k\} \cup [\frac{1}{4}, \infty\,)\;,\;\; 0 < E_1 < E_2 \leq \cdots E_k \leq \frac{1}{4}.$$

The first term in the union constitutes the pure point spectrum and the second the absolutely continuous one. We recall from [23],[33] that for $\delta = \dim \Lambda(\Gamma) > \frac{1}{2}$ we have $E_1 = \delta(1-\delta)$ while for $\delta < \frac{1}{2}$ there is no point spectrum. Hence the resolvent of the Laplacian has the following mapping property:

$$R_X(s) \stackrel{\mathrm{def}}{=} (\Delta_X - s(1-s))^{-1} \;:\; L^2(X, d\mathrm{vol}_g) \longrightarrow H^2(X, d\mathrm{vol}_g)\,,\;\; \mathrm{Re}\, s > \frac{1}{2}\,,\;\; s(1-s) \notin \mathrm{spec}\,(\Delta_X)\,.$$

By shrinking the source space and enlarging the target space the operator R_X continues meromorphically to all of $\mathbb{C}$:

$$R_X(s) : L^2_{\mathrm{comp}}(X, d\mathrm{vol}_g) \longrightarrow H^2_{\mathrm{loc}}(X, d\mathrm{vol}_g)\,,$$

with poles of finite rank. Following [22] (see also [36]) we define the multiplicity of a pole of R_X as the rank of the full polar part of R_X:

$$m_s(R_X) = \epsilon(s) \dim \sum_{j=1}^{k} \{\mathrm{Im} A_j \;:\; R_X(s') = \sum_{j=1}^{k} \frac{A_j}{(s'-s)^j} + H(s')\,,\;\; H(s') \;\text{ is holomorphic in } s' \text{ near } s\}\,.$$

$$\epsilon(s) = \begin{cases} 2 & s = 1/2 \\ 1 & s \neq 1/2 \end{cases}$$

By Lemma 2.3 of [12] this multiplicity is equal to the rank of the residue of R_X as a function of $s(1-s)$:

$$m_s(R_X) = \mathrm{rank} \int_{\gamma_{s,\epsilon}} R_X(s')(1-2s')ds'\,,\;\; s \neq \frac{1}{2}\,,\;\; \gamma_{s,\epsilon}(t) = s + \epsilon e^{it}\,, t \in [0, 2\pi)\,. \tag{3.9}$$

The poles of R_X are called *resonances* and their set, with multiplicities given by $m_s(R_X)$ is going to be denoted by $\mathcal{R}_X$.

A more complicated but in many ways more natural (and essentially equivalent) definition of resonances comes from considering the scattering matrix. To introduce it we need to recall the definition of the Eisenstein series of X. The classical definition is as follows:

$$E(s; z, \xi) = \sum_{\gamma \in \Gamma} P(\gamma z; \xi)^s\,,\quad P(s, \xi) = \begin{cases} y & \xi = \infty \\ y/|z-\xi|^2 & \xi \neq \infty \end{cases}\,,\; z = x + iy \in \mathbb{H}^2\,,\; \xi \in \partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}\,. \tag{3.10}$$

The surface X has a natural compactification obtained by using the fundamental domains of Y_j^0, $j = 1, \cdots, n$. Its boundary (at infinity), $\partial_F X = \bigsqcup_{j=1}^n \partial_F Y_j^0$, is also given by

$$\partial_F X \simeq \Gamma \backslash \left(\partial\mathbb{H}^2 \setminus \Lambda(\Gamma)\right)$$

which then allows an identification of $E(s; z, \xi)$ with sections of an s-dependent line bundle over $X \times \partial_F X$ (see [24],[25]). From the more general point of view of geometric scattering theory adopted in [12] we have

$$E_X(s) \in \mathcal{C}^\infty(X^o \times \partial_F X; {}^0\Omega^{\frac{1}{2}}_{X \times \partial_F X} \otimes |N^* \partial_F X|^{s-\frac{1}{2}})$$
$$E_X(s; z, \xi) = x'^{\,s-\frac{1}{2}} R_X(s)(z', z)\restriction_{x'=0}\,,\;\; z' \simeq (x', \xi)\,,$$

where we refer to [12] for the definitions of the geometrically defined line bundles and where we now took R_X to be the resolvent of the Laplacian acting on the 0-half-densities, $\mathcal{C}^\infty(X; {}^0; \Omega_X^{\frac{1}{2}})$. The variable z' is taken in a neighbourhood of $\partial_F X$ which can be parametrized by (x', ξ), $x' \in [0, \epsilon)$, $\xi \in \partial_F X$, $x'\restriction_{\partial_F X} = 0$, $dx'\restriction_{\partial_F X} \neq 0$.

The Eisenstein series, $E_X(s)$, are generalized eigenfunctions of Δ_X in the following standard sense: if P_X^{c} is the orthogonal projection on the orthogonal complement of the space of L^2-eigenfunctions then

$$P_X^{\mathrm{c}}\Delta_X = \int_{\mathrm{Re}\, s=\frac{1}{2}} s(1-s)d\mathcal{E}_s\,, \quad d\mathcal{E}_s = \frac{1}{2\pi i}(2s-1)(R_X(s)-R_X(1-s))ds\,,$$

$$R_X(s)(z,z') - R_X(1-s)(z,z') = (1-2s)\int_{\partial_F X} E_X(s;z,\xi)E_X(s;z',\xi)\,.$$

The scattering matrix is the operator which intertwines the generalized eigenfunctions:

$$S_X(s) : \mathcal{C}^\infty(\partial_F X; \Omega^{\frac{1}{2}}_{\partial_F X}\otimes |N^*\partial_F X|^{-s+\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(\partial_F X; \Omega^{\frac{1}{2}}_{\partial_F X}\otimes |N^*\partial_F X|^{s-\frac{1}{2}})\,,$$
$$S_X(s)E_X(1-s;z,\bullet) = E_X(s;z,\bullet)\,.$$

In more general and geometric terms it can be defined as the operator intertwining the incoming and outgoing boundary data at infinity (see Proposition 8.2 of [22]).

The operator $S_X(s)$ is meromorphic in $s\in\mathbb{C}$ with poles of infinite rank at $s\in\mathbb{N}+\frac{1}{2}$ and poles of finite rank at $s\in\mathcal{R}_X$. It is a classical elliptic pseudodifferential operator of order depending on s:

$$S_X(s)\in\Psi^{2s-1}_{\mathrm{phg}}(\partial_F X; |N^*\partial_F X|^{-s+\frac{1}{2}}, |N^*\partial_F X|^{s-\frac{1}{2}})\,, \quad s\notin\mathcal{R}_X\cup\mathbb{N}+\frac{1}{2}\,.$$

For $s\in\mathcal{R}_X$ the multiplicity of a pole of S_X was defined in [12] as

$$v_s(S_X) = -\frac{1}{2\pi i}\mathrm{tr}\int_{\gamma_{s,\epsilon}} S_X(s')^{-1}\frac{d}{ds}S_X(s')ds'\,,$$

where $\gamma_{s,\epsilon}$ is as in (3.9) and ϵ is small (we recall that by Lemma 2.5 of [12] we have $\mathcal{R}_X\cap(-\mathbb{N}+\frac{1}{2})=\emptyset$ and hence the infinite rank zeros of $S_X(s)$ do not contribute). We then showed that S_X can have a pole only when R_X does and that the two notions of multiplicity essentially agree:

$$v_s(S_X) = m_s(R_X) - m_{1-s}(R_X)\,,$$

with the second term present only when $s(1-s)\in\mathrm{spec}_{\mathrm{pp}}(\Delta_X)$.

Patterson and Perry [25] showed, in the generality including all even dimensions, that the non-trivial zeros of the dynamical zeta function, Z_X, given by (1.3) and continued meromorphically to $\mathbb{C}$ (that continuation was proved in [9] for more general groups and with an order estimate in [25]) coincide with the poles of S_X with multiplicities defined by $v_s(S_X)$. Hence by the discussion above the non-trivial zeros of the zeta function coincide with the resonances.

We end this section by recalling some explicit formulæ in the funnels. From (3.2) we have

$$\Delta_X\restriction_{Y_j^0} = D_r^2 - i\tanh D_r + (\cosh r)^{-2}D_y^2\,, \quad \mathrm{dvol}_g\restriction_{Y_j^0} = \cosh r dr dy\,, r\in[0,\infty)\,, y\in\mathbb{R}/\ell_j\mathbb{Z}\,. \tag{3.11}$$

We can eliminate the first order term by conjugation with a unitary operator:

$$U_j \;:\; L^2([0,\infty)\times\mathbb{R}/\ell_j\mathbb{Z}, drdy)\longrightarrow L^2(Y_j^0,\mathrm{dvol}_g)\,,\; U_j \;:\; f(r,y)\longmapsto(\cosh r)^{-\frac{1}{2}}f(r,y)\,,$$
$$U_j^*\Delta_X\restriction_{Y_j^0} U_j = D_r^2 + (\cosh r)^{-2}\left(D_y^2+\frac{1}{4}\right)+\frac{1}{4}\,. \tag{3.12}$$

We note that the last computation shows why the continuous spectrum of Δ_X starts at $\frac{1}{4}$. From the point of view of the wave equation and the resonances a more natural operator to consider is

$$P = \Delta_X - \frac{1}{4}\,.$$

To work near a fixed energy level we introduce the semi-classical operator

$$P(h) = h^2 P\,.$$

4. **Complex scaling for a family of ordinary differential equations**

We will consider a general family of semi-classical ordinary differential operators with an additional large parameter:

$$Q(\alpha) = (hD_r)^2 + e^{-2r}a(\alpha, r) \ \text{ on } [0,\infty)\,, \tag{4.1}$$

where $\alpha \in \mathbb{R}$ and for r in a conic neighbourhood of the positive real axis, a is holomorphic in r with values in $S^2([0,\infty)_\alpha)$. We assume in addition that a is real valued for r real, and that it is elliptic:

$$\alpha^2/C \leq |a(\alpha, r)| \leq C\alpha^2\,, \quad |\arg(r)| < \gamma_0\,,$$

with C independent of r, and that has a limit at infinity:

$$\frac{a(\alpha,r)}{\alpha^2} \to \frac{a_0(\alpha)}{\alpha^2} \ \text{ as } \ |r| \longrightarrow \infty\,, \ \ a_0(\alpha) > 0\,, \quad |\arg(r)| < \gamma_0\,,$$

uniformly in α. The potential $e^{-2r}a(\alpha, r)$ comes from (3.12) and α is essentially the symbol (or spectral parameter) of D_y.

Let R and $\widetilde{R}$ be rectangles in $\mathbb{C}$

$$R = \{z \ : \ a \leq \operatorname{Re} z \leq b\,, \ |\operatorname{Im} z| < c\}\,, \quad \widetilde{R} = \{z \ : \ \tilde{a} \leq \operatorname{Re} z \leq \tilde{b}\,, \ |\operatorname{Im} z| < \tilde{c}\}\,, \ \tilde{c} < 1\,, \tag{4.2}$$

such that if $R_\theta = \widetilde{R} \cap \{z \ : \ \operatorname{Im} z > -\theta \operatorname{Re} z/C^\sharp\}$, $C^\sharp \gg 1$, then $R \subset R_{\theta_0}$ for some fixed $\theta_0 > 0$. If $q(\alpha)$ is the symbol of $Q(\alpha)$ extended holomorphically to a conic neighbourhood of $T^*\mathbb{R}$ in $T^*\mathbb{C}$, we would now like to find a family of contours in $\mathbb{C}$, $\Gamma_\theta(\alpha)$, such that

$$q_\theta(\alpha) - z \stackrel{\text{def}}{=} q(\alpha)\restriction_{T^*\Gamma_\theta(\alpha)} - z\,,$$

is uniformly elliptic for $z \in R_\theta$, $0 \leq \theta \leq 1/C$. Here $T^*\Gamma_\theta(\alpha) \subset T^*\mathbb{C}$ is an $\mathbb{R}$-symplectic and Im-Lagrangian submanifold of the complex symplectic manifold $T^*\mathbb{C}$ (that is, symplectic with respect to the real part of and Lagrangian with respect to the imaginary part of the complex symplectic form).

The contours $\Gamma_\theta(\alpha)$ are given by

$$\Gamma_\theta(\alpha) = \{r + if_\theta(\alpha, r) \ : \ r \in [0,\infty)\}$$

with $f_\theta \in \mathcal{C}^\infty([0,\infty) \times [1,\infty))$ of the form

$$f_\theta(\alpha, r) = \mathbf{1}_{|\alpha|<2A} f_\theta^\sharp(r) + \mathbf{1}_{|\alpha|\geq 2A} f_\theta^\flat(r, \log a_0(\alpha))\,. \tag{4.3}$$

The resulting f_θ is clearly not a smooth function of α but that will not cause any difficulties (we could make f_θ smooth in α but will avoid that for the sake of simplicity). For the $f_\theta^\sharp$ we require that

$$f_\theta^\sharp(r) = \begin{cases} 0 & r < 2 \\ \theta r & r > 3 \end{cases}\,, \quad 0 \leq \partial_r f_\theta^\sharp(r) \leq 4\theta\,,$$

and we choose the function $f_\theta^\flat(r, \kappa)$ so that the following conditions hold: for κ large enough

$$f_\theta^\flat(r,\kappa) = \begin{cases} 0 & r \leq \kappa/2 - C_1 \\ \theta(r - \kappa/2) & r \geq \kappa/2 + C_2 \end{cases}\,, \tag{4.4}$$

for all r,

$$0 \leq \partial_r f_\theta^\flat \leq 2\theta\,,$$

and for $r > \kappa/2 - C_1/2$

$$f_\theta^\flat(r,\kappa) \geq \theta/C_3\,, \quad \partial_r f_\theta^\flat(r,\kappa) \geq \theta/C_3\,, \tag{4.5}$$

where the constants C_i, $i = 1, 2, 3$ will be chosen noting that (4.4) forces $C_3 > 1 + C_1/(2C_2)$. We can also arrange that $\partial_r^k f_\theta^\sharp \leq C_k\theta$.

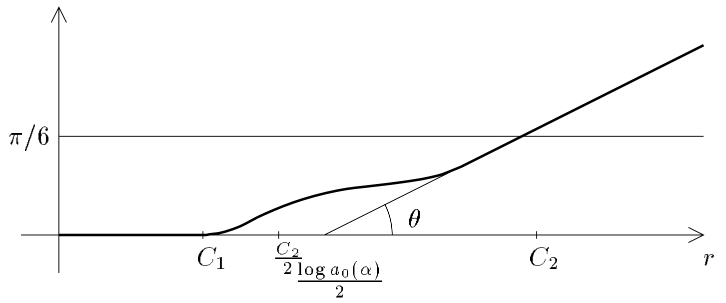

FIGURE 5. The contour $\Gamma_\theta(\alpha)$ for large values of α.

Lemma 4.1. *For any fixed $\tilde{a}$ and $\tilde{b}$, $0 < \tilde{a} < \tilde{b}$, in (4.2) we can find $\tilde{c}$ and $A > 0$, $\theta_0 > 0$ so that for $0 < \theta < \theta_0$ and C_0 sufficiently large we have*

$$|q_\theta(\alpha) - z| > \theta(1 + |\rho|^2)/C, \quad (r, \rho) \in T^*\Gamma_\theta(\alpha), \; |\alpha| < A/2, \; |r| > C_0, \quad z \in R_\theta .$$

The constant $C_0 = C_0(A, \tilde{a}, \tilde{b})$ can be chosen so that for $|r| < 2C_0$, $\operatorname{Re} q_\theta(\alpha)$ is positive definite in (ρ, α), that is

$$\operatorname{Re} q_\theta(\alpha)(r, \rho) \geq \frac{1}{C}(|\rho|^2 + |e^{-2r}|\alpha^2), \quad |r| < 2C_0$$

where C is independent of θ.

Proof. We observe that $\Gamma_\theta(\alpha) \cap \{\operatorname{Re} r > 3\} = \{(1 + i\theta)r : r > 3\}$ and using $\operatorname{Re} r$ as a parameter (which we will denote by r as well, with its dual variable denoted by ρ) we have

$$\begin{aligned} q_\theta(\alpha) &= (1 + i\theta)^{-2}\rho^2 + e^{-2r-2i\theta r} a(\alpha, (1 + i\theta)r) \\ &= \rho^2(1 + \mathcal{O}(\theta^2)) + e^{-2r}\cos(2\theta r)(a_0(\alpha) + o(1)|\alpha|^2) \\ &\quad - i[2\theta\rho^2(1 + \mathcal{O}(\theta^2)) + e^{-2r}\sin(2r\theta)(a_0(\alpha) + o(1)|\alpha|^2)] , \end{aligned}$$

where $o(1) \to 0$ as $r \to \infty$. Hence for $C_0 < r < \pi/(2\theta)$ with C_0 sufficiently large (independent of $\theta < \theta_0$ with θ_0 sufficiently small) we have

$$\begin{aligned} \operatorname{Re} q_\theta(\alpha) &= (1 + \mathcal{O}(\theta^2))\rho^2 + \mathcal{O}(e^{-C_0}A^2) , \\ \operatorname{Im} q_\theta(\alpha) &\leq -2\theta\rho^2(1 + \mathcal{O}(\theta^2)) . \end{aligned}$$

Choosing C_0 possibly yet larger depending on $A, \tilde{a}, \tilde{b}$ and θ_0 we see that $|\operatorname{Re} q_\theta - z| < 1/C$, $z \in \widetilde{R}$, implies that $\rho^2 > 1/C$ and that gives the estimate for that range of r's. If $r > 1/\theta$ then

$$q_\theta(\alpha) = (1 + \mathcal{O}(\theta^2))\rho^2 - 2i\theta\rho^2(1 + \mathcal{O}(\theta^2)) + \mathcal{O}(e^{-1/\theta}A^2) ,$$

and the estimate holds as well.

The second part follows from imposing the condition that $C_0 \ll 1/\theta$ so that $\cos(2r\theta) > 1/2$, and so that can be done as long as θ_0 is small enough depending on A. ☐

We now turn to the intermediate range of α's:

Lemma 4.2. *For any fixed $\tilde{a}$ and $\tilde{b}$, $0 < \tilde{a} < \tilde{b}$, in (4.2) we can find $\tilde{c} > 0$, a sufficiently large $A > 0$ and $\theta_0 = \theta_0(A) > 0$ such that for $0 < \theta < \theta_0$, $z \in R_\theta$ and $A/2 < |\alpha| < 2A$ we have*

$$|q_\theta(\alpha) - z| \geq \theta(1 + |\rho^2|)/C , \quad (r, \rho) \in T^*\Gamma_\theta(\alpha) .$$

Proof. For $0 < r < 2$ we have

$$q_\theta(\alpha) = q_0(\alpha) = \rho^2 + e^{-2r}a(\alpha, r) \geq \rho^2 + e^{-4}\frac{A^2}{4}$$

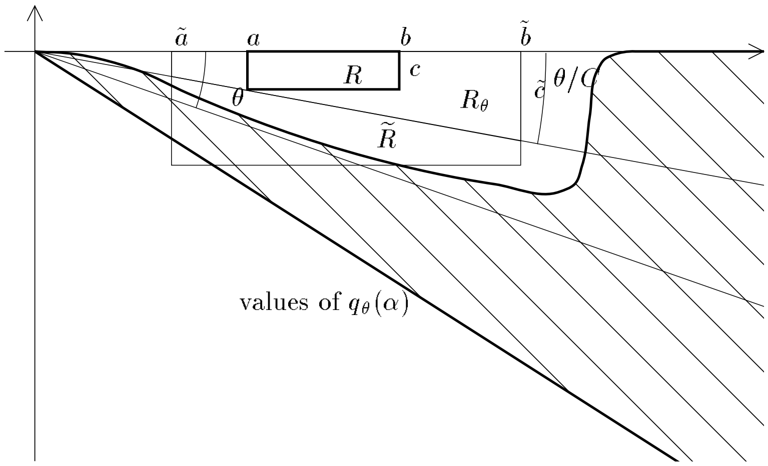

FIGURE 6. The region $\widetilde{R}$, $\overline{R}$, R_θ and the range of values of $q_\theta(\alpha) = q(\alpha)\restriction_{\Gamma_\theta(\alpha)}$.

and hence we get the needed estimate once A is large enough. This argument applies also in the transitional zone $2 < r < 3$. When $2 \leq r \leq C(\theta)\log A$ then changing $\operatorname{Re} r$ to r as a variable on $\Gamma_\theta(\alpha)$ we get

$$\begin{aligned} q_\theta(\alpha) =& \rho^2(1+\mathcal{O}(\theta^2)) + e^{-2r}\left(\cos(2\theta r)\operatorname{Re} a(\alpha, r(1+i\theta)) + \mathcal{O}(\theta C(\theta)A^2\log A)\right) \\ & - i(2\theta\rho^2(1+\mathcal{O}(\theta^2)) + e^{-2r}\left(\sin(2r\theta)\operatorname{Re} a(\alpha, r(1+i\theta)) + \cos(2r\theta)\theta A^2 o_{r\to\infty}(1)\right) , \end{aligned}$$

where we noted that $a(\alpha, r)$ is real for r real and that it has a real limit in a conic neighbourhood of the real axis. We can now take $\theta C(\theta)\log A \ll 1$, so that $\sin 2r\theta \geq 0$, $\cos(2r\theta) > 1/2$. Hence

$$\begin{aligned} (1+\mathcal{O}(\theta^2))\rho^2 + Ce^{-2r}A^2 \geq \operatorname{Re} q_\theta(\alpha) \geq (1+\mathcal{O}(\theta^2))\rho^2 + \frac{1}{C}e^{-2r}A^2 , \\ \operatorname{Im} q_\theta(\alpha) \leq -2\theta\rho^2(1+\mathcal{O}(\theta^2)) - \frac{1}{C}e^{-2r}\sin 2r\theta A^2 + \theta A^2 o_{r\to\infty}(1) . \end{aligned}$$

When $\rho \geq 1/C$ we get the desired bound and when $\rho^2 \leq 1/C$ then $|q_\theta(\alpha) - z| \leq 1/C$ implies (here C is very large) that $r \sim \log A$ by looking at the real part. Then $\operatorname{Im} q_\theta \leq -\theta \log A/C$ provided that $\theta \log A \ll 1$ and that also gives the desired bound.

For $r \geq C(\theta)\log A$ we have

$$q_\theta(\alpha) = \rho^2(1+\mathcal{O}(\theta^2)) - i2\theta\rho^2(1+\mathcal{O}(\theta^2)) + \mathcal{O}(e^{(-C(\theta)+2)\log A}) .$$

Thus we get the needed estimate, if $(C(\theta)-2)\log A \gg -\log\theta$, which can be arranged concurrently with the previous conditions provided that $\theta_0 = \theta_0(A)$ is small enough.

□

So far we were only reviewing the rather standard aspects of complex scaling. A difference occurs in the uniform treatment of large values of α:

Lemma 4.3. *For any fixed $\tilde a$ and $\tilde b$, $0 < \tilde a < \tilde b$, in (4.2) we can find $\tilde c$ and constants C_i, $0 \leq i \leq 3$, such that for $|\alpha| \geq 2A$, f_θ given by (4.3) and satisfying (4.4) and (4.5), we have for some $\theta_0 > 0$ and $z \in R_\theta$, $0 \leq \theta \leq \theta_0$,*

$$|q_\theta(\alpha) - z| > \theta(1 + |\rho|^2 + |e^{-2r}|\langle\alpha\rangle^2)/C , \quad (r,\rho) \in T^*\Gamma_\theta(\alpha) . \tag{4.6}$$

Proof. We will discuss the different ranges of r given in (4.4) and we start with $r < \log a_0(\alpha)/2 - C_1$. Then for $z \in \widetilde{R}$

$$\operatorname{Re}(\rho^2 + e^{-2r}a(\alpha, r) - z) \geq \rho^2 + e^{-2r}a(\alpha,r) - \tilde b > \rho^2 + e^{2C_1}\frac{a(\alpha,r)}{a_0(\alpha)} - \tilde b ,$$

and hence we can take

$$C_1 > \frac{1}{2}\left(-\log\min_{r>1}\frac{a(\alpha,r)}{a_0(\alpha)} + \log(1+\tilde{b})\right) .$$

When $r > \log a_0(\alpha)/2 + C_2$ we first consider $\rho^2 \in [1,\infty) \setminus [\tilde{a} - e^{-2C_2} - 1/\widetilde{C}, \tilde{b} + e^{-C_2} + 1/\widetilde{C}]$ so that

$$\left|\mathrm{Re}\left((1+i\theta)^{-2}\rho^2 + e^{-2(r+i\theta(r-\log a_0(\alpha)))}a(\alpha, r+i\theta(r-\log a_0(\alpha))) - z\right)\right| > \frac{1}{C}\langle\rho\rangle^2 ,$$

provided that A is large enough so that for $2r > \log a_0(\alpha)$, $|\alpha| \geq 2A$, $a(\alpha,r)$ is close to $a_0(\alpha)$

For $\rho^2 \in [\tilde{a} - e^{-2C_2} - 1/\widetilde{C}\tilde{b} + e^{-2C_2} + 1/\widetilde{C}]$ we have

$$\begin{aligned}\mathrm{Im}\left((1+i\theta)^{-2}\rho^2 + e^{-2(r+i\theta(r-\log a_0(\alpha)))}a(\alpha, r+i\theta(r-\log a_0(\alpha))) - z\right) < \\ -\theta\rho^2 + \theta/C' + \theta\tilde{b}/C^\sharp < -\theta\langle\rho\rangle^2/C ,\end{aligned}$$

where $C^\sharp$ is as in the definition of R_θ (chosen large depending on $\tilde{a},\tilde{b}$), and where we used the elementary inequality $|e^{-2t}\sin\theta t| < (1+t^2)^{-1}t\theta$ with $t = r - \log a_0(\alpha)/2 > C_2$ (again, once A is large enough, a is close to a_0 for r in a conic neighbourhood of $[1,\infty)$).

In the first intermediate region $\log a_0(\alpha)/2 - C_1 \leq r \leq \log a_0(\alpha)/2 - C_1/2$ we have

$$\begin{aligned}&\mathrm{Re}\left((1+i\partial_r f_\theta(\alpha,r))^{-2}\rho^2 + e^{-2r}e^{-2if_\theta(\alpha,r)}a(\alpha, r+if_\theta(\alpha,r)) - z\right) \geq \\ &\quad\left(1-\mathcal{O}(\theta^2)\right)\rho^2 + e^{-2r}\cos f_\theta(\alpha,r)a_0(\alpha) - e^{-2r}a_0(\alpha)/C - \mathrm{Re}\, z \geq \frac{1}{2}\rho^2 + \frac{1}{2}e^{C_1} - \tilde{b} > \langle\rho\rangle^2/C ,\end{aligned}$$

if $C_1 > 1 + \log\tilde{b}$.

Finally, we use (4.5) in the critical region $(\log a_0(\alpha) - C_1)/2 < r < \log a_0(\alpha)/2 + C_2$:

$$\begin{aligned}&\left|\mathrm{Im}\left((1+i\partial_r f_\theta(\alpha,r))^{-2}\rho^2 + e^{-2(r+if_\theta(\alpha,r))}a(\alpha, r+if_\theta(\alpha,r)) - z\right)\right| = \\ &\quad\big|(-2\partial_r f_\theta(\alpha,r) + \mathcal{O}(\theta^2))\rho^2 - \sin(2f_\theta(\alpha,r))e^{-2r}\,\mathrm{Re}\, a(\alpha, r+if_\theta(\alpha,r)) + \\ &\quad\cos(2f_\theta(\alpha,r))e^{-2r}\,\mathrm{Im}\, a(\alpha, r+if_\theta(\alpha,r)) - \mathrm{Im}\, z\big| > \\ &\quad\frac{\theta}{C_3}\rho^2 + \frac{\theta}{C_3}e^{-2r}a_0(\alpha) - \frac{\theta}{C^\sharp}\tilde{b} ,\end{aligned} \tag{4.7}$$

since $\mathrm{Re}\, a = a_0(1+o(1))$ and $\mathrm{Im}\, a = \theta o(a_0))$, $|\alpha| \to \infty$ ($|\alpha| \geq 2A$ and A is large), provided that

$$f_\theta(\alpha,r) < \pi/2 \ \text{ for } \ r < \frac{1}{2}a_0(\alpha) + C_2 .$$

This can be arranged by demanding that $2C_2\theta < \pi/3$. We also have

$$\begin{aligned}&\left|\mathrm{Re}\left((1+i\partial_r f_\theta(\alpha,r))^{-2}\rho^2 + e^{-2(r+if_\theta(\alpha,r))}a(\alpha, r+if_\theta(\alpha,r)) - z\right)\right| \geq \\ &\quad\left|\mathrm{Re}\, z - \left(\cos(2f_\theta(\alpha,r))e^{-2r}(a_0(\alpha)+o(1)) + (1+\mathcal{O}(\theta^2))\rho^2\right)\right|\end{aligned}$$

and consequently we need only consider the case $\rho^2 < 2\tilde{b}$. Then (4.6) follows from the lower bound on the real part unless $\mathrm{Re}\, z - e^{-2r}\cos(2f_\theta(\alpha,r))a_0(\alpha) - \rho^2 < 1/C$, that is, $\rho^2 + e^{-2r}a_0(\alpha) > \tilde{a} - 1/C$. In that case we can use (4.7) provided that $C^\sharp$ in the definition of R_θ is chosen much larger than C_3. □

We remark that an infinitesimal version of a similar argument was present already in [5] where a positive commutator method was used to prove absence of embedded eigenvalues for some mixed rank quotients.

For future reference we also note the following estimate which follows from the proofs of the above lemmas:

$$\text{If} \quad r > C_0 \quad \text{or} \quad |\alpha| > A/2 \quad \text{then} \quad \mathrm{Im}\, q_\theta(\alpha) \leq e^{-\frac{1}{C_A\theta}} , \qquad 0 < \theta \leq \theta_0(A) , \tag{4.8}$$

that is the imaginary part of q_θ is "almost" non-positive and the positive part can only come from the potential term. In fact, in the regions treated by Lemmas 4.1 and 4.3 ($r > C_0$ and $|\alpha| > 2A$ respectively) the imaginary part is $\mathcal{O}(\exp(-\theta/C))$: that is clear in Lemma 4.1 and in Lemma 4.3 we observe that $-\sin(2f_\theta)(1+o(1))+\theta\cos(2f_\theta)o(1)\leq 0$ for $r\leq 1/(C\theta)$ (see (4.7)). In the proof of Lemma 3.2 we have $\operatorname{Im} q_\theta(\alpha)<0$ for $r<C(\theta)\log A$ and $\operatorname{Im} q_\theta(\alpha)\leq\exp(-1/(C_A\theta))$ for large r if we take $C(\theta)\log A>1/(\widetilde{C}_A\theta)$ as we can.

In what follows we will choose the constants a,b to be close to 1 and consequently assume that

$$R=[1-\epsilon_0,1+\epsilon_0]-i[0,\delta_0]\,,\quad \epsilon_0,\delta_0>0\,,\quad R\subset R_{\theta_0}\,.$$

The following elementary lemma will allow us to use (4.6) in order to invert $Q_\theta(\alpha)-z$ for $z\in R_\theta$. We present it to avoid the need to develop a semi-classical calculus with an additional large parameter α.

Lemma 4.4. *If $V\in\mathcal{C}^2(\mathbb{R};\mathbb{C})$ satisfies*

$$d(V(r),(-\infty,0])>a|V(r)|\,,\quad |V(r)|>b\,,\quad |V^{(k)}(r)|\leq A|V(r)|\,,\ k\leq 2\,,\tag{4.9}$$

for all $r\in\mathbb{R}$ and some $a,b,A>0$, then

$$\|\left((hD_r)^2+V(r)\right)u\|_{(0)}\geq\frac{ab}{3}\|u\|_{(0)}\,,\quad u\in\mathcal{C}^\infty_{\rm c}(\mathbb{R})\,,$$

for $0<h<h_0=h_0(a,b,A)$ and C independent of a,b and A.

Proof. We write

$$\|\left((hD_r)^2+V(r)\right)u\|^2_{(0)}=\langle\widetilde{P}u,u\rangle\,,$$

where $\widetilde{P}=(hD_r)^4+2\operatorname{Re}V(hD_r)^2+|V|^2+[(hD_r)^2,V]$. We then note that

$$\langle[(hD_r)^2,V]u,u\rangle=\mathcal{O}(h)\|u\|_{(1)}\|V(r)u\|_{(0)}\,,$$

(where we use the h dependent Sobolev norms) and

$$\begin{aligned}\langle 2\operatorname{Re}V(hD_r)^2u,u\rangle&\geq -2\|(\operatorname{Re}V)_-u\|_{(0)}\|\|(hD_r)^2u\|_{(0)}-\mathcal{O}(h)\|V(r)u\|_{(0)}\|u\|_{(1)}\\&\geq-(1+\epsilon)\|(\operatorname{Re}V)_-u\|^2_{(0)}-(1+\epsilon)^{-1}\|(hD_r)^2u\|^2_{(0)}-\mathcal{O}(h)\|V(r)u\|_{(0)}\|u\|_{(1)}\,.\end{aligned}$$

Here we denoted by p_- the negative part of a function p and used the second part of (4.9) together with the following estimate: if p is a real valued function in $\mathcal{C}^1$ then for $u\in\mathcal{C}^\infty_{\rm c}(\mathbb{R})$

$$\begin{aligned}\langle p(hD_r)^2u,u\rangle&=\langle phD_ru,hD_ru\rangle+\langle[p,hD_r]hD_ru,u\rangle\\&\geq\langle p_-hD_r,hD_r\rangle-\mathcal{O}(h)\|p'u\|\|hD_ru\|\\&\geq-\|p_-u\|\|(hD_r)^2u\|-\mathcal{O}(h)\|p'u\|\|hD_ru\|\,.\end{aligned}$$

This holds because in dimension one the open set $\{r:p(r)<0\}$ is a countable union of disjoint open intervals and consequently we have

$$\int p_-v_1v_2'=\int_{p<0}pv_1v_2'=\int_{p<0}(pv_1)'v_2=\int p_-v_1'v_2+\mathcal{O}(1)\int|p'v_1||v_2|\,.$$

The second part of (4.9) shows that for ϵ sufficiently small

$$\|Vu\|^2_{(0)}-(1+\epsilon)\|(\operatorname{Re}V)_-u\|^2_{(0)}\geq a^2\|Vu\|^2_{(0)}\,.$$

In fact, for $z\in\mathbb{C}$ with $d(z,(-\infty,0])>a|z|$ we have $|z|^2-(1+\epsilon)(\operatorname{Re}z)_-^2>a^2|z|^2/2$ if $\epsilon<a^2/2$. This estimate is obvious when $\operatorname{Re}z\geq 0$. Otherwise $d(z,(-\infty,0])=|\operatorname{Im}z|$ and

$$|z|^2-(1+\epsilon)(\operatorname{Re}z)^2=(\operatorname{Im}z)^2-\epsilon(\operatorname{Re}z)^2\geq(\operatorname{Im}z)^2\left(1-\frac{\epsilon}{a^2}\right)\geq\frac12(\operatorname{Im}z)^2>\frac12a^2|z|^2\,.$$

Hence for $u\in\mathcal{C}^\infty_{\rm c}(\mathbb{R})$ we have

$$\|\left((hD_r)^2+V(r)\right)u\|^2_{(0)}\geq\frac1C\|(hD_r)^2u\|^2_{(0)}+\frac12a^2\|Vu\|^2_{(0)}-\mathcal{O}(h)\|u\|_{(1)}\|Vu\|_{(0)}\,.$$

Since by (4.9), $|V(r)|>b$, we obtain the desired lower bound by taking h sufficiently small. $\square$

We will now assume that $Q(\alpha)$ is defined on $\mathbb{R}$, $Q(\alpha) = (hD_r)^2 + v(\alpha, r)$, where $v(\alpha, \bullet)$ is even and it satisfies the conditions in (4.1). We then redefine the contours to be $\Gamma_\theta(\alpha) \cup -\Gamma_\theta(\alpha)$ still denoting them by $\Gamma_\theta(\alpha)$. The estimates of Lemmas 4.1, 4.2 and 4.3 hold now with the new $\Gamma_\theta(\alpha)$. Considering $Q(\alpha)$ as a holomorphic differential operator in a neighbourhood of $\mathbb{R}$ in $\mathbb{C}$ we now define the *deformed* operator

$$Q_\theta(\alpha) = Q(\alpha)\restriction_{\Gamma_\theta(\alpha)} \,.$$

The lemmas above give the following

Proposition 4.1. *If $z \in R_\theta$, $0 < h < h_0$ (with h_0 independent of α) and if we consider $Q_\theta(\alpha)$ on $\Gamma_\theta(\alpha)$ then for $|\alpha| > A/2$*

$$G_\theta^\flat(\alpha, z) \stackrel{\text{def}}{=} (Q_\theta(\alpha) - z)^{-1} = \mathcal{O}\left(\frac{1}{\theta}\right) : L^2(\Gamma_\theta(\alpha), dr) \longrightarrow H^2(\Gamma_\theta(\alpha))$$

exists and its kernel satisfies the estimates

$$\left|(h\partial_r)^k (h\partial_{r'})^l G_\theta^\flat(\alpha, z)(r, r')\right| \leq C_{kl} \exp\left(-\frac{|r - r'|}{C_{kl} h}\right) \,, \text{ for } \ |r - r'| > c > 0 \,, \ C_{kl} = C_{kl}(\theta) \,. \tag{4.10}$$

For $|\alpha| < A/2$ we can find a second order operator $Q_\theta^\sharp(\alpha)$ on $\Gamma(\alpha)$ which has the same form as $Q_\theta(\alpha)$ and

$$Q_\theta^\sharp(\alpha) \equiv Q_\theta(\alpha) \quad on \quad \Gamma_\theta(\alpha) \cap \{|r| > C_0\} \,,$$

where C_0 is as in Lemma 4.1, such that

$$G_\theta^\sharp(\alpha, z) \stackrel{\text{def}}{=} (Q_\theta^\sharp(\alpha) - z)^{-1} = \mathcal{O}\left(\frac{1}{\theta}\right) : L^2(\Gamma_\theta(\alpha), dr) \longrightarrow H^2(\Gamma_\theta(\alpha))$$

exists and its kernel satisfies the estimates (4.10).

Proof. The existence of the inverses follows directly from Lemmas 4.1, 4.2, 4.3 and 4.4 applied with

$$V(r) = (1 + i\partial_r f_\theta(\alpha, r))^{-2} \left[\exp(-2r - 2if_\theta(\alpha, r)) a(\alpha, r + if_\theta(\alpha, r)) - z\right] \,,$$
$$r + if_\theta(\alpha, r) \in \Gamma_\theta(\alpha) \,, \quad z \in R_\theta \,,$$

noting that all lower order terms produce contributions which are $\mathcal{O}(h)$. To see (4.10) we conjugate $Q_\theta(\alpha)$ by $\exp(\pm cr/h)$ which, in the leading order term, changes $V(r)$ to $V(r) - c^2$. As long as c is small enough (independently of α, $z \in R_\theta$ and h) we can still apply Lemma 4.4 to $\exp(\pm cr/h)(Q_\theta(\alpha) - z)\exp(\mp cr/h)$. From that an L^2 estimate on $\exp(\pm cr/h) G_\theta(\alpha, z) \exp(\mp cr/h)$ follows. As for instance in [26] the estimates on the derivatives and the Sobolev embedding give the pointwise bound away from the diagonal. ☐

We shall now review, in the simpler setting of ordinary differential operators, the deformation method leading to holomorphic continuation of solutions (see [14],[30] and references given there):

Proposition 4.2. *Suppose that $0 < \theta_1 < \theta_2 < \theta_0$ and that $z \in R_{\theta_1}$. If*

$$(Q_{\theta_1}(\alpha) - z)u = v \in \mathcal{C}_c^\infty([0, 1)) \,, \quad u(1) = 0 \,, \tag{4.11}$$

then $u = U\restriction_{\Gamma_{\theta_1}(\alpha)}$ where U is holomorphic in a neighbourhood of $\bigcup_{\theta_1 < \theta < \theta_2} \Gamma_\theta(\alpha) \setminus [0, 2]$. Moreover, if $u \in L^2(\Gamma_{\theta_1}(\alpha))$ then $U\restriction_{\Gamma_{\theta_2}(\alpha)} \in L^2(\Gamma_{\theta_2}(\alpha))$.

Proof. If we rewrite the operator on the left hand side of (4.11) as $((hD_\zeta)^2 + V(\zeta))\restriction_{\Gamma_\theta(\alpha)}$, that is as a restriction of a holomorphic differential operator to a totally real submanifold, then $V(\zeta)$ is holomorphic in a neighbourhood of $\bigcup_{\theta_1 < \theta < \theta_2} \Gamma_\theta(\alpha)$. Hence the Cauchy-Riemann estimates on V imply that u extends to a holomorphic function in a neighbourhood of $\Gamma_{\theta_1}(\alpha) \setminus [0, 2]$. Iterating a finite number of times (depending on h) we obtain a holomorphic extension to a neighbourhood of $\bigcup_{\theta_1 < \theta < \theta_2} \Gamma_\theta(\alpha) \setminus [0, 2]$.

The second part of the lemma is proved by adapting the argument of Lemma 3.4 of [30]. We first recall the standard estimate which holds when the assumptions of Lemma 4.4 are satisfied:

$$\sum_{l\leq 2} \|(\tilde{h}D_r)^l u\|_{L^2(I_2)} \leq C\left(\| \left((\tilde{h}D_r)^2 u + \widetilde{V}(r)\right) u\|_{L^2(I_1)} + \|u\|_{L^2(I_1\setminus I_2)}\right)\,, \quad 0<\tilde{h}<\tilde{h}_0\,, \tag{4.12}$$

where $I_1, I_2 \subset \bar{I}_1$ are open intervals and C is independent of $\tilde{h}$. In fact, we can apply Lemma 4.4 to χu where $\chi \in \mathcal{C}^\infty_{\rm c}(I_1)$, $\chi \equiv 1$ on I_2 and then estimate the commutator term.

To apply (4.12) we introduce a new contour, $\Gamma_{\theta_1,\theta_2,T}(\alpha)$ obtained by replacing $f_\theta(\alpha, r)$ by

$$F_{\theta_1,\theta_2,T}(\alpha,r) = f_{\theta_1}(\alpha,r) + \chi(\alpha,r,T)\left(f_{\theta_2}(\alpha,r) - f_{\theta_1}(\alpha,r)\right)\,,$$

$$\chi(\alpha,r,T) = \mathbf{1}_{|\alpha|<2A}\chi_0\left(\frac{r}{T}\right) + \mathbf{1}_{|\alpha|\geq 2A}\chi_0\left(\frac{2r-\log a_0(\alpha)}{2T}\right)\,,$$

where T is large and $\chi_0 \in \mathcal{C}^\infty_{\rm c}((1/2,3/2);[0,1])$, and $\chi_0 \equiv 1$ near 1 (see Fig.7).

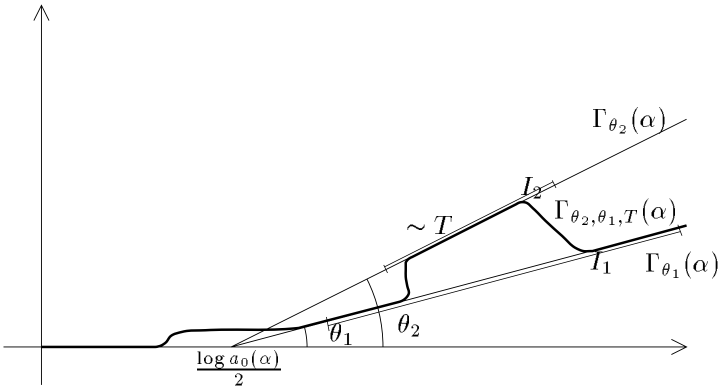

FIGURE 7. The contour $\Gamma_{\theta_1,\theta_2,T}(\alpha)$.

We now apply (4.12) with I_1 and I_2 as shown in Fig.7, $\tilde{h} = h/T$ and

$$\widetilde{V}(r) = \widetilde{V}_T(r) = (e^{-2\zeta}a(\alpha,\zeta) - z)\,, \quad \zeta = Tr + iF_{\theta_1,\theta_2,T}(\alpha, Tr)\,.$$

By letting $T\to\infty$ we obtain the desired estimate:

$$\|U\restriction_{\Gamma_{\theta_2}(\alpha)}\|_{L^2(\Gamma_{\theta_2}(\alpha))} \leq C\|U\restriction_{\Gamma_{\theta_1}(\alpha)}\|_{L^2(\Gamma_{\theta_1}(\alpha))}\,.$$

$\square$

5. Effective meromorphic continuation

By effective meromorphic continuation of the resolvent we mean a continuation obtained by deforming the operator P and the Hilbert space on which it acts in such a way that $P-z$ becomes a Fredholm operator and consequently the poles of the meromorphic continuation of the resolvent are the same as the eigenvalues of the (non-self-adjoint) deformed operator.

We will now use the results of Sect.4 to construct a space on which the deformation of $P(h) - z = h^2(\Delta_X - \frac14) - z$ is a Fredholm operator with eigenvalues appearing as resonances. For that we write

$$Y^a = \bigsqcup_{j=1}^n Y_j^a\,,\ Y_j^a \simeq [a,\infty)\times\mathbb{R}/\ell_j\mathbb{Z}\,,$$

and we introduce cut-off functions $\chi_i \in \mathcal{C}^\infty(X;[0,1])$ such that

$$\chi_0^2 + \chi_1^2 = 1\,,\ \chi_0\in\mathcal{C}^\infty_{\rm c}(X,[0,1])\,,\ \chi_1\in\mathcal{C}^\infty(Y^0,[0,1])\,,\ \operatorname{supp}\chi_0 \subset X\setminus Y^1\,.$$

For $A > 0$ large and $h > 0, \theta$ small, we will define the space $\mathcal{H}_{\theta,h,A}(Y^0)$ and then put

$$\mathcal{H}_{\theta,h,A}(X) = \{u \in \mathcal{D}'(X) \; : \; \chi_0 u \in L^2_{\rm loc}(X), \; \chi_1 u \in \mathcal{H}_{\theta,h,A}(Y^0)\} \,, \tag{5.1}$$

and where no confusion is likely to arise we will simply write $\mathcal{H}_{\theta,h,A}$ for $\mathcal{H}_{\theta,h,A}(X)$.

To define $\mathcal{H}_{\theta,h,A}(Y^0)$ we introduce the decomposition

$$L^2(Y^0) \subset \bigoplus_{j=1}^N \bigoplus_{n\in\mathbb{Z}} E^j_n \otimes U_j L^2([0,\infty), dr) \,, \quad L^2(Y^0) \simeq \bigoplus_{j=1}^N \ell^2(\mathbb{Z}; U_j L^2([0,\infty), dr)) \,,$$

where $E^j_n \simeq \mathbb{C}$ is spanned by $\exp(2\pi i n t/\ell_j)$ and U_j is given by (3.12). Using the contours $\Gamma_\theta(\alpha)$ given in Sect.4 we define

$$\mathcal{H}_{\theta,h,A}(Y^0) \subset \bigoplus_{j=1}^N \bigoplus_{n\in\mathbb{Z}} E^j_n \otimes U_{j,\theta} L^2(\Gamma_\theta(\ell_j^{-1} nh), dr) \,, \quad \mathcal{H}_{\theta,h,A}(Y^0) \simeq \bigoplus_{j=1}^N \ell^2(\mathbb{Z}; U_{j,\theta} L^2(\Gamma_\theta(\ell_j^{-1} nh), dr)) \,, \tag{5.2}$$

where $U_{j,\theta} : L^2(\Gamma_\theta(\ell_j^{-1}nh), dr) \ni f \mapsto (\cosh\restriction_{\Gamma_\theta(\ell_j^{-1}nh)})^{-\frac{1}{2}} f \in L^2(\Gamma_\theta(\ell_j^{-1}nh), |\cosh r|\restriction_{\Gamma_\theta(\ell_j^{-1}nh)} \, dr)$.

We can also consider the same decomposition for the *full* hyperbolic cylinder,

$$Y = \bigsqcup_{j=1}^n Y_j \,, \;\; Y_j \simeq \mathbb{R} \times \mathbb{R}/\ell_j\mathbb{Z} \,,$$

and denote the corresponding space by $\mathcal{H}_{\theta,h,A}(Y)$.

The deformed operator, $P^0_{\theta,A}(h)$, is an unbounded operator on $\mathcal{H}_{\theta,h,A}(Y)$ defined as

$$P^0_{\theta,A}\restriction_{E^j_n \otimes U_{j,\theta} L^2(\Gamma_\theta(\ell_j^{-1}nh), dr)} = I \otimes U_{j,\theta}((hD_r)^2 + (n^2 + 1/4)h^2 \ell_j^{-2} \cosh^{-2} r)\restriction_{\Gamma_\theta(\ell_j^{-1}nh)} \, U_{j,\theta}^{-1} \,, \tag{5.3}$$

where we treat the term $h^2(\cosh r)^{-2}/4$ as a lower order term (so that in applying results of Sect.4 we put $\alpha = \ell_j^{-1} nh$ which simplifies the notation).

To define the domain of $P^0_{\theta,A}(h)$ we observe that an adaptation of the argument of Proposition 3.1 gives uniformly bounded operators

$$G^\flat_\theta(\ell_j^{-1}nh, z_0, h) : L^2(\Gamma_\theta(\ell_j^{-1}nh)) \longrightarrow H^2(\Gamma_\theta(\ell_j^{-1}nh)) \cap H^1_0((\Gamma_\theta(\ell_j^{-1}nh)) \,, \quad \operatorname{Im} z_0 > 0 \,, \quad n \in \mathbb{Z} \,,$$

which invert the Dirichlet realizations of ordinary differential operators $U_{j,\theta}^{-1} P_{\theta,A}\restriction_{E^j_n \otimes U_{j,\theta} L^2(\Gamma_\theta(\ell_j^{-1}nh), dr)} U_{j,\theta}$. We then define $G^0_{\theta,A}(z_0, h)$ by

$$G^0_{\theta,A}(z_0,h)\restriction_{E^j_n \otimes U_{j,\theta} L^2(\Gamma_\theta(\ell_j^{-1}nh), dr)} = I \otimes U_{j,\theta} G^\flat_\theta(\ell_j^{-1}nh, z_0, h) U_{j,\theta}^{-1}$$

which gives a bounded operator on $\mathcal{H}_{\theta,h,A}(Y)$. For elements $u \in \mathcal{H}_{\theta,h,A}(Y^0)$ which have only finitely many n modes and are smooth and compactly supported in r we clearly have

$$(P^0_{\theta,A}(h) - z_0) G^0_{\theta,A}(z_0,h) u = u$$
$$G^0_{\theta,A}(z_0,h)(P^0_{\theta,A}(h) - z_0) u = u \,.$$

The domain of $P^0_{\theta,A}(h)$ can be given as

$$\mathcal{D}^0_{\theta,A} = G^0_{\theta,A}(z_0,h)\mathcal{H}_{\theta,h,A}(Y) \,, \quad \operatorname{Im} z_0 > 0 \,,$$

with the obvious norm $\|u\|_{\mathcal{D}^0_{\theta,h,A}} = \|(P_{\theta,A}(h) - z_0)u\|_{\mathcal{H}_{\theta,h,A}(Y)}$.

We now define the deformed operator on $\mathcal{H}_{\theta,h,A}(X)$ by

$$\mathbf{1}_{N_2} P_{\theta,A}(h) u = P(h)(u\restriction_{N_2}) \,, \quad N_2 = X \setminus Y_2 \,, \;\; u \in H^2(X)$$
$$\mathbf{1}_{(Y^1)^\circ} P_{\theta,A}(h) u = P^0_{\theta,A}(h)(\mathbf{1}_{(Y^1)^\circ} u) \,, \quad \mathbf{1}_{(Y^1)^\circ} u \in \mathbf{1}_{(Y^1)^\circ} \mathcal{D}^0_{\theta,h,A} \,.$$

Since for small values of r in Y^0, the operators $P^0_{\theta,A}$ and $P(h)$ coincide, $P_{\theta,A}(h)$ is well defined. Its domain is given by

$$\mathcal{D}_{\theta,h,A} = \{u \in \mathcal{H}_{\theta,h,A} \ : \ \chi u \in H^2(X)\,, \ (1-\chi)u \in \mathcal{D}^0_{\theta,h,A}\}\,,$$
$$\chi \in \mathcal{C}^\infty_{\rm c}(X)\,, \ \text{supp}\,(\chi) \subset X \setminus Y^1 \quad \text{supp}\,(1-\chi) \subset Y^0$$

and the definition is independent of the choice of a specific χ. Since $1-\chi = 1$ on Y^1, multiplication by $1-\chi$ is well defined on $\mathcal{H}_{\theta,h,A}$.

Proposition 5.1. *If R_θ is a small neighbourhood of* 1 *given by Proposition 3.1 and $0 < h < h_0$ then for $z \in R_\theta$ and A large enough the operator*

$$P_{\theta,A}(h) \ : \ \mathcal{D}_{\theta,h,A} \longrightarrow \mathcal{H}_{\theta,h,A}$$

is a Fredholm operator of index 0*.*

Proof. We can use a local parametrix construction on a neighbourhood of N, N_1, together with Proposition 3.1. Except for some technical complications the argument follows [30], Lemma 3.2. Let $G^\flat_\theta(\alpha, z, h)$ and $G^\sharp_\theta(\alpha, z, h)$ be as in Proposition 3.1 with $a(\alpha, r) = \alpha^2 e^{2r}(\cosh r)^{-2}$. We also define

$$G^{\sharp,R}_\theta(\ell_j^{-1}nh, z, h) = \tilde\chi_0 \widetilde{Q}_\theta(\ell_j^{-1}nh, z, h)\tilde\theta_0 + \tilde\chi_1 G^\sharp_\theta(\ell_j^{-1}nh, z, h)\tilde\theta_1\,,$$
$$G^{\sharp,L}_\theta(\ell_j^{-1}nh, z, h) = \tilde\theta_0 \widetilde{Q}_\theta(\ell_j^{-1}nh, z, h)\tilde\chi_0 + \tilde\theta_1 G^\sharp_\theta(\ell_j^{-1}nh, z, h)\tilde\chi_1\,,$$

where $\widetilde{Q}_\theta(\ell_j^{-1}nh, z, h)$ is a properly supported parametrix of the operator $(hD_r)^2 + (n^2+1/4)h^2\ell_j^{-2}(\cosh r)^{-2}$ on $[0,\infty)$ with the Dirichlet boundary condition. The cut-off functions satisfy $\tilde\theta_j \in \mathcal{C}^\infty([0,\infty))$, $\tilde\theta_0 = 1$ on $[0, 2C_0]$, supp $\tilde\theta_0 \subset [0, 3C_0]$, $\tilde\theta_0 + \tilde\theta_1 = 1$ and $\tilde\chi_0 \in \mathcal{C}^\infty_{\rm c}([0,\infty))$, $\tilde\chi_1 \in \mathcal{C}^\infty([0,\infty))$ are equal to 1 on neighbourhoods of the supports of $\tilde\theta_0$ and $\tilde\theta_1$ respectively.

We then put

$$G^\bullet_\theta(z, h) \ : \ \mathcal{H}_{\theta,h,A}(Y^0) \longrightarrow \mathcal{D}^0_{\theta,h,A}$$

to be

$$G^\bullet_\theta(z,h)\restriction_{E^j_n \otimes U_{j,\theta} L^2(\Gamma_\theta(\ell_j^{-1}nh), dr)} = \begin{cases} I \otimes U_{j,\theta} G^\flat_\theta(\ell_j^{-1}nh, z, h) U^{-1}_{j,\theta} & |nh| \geq A/2 \\ I \otimes U_{j,\theta} G^{\sharp,\bullet}_\theta(\ell_j^{-1}nh, z, h) U^{-1}_{j,\theta} & |nh| < A/2 \end{cases}, \quad \bullet = L, R,$$

Using the classical ellipticity of $P(h) - z$ we can construct its properly supported parametrix $Q(z,h)$ on X. If $\theta_j \in \mathcal{C}^\infty(X)$ satisfy $\theta_0 = 1$ on N, supp $\theta_0 \subset N_1$, $\theta_0 + \theta_1 = 1$ and if $\chi_0 \in \mathcal{C}^\infty_{\rm c}(X)$, $\chi_1 \in \mathcal{C}^\infty(X)$ are equal to 1 on neighbourhoods of the supports of θ_0 and θ_1 respectively, we define

$$\widetilde{E}_\theta(z,h) = \chi_0 Q(z,h)\theta_0 + \chi_1 G^R_\theta(z,h)\theta_1\,,$$
$$\widetilde{F}_\theta(z,h) = \theta_0 Q(z,h)\chi_0 + \theta_1 G^L_\theta(z,h)\chi_1\,.$$

We can also demand that $\chi_1 \equiv 1$ on a neighbourhood of the support of $\widetilde{\chi}_1$. Then

$$(P_{\theta,A}(h) - z)\widetilde{E}_\theta(z,h) = I + \widetilde{K}_\theta(z,h)\,,$$
$$\widetilde{F}_\theta(z,h)(P_{\theta,A}(h) - z) = I + \widetilde{L}_\theta(z,h)\,,$$

where (we will discuss the right inverse first)

$$\widetilde{K}_\theta(z,h) = \widetilde{K}^0_\theta(z,h) + [P(h), \chi_0]Q(z,h)\theta_0 + [P_{\theta,A}(h), \chi_1]G^R_0(z,h)\theta_1$$

with $\widetilde{K}^0_\theta(z,h)$ coming from the local parametrix errors and hence compact. We rewrite $\widetilde{K}_\theta(z,h)$ as follows

$$\widetilde{K}_\theta(z,h) = K^0_\theta(z,h) + K^1_\theta(z,h) + K^2_\theta(z,h)\,,$$

$$
\begin{aligned}
&K^0_\theta(z,h) = \widetilde{K}^0_\theta(z,h) + [P(h),\chi_0]Q(z,h)\theta_0 \\
&K^1_\theta(z,h)\restriction_{E^j_n\otimes U_{j,\theta}L^2(\Gamma_\theta(\ell_j^{-1}nh),dr)} = \begin{cases} 0 & |nh|\geq A/2 \\ I\otimes [P^0_{\theta,A}(h),\chi_1\tilde\chi_0]U_{j,\theta}\widetilde{Q}_\theta(\ell_j^{-1}nh,z,h)\tilde\theta_0 U_{j,\theta}^{-1} & |nh|<A/2 \end{cases} \\
&K^2_\theta(z,h)\restriction_{E^j_n\otimes U_{j,\theta}L^2(\Gamma_\theta(\ell_j^{-1}nh),dr)} = \begin{cases} I\otimes [P^0_{\theta,A}(h),\chi_1]U_{j,\theta}G^\flat_\theta(\ell_j^{-1}nh,z,h)\tilde\theta_1 U_{j,\theta}^{-1} & |nh|\geq A/2 \\ I\otimes [P^0_{\theta,A}(h),\tilde\chi_1]U_{j,\theta}G^\sharp_\theta(\ell_j^{-1}nh,z,h)\tilde\theta_1\tilde\theta_1 U_{j,\theta}^{-1} & |nh|<A/2\,. \end{cases}
\end{aligned}
\tag{5.4}
$$

The operators $K^0_\theta(z,h), K^1_\theta(z,h)$ are clearly compact and we claim that

$$K^2_\theta(z,h) = \mathcal{O}(h) : \mathcal{H}_{\theta,h,A} \longrightarrow \mathcal{H}_{\theta,h,A}\,.$$

In fact, we only need to show that each term in the decomposition given in (5.4) is $\mathcal{O}(h)$ as an operator on $U_{j,\theta}L^2(\Gamma_\theta(\ell_j^{-1}nh),dr)$, uniformly with respect to n. The commutators appearing there are of the form $h(\psi(r)hD_r)$ where the functions ψ have supports disjoint from those of θ_1 and $\tilde\theta_1\theta_1$. The estimate (4.10) then gives uniform boundedness of each term.

Hence, for h small enough $I+K^2_\theta(z,h)$ is invertible on $\mathcal{H}_{\theta,h,A}$ and we can write

$$\widetilde{K}_\theta(z,h) = (I+K_\theta(z,h))(I+K^2_\theta(z,h))\,,\quad K_\theta(z,h) = (K^0_\theta(z,h)+K^\theta_1(z,h))(I+K^2_\theta(z,h))^{-1}\,.$$

The operator $K_\theta(z,h)$ is compact and $E_\theta(z,h) = \widetilde{E}_\theta(z,h)(I+K^2_\theta(z,h))^{-1}$ satisfies

$$(P_{\theta,A}(h)-z)E_\theta(z,h) = I + K_\theta(z,h)\,.$$

We proceed similarly to construct the left approximate inverse. Writing out the corresponding terms (with L replacing K) we now need

$$L^2_\theta(z,h) = \mathcal{O}(h) : \mathcal{D}_{\theta,h,A} \longrightarrow \mathcal{D}_{\theta,h,A}$$

and that amounts to showing that for a fixed z_0 with $\operatorname{Im} z_0 > 0$

$$
\begin{gathered}
(P^0_{\theta,A}(h)-z_0)\theta_1 U_{j,\theta}G^\flat_\theta(\ell_j^{-1}nh,z,h)U_{j,\theta}^{-1}[P^0_{\theta,A}(h),\chi_1]U_{j,\theta}G^\flat_\theta(\ell_j^{-1}nh,z_0,h)U_{j,\theta}^{-1} = \\
\mathcal{O}(h) : U_{j,\theta}L^2(\Gamma_\theta(\ell_j^{-1}nh),dr) \longrightarrow U_{j,\theta}L^2(\Gamma_\theta(\ell_j^{-1}nh),dr)\,,\ |nh|\geq A/2 \\
(P^0_{\theta,A}(h)-z_0)\theta_1\tilde\theta_1 U_{j,\theta}G^\sharp_\theta(\ell_j^{-1}nh,z,h)U_{j,\theta}^{-1}[P^0_{\theta,A}(h),\chi_1]U_{j,\theta}G^\flat_\theta(\ell_j^{-1}nh,z_0,h)U_{j,\theta}^{-1} = \\
\mathcal{O}(h) : U_{j,\theta}L^2(\Gamma_\theta(\ell_j^{-1}nh),dr) \longrightarrow U_{j,\theta}L^2(\Gamma_\theta(\ell_j^{-1}nh),dr)\,,\ |nh|< A/2\,.
\end{gathered}
$$

We write

$$
\begin{gathered}
(P^0_{\theta,A}(h)-z_0)\theta_1\tilde\theta_1 U_{j,\theta}G^\sharp_\theta(\ell_j^{-1}nh,z,h)U_{j,\theta}^{-1}[P^0_{\theta,A}(h),\chi_1]U_{j,\theta}G^\flat_\theta(\ell_j^{-1}nh,z_0,h)U_{j,\theta}^{-1} = \\
([P_{\theta,A}(h)^0(h),\theta_1] + (z-z_0)\theta_1)U_{j,\theta}G^\sharp_\theta(\ell_j^{-1}nh,z,h)U_{j,\theta}^{-1}[P^0_{\theta,A}(h),\chi_1]U_{j,\theta}G^\flat_\theta(\ell_j^{-1}nh,z_0,h)U_{j,\theta}^{-1}\,,
\end{gathered}
$$

as one term drops out since $\theta_1 = 0$ on the support of the commutator on the right side. The $G^\sharp_\theta$ terms is analysed similarly and remaining terms are analysed as before. Finally, continuous deformation of z from the lower to the upper half-plane and then of θ to 0 shows that the index is equal to the index of $P(h)-z_0$, $\operatorname{Im} z_0 > 0$, hence it is zero. $\square$

The proposition implies that for $z_0 \in R_\theta$

$$z_0 \in \operatorname{spec}\,(P_{\theta,A}(h)) \Longleftrightarrow \ker\,(P_{\theta,A}(h)-z_0) \neq 0$$

and through the application of analytic Fredholm theory and of the Jordan normal form theorem we can obtain a precise description of $(P_{\theta,A}(h)-z)^{-1}$ near z_0 (see [30], p.744 and [17]). This motivates the following definition:

$$\pi_{\theta,z_0} = \frac{1}{2\pi i}\int_{\gamma_{\epsilon,z_0}} (z-P_{\theta,A}(h))^{-1}dz\,,\quad \gamma_{\epsilon,z_0}(t) = z_0 + \epsilon e^{it}\,, 0\leq t<2\pi\,,\quad \epsilon \ll 1\,.$$

The standard argument gives $\pi^2_{\theta,z_0} = \pi_{\theta,z_0}$. We next show

Lemma 5.1. *For $z \in R_\theta$ we have*

$$\pi_{\theta,z_0}(\mathcal{C}_c^\infty(N)) = \pi_{\theta,z_0}(\mathcal{H}_{\theta,h,A}) \,. \tag{5.5}$$

Proof. We modify slightly the argument of Proposition 3.6 of [30]. For that we first notice that the dual space of $\mathcal{H}_{\theta,h,A}$ can be identified with $\mathcal{H}_{-\theta,h,A}$ and this needs to be checked for $\mathcal{H}^0_{\theta,h,A}$ only. We will find a pairing on each term of the direct sum decomposition (5.2). For that we define a bilinear form

$$L^2(\Gamma_\theta(\ell_j^{-1}nh), dr) \times L^2(\Gamma_{-\theta}(\ell_j^{-1}nh), dr) \ni (u, v) \longmapsto \int_{\Gamma_\theta(\ell_j^{-1}nh)} u(r)\overline{v(\bar{r})} dr \,, \tag{5.6}$$

which gives $L^2(\Gamma_\theta, dr)^* \simeq L^2(\Gamma_{-\theta}, dr)$. The corresponding pairing on $\mathcal{H}_{\theta,h,A} \times \mathcal{H}_{-\theta,h,A}$ will be denoted by $(\bullet, \bullet)_\theta$. The adjoint to $P_{\theta,A}(h)$ is then checked to be $P_{-\theta,A}(h)$ so that $((P_{\theta,A}(h) - z)^{-1})^* = (P_{-\theta,A}(h) - \bar{z})^{-1}$ and consequently $\pi^*_{\theta,z_0} = \pi_{-\theta,\bar{z}_0}$. If (5.5) did not hold then there would exist $0 \neq u \in \pi_{-\theta,\bar{z}_0}(\mathcal{H}_{-\theta,h,A}) = (\pi_{\theta,z_0}(\mathcal{H}_{\theta,h,A}))^*$ (since $\pi^2_{\theta,z_0} = \pi_{\theta,z_0}$ implies that the images of π_{θ,z_0} and $1 - \pi_{-\theta,\bar{z}_0}$ are orthogonal) such that $(\pi_{\theta,z_0} v, u)_\theta = 0$ for all $v \in \mathcal{C}_c^\infty(N)$. We would then have

$$0 = (\pi_{\theta,z_0} v, u)_\theta = (v, \pi_{-\theta,\bar{z}_0} u)_\theta = (v, u)_\theta$$

which implies that $\mathbf{1}_N u \equiv 0$. At the same time, as an element of $\pi_{-\theta,\bar{z}_0}(\mathcal{H}_{-\theta,h,A})$, u satisfies the equation $(P_{-\theta,A}(h) - \bar{z}_0)^k u = 0$ for some $k \in \mathbb{N}$. Thus, by unique continuation, $\mathbf{1}_{N_1} u \equiv 0$ (on N_1 we have $P_{\theta,A}(h) = P(h)$). Hence in the direct sum all components vanish for r small. If we recall the action of the operator on the components, (5.3), we see that $u \equiv 0$ in $\mathcal{H}_{-\theta,h,A}$. □

This lemma implies that to study the relation between the spectrum of $P_{\theta,A}(h)$ and the poles of the meromorphic continuation of $(P(h) - z)^{-1}$ it is enough to study the relation between the meromorphic functions $((P_{\theta,A}(h) - z)^{-1} v_1, v_2)_\theta$ and $\langle (P(h) - z)^{-1} v_1, v_2 \rangle_{L^2(X)}$ for $v_i \in \mathcal{C}_c^\infty(N)$ (see Sect.5 of [10] of the discussion of the meromorphic continuation of $(P(h) - z)^{-1}$). They are in fact equal:

Lemma 5.2. *For $v_1, v_2 \in \mathcal{C}_c^\infty(N)$ and $z \in R_\theta$ we have*

$$\langle (P(h) - z)^{-1} v_1, v_2 \rangle_{L^2(X)} = ((P_{\theta,A}(h) - z)^{-1} v_1, v_2)_\theta \tag{5.7}$$

where $(P(h) - z)^{-1}$ denote the meromorphic continuation of the resolvent of $P(h)$ from $\operatorname{Im} z > 0$ to R_θ and $(\bullet, \bullet)_\theta$ is defined by using (5.6).

Proof. By a slight abuse of notation we denote by $(P(h)-z)^{-1}$ the meromorphic continuation of the resolvent as an operator from $L^2_{\mathrm{comp}}(X)$ to $H^2_{\mathrm{loc}}(X)$. We claim that for $v \in \mathcal{C}_c^\infty(N)$ we have

$$(P_{\theta,A}(h) - z)^{-1} v \restriction_N = (P(h) - z)^{-1} v \restriction_N \ .$$

Let us first take $\operatorname{Im} z > 0$. Then by Proposition 3.2 (applied with $\theta_1 = 0$ and $\theta_2 = \theta$) we obtain that $((P(h) - z)^{-1} v)^j_n \in U_j L^2([0, \infty))$, (where we denote by u^j_n the components of u in the decomposition (5.2)) extends holomorphically to a neighbourhood of $\bigcup_{0 \leq \theta' \leq \theta} \Gamma_{\theta'}(\ell_j^{-1} nh)$. The restriction of this extension to $\Gamma_\theta(\ell_j^{-1}nh)$ is in $U_{j,\theta} L^2(\Gamma_\theta(\ell_j^{-1}nh))$. Hence that restriction has to be equal to $((P_{\theta,A}(h) - z)^{-1} v)^j_n$.

By unique continuation for holomorphic functions on a neighbourhood of $\bigcup_{0 \leq |\theta'| \leq \theta} \Gamma_{\theta'}(\ell_j^{-1} nh)$ and for solutions of second order elliptic equation on $N_1 \setminus N$ we conclude that $(P_{\theta,A}(h) - z)^{-1} v \restriction_{N_1}$ is independent of θ and A. From this (5.7) follows. □

We remark that the proof of the lemma shows that the spectrum of $P_{\theta,A}(h)$ in R_θ is independent of θ, that is it is contained in the spectrum of $P_{\theta',A}(h)$ for $\theta' \geq \theta$. The argument was based however on knowing already that $(P(h) - z)^{-1}$ continues meromorphically and a direct approach using our complex scaling would require more care.

Lemmas 4.1 and 4.2 reduce the study of resonances to the study of the spectrum of $P_{\theta,A}(h)$ as described in the following

Proposition 5.2. *The poles of the meromorphic continuation of* $(P(h)-z)^{-1}$ *from* $\operatorname{Im} z > 0$ *to* R_θ *coincide with the eigenvalues of* $P_{\theta,A}(h)$ *provided that* $0 < \theta < \theta_0$, $0 < h < h_0$ *and that* A *is large enough. The multiplicities agree and more precisely*

$$m_s(R_X) = \dim \pi_{\theta,z}(\mathcal{H}_{\theta,h,A}) \,, \quad s(1-s) = h^{-2}z \;\; z \in R_\theta \,,$$

where $m_s(R_X)$ *is given by* (3.9).

Although the construction of the space $\mathcal{H}_{\theta,h,A}$ is very non-geometric we can, roughly speaking, identify it with an L^2 space of a manifold on a subset of the phase space. For that let $\widetilde{X}$ be a Grauert tube of X (see for instance [7]).

We define a smooth family of totally real submanifolds of $\widetilde{X}$

$$X_\theta^\sharp \subset \widetilde{X} \,, \quad X_0^\sharp = X \,,$$

defined by deforming the variable r in the funnels:

$$X_\theta^\sharp = N \cup Y_\theta^0 \,, \quad Y_\theta^0 = \bigsqcup_{j=1}^{N} Y_{j,\theta}^0 \,, \quad Y_{j,\theta}^0 \simeq \{(r + i f_\theta^\sharp(r), y) \;:\; r \in [0,\infty) \,,\; y \in \mathbb{R}/\ell_j\mathbb{Z}\} \,. \tag{5.8}$$

For $0 < \theta < \theta_0$ and with C_0 of Lemma 4.1 we then put

$$X_\theta = N \cup Y_\theta^0 \setminus Y_\theta^{2C_0} \,, \tag{5.9}$$

where $r \geq 2C_0$ in $Y_\theta^{2C_0}$ in the above identification. We then let $\widetilde{X}_\theta$ to be a precompact open subset of $\widetilde{X}$ which is a complex neighbourhood of X_θ. We note that as X_θ are totally real submanifolds of $\widetilde{X}_\theta$ which by choosing f_θ (satisfying the conditions required in (4.3)) and C_0 can be made close to $X \times \widetilde{X}_\theta$ in $\mathcal{C}^\infty$. The metric g_X continues holomorphically to a $T^{2,0}$ form $\tilde{g}_X$ on $\widetilde{X}$ and, by Lemma 4.1, its real part restricts to a Riemannian metric on X_θ:

$$g_\theta = \operatorname{Re} \tilde{g}_X \restriction_{T^* X_\theta} \,. \tag{5.10}$$

This gives us the Riemannian density, $d\mathrm{vol}_\theta$ on X_θ but we could have made any other choice. The operator $P(h)$ extends to a holomorphic differential operator on $\widetilde{X}_\theta$ which restricts (see for instance Sect.3 of [30]) to a classically elliptic operator on X_θ. Its symbol is given by $\tilde{g}_X$.

To obtain a map from $\mathcal{H}_{\theta,h,A}$ to $L^2(X_\theta, d\mathrm{vol}_\theta)$ we need some additional cut-off functions:

$$\begin{gathered} \psi_{\theta,A} = 1 - \chi_{\theta,A} \,, \quad \chi_{\theta,A} = \chi_{\theta,A}(r) \,, \quad , \\ \chi_{\theta,A}(r) \equiv 0 \,, r < 5C_0/3 \,, \quad \chi_{\theta,A}(r) \equiv 1 \,, r > 2C_0 \,, \end{gathered} \tag{5.11}$$

and

$$\rho_A \in \mathcal{C}^\infty(\mathbb{R};[0,1]) \,, \quad \rho_A = \begin{cases} 0 & |\eta| < 2A/3 \\ 1 & |\eta| > A \end{cases} \,.$$

As before $\chi_1 \in \mathcal{C}^\infty(Y^0;[0,1])$, $\chi_1 = 1$ on Y^2 and $\operatorname{supp}\chi_1 \subset Y^1$. In view of (4.3) multiplication by χ_1 is well defined on $\mathcal{H}_{\theta,h,A}$. We then observe that since ρ_A is equal to 1 for $|\alpha| < 2A$, we can define

$$\begin{gathered} \psi_{\theta,A}(1 - \chi_1\rho_A(hD_y)) \;:\; \mathcal{H}_{\theta,h,A} \longrightarrow L^2(X_\theta; d\mathrm{vol}_{g_\theta}) \,, \\ \|\psi_{\theta,A}(1 - \chi_1\rho_A(hD_y))u\|_{\mathcal{H}_{\theta,h,A}} \sim \|\psi_{\theta,A}(1 - \chi_1\rho_A(hD_y))u\|_{L^2(X_\theta; d\mathrm{vol}_{g_\theta})} \,. \end{gathered} \tag{5.12}$$

This provides a needed identification of a subset of elements of $\mathcal{H}_{\theta,h,A}$ with functions on X_θ. For later convenience we choose smooth cut-off functions: replacing them by characteristic functions would give a projection on a subspace of $L^2(X_\theta)$ and an identification of a closed subspace of $\mathcal{H}_{\theta,h,A}$ with that subspace.

6. Review of the construction of the escape function

The escape functions, also known as Lyapunoff functions, have a long tradition in the study of dynamical systems. Roughly speaking they are defined by demanding that they grow uniformly along the flow. We will now recall from [28] the construction of the escape function adapted to hyperbolic flows, modifying it slightly to our setting.

For $0 < \theta < \theta_0$ and $0 < \delta < 1$ we define

$$S^*_\delta X_\theta = \{ m \in T^* X_\theta \setminus 0 \; : \; 1 - \delta < g_\theta(m) < 1 + \delta \} \,,$$

where g_θ is the metric given by (5.10). We will construct the escape function near $S^*_{\epsilon_0} X_\theta$ for some small $\epsilon_0 > 0$. Away from N that is carried out rather easily:

Lemma 6.1. *There exists $G^\sharp_\theta \in \mathcal{C}^\infty(T^* X_\theta ; \mathbb{R})$ such that* $\operatorname{supp} G^\sharp_\theta \subset S^*_{\frac12} X_\theta$ *and for some $\epsilon_0 > 0$*

$$H_{g_\theta} G^\sharp_\theta \geq 1/C \quad on \quad S^*_{\epsilon_0} X_\theta \cap \pi^{-1}(\{ r > 1 \}) \,, \tag{6.1}$$

where r is as in (5.8) and C is independent of θ.

Proof. Let us recall that

$$g_\theta = \operatorname{Re} \left((1 + i\partial_r f^\sharp_\theta(r))^{-2} \rho^2 + (\cosh(r + i f^\sharp_\theta(r)))^{-2} \eta^2 \right)$$

and hence

$$\begin{aligned} H_{g_\theta} = \operatorname{Re} \Big(& 2(1 + i\partial_r f^\sharp_\theta(r))^{-2} \rho \partial_r + 2(\cosh(r + i f^\sharp_\theta(r)))^{-2} \eta \partial_y + \\ & 2 \left(i\partial_r^2 f^\sharp_\theta (1 + i\partial_r f^\sharp_\theta(r))^{-3} \rho^2 + (1 + i\partial_r f^\sharp_\theta(r))(\cosh(r + i f^\sharp_\theta(r)))^{-2} \tanh(r + i f^\sharp_\theta(r)) \eta^2 \right) \partial_\rho \Big) = \\ & 2(1 + \mathcal{O}(\theta^2)) \rho \partial_r + 2(\cosh r)^{-2} (1 + \mathcal{O}(\theta^2)) \eta \partial_y + 2(\mathcal{O}(\theta^2) + 2(\cosh r)^{-2} \tanh r (1 + \mathcal{O}(\theta^2) \eta^2) \partial_\rho \,, \end{aligned}$$

where the error terms depend on C_0 (which is independent of θ) in the definition of X_θ (see Lemma 3.1).

To define $G^\sharp_\theta$ we introduce cut-off functions $\chi \in \mathcal{C}^\infty(\mathbb{R}; [0,1])$, $\chi_0 \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}; [0,1])$:

$$\begin{aligned} & \chi(t) = 0 \quad \text{for} \quad t < -2 \,, \qquad \chi(t) = 1 \quad \text{for} \quad t > -1 \,, \quad \chi'(t) \geq 0 \,, \\ & \chi_0(t) = 0 \quad \text{for} \quad |t| > \frac{1}{2} \,, \qquad \chi_0(t) = 1 \quad \text{for} \quad |t| < \frac{1}{4} \,. \end{aligned}$$

Using $(r, y; \rho, \eta)$ as coordinates on $T^* X_\theta$ we put

$$G^\sharp_\theta(r; \rho, \eta) = \chi_0(\rho^2 + (\cosh r)^{-2} \eta^2 - 1) \chi(\delta^{-1} \log(\cosh(r)/|\eta|)) r \rho \,,$$

where $\delta > 0$ will be chosen and $\chi(\delta^{-1} \log(\cosh(r)/|\eta|))$ is smooth as for $|\eta| < 1$ it is identically one. Applying the Hamilton vector field we get

$$\begin{aligned} H_{g_\theta} G^\sharp_\theta = & 2(\rho^2 + r(\cosh r)^{-2} \tanh r \eta^2 + \mathcal{O}(\theta)) \chi(\delta^{-1} \log(\cosh r/|\eta|)) \\ & + 2\delta^{-1}(1 + \mathcal{O}(\theta^2)) r \rho^2 \tanh r \chi'(\delta^{-1} \log(\cosh r/|\eta|)) \,. \end{aligned}$$

When $|\rho^2 + (\cosh r)^{-2} \eta^2 - 1| \leq \epsilon_0$ then $0 \leq (\cosh r)^{-2} \eta^2 \leq 1 + \epsilon_0$ and hence $(r; \rho, \eta)$ are in the set where the cut-off $\chi \geq 1/2$, provided that $\delta > \delta_0(\epsilon_0)$. If $\rho^2 \leq 1/C$ then $|(\cosh r)^{-2} \eta^2 - 1| \leq \epsilon_0 + 1/C$ and consequently $2r(\cosh r)^{-2} \tanh r \eta^2 \geq 1/\widetilde{C}$. Since the term involving χ' is positive we obtain the desired lower bound for $H_{g_\theta} G^\sharp_\theta$. ∎

Remark. If C_0 is fixed and θ is sufficiently small depending on C_0 then the constant C in the lower bound (6.1) is independent of C_0 and θ. We actually do not need that here and could have instead used a simpler escape function defined for $r > 1$

$$G^\sharp_\theta(r; \rho, \eta) = \chi_0(\rho^2 + (\cosh r)^{-2} \eta^2 - 1) r \rho \,,$$

for which (6.3) would still hold with a constant C depending on C_0. The construction in the proof of Lemma 6.1 is nevertheless instructive as it is related to the complex scaling method used in Sect.4 and to the construction of the operator A in [5].

If we put

$$\Gamma^\theta_{\epsilon_0,\pm} = \{m \in S^*_{\epsilon_0} X_\theta \setminus 0 \;:\; \exists T(m) > 0 \; \forall \mp t > T(m) \quad \pi(\exp tH_{g_\theta}(m)) \in \overline{N} \subset X_\theta\}$$

then by Lemma 6.1

$$K_{\epsilon_0} = \Gamma^\theta_{\epsilon_0,+} \cap \Gamma^\theta_{\epsilon_0,-} \,. \tag{6.2}$$

is independent of θ, that is, $K_{\epsilon_0} = K \cap S^*_{\epsilon_0} X_\theta$.

The escape function, G, will be defined on a neighourhood of $S^*_{\epsilon_0} X_\theta$ and we demand that it satisfies the following properties:

$$\begin{gathered} G\,, H_{g_\theta} G \in \mathcal{C}^{1,1}_{\mathrm{c}}(S^*_{2\epsilon_0} X_\theta; \mathbb{R})\,,\; G\restriction_{S^*_{2\epsilon_0} X_\theta \cap T^* Y^0_\theta} \in \mathcal{C}^\infty(S^*_{2\epsilon_0} X_\theta \cap T^* Y^0_\theta; \mathbb{R})\,, \\ H_{g_\theta} G \geq \|\nabla G\|^2/C\,,\; H_{g_\theta} G \geq d(\bullet, K_{\epsilon_0})^2 \;\text{ on an open neighbourhood } W \text{ of } K_{\epsilon_0} \text{ in } S^*_{\epsilon_0} X_\theta\,, \\ H_{g_\theta} G \geq 1/C \;\text{ on }\; S^*_{\epsilon_0} X_\theta \setminus W\,. \end{gathered} \tag{6.3}$$

To construct G on a neighbourhood of K_{ϵ_0} in $S^*_{\epsilon_0} X_\theta$ we follow Sect.5 of [28] with a simplification due to the homegeneity of the Hamiltonian. Thus, let Ω be a small neighbourhood of $\widetilde{K} = K \cap S^* X_\theta$ in $T^* X_\theta \setminus 0$. Then there exist non-negative functions $\varphi_\pm \in \mathcal{C}^{1,1}(\Omega)$ such that for all compact subsets $B \subset \Omega$

$$\frac{1}{C_B} d(\rho, \widetilde{\Gamma}^\theta_\pm)^2 \leq \varphi_\pm(\rho) \leq C_B d(\rho, \widetilde{\Gamma}^\theta_\pm)^2\,,\; \rho \in B\,,$$

see Lemma 5.3 of [28]. The hyperbolicity of the flow, (3.3), gives for $\rho \in \Omega$ and a large $T > 0$ (arbitrarily large if we make Ω small enough)

$$d(\exp(\pm T H_{g_\theta})(\rho), \widetilde{\Gamma}^\theta_\pm) \leq C e^{-T/C} d(\rho, \widetilde{\Gamma}^\theta_\pm)\,,\quad d(\rho, \widetilde{\Gamma}^\theta_+)^2 + d(\rho, \widetilde{\Gamma}^\theta_-)^2 \sim d(\rho, \widetilde{K})^2\,.$$

Hence we put

$$\widetilde{G}(\rho) = \int_0^T (\varphi_-(\exp tH_{g_\theta}(\rho)) - \varphi_+(\exp tH_{g_\theta}(\rho)))dt\,, \tag{6.4}$$

so that for $\rho \in \Omega$

$$\begin{gathered} \widetilde{G} \in \mathcal{C}^{1,1}(\Omega)\,,\; H_{g_\theta}\widetilde{G} \in \mathcal{C}^{1,1}(\Omega)\,,\; \widetilde{G}(\rho) = \mathcal{O}(d(\rho, \widetilde{K})^2) \\ \nabla\widetilde{G} = \mathcal{O}(d(\rho, \widetilde{K}))\,,\; H_{g_\theta}\widetilde{G}(\rho) \sim d(\rho, \widetilde{K})^2\,. \end{gathered}$$

We then extend $\widetilde{G}$ by homogeneity and a cut-off to a neighbourhood of $S^* X_\theta \cap \Omega$ in $T^*_{\pi(\Omega)} X_\theta$.

To obtain a global escape function with properties given by (6.3) we use the appendix to [6] and Lemma 6.1 to extend $\widetilde{G}$ to G.

The actual weight function used in the microlocal estimates of the next section, G_θ, will be a truncation of G – we will give up the positivity on of $H_{g_\theta} G$ in the region where the operator $P_{\theta,A}(h)$ is uniformly elliptic. Thus we first put

$$G^\sharp_\theta = \widetilde{\psi}_{\theta,A} G\,,$$

where $\widetilde{\psi}_{\theta,A}$ is defined as $\psi_{\theta,A}$ in (5.11) with $\frac{5}{3}$ and 2 replaced by $\frac{4}{3}$ and $\frac{5}{3}$ respectively. Since our deformation does not coincide with X_θ for all values of η (roughly speaking) we need a yet another cut-off and we put

$$G_\theta = (1 - \widetilde{\chi}_1 \widetilde{\rho}_A) G^\sharp_\theta\,, \tag{6.5}$$

where $\widetilde{\rho}_A$ is defined the same way as ρ_A but with $2A/3$ and A defined by $A/2$ and $2A/3$ respectively and $\widetilde{\chi}_1$ same as χ_1 but with Y^2 and Y^1 replaced by Y^1 and $Y^{\frac{1}{2}}$ respectively. In particular we get $(1 - \widetilde{\chi}_1 \widetilde{\rho}_A)(1 - \chi_1 \rho_A)(1 - \widetilde{\chi}_1 \widetilde{\rho}_A)$.

7. FBI transformations and microlocally weighted spaces

We now fix θ small and introduce a new parameter t, $0 < t \ll \theta$. Using the escape function, G_θ, constructed in Sect.6 we will now define a new norm, $|||\bullet|||_{tG_\theta}$, on $\mathcal{H}^G_{\theta,h,A}$ which for each fixed $h > 0$ is equivalent to the old norm but without uniformity in h:

$$e^{-\frac{1}{Ch}}\|u\|_{\mathcal{H}_{\theta,h,A}} \leq |||u|||_{tG_\theta} \leq e^{\frac{1}{Ch}}\|u\|_{\mathcal{H}_{\theta,h,A}}\,.$$

For the definition we first need to consider the Fourier-Bros-Iagolnitzer (FBI) transform on X_θ.

We start with the construction of an appropriate phase function. For that let D be a small conic neighbourhood of

$$\Delta_0^* = \{(\alpha, z) \in T^*X \times X \ : \ \pi(\alpha) = z\}\,,$$

and let $\widetilde{D}$ be a conic complex neighbourhood of Δ_0^* in $\widetilde{T^*X} \times \widetilde{X}$ (here we can identify $\widetilde{T^*X}$ with a subset of $T^*\widetilde{X}$), $\widetilde{D} \cap (T^*X \times X) = D$. We define the following real analytic function on D:

$$\begin{aligned} &\varphi(\alpha, z) = -\langle \alpha_\xi, \exp^{-1}_{\alpha_x}(z)\rangle + \frac{i}{2}\langle\alpha_\xi\rangle d(\alpha_x, z)^2\,, \\ &\alpha = (\alpha_x, \alpha_\xi) \in T^*X\,, \quad \langle\alpha_\xi\rangle = (1 + |\alpha_\xi|^2_{\alpha_x})^{\frac{1}{2}}\,, \end{aligned} \tag{7.1}$$

where the exponential map and the norm, $|\bullet|^2_z$, are defined using the metric on X.

Provided that $\widetilde{D}$ is sufficiently small we can continue φ holomorphically to $\widetilde{D}$ where it becomes an *admissible phase function* in the sense of [29], that is

$$\begin{gathered} |\varphi(\alpha)| \leq C|\langle\alpha_\xi\rangle| \\ \varphi(\alpha, \pi(\alpha)) = 0\,, \ \alpha \in \widetilde{D} \\ -d_z\varphi(\alpha, \pi(\alpha)) = \alpha\,, \end{gathered} \tag{7.2}$$

and for $\alpha \in D$, that is in the real domain,

$$\operatorname{Im} d_z^2\varphi(\alpha, \pi(\alpha)) \sim \langle\alpha_\xi\rangle \mathrm{Id}\,, \quad \alpha \in D\,. \tag{7.3}$$

We note that now the Hessian is well defined as $\operatorname{Im} d_z\varphi = \operatorname{Im}(-\alpha_\xi) = 0$.

Considering $T^*X_\theta \times X_\theta$ as a subset of $\widetilde{T^*X} \times \widetilde{X}$ (which is possible for θ small) we can take $\widetilde{D}$ to be also a neighbourhood of

$$\Delta_\theta^* = \{(\alpha, z) \in T^*X_\theta \times X_\theta : \pi(\alpha) = z\}\,.$$

We then define

$$\varphi_\theta = \varphi\restriction_{\widetilde{D}\cap(T^*X_\theta\times X_\theta)}\,. \tag{7.4}$$

From (7.2) and (7.3) we have

$$\begin{gathered} |\partial^k_{\alpha_x}\partial^l_{\alpha_\xi}\partial^m_z\varphi_\theta| = \mathcal{O}_{k,l,m}(1)|\langle\alpha_\xi\rangle|^{1-|l|} \\ \varphi_\theta(\alpha, \pi(\alpha)) = 0\,, \ \alpha \in T^*X_\theta \subset T^*\widetilde{X} \\ -d_z\varphi_\theta(\alpha, \pi(\alpha)) = \alpha \end{gathered} \tag{7.5}$$

and

$$\operatorname{Im}\varphi_\theta(\alpha, z) \sim |\langle\alpha_\xi\rangle| d_\theta(\alpha_x, z)^2\,, \tag{7.6}$$

where d_θ is the distance on X_θ given by the metric g_θ, see (5.10). To see this we recall that X_θ is close to X in $\mathcal{C}^1$. More precisely we can express X_θ as a graph over $X_0 = N \cup Y^0 \setminus Y^{C_0} \subset X$ in $\widetilde{X}$. For that we can identify $\widetilde{X}$ with a neighbourhood of the zero section in T^*X (see for instance [7]) and introduce

$$X \supset X_0 \ni z \longmapsto F_\theta(z) = (z, f_\theta(z)) \in X_\theta \subset \widetilde{X} \subset T^*X\,.$$

where $\|\partial_z^k f_\theta\| \to 0$ as $\theta \to 0$, uniformly on X_0 for some fixed family of norms $\|\bullet\|$. This and the last part of Lemma 4.1 show that

$$d_\theta(F_\theta(z), F_\theta(w)) \sim d(z,w)\,, \quad z, w \in X^\theta\,, \tag{7.7}$$

that is the distances are uniformly comparable. Since X_θ is totally real, $\mathrm{Im}\langle \alpha_\xi, \exp_{\alpha_x}^{-1} z\rangle$ vanishes to second order on $\widetilde{D} \cap (T^*X_\theta \times X_\theta)$. In fact, the map $TX \ni (z,\zeta) \mapsto \exp_z(\zeta) \in X$ extends to a holomorphic map on $\widetilde{TX}$. Locally we can represent that Grauert tube as a subset of $\mathbb{C}^2 \times \mathbb{C}^2$ (that is, we introduce some holomorphic coordinates and trivialize the tangent bundle) and by continuation from TX we have that for $(z,\zeta) \in \mathbb{C}^2 \times \mathbb{C}^2$, $d_\zeta \exp_z(\bullet)|_{\zeta=0} = Id$. For a small neighbourhood U of α_z in X_θ we consider the map $X_\theta \times U \ni (\alpha_z, z) \mapsto \exp_{\alpha_z}^{-1}(z) \in T_{\alpha_z}\widetilde{X}$. From the discussion above, the differential with respect to z at $z = \alpha_z$ maps into $T_z X_\theta \subset T_z\widetilde{X}$. Hence $\langle d_z \exp_{\alpha_z}^{-1}(z), \alpha_\zeta\rangle|_{z=\alpha_z}$ is real.

We conclude that

$$\mathrm{Im}\,\varphi_\theta(\alpha, z) = \mathcal{O}(\langle \alpha_\xi\rangle) d_\theta(\alpha_x, z)^2\,, \quad (\alpha, z) \in \widetilde{D} \cap (T^*X_\theta \times X_\theta) \tag{7.8}$$

Directly from the definition of φ, (7.1),

$$\mathrm{Im}\,\varphi(\alpha, z) \sim \langle \alpha_\xi\rangle d(\alpha_x, z)^2\,, \quad (\alpha, z) \in D = \widetilde{D} \cap (T^*X \times X)\,, \tag{7.9}$$

and by a perturbations argument

$$\mathrm{Im}\,\varphi(\alpha, z) \sim |\langle \alpha_\xi\rangle| d_\theta(\alpha_x, z)^2\,, \quad (\alpha, z) \in \widetilde{D} \cap (T^*X_\theta \times X_\theta)\,,$$

provided that we take $\widetilde{D}$ small enough (as the neighbourhood of the lift of the diagonal).

Thus for $u \in L^2_{\mathrm{comp}}(X_\theta)$ we define

$$T_\theta u(\alpha; h) = \int_{X_\theta} e^{i\varphi_\theta(\alpha,z)/h} a_\theta(\alpha, z; h)\chi_\theta(\alpha_z, z) u(z) d\mathrm{vol}_{g_\theta}\,, \quad \alpha = (\alpha_z, \alpha_\zeta) \in T^*X_\theta \subset T^*\widetilde{X}\,,$$

where φ_θ is given by (7.4). The amplitude a_θ is obtained by restricting an elliptic holomorphic symbol on $\widetilde{D}$, to $\widetilde{D} \cap (T^*X_\theta \times X_\theta)$:

$$a_\theta = a\restriction_{\widetilde{D}\cap(T^*X_\theta\times X_\theta)}\,, \quad a \in S^{\frac{3}{2},\frac{1}{2}}_{\mathrm{phg}}(\widetilde{D})\,.$$

Here $S^{m,k}_{\mathrm{phg}}$ denotes the class of semi-classical poly-homogeneous holomorphic symbols bounded by $h^{-m}\langle\alpha_\zeta\rangle^m$ – see [29], (1.6)-(1.8). The cut-off function χ_θ is obtained by restricting a cut-off function on $\widetilde{X}\times\widetilde{X}$ to $X_\theta \times X_\theta$:

$$\chi_\theta = \chi\restriction_{X_\theta\times X_\theta}\,, \ \chi \in \mathcal{C}^\infty(\widetilde{X}\times\widetilde{X})\,, \ \mathrm{supp}\,\chi \subset \pi(\widetilde{D})\,,$$

where $\pi : \widetilde{T^*X}\times\widetilde{X} \to \widetilde{X}\times\widetilde{X}$. We assume that $\chi \equiv 1$ on a neighbourhood of $\{(\alpha, z) \in \widetilde{T^*X}\times\widetilde{X} \ : \ \pi(\alpha) = x\}$.

The restriction to $L^2_{\mathrm{comp}}(X_\theta)$ comes from openness of X_θ: its closure is a manifold with boundary so the study of $u \in L^2(X_\theta)$ would introduce new and irrelevant difficulties. Below we shall only apply the FBI transformation to functions which are compactly supported in X_θ so that restriction is indeed of no consequence.

The transformation we introduced has all the essential properties of the global FBI transformation discussed in Sect.1 of [29]. Thus the same argument as there shows that there exists $b \in S^{\frac{3}{2},\frac{1}{2}}(\widetilde{D})$ such that we have an approximate inverse of T:

$$S_\theta v(z) = \int_{T^*X_\theta} e^{-i\varphi^*_\theta(\alpha,z)} b_\theta(\alpha, z, h)\chi_\theta(\alpha_z, z) d\alpha\,, \quad b_\theta = b\restriction_{\widetilde{D}\cap T^*X_\theta\times X_\theta}\,,$$

where φ^*_θ is the restriction to $T^*X_\theta \times X_\theta$ of the holomorphic continuation of $\bar\varphi\restriction_{\widetilde{D}\cap T^*X\times X}$ and $d\alpha$ is the canonical measure on T^*X_θ. We have

$$S_\theta T_\theta u = u + R_\theta u\,, \quad u \in L^2_{\mathrm{comp}}(X_\theta)\,, \quad |\partial_z^k \partial_{z'}^l R_\theta(z, z'; h)| \le C_{k,l}\exp(-1/Ch)\,,$$

where the constants depend on $d(\mathrm{supp} u,\ \partial X_\theta)$.

For $G_\theta \in \mathcal{C}^{1,1}_{\rm c}(T^*X_\theta)$ constructed in Sect.6 we obtain its $\mathcal{C}^{1,1}$ extension to the Grauert tube of T^*X_θ with the property that

$$(dG_\theta)_\rho \restriction_{JT_\rho(T^*X_\theta)} = 0 \tag{7.10}$$

where $J : T_\rho(\widetilde{T^*X_\theta}) = T_\rho(\widetilde{T^*X}) \to T_\rho(\widetilde{T^*X}) = T_\rho(\widetilde{T^*X_\theta})$ is the complex involution. Over T^*N_1 one way to obtain this is by identifying $\widetilde{T^*N_1}$ with a neighbourhood of the zero section in $T^*(T^*N_1)$ in which J and the metric give the canonical symplectic structure (see for instance [7]) and then making G constant on the fibers and over T^*Y_θ the topology is trivial so the extension is easy.

Denoting the extension by the same symbol, $G \in \mathcal{C}^{1,1}(\widetilde{T^*X_\theta})$, we now define a Lipschitz I-Lagrangian, $\mathbb{R}$-symplectic submanifold of $\widetilde{T^*X_\theta}$:

$$\Lambda_{tG_\theta} = \exp(tH^{\operatorname{Im}\sigma}_{G_\theta})(T^*X_\theta) \subset \widetilde{T^*X_\theta}\,, \quad \operatorname{Im}\sigma(X, H^{\operatorname{Im}\sigma}_{G_\theta}) = dG_\theta(X)\,, \quad \sigma = d\alpha_\zeta \wedge d\alpha_z\,.$$

Here by a Lipschitz manifold we mean a graph of a Lipschitz function. We remark that a Lipschitz graph in $\widetilde{T^*X}$ over T^*X_θ (and hence for small θ over T^*X_θ) which is I-Lagrangian (i.e. its almost everywhere defined tangent plane is Lagrangian with respect to $\operatorname{Im}\sigma$) can be locally written as a graph of a differential of a $\mathcal{C}^{1,1}$ function, $\widetilde{G}$, on $T^*X \simeq T^*\mathbb{R}^2$ (locally):

$$\begin{gathered}\{(x,\xi) \in T^*\mathbb{C}^n \ : \operatorname{Im} x = \frac{\partial \widetilde{G}}{\partial \operatorname{Re}\xi}(\operatorname{Re} x, \operatorname{Re}\xi)\,,\ \operatorname{Im}\xi = -\frac{\partial \widetilde{G}}{\partial \operatorname{Re} x}(\operatorname{Re} x, \operatorname{Re}\xi)\,\}\,, \\ (\operatorname{Re} x, \operatorname{Re}\xi) \in T^*\mathbb{R}^n \simeq T^*X\,,\ n = 2\,,\end{gathered} \tag{7.11}$$

(this is the same as showing that $\Lambda = (x, \xi(x)) \subset T^*\mathbb{R}^n$, $\xi(x)$ Lipschitz, $T_{x,\xi(x)}\Lambda$ Lagrangian for almost all x, is given by $\xi(x) = d\phi(x)$, with $\phi \in \mathcal{C}^{1,1}(\mathbb{R}^n)$ and that in turn follows from the Fundamental Theorem of Calculus of Lebesgue integration – see for instance [4], (3.36)).

The form $-\operatorname{Im}\zeta dz\restriction_{\Lambda_{tG_\theta}}$ is (formally) closed and, as Λ_{tG_θ} is close to T^*X_θ, it has a primitive which is a Lipschitz function on Λ_{tG_θ} (to see that it is Lipschitz we can for instance use the exact local formula based on (7.11), see Lemma 7.1 below, or a global formula (1.17) from [29]). We denote it by $H_{\Lambda_{tG_\theta}} \in \operatorname{Lip}(\Lambda_{tG_\theta}; \mathbb{R})$. We normalize $H_{\Lambda_{tG_\theta}}$ by demanding that it is equal to 0 for $|\alpha_\zeta|$ large enough.

Thanks to the global properties of the amplitude and the phase function of T_θ we can continue it in α to a neighbourhood of T^*X_θ, $\widetilde{T^*X_\theta}$, in $T^*\widetilde{X}$ and in particular we can define

$$T_{\Lambda_{tG_\theta}} u(\alpha; h) = T_\theta u\restriction_{\Lambda_{tG_\theta}} (\alpha; h)\,, \quad \alpha \in \Lambda_{tG_\theta}\,.$$

A deformation agument which despite the irregularity of Λ_{tG_θ} proceeds as in Sect.1 of [29] gives an approximate inverse, $S_{\Lambda_{tG_\theta}}$, with the same properties as S_θ above.

Following [29] (see also [15] and [28]) we can now put, for $u \in \mathcal{C}^\infty_{\rm c}$,

$$\|u\|^2_{H(X_\theta;\Lambda_{tG_\theta},\langle\alpha_\zeta\rangle^m)} = \int_{\Lambda_{tG_\theta}} |T_{\Lambda_{tG_\theta}} u(\alpha; h)|^2 |\langle\alpha_\zeta\rangle|^{2m} e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} d\alpha\,,$$

where again $d\alpha$ is the canonical measure on the totally real (as a submanifold of $T^*\widetilde{X}$), Re-symplectic manifold Λ_{tG_θ}. Using $S_{\Lambda_{tG_\theta}}$ we can show the independence of this norm of the choice of a specific phase function (see the proof of Lemma 7.2 below for a similar argument). We will eventually replace $\| \bullet \|_{H(X_\theta;\Lambda_{tG_\theta},\langle\alpha_\zeta\rangle^m)}$ by an equivalent norm obtained by using a decomposition of T^*X_θ.

To describe this decomposition we first need to introduce some local coordinates. Let X_1 be a neighbourhood of N in X such that $X_1 \subset X_\theta$ for all $0 \leq \theta \leq \theta_0$. We denote by $\widetilde{X}_1$ a neighbourhood of X in $\widetilde{X}$ and we let $\widetilde{X}_1 \subset \bigcup_{q=1}^Q \Omega_q$ to be a covering of $\widetilde{\widetilde{X}}_1$ by open charts with holomorphic coordinate maps, $\chi_q : \Omega_q \to \mathbb{C}^2$, such that $\chi_q : \Omega_q \cap X \to \mathbb{R}^2$. We also assume that there exist Ω'_q, $\widetilde{X} \supset \Omega'_q \Supset \Omega_q$, such that χ_q extends to a coordinate map on Ω'_q.

Let $T_0 : L^2(\mathbb{R}^2) \to H^2_{\Phi_0}(\mathbb{C}^2)$ be the unitary operator given by the Bargman type FBI transformation (here H^2_Φ denotes the spaces of holomorphic functions in $\mathbb{C}^2$ which are square integrable with respect to

$\exp(-2\Phi(w)/h)\mathcal{L}(dw)$, $\mathcal{L}(dw) = (2i)^{-1}d\bar{w} \wedge w$; the norm on H^2_Φ is denoted by $\| \bullet \|_\Phi$; here $\Phi(w) = \Phi_0(w) = (\operatorname{Im} w)^2/2)$:

$$T_0 u(w,h) = \frac{\sqrt{2}}{(2\pi h)^{\frac{3}{2}}} \int_{\mathbb{R}^2} e^{-(x-w)^2/2h} u(x)dx \,.$$

Let $\Omega_q^\sharp$ be an open precompact subset of $\chi_q(\Omega_q \cap X) - i\mathbb{R}^2 \subset \mathbb{C}^2$. Then $\Omega_q^\sharp$ can be identified with a subset of $T^*_{\Omega_q} X$:

$$\alpha_q^{-1} \;:\; (\alpha_z, \alpha_\zeta) \longmapsto (\chi_q(\alpha_z), {}^t(d\chi_q(\alpha_z))^{-1}(\alpha_\zeta)) \in \chi_q(\Omega_q \cap X) \times \mathbb{R}^2 \simeq \chi_q(\Omega_q \cap X) - i\mathbb{R}^2 \,.$$

We now define a new transformation T_q by

$$T_q u(w,h) = T_0((\chi_q^{-1})^*(u\restriction_{\Omega_q' \cap X}))\restriction_{\Omega_q^\sharp} \,, \tag{7.12}$$

which is independent of the extensions of $u\restriction_{\Omega_q' \cap X}$ up to exponentially small errors.

The phase of T_q is given by

$$\phi_q(w,z) = \frac{i}{2}(\chi_q(z) - w)^2 \tag{7.13}$$

which is holomorphic in $z \in \Omega_q'$ and gives an admissible phase function once $-i\Phi_0(w) = -i(\operatorname{Im} w)^2/2$ is subtracted. Thus T_q can be considered as a local FBI transformation. Before considering the mapping properties of T_q we need the following

Lemma 7.1. *Let ϕ_1 and ϕ_2 be admissible phase functions in a neighbourhood $U \times V \subset \mathbb{C}^{2n} \times \mathbb{C}^n$ of $(\alpha_0, (\alpha_0)_x) \in \mathbb{R}^{2n} \times \mathbb{R}^n$, $\alpha_0 = ((\alpha_0)_x, (\alpha_0)_\xi)$, that is, ϕ_j are holomorphic in all variables and*

$$\phi_j(\alpha, \alpha_x) = 0 \,,\; \partial_z \phi_j(\alpha, \alpha_x) = -\alpha_\xi \,,\; \alpha \in U \,,\; \operatorname{Im} \partial_z^2 \phi_j(\alpha, \alpha_x) \sim I \,,\; \alpha \in U \cap \mathbb{R}^{2n} \,, \quad j = 1, 2 \,. \tag{7.14}$$

Let Λ_ϵ be given by (7.11) with $\widetilde{G} = \epsilon G$, $G \in \mathcal{C}^{1,1}(\mathbb{R}^{2n}; \mathbb{R})$ and let $H(\alpha) = -\operatorname{Re}\alpha_\xi \cdot \operatorname{Im}\alpha_\xi + \epsilon G(\operatorname{Re}\alpha_x, \operatorname{Re}\alpha_\xi)$, so that $d_\alpha H\restriction_{\Lambda_\epsilon} = -\operatorname{Im}\alpha_\xi d\alpha_x\restriction_{\Lambda_\epsilon}$. Let us also put $\phi_j^*(\alpha, z) = \overline{\phi_j(\bar{\alpha}, \bar{z})}$.

Then for U and ϵ small enough

$$\psi(\alpha, \beta) = \text{critical value}_z(\phi_1(\alpha, z) - \phi_2^*(\beta, z)) \tag{7.15}$$

satisfies

$$-H(\alpha) - \operatorname{Im}\psi(\alpha, \beta) + H(\beta) \leq -\frac{1}{C}(|\alpha - \beta|^2) \,,\; (\alpha, \beta) \in (U \cap \Lambda_\epsilon) \times (U \cap \Lambda_\epsilon) \,. \tag{7.16}$$

Proof. When G is smooth this follows from Proposition 4.6 of [15] but we need to be more careful due to the low regularity of G (which in fact is minimal). We start by recalling that for $\alpha = \beta$ the critical value is $z = \alpha_x$ and that

$$\psi(\alpha, \alpha) = 0 \,, \;\; d_\alpha \psi(\alpha, \alpha) = -d_\beta \psi(\alpha, \alpha) = \alpha_\xi \,.$$

Hence Taylor's formula gives

$$\psi(\alpha, \beta) = (\alpha_x - \beta_x)\alpha_\xi + \mathcal{O}(|\alpha - \beta|^2) \,, \tag{7.17}$$

and in the real domain we have $\operatorname{Im}\psi(\alpha, \beta) \sim |\alpha - \beta|^2$. Since the derivatives of G are Lipschitz, the behaviour of the quadratic term in (7.17) is, for small ϵ, controlled by the behaviour of ψ in the real domain:

$$\operatorname{Im}(\psi(\alpha, \beta) - (\alpha_x - \beta_x)\alpha_\xi) \geq \frac{1}{C}|\alpha - \beta|^2 \,, \;\; (\alpha, \beta) \in (U \cap \Lambda_\epsilon) \times (U \cap \Lambda_\epsilon) \,,$$

provided that we take U small (independently of ϵ). Hence we need to show that

$$|\operatorname{Im}((\alpha_x - \beta_x)\alpha_\xi) + H(\alpha) - H(\beta)| \leq C\epsilon|\alpha - \beta|^2 \,, \;\; (\alpha, \beta) \in (U \cap \Lambda_\epsilon) \times (U \cap \Lambda_\epsilon) \,. \tag{7.18}$$

Putting $\mathrm{Re}\,\alpha = (x,\xi)$ and $\mathrm{Re}\,\beta = (y,\eta)$ and using the definitions of Λ_ϵ, (7.11), and of $H(\alpha)$ above, we divide the left hand side of (7.18) by ϵ and expand it to obtain:

$$\begin{aligned}
&- G'_{\mathrm{Re}\,\alpha_x}(x,\xi)(x-y) + (G'_{\mathrm{Re}\,\alpha_\xi}(x,\xi) - G'_{\mathrm{Re}\,\alpha_\xi}(y,\eta))\xi \\
&\qquad - G'_{\mathrm{Re}\,\alpha_\xi}(x,\xi)\xi + G'_{\mathrm{Re}\,\alpha_\xi}(y,\eta)\eta + G(x,\xi) - G(y,\eta) \\
&\quad = -G'_{\mathrm{Re}\,\alpha_x}(x,\xi)(x-y) - G'_{\mathrm{Re}\,\alpha_\xi}(y,\eta)\xi + G'_{\mathrm{Re}\,\alpha_\xi}(y,\eta)\eta \\
&\qquad + G'_{\mathrm{Re}\,\alpha_x}(x,\xi)(x-y) + G'_{\mathrm{Re}\,\alpha_\xi}(x,\xi)(\xi-\eta) + \mathcal{O}(|x-y|^2 + |\xi-\eta|^2) \\
&\quad = (G'_{\mathrm{Re}\,\alpha_\xi}(x,\xi) - G'_{\mathrm{Re}\,\alpha_\xi}(y,\eta))(\xi-\eta) + \mathcal{O}(|x-y|^2 + |\xi-\eta|^2) \\
&\quad = \mathcal{O}(|x-y|^2 + |\xi-\eta|^2)\,.
\end{aligned}$$

Bringing back the factor ϵ and noting again that the Lipschitz behaviour of the derivative of G gives $|x-y|^2 + |\xi-\eta|^2 = \mathcal{O}(|\alpha-\beta|^2)$, for ϵ small enough, we obtain (7.18). $\square$

We can now describe the mapping properties of T_q:

Lemma 7.2. *If $m \in \mathbb{R}$, θ is small enough and T_q is defined by (7.12) then*

$$T_q \ : \ H(X_\theta; \Lambda_{tG_\theta}, \langle\alpha_\zeta\rangle^m) \longrightarrow H^2_{\Phi_{q,t}}(\Omega_q^\sharp)\,, \quad \Omega_q^\sharp \Subset \chi_q(\Omega_q \cap X) - i\mathbb{R}^2 \subset \mathbb{C}^2\,,$$

where

$$\Phi_{t,q}(w) = \frac{1}{2}(\mathrm{Im}\,w)^2 + F_{q,t}\,, \quad F_{q,t} \in \mathcal{C}^{1,1}(\Omega_q^\sharp)$$

is given by

$$\kappa_q(\Lambda_{tG_\theta}) = \Lambda_{\Phi_{q,t}}\,, \quad \Lambda_\Phi = \left\{ \left(w, \frac{2}{i}\partial_w \Phi(w) \right) \ : \ w \in \mathbb{C}^2 \right\}\,,$$

$$\kappa_q \ : \ \widetilde{T^*X_1} \ni (z, -\partial_z\phi_q(z)) \longmapsto (w, \partial_w\phi_q(w)) \in T^*\mathbb{C}^2\,,$$

where ϕ_q is given by (7.13). The normalization of $F_{q,t}$ is provided by requiring that $F_{q,t} = 0$ for $|\alpha_\zeta| \geq C$.

Proof. The lemma is a consequence of a local version of the independence of the space $H(X_\theta; \Lambda_{tG_\theta}, \langle\alpha_\zeta\rangle^m)$ from a particular choice of an FBI transformation, and is similar to the arguments of [15], Sect.5.

We need to show that

$$\int_{\Omega_q^\sharp} |T_q u(w,h)|^2 e^{-(\mathrm{Im}\,w)^2/h} e^{-2F_{q,t}(w)/h} \mathcal{L}(dw) \leq C \int_{\Lambda_{tG_\theta}} |T_{\Lambda_{tG_\theta}}(\alpha,h)|^2 |\langle\alpha_\zeta\rangle|^{2m} e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} d\alpha\,, \tag{7.19}$$

and for that we need to move on the left hand side to the I-Lagrangian Λ_{tG_θ}.

For that we define

$$\Lambda_{q,w} = \{(z, -d_z\phi_q(w,z)) \ : \ z \in \Omega_q'\} \subset T^*\Omega_q'\,,$$

so that $\Lambda_{q,w} \cap T^*X_1 = \{\alpha_q(w)\}$, with a transversal intersection which is particularly easy to see here as ϕ_q is quadratic in the coordinates on Ω_q'. Since Λ_{tG_θ} is close to T^*X_θ we also get a transversal intersection $\Lambda_{tG_\theta} \cap \Lambda_{q,w} = \{\beta_q(w)\}$ and a bi-Lipschitz map $\alpha_q(w) \mapsto \beta_q(w)$. We can assume that the image of $\Omega_q^\sharp$ under β_q is contained in a compact subset of $\Lambda_{tG_\theta} \cap T^*\Omega_q'$ (again, since Λ_{tG_θ} is close to T^*X_θ). We then define

$$T_{q,\Lambda_{tG_\theta}} u(\beta;h) = T_q u(w;h) e^{-\frac{i}{h}\phi_q(w,\beta_z)}\,, \quad \beta = \beta_q(w) \in \Lambda_{tG_\theta} \cap T^*\Omega_q\,.$$

In the coordinates $(\alpha_x, \alpha_\xi) = \alpha_q^{-1}(\beta) \in \mathbb{C}^4$, $x = \chi_q(z) \in \mathbb{C}^2$, the new transformation takes a particularly simple form

$$T_{q,\Lambda_{tG_\theta}} u(\alpha;h) = \frac{\sqrt{2}}{(2\pi h)^{\frac{3}{2}}} \int_{\chi_q(\Omega_q')} e^{\frac{i}{h}((\alpha_x - x)\alpha_\xi + i(x-\alpha_x)^2/2)} u(x)dx\,, \ \alpha \in \alpha_q^{-1}(\Lambda_{tG_\theta} \cap T^*\Omega_q)\,,$$

that is, we obtain the Bargman-FBI transformation adapted to Λ_{tG_θ} in the sense of [15]. The simple form of the phase is a consequence of general considerations: if $\psi_0(\alpha,x) = (\alpha_x - x)\alpha_\xi + i(\alpha_x - x)^2/2$ and

$\Lambda_\alpha = \{(x, -\partial_x \psi_0(\alpha, x) : x \in \mathbb{C}^n\}$ then for $\beta \in \Lambda_\alpha$, $\psi_0(\alpha, x) - \psi_0(\alpha, \beta_x) = \psi_0(\beta, x)$. In the coordinates α, x above the phases are precisely of this form and consequently we have this special form of the FBI transform.

The analyticity of the phase in α allows now a construction of an approximate inverse (we will use it later in the proof of Lemma 7.3)

$$S_{q,\Lambda_{tG_\theta}} v(x,h) = \frac{\sqrt{2}}{(2\pi h)^{\frac{3}{2}}} \int_{\alpha_q^{-1}(\Lambda_{tG_\theta} \cap T^*\Omega_q')} e^{\frac{i}{h}(-(\alpha_x - x)\alpha_\xi + i(x-\alpha_x)^2/2)} \chi_\alpha(x) v(\alpha) d\alpha \,,$$

where $\chi_\alpha(x)$ is a cut-off to a neighbourhood of $\operatorname{Re}\alpha_x$. The operator we constructed satisfies

$$S_{q,\Lambda_{tG_\theta}} T_{q,\Lambda_{tG_\theta}} u\restriction_{\Omega_q \cap X} = u\restriction_{\Omega_q \cap X} + R_{q,\Lambda_{tG_\theta}} u \,,$$

where the kernel of $R_{q,\Lambda_{tG_\theta}}$ satisfies

$$\partial_x^l \partial_y^k R_{q,\Lambda_{tG_\theta}}(x,y) = \mathcal{O}_{kl}(1) \exp(-1/Ch) \,, \quad x, y \in \Omega_q \cap X \,.$$

We remark that we could restrict the integration in the definition of $S_{q,\Lambda_{tG_\theta}}$ to a compact subset in α_ξ which would result in a microlocal inverse for the relevant region.

We now give a formula for $F_{q,t}$, which was the weight defined in the lemma using a canonical transformation:

$$F_{q,t}(w) + \frac{1}{2}(\operatorname{Im} w)^2 = H_{\Lambda_{tG_\theta}}(\beta_q(w)) + \operatorname{Im}\phi_q(w, (\alpha_q(w))_x) - \operatorname{Im}\phi_q(w, (\beta_q(w))_x) \,.$$

Again, this follows from general considerations which are implicit in [15]. Let $\psi(\alpha, x)$, $\alpha = (\alpha_x, \alpha_\xi) \in \mathbb{C}^{2n}$, $x \in \mathbb{C}^n$ be a complex phase function satisfying $\psi(\alpha, \alpha_x) = 0$, $\psi_x'(\alpha, \alpha_x) = -\alpha_\xi$, and for $\alpha \in \mathbb{R}^{2n}$, $\operatorname{Im}\psi_{xx}''(\alpha, \alpha_x) \gg 0$. We associate to it a canonical transformation to

$$T^*\mathbb{C}^n \xrightarrow{\kappa_\psi} T^*\mathbb{R}^{2n} \,, \quad \kappa_\psi \;:\; (x, -\partial_x \psi) \longmapsto (\alpha, -\operatorname{Im}\partial_\alpha \psi) \,, \tag{7.20}$$

where the source space is equipped with the symplectic form $-\operatorname{Im}\sigma$ and the target space with the standard canonical form. If $\Lambda \subset T^*\mathbb{C}^n$ is a Lipschitz I-Lagrangian submanifold close to $T^*\mathbb{R}^n$ we define a bi-Lipschitz map $T^*\mathbb{R}^n \ni \alpha \mapsto \beta \in \Lambda$, by demanding that $\Lambda_\alpha \cap \Lambda = \{\beta\}$, $\Lambda_\alpha = \{(x, -\partial_x \psi(\alpha, x)) : x \in \mathbb{C}^n\}$. If $H \in \operatorname{Lip}(\Lambda)$ is the primitive of $-\operatorname{Im}\alpha_\xi d\alpha_x\restriction_\Lambda$ (see Lemma 7.1 above) we put

$$F(\alpha) = H(\beta) - \operatorname{Im}\psi(\alpha, \beta_x)$$

and we claim that

$$T^*\mathbb{R}^{2n} \supset \lambda = \{\alpha \;:\; d_\alpha F\} = \kappa_\psi(\Lambda) \,.$$

In fact, if we put $\psi_\Lambda(\beta, x) = \psi(\alpha, x) - \psi(\alpha, \beta_x)$ and $h(\beta) = \psi(\alpha, \beta_x)$ then

$$(\alpha, -\operatorname{Im}(d_\beta \psi_\Lambda(\beta, x) + d_\beta h(\beta))) = (\alpha, -\operatorname{Im} d_\alpha \psi(\alpha, x)) \stackrel{\kappa_\psi^{-1}}{\longmapsto} (x, -d_x \psi_\Lambda \psi(\alpha, x)) \in \Lambda_\alpha \,.$$

If x is such that $(x, -d_x\psi) \in \Lambda$ then $(x, -d_x\psi) = \beta$, by the definition of β, $\Lambda_\alpha \cap \Lambda = \{\beta\}$. On the other hand $\psi_\Lambda(\beta, \beta_x) = 0$ and consequently

$$\operatorname{Im} d_\beta \psi_\Lambda(\beta, \beta_x) = -\operatorname{Im} d_x \psi_\Lambda(\beta, \beta_x) = \operatorname{Im}\beta_\xi d\beta_\xi = -d_\beta H(\beta) \,,$$

so that $\kappa_\psi(\beta) = (\alpha, d_\beta(H - \operatorname{Im} h)) = (\alpha, d_\alpha F)$, which proves the claim. To see that $F \in \mathcal{C}^{1,1}$ we can use either the fact that λ is an image of a Lipschitz graph or an explicit formula for F based on the representation of Λ given by (7.11) (see (4.40) in [15]):

$$F(\alpha) = \text{critical value}_{x, \operatorname{Re}\zeta}(-\operatorname{Im}\psi(\alpha, x) - \operatorname{Im} x \cdot \operatorname{Re}\zeta + \widetilde{G}(\operatorname{Re} x, \operatorname{Re}\zeta)) \,.$$

Hence if $\widetilde{G} \in \mathcal{C}^{1,1}$ is small then $F_\theta \in \mathcal{C}^{1,1}$. (In fact, in $\partial_y^2 f$ is non-degenerate and G is a $\mathcal{C}^{1,1}$ function, then $F(\alpha)$ defined as the critical value in y of $f(\alpha, y) + \epsilon G(y)$ is a well defined $\mathcal{C}^{1,1}$ function for ϵ small.)

All of these considerations are local and in our case we can take $\psi(\alpha, x) = \phi_q(w, \chi_q^{-1}(x)) - \phi_q(w, \chi_q^{-1}(\alpha_x))$, $w = \alpha_x - i\alpha_\xi$. We then obtain $\Lambda_{\Phi_{q,t}} = \kappa_q \circ \kappa_\psi^{-1}(\lambda)$ and the agreement of F's can be checked by a direct computation.

The discussion above shows that to obtain (7.19) we need to prove that

$$\int_{\beta_q(\Omega_q^\sharp)\subset\Lambda_{tG_\theta}\cap T^*\Omega_q'} |T_{q,\Lambda_{tG_\theta}}u(\beta;h)|^2 e^{-2H_{\Lambda_{tG_\theta}}(\beta)/h}d\beta \leq C\int_{\Lambda_{tG_\theta}} |T_{\Lambda_{tG_\theta}}u(\beta;h)|^2 e^{-2H_{\Lambda_{tG_\theta}}(\beta)/h}d\beta\,.$$

For that we write

$$u = S_{\Lambda_{tG_\theta}}T_{\Lambda_{tG_\theta}}u - R_{\Lambda_{tG_\theta}}u$$

and due to nice properties of $R_{\Lambda_{tG_\theta}}$ we need to concern ourselves with the first term only. The inequality we need will follow from

$$T_{q,\Lambda_{tG_\theta}}S_{\Lambda_{tG_\theta}} : L^2(\Lambda_{tG_\theta}, |\langle\beta_\xi\rangle|^{2m}e^{-2H_{\Lambda_{tG_\theta}}(\beta)/h}) \longrightarrow L^2(\beta_q(\Omega_q^\sharp), e^{-2H_{\Lambda_{tG_\theta}}/h})\,.$$

The analysis of $T_{q,\Lambda_{tG_\theta}}S_{\Lambda_{tG_\theta}}$ proceeds almost exactly as the analysis of $\widetilde{T}_\Lambda S_\Lambda$ in Sect.1 of [29] but we now use Lemma 7.1 to describe the behaviour of the resulting phase function. Thus for $\beta\in\Lambda_{tG_\theta}$ and $\alpha\in\beta_q(\Omega_q^\sharp)\subset\Lambda_{tG_\theta}$ the kernel of $T_{q,\Lambda_{tG_\theta}}S_{\Lambda_{tG_\theta}}$ has the form

$$T_{q,\Lambda_{tG_\theta}}S_{\Lambda_{tG_\theta}}(\alpha,\beta) = c(\alpha,\beta)e^{\frac{i}{h}\psi_{q,\Lambda_{tG_\theta}}(\alpha,\beta)} + k(\alpha,\beta;h)\,,$$

where $c(\alpha,\beta;h)=\mathcal{O}(h^{-n})$ is supported near $\alpha=\beta$, $k(\alpha,\beta;h)=\mathcal{O}_N(1)\exp(-1/Ch)|\langle\beta_\zeta\rangle|^{-N}$ and

$$-H_{\Lambda_{tG_\theta}}(\alpha) - \operatorname{Im}\psi_{q,\Lambda_{tG_\theta}}(\alpha,\beta) + H_{\Lambda_{tG_\theta}} \leq -\frac{1}{C}d(\alpha,\beta)^2\,,\quad \beta\sim\alpha\in\beta_q(\Omega_q^\sharp)\,,$$

where we can use any non-degenerate distance, d, on a neighbourhood of the compact set $\beta_q(\Omega_q^\sharp)$. The structure of the kernel implies now the desired mapping property (by Schur's Lemma) and then the inequality

$$\begin{aligned}&\int_{\Omega_q^\sharp}|T_qu(w;h)|^2e^{-2\Phi_{q,t}(w)/h}\mathcal{L}(dw)\\ &\quad\leq C\int_{\Lambda_{tG_\theta}\cap\{|\beta_\zeta|\leq C\}}|T_{\Lambda_{tG_\theta}}u(\beta;h)|^2e^{-2H_{\Lambda_{tG_\theta}}(\beta)/h}d\beta + \mathcal{O}(e^{-\frac{1}{Ch}})\|u\|_{H(X_\theta,\Lambda_{tG_\theta},\langle\alpha_\zeta\rangle^m)}\,,\end{aligned}$$

from which the lemma follows. ☐

In the proof of Lemma 7.2 we reviewed techniques which can also be used to prove its local converse. To describe it let $\tilde\psi\in\mathcal{C}_c^\infty(T^*X_\theta;[0,1])$ be identically equal to one in a neighbourhood of $S^*_{2\epsilon_0}N$ (defined as in Sect.6) and satisfy supp $\tilde\psi\subset T^*X_1$. We can lift $\tilde\psi$ to Λ_{tG_θ} to obtain a "smooth" function on Λ_{tG_θ} (our Lagrangian is a graph over T^*X_θ). Let $\{\phi_q\}_{q=1}^Q$ be a partition of unity on X_1 subordinate to $\{\Omega_q\cap X\}_{q=1}^Q$:

$$\sum_{q=1}^Q\phi_q = 1 \text{ in } X_1\,,\quad \operatorname{supp}\phi_q\subset\Omega_q\,.$$

From this we obtain $\tilde\psi_q=(\pi^*\phi_q)\tilde\psi\in\mathcal{C}_c^\infty(T^*X_\theta;[0,1])$ and the corresponding functions on Λ_{tG_θ}. We then have

Lemma 7.3. *For the choice of $\{\phi_q\}_{q=1}^Q\subset\mathcal{C}_c^\infty(X_1;[0,1])$ and with the corresponding $\{\tilde\psi_q\}_{q=1}^Q$ defined above we can choose $\Omega_q^\sharp\Subset\chi_q(\Omega_q\cap X)-i\mathbb{R}^2$ and $\psi_q\in\mathcal{C}_c^\infty(\Omega_q^\sharp;[0,1])$ so that*

$$\begin{aligned}&\int_{\Lambda_{tG_\theta}}\tilde\psi_q(\alpha)|T_{\Lambda_{tG_\theta}}u(\alpha;h)|^2e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h}|\langle\alpha_\zeta\rangle|^{2m}d\alpha\\ &\quad\leq C\int_{\Omega_q^\sharp}\psi_q(w)|T_qu(w;h)|^2e^{-2\Phi_{q,t}(w)/h}\mathcal{L}(dw)+\mathcal{O}(e^{-\frac{1}{Ch}})\|u\|^2_{H(X_\theta;\Lambda_{tG_\theta},\langle\alpha_\zeta\rangle^m)}\,.\end{aligned}\tag{7.21}$$

Proof. We reverse the argument of Lemma 7.2 but with more attention given to localization properties of the FBI transformations. We start by observing that if $\tilde{\phi}_q \in \mathcal{C}^\infty_{\rm c}(\Omega_q \cap X)$ is equal to one on a neighbourhood of the support of ϕ_q then, for t small enough

$$\begin{aligned}\int_{\Lambda_{tG_\theta}} & \tilde{\psi}_q |T_{\Lambda_{tG_\theta}} u(\alpha;h)|^2 e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} |\langle \alpha_\zeta \rangle|^{2m} d\alpha \\ &= \int_{\Lambda_{tG_\theta}} \tilde{\psi}_q |T_{\Lambda_{tG_\theta}}(\tilde{\phi}_q u)(\alpha;h)|^2 e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} |\langle \alpha_\zeta \rangle|^{2m} d\alpha + \mathcal{O}(e^{-\frac{1}{Ch}}) \|u\|^2_{H(X_\theta, \Lambda_{tG_\theta}, \langle \alpha_\zeta \rangle^m)} ,\end{aligned}$$

which follows from the Gaussian behaviour of the phases (in fact, by choosing the cut-off function $\chi(\alpha_z, z)$ in the amplitude of $T_{\Lambda_{tG_\theta}}$ well we could arrange an equality without an error). Similarly we can choose $\Omega_q^\sharp$ so that $\operatorname{supp}\phi_q \Subset \chi_q^{-1}(\operatorname{Re}\Omega_q^\sharp) \Subset \operatorname{supp}\tilde{\phi}_q$ and

$$\begin{aligned}\int_{\Omega_q^\sharp} & \psi_q |T_q(\tilde{\phi}_q u)(w;h)|^2 e^{-2\Phi_{q,t}(w)/h} \mathcal{L}(dw) \\ &= \int_{\Omega_q^\sharp} \psi_q |T_q u(w;h)|^2 e^{-2\Phi_{q,t}(w)/h} \mathcal{L}(dw) + \mathcal{O}(e^{-\frac{1}{Ch}}) \|u\|^2_{H(X_\theta;\Lambda_{tG_\theta}, \langle \alpha_\zeta \rangle^m)} ,\end{aligned}$$

where we used Lemma 7.2 to obtain the norm in the last term on the right. Hence we only need to prove (7.21) with u replaced by $\tilde{\phi}_q u$, that is, for $u \in \mathcal{C}^\infty_{\rm c}(\Omega_q \cap X)$. In that case we can use the approximate inverse constructed in the proof of Lemma 7.2:

$$u = S_{q,\Lambda_{tG_\theta}} T_{q,\Lambda_{tG_\theta}} u + R_{q,\Lambda_{tG_\theta}} u \,.$$

Using the transplantation from $\Omega_q^\sharp$ to Λ_{tG_θ} described there we have to prove that

$$\begin{aligned}\int_{\operatorname{supp}\tilde{\psi}_q \subset \Lambda_{tG_\theta}} & |T_{\Lambda_{tG_\theta}} S_{q,\Lambda_{tG_\theta}} T_{q,\Lambda_{tG_\theta}} u(\alpha;h)|^2 e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} |\langle \alpha_\zeta \rangle|^{2m} d\alpha \\ &\leq C \int_{\{\psi_q = 1\} \subset \Omega_q^\sharp} |T_{q,\Lambda_{tG_\theta}} u(\alpha;h)|^2 e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} |\langle \alpha_\zeta \rangle|^{2m} d\alpha + \mathcal{O}(e^{-\frac{1}{Ch}}) \|u\|^2_{H(X_\theta;\Lambda_{tG_\theta}, \langle \alpha_\zeta \rangle^m)} ,\end{aligned}$$

where the choices are made so that

$$\operatorname{supp}\tilde{\psi}_q \Subset \beta_q(\{\psi_q = 1\}) \Subset \beta_q(\Omega_q^\sharp) \subset \Lambda_{tG_\theta} \,.$$

The analysis of the kernel of $T_{\Lambda_{tG_\theta}} S_{q,\Lambda_{tG_\theta}}$ based on Lemma 7.1 proceeds as in the proof of Lemma 7.2 following Sect.1 of [29] and gives the desired estimate by the same argument. □

The two lemmas show that if we put

$$\begin{aligned}||| u |||^2_{H(X_\theta, \Lambda_{tG_\theta}; \langle \alpha_\zeta \rangle^m)} & \\ = \sum_{q=1}^{Q} \int_{\Omega_q^\sharp} & \psi_q |T_q u(w;h)|^2 e^{-2\Phi_{q,t}(w)/h} \mathcal{L}(dw) + \int_{\Lambda_{tG_\theta}} (1 - \tilde{\psi}) |T_{\Lambda_{tG_\theta}} u(\alpha;h)|^2 e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} d\alpha \,,\end{aligned}$$

then we obtain a norm (and the corresponding inner product) uniformly equivalent to the old one:

$$\frac{1}{C} \|u\|_{H(X_\theta, \Lambda_{tG_\theta}; \langle \alpha_\zeta \rangle^m)} \leq ||| u |||_{H(X_\theta, \Lambda_{tG_\theta}; \langle \alpha_\zeta \rangle^m)} \leq C \|u\|_{H(X_\theta, \Lambda_{tG_\theta}; \langle \alpha_\zeta \rangle^m)} \,.$$

We can now define the h dependent norm on $\mathcal{H}_{\theta,h,A}$:

$$||| u |||^2_{tG_\theta} = ||| \psi_{\theta,A} (1 - \chi_1 \rho_A(hD_y) u |||^2_{H(X_\theta; \Lambda_{tG_\theta}, 1)} + \|\chi_{\theta,A} u\|^2_{\mathcal{H}_{\theta,h,A}} + \|\chi_1 \rho_A(hD_y) u\|^2_{\mathcal{H}_{\theta,h,A}} \,,$$

where $\psi_{\theta,A}, \chi_{\theta,A}, \rho_A$ and χ_1 are as in (5.11) and (5.12). That the norm is essentially independent of the choice of the cuf-off functions and that it is equivalent (in an h-dependent way) to the norm on $\mathcal{H}_{\theta,h,A}$.

8. Estimates on the number of resonances

To obtain bounds on the number of resonances of $P(h)$, that is on the number of eigenvalues of the non-self-adjoint operator $P_\theta(h)$ (for simplicity we will denote the operator $P_{\theta,A}(h)$ by $P_\theta(h)$, A being fixed), we need to estimate the number of its characteristic values. More precisely we need to estimate the number of eigenvalues of $[(P_\theta(h)-\omega_0)(P_\theta(h)-\omega_0)^*]^{\frac{1}{2}}$ for a suitably chosen $\omega_0 \in \mathbb{C}$. That in turn amounts to obtaining lower bounds for $|||(P_\theta(h)-\omega_0)u|||^2_{tG_\theta}$. We start with the easier regions:

Lemma 8.1. *Suppose that $u \in \mathcal{D}_{\theta,h,A}$ and that $\chi_{\theta,A}u = u$ where $\chi_{\theta,A}$ is given by (5.11). If $R_\theta \subset \mathbb{C}$ is the same as in Proposition 3.1, then for $\omega_0 \in \mathbb{C}$, $0 < \mu < \mu_0 \ll 1$ satisfying*

$$\operatorname{Re}\omega_0 > 0\,, \ \operatorname{Im}\omega_0 = r_0 > 0\,, \ \operatorname{Re}\omega_0 - i\mu \in R_\theta\,,$$

we have a lower bound

$$\|(P_\theta(h)-\omega_0)u\|^2_{\mathcal{H}_{\theta,h,A}} \geq ((r_0+\mu)^2 - \mathcal{O}(h))\|u\|^2_{\mathcal{H}_{\theta,h,A}}\,.$$

Proof. Since u satisfying the assumptions of the lemma can be identified with an element of $\mathcal{H}^0_{\theta,h,A}$, both the norm and the operator can be decomposed according to (5.2). Hence we only need to check the inequality for each component of the direct sum. Each of these components is supported in $r > C_0$ and the estimate follows from Proposition 3.1 and (4.8). In fact, Proposition 4.1 shows that $\|(P_\theta(h) - \operatorname{Re}\omega_0)u\|^2_{\mathcal{H}_{\theta,h,A}} \geq (\theta/C)^2\|u\|^2_{\mathcal{H}_{\theta,h,A}}$ so that

$$\begin{aligned}\|(P_\theta(h)-\omega_0)u\|^2_{\mathcal{H}_{\theta,h,A}} &= \|(P_\theta(h)-\operatorname{Re}\omega_0)u\|^2_{\mathcal{H}_{\theta,h,A}} + r_0^2\|u\|^2_{\mathcal{H}_{\theta,h,A}} - 2r_0\operatorname{Im}\langle P_\theta(h)u,u\rangle_{\mathcal{H}_{\theta,h,A}} \\ &\geq \left(\frac{\theta^2}{C^2} + r_0^2 - r_0 e^{-\frac{1}{C_A\theta}} - \mathcal{O}(h)\right)\|u\|^2_{\mathcal{H}_{\theta,h,A}}\,,\end{aligned}$$

where we used (4.8) to bound the imaginary part (only the potential term contributes to the positive imaginary part and the lower order terms give a uniform $\mathcal{O}(h)$ term). When μ_0 is small then this gives the lemma. $\square$

Lemma 8.2. *Suppose that $u \in \mathcal{D}_{\theta,h,A}$ and $\chi_1\rho_A(hD_y)u = u$. If ω_0 and μ are as in Lemma 8.1 then*

$$\|(P_\theta(h)-\omega_0)u\|^2_{\mathcal{H}_{\theta,h,A}} \geq ((r_0+\mu)^2 - \mathcal{O}(h))\|u\|^2_{\mathcal{H}_{\theta,h,A}}\,.$$

Proof. Since $\chi_1 u = u$ we can apply the same argument as in the proof of Lemma 8.1 and since $\rho_A(hD_y)u = u$ we only have components for which $|nh| > A/2$. Hence by the first part of Proposition 3.1 we obtain the desired estimate as in the previous lemma. $\square$

In the estimates involving the microlocally weighted norm, $\|\bullet\|_{H(X_\theta,\Lambda_{tG_\theta},1)}$, we need to understand the value of the symbol p on the I-Lagrangian Λ_{tG_θ}:

Lemma 8.3. *Let p denote the holomorphic extension of the symbol of $P(h)$ to $T^*\widetilde{X}$ and p_θ its restriction to T^*X_θ. Then*

$$p(\exp tH^{\operatorname{Im}\sigma}_{G_\theta}(m)) = p_\theta(m) - itH_{g_\theta}G_\theta(m) + \mathcal{O}(t^2) + \mathcal{O}(t\theta)\,, \quad m \in T^*X_\theta\,, \tag{8.1}$$

where $g_\theta = \operatorname{Re}p_\theta$ and the last error term can be omitted in $T^*X_1 \subset T^*X_\theta \cap T^*X$. Moreover, in a neighbourhood W of K in $S^*_{\epsilon_0}X$ we have

$$p(\exp tH^{\operatorname{Im}\sigma}_{G_\theta}(m)) = p(m) - i(1+\mathcal{O}(t))tH_{g_\theta}G_\theta(m)\,, \quad m \in W\,. \tag{8.2}$$

Proof. The equation (8.1) is the now standard formula which is behind the analysis of [15],[28],[29]. We briefly recall the main point: for $m \in T^*X_\theta$

$$
\begin{aligned}
p(\exp tH_{G_\theta}^{\operatorname{Im}\sigma}(m)) &= p_\theta(m) + \langle dp_\theta(m), tH_{G_\theta}^{\operatorname{Im}\sigma}(m)\rangle + \mathcal{O}(t^2)\\
&= p_\theta(m) + \langle d\operatorname{Re} p_\theta(m), tH_{G_\theta}^{\operatorname{Im}\sigma}(m)\rangle + i\langle d\operatorname{Im} p_\theta(m), tH_{G_\theta}^{\operatorname{Im}\sigma}(m)\rangle + \mathcal{O}(t^2)\\
&= p_\theta(m) - \langle tH_{\operatorname{Re} p_\theta}^{\operatorname{Im}\sigma}(m), dG_\theta(m)\rangle - i\langle tH_{\operatorname{Im} p_\theta}^{\operatorname{Im}\sigma}(m), dG_\theta(m)\rangle + \mathcal{O}(t^2)\\
&= p_\theta(m) - \langle J(tH_{p_\theta}^{\sigma}), dG_\theta\rangle - i\langle tH_{p_\theta}^{\sigma}(m), dG_\theta(m)\rangle + \mathcal{O}(t^2)\\
&= p_\theta(m) - itH_{g_\theta}G_\theta(m) + \mathcal{O}(t^2) + \mathcal{O}(t\theta)\,,
\end{aligned}
$$

where to obtain the second equation we used (7.10) and for the last one $g_\theta = \operatorname{Re} p_\theta = p_\theta + \mathcal{O}(\theta)$ (and the same formula for derivatives). This gives (8.1) and to obtain (8.2) we note that in any local holomorphic coordinates on $\widetilde{T^*X}$ in which T^*X is locally identified with $T^*\mathbb{R}^2$, the extension of G satisfies (because of (7.10))

$$
G(x+iy, \xi + i\eta) = G(x + f(x,y,\xi,\eta), \xi + h(x,y,\xi,\eta))\,, \quad f,h \in \mathcal{C}^\infty\,, \quad f,h = \mathcal{O}(y^2+\eta^2)\,.
$$

Hence, using the fact that ∇G is Lipschitz,

$$
\begin{aligned}
H_G^{\operatorname{Im}\sigma} =& (G'_\xi + \mathcal{O}(\|\nabla G\|^2 + y^2 + \eta^2))\partial_y + (-G'_x + \mathcal{O}(\|\nabla G\|^2 + y^2 + \eta^2))\partial_\eta \\
&+ \mathcal{O}(\|\nabla G\|^2 + y^2 + \eta^2)\partial_x + \mathcal{O}(\|\nabla G\|^2 + y^2 + \eta^2)\partial_\xi\,.
\end{aligned}
$$

From this it is easy to see that

$$
\exp tH_G^{\operatorname{Im}\sigma}(x,\xi) = (x + itG'_\xi(x,\xi) + \mathcal{O}(t^2)\|\nabla G(x,\xi)\|^2, y - itG'_x(x,\xi) + \mathcal{O}(t^2)\|\nabla G(x,\xi)\|^2)\,.
$$

Recalling that in W, $|\nabla G_\theta|^2 = \mathcal{O}(H_{g_\theta}G_\theta)$ gives (8.2). □

We now recall Theorem 1.3 of [28]. Let $\widetilde{Q}$ be an analytic pseudodifferential operator on an open neighbourhood Ω_2, of $\overline{\Omega_1} \Subset \mathbb{C}^n$, associated to Λ_Φ where $\Phi \in \mathcal{C}^{1,1}(\Omega_2;\mathbb{R})$ (see (1.1)-(1.3) of [28] for the definition of $\widetilde{Q}$) and let $\psi \in \mathcal{C}_c^\infty(\Omega_1;\mathbb{R})$. Then for $v_1, v_2 \in H_\Phi(\Omega_2)$ (see the discussion before Lemma 7.3 for the definition of that space) we have

$$
\langle \psi\widetilde{Q}v_1, \widetilde{Q}v_2\rangle_\Phi = \int_{\Omega_1} |\tilde{q}(z, -2i\partial_z\Phi(z)))|^2 v_1(z)\overline{v_2(z)}\psi(z)e^{-2\Phi(z)/h}\mathcal{L}(dz) + \mathcal{O}(h)\|v_1\|_\Phi\|v_2\|_\Phi\,, \tag{8.3}
$$

where $(\bullet,\bullet)_\Phi$ is the inner product on $L^2(\Omega_2; e^{-2\Phi(z)/h}\mathcal{L}(dz))$ and $\tilde{q}(z,\zeta)$ is the semi-classical symbol of $\widetilde{Q}$.

We also recall from [27],[28] (see (2.16) in [28]) that if Q is a differential operator on $V = \operatorname{Re}\Omega_3 \Subset \mathbb{R}^n$, $\Omega_2 \Subset \Omega_3 \Subset \mathbb{C}^n$ with coefficients holomorphic in a complex neighbourhood, Ω_3, of V then, assuming that $\Phi - (\operatorname{Im} z)^2/2$ is small enough,

$$
T_0 Qu = \widetilde{Q}T_0 u + Ku\,, \quad u \in \mathcal{C}_c^\infty(V)\,, \quad \|Ku\restriction_{\Omega_1}\|_\Phi = \mathcal{O}_N(h^N)\|T_0 u\restriction_{\Omega_2}\|_\Phi + \mathcal{O}(e^{-1/Ch})\|u\|_{L^2}\,, \tag{8.4}
$$

where $\widetilde{Q}$ is a holomorphic differential operator associated to Λ_Φ and its principal symbol is given by $\tilde{p} = (\kappa_0^{-1})^* q$, where q is the holomorphic continuation of the semi-classical principal symbol of Q and κ_0 is the complex canonical transformation associated to T_0 (as in Lemma 7.2 with $\phi_0(w,z) = i(w-z)^2/2$).

Using this we can finally estimate $|||(P_\theta(h) - \omega_0)u|||_{tG_\theta}$ in the interesting region:

Lemma 8.4. *If $u \in \mathcal{C}_c^\infty(X_\theta)$, $\psi_{\theta,A}(1 - \chi_1\rho_A(hD_y))u = u$, and ω_0 and μ are as in Lemma 6.1 then for t small enough*

$$
|||(P_\theta(h) - \omega_0)u|||^2_{H(X_\theta;\Lambda_{tG_\theta},1)} \geq ((r_0 + \mu)^2 - \mathcal{O}(h))|||u|||^2_{H(X_\theta;\Lambda_{tG_\theta},1)} - \mathcal{O}(\mu)\sum_{q=1}^{Q}\|f_{q,t}^\mu T_q u\|^2_{\Phi_{q,t}}\,, \tag{8.5}
$$

where $\Phi_{q,t}$ is as in Lemma 7.2 and $f^{\mu}_{q,t}$ is the characteristic function of

$$\pi \circ \kappa_q \left(\{ \alpha \in \Lambda_{tG_\theta} \cap \widetilde{T^*N} \; : \; |p\restriction_{\Lambda_{tG_\theta}}(\alpha) - \omega_0|^2 \leq (r_0 + \mu)^2 \} \right) , \tag{8.6}$$

where $\widetilde{T^*N}$ is a complex neighbourhood of T^*N in $\widetilde{T^*X}$, p is the holomorphic continuation of the principal symbol of $P(h)$ and κ_q is as in Lemma 7.2 and $\pi : T^*\mathbb{C}^2 \to \mathbb{C}^2$ is the natural projection.

Proof. We put $Q = P - \omega_0$ and using the definition of $|||\bullet|||_{H(X_\theta;\Lambda_{tG_\theta},1)}$ we write

$$\begin{aligned} &|||(P-\omega_0)u|||_{H(X_\theta;\Lambda_{tG_\theta},1)} \\ &= \sum_{q=1}^{Q} \langle \psi_q T_q Q u, T_q Q u \rangle_{\Phi_q,t} + \langle (1-\tilde\psi) T_{\Lambda_{tG_\theta}} Q u, T_{\Lambda_{tG_\theta}} Q u \rangle_{L^2(\Lambda_{tG_\theta} e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} d\alpha)} . \end{aligned}$$

Using (8.4) and Lemma 7.2 we have

$$\langle \psi_q T_q Q u, T_q Q u \rangle_{\Phi_{q,t}} = \langle \psi_q \widetilde{Q} T_q u, \widetilde{Q} T_q u \rangle_{\Phi_{q,t}} + \mathcal{O}_N(h^N) |||u|||_{H(X_\theta;\Lambda_{tG_\theta},1)} .$$

Then (8.3) gives

$$\begin{aligned} &\langle \psi_q T_q Q u, T_q Q u \rangle_{\Phi_{q,t}} \\ &= \int_{\Omega_q^\sharp} \psi_q(w) |\tilde q(w, -2i\partial_w \Phi_{q,t})|^2 |T_q u(w;h)|^2 e^{-2\Phi_{q,t}(w)/h} \mathcal{L}(dw) + \mathcal{O}(h) |||u|||^2_{H(X_\theta;\Lambda_{tG_\theta},1)} \\ &\geq (r_0+\mu)^2 \int_{\Omega_q^\sharp} \psi_q(w) |T_q u(w;h)|^2 e^{-2\Phi_{q,t}(w)/h} \mathcal{L}(dw) \\ &\quad - C\mu \int_{\Omega_q^\sharp} \mathbf{1}_{\{|\tilde q(w,-2i\Phi(w,-2i\partial_w\Phi(w))| \leq r_0+\mu\}} |T_q u(w;h)|^2 e^{-2\Phi_{q,t}(w)/h} \mathcal{L}(dw) - \mathcal{O}(h) |||u|||^2_{H(X_\theta;\Lambda_{tG_\theta},1)} , \end{aligned} \tag{8.7}$$

where in the last inequality we used the fact that $\operatorname{Im} p\restriction_{\Lambda_{tG_\theta}} \leq 0$ (see Lemma 8.3) so that when $|\tilde q| = |(\kappa_q^{-1})^*(p\restriction_{\Lambda_{tG_\theta}} - \omega_0)| \leq r_0 + \mu$ then $(r_0+\mu)^2 - |\tilde q|^2 \leq C\mu$ (since $|z - ir_0| \leq r_0 + \mu$ and $\operatorname{Im} z \leq 0$ imply $(r_0+\mu)^2 - |z - ir_0|^2 \leq C\mu$ for μ small).

On the support of $(1-\tilde\psi)$, Λ_{tG_θ} is a smooth submanifold of $\widetilde{T^*X}$ and on a neighbourhood of the support of the symbol of $\psi_{\theta,A}(1-\chi_1\rho_A(hD_y))$ the absolute value of the symbol of $P_\theta(h) - \omega$ is bounded from below by $r_0 + 1/C$. This follows from Lemmas 8.3, 6.1, 4.1 and 4.2. We used here the fact that the symbol of the cut-off $\psi_{\theta,A}(1-\chi_1\rho_A(hD_y))$ is equal to one on a neighbourhood of the support of G (see the proof of Proposition 8.1). Hence we can use Proposition 1.3 of [29] (with a modification allowing the Toeplitz operator $P_{\Lambda_{tG_\theta}}(1-\tilde\psi)P_{\Lambda_{tG_\theta}}$ – compare [28], (2.22)-(2.24)):

$$\begin{aligned} &\langle (1-\tilde\psi) T_{\Lambda_{tG_\theta}} Q u, T_{\Lambda_{tG_\theta}} Q u \rangle_{L^2(\Lambda_{tG_\theta}, e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} d\alpha)} \\ &= \langle (1-\tilde\psi)|q|^2\restriction_{\Lambda_{tG_\theta}} T_{\Lambda_{tG_\theta}} u, T_{\Lambda_{tG_\theta}} u \rangle_{L^2(\Lambda_{tG_\theta}, e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} d\alpha)} + \mathcal{O}(h) \|u\|^2_{H(X_\theta;\Lambda_{tG_\theta},\langle\alpha_\zeta\rangle^2)} \\ &= \langle (1-\tilde\psi)|q|^2\restriction_{\Lambda_{tG_\theta}} T_{\Lambda_{tG_\theta}} u, T_{\Lambda_{tG_\theta}} u \rangle_{L^2(\Lambda_{tG_\theta}, e^{-2H_{\Lambda_{tG_\theta}}(\alpha)/h} d\alpha)} + \mathcal{O}(h) |||u|||^2_{H(X_\theta;\Lambda_{tG_\theta},1)} , \end{aligned}$$

where to change the last norm we used the ellipticity of $(1-\tilde\psi)|q|^2$ for $|\alpha_\zeta| \geq C$.

Summing up all the contributions gives the lower bound (8.6). □

By combining the three lemmas we obtain a global lower bound:

Proposition 8.1. *Let h, θ, A and ω_0, μ be as in Lemma 8.1. Then with G given by (6.3) we have for $u \in \mathcal{C}^\infty_c(X)$,*

$$|||(P_\theta(h) - \omega_0)u|||^2_{tG_\theta} \geq \left((r_0+\mu)^2 - \mathcal{O}(h) \right) |||u|||^2_{tG_\theta} - \mathcal{O}(\mu) \sum_{q=1}^{Q} \|f^\mu_{q,t} T_q u\|^2_{\Phi_{q,t}} , \tag{8.8}$$

where $f^{\mu}_{q,t}$ *is as in Lemma 8.4.*

Proof. To apply the previous lemmas we need to estimate commutators. The main point is that the support of the weight function (which would produce exponentially large errors in the commutator estimates) is away from the support of the symbols of the commutators. In fact, for the cut-offs in the definition of the norm $|||\bullet|||_{tG_\theta}$, the estimates

$$\begin{aligned}
\|[P_\theta(h),\psi_{\theta,A}]u\|_{H(X_\theta;\Lambda_{tG_\theta},1)} &= \mathcal{O}(h)(|||u|||_{tG_\theta} + |||(P_\theta(h)-\omega_0)u|||_{tG_\theta})\,, \quad u\in \mathcal{C}^\infty_{\rm c}(X_\theta)\,,\\
|||[P_\theta(h),\psi_{\theta,A}\rho_A(hD_y)]u|||_{H(X_\theta;\Lambda_{tG_\theta},1)} &= \mathcal{O}(h)(|||u|||_{tG_\theta} + |||(P_\theta(h)-\omega_0)u|||_{tG_\theta})\,, \quad u\in \mathcal{C}^\infty_{\rm c}(X_\theta)\,,\\
\|[P_\theta(h),\chi_1\rho_A(hD_y)]u\|_{\mathcal{H}_{\theta,A,h}} &= \mathcal{O}(h)(|||u|||_{tG_\theta} + |||(P_\theta(h)-\omega_0)u|||_{tG_\theta})\,, \quad u\in \mathcal{D}_{\theta,A,h}\,,\\
\|[P_\theta(h),\chi_{\theta,A}]u\|_{\mathcal{H}_{\theta,A,h}} &= \mathcal{O}(h)(|||u|||_{tG_\theta} + |||(P_\theta(h)-\omega_0)u|||_{tG_\theta})\,,
\end{aligned}$$

are a consequence of the independence of the norm $|||\bullet|||$ of the choice of the cut-off functions and the classical ellipticity to $P_\theta(h)$. For the space $\mathcal{H}_{\theta,A,h}$ this follows from the standard estimates and for $H(X_\theta;\Lambda_{tG_\theta},1)$ from the exponential decay in $1/h$ of $a(\alpha)T(b(z,hD_z)u)(\alpha,h)$ when the supports of a and b are disjoint (essentially a consequence of the Gaussian character of the phase – since the operator $P_\theta(h)$ commutes with the phase space cut-off $\rho_A(hD_y)$ it is even easier here as only disjoint spatial supports need to be considered).

With $\chi_1,\chi_{\theta,A}$ and $\psi_{\theta,A},\rho_A$ as in the definition of $|||\bullet|||_{tG_\theta}$ we can rewrite the left hand side of (8.8) as

$$\begin{aligned}
|||\psi_{\theta,A}(1-\chi_1\rho_A(hD_y))(P(h)-\omega_0)u|||^2_{H(X_\theta,\Lambda_{tG_\theta},1)} + \|\chi_{\theta,A}(P_\theta(h)-\omega_0)u\|^2_{\mathcal{H}^0_{\theta,h,A}}\\
+\|\chi_1\rho_A(hD_y)(P_\theta(h)-\omega_0)u\|^2_{\mathcal{H}^0_{\theta,h,A}}\,.
\end{aligned}$$

If ψ denotes a cut-off operator and $\|\bullet\|$ one of the norms we then use the following inequalities:

$$\begin{aligned}
\|\psi(P_\theta(h)-\omega_0)u\|^2 &\geq \|(P_\theta(h)-\omega_0)\psi u\|^2 + \|[\psi,P_\theta(h)u\|^2 - 2\|(P_\theta(h)-\omega_0)\psi u\|\|[\psi,P_\theta(h)]\|\\
&\geq \|(P_\theta(h)-\omega_0)\psi u\|^2 - \|[\psi,P_\theta(h)]u\|^2 - 2\|\psi(P_\theta(h)-\omega_0)u\|\|[\psi,P_\theta(h)]u\|\,.
\end{aligned}$$

After summing over the different cut-off operators and using the estimates on the commutators, the last two terms can be moved to the right hand side which gives

$$\begin{aligned}
|||(P_\theta(h)-\omega_0)u|||^2_{tG_\theta} \geq (1-\mathcal{O}(h))\Big(|||(P_\theta(h)-\omega_0)\psi_{\theta,A}(1-\chi_1\rho_A(hD_y))u|||^2_{H(X_\theta,\Lambda_{tG_\theta},1)}\\
+\|(P_\theta(h)-\omega_0)\chi_{\theta,A}u\|^2_{\mathcal{H}_{\theta,h,A}} + \|(P_\theta(h)-\omega_0)\chi_1\rho_A(hD_y)u\|^2_{\mathcal{H}_{\theta,h,A}}\Big) - \mathcal{O}(h)|||u|||^2_{tG_\theta}\,.
\end{aligned}$$

The lemmas 8.1,8.2 and 8.4 (applied with modified cut-off operators which preserve the cut-offs in the definition of $|||\bullet|||_{tG_\theta}$) conclude the proof of (8.8). ☐

We now want to study the set (8.6) and the construction of G_θ and Lemma 8.3 give

Proposition 8.2. *If the trapped set K given by (3.5) is of pure dimension m and ω_0, r_0 and $0<\mu<\mu_0\ll 1$ (where μ_0 is chosen depending on t) are as in Proposition 8.1 then*

$$\mathrm{vol}_{\mathbb{C}^2}\left(\pi\circ\kappa_q\left(\left\{\alpha\in\Lambda_{tG_\theta}\cap\widetilde{T^*N} : \left|p\restriction_{\Lambda_{tG_\theta}}-\omega_0\right|^2\leq (r_0+\mu)^2\right\}\right)\right)\leq C\mu^{\frac{5}{2}-\frac{m}{2}}\,.$$

Proof. We first observe that for μ small enough Lemma 8.3 and the construction of G (only the region where $G=G_\theta$ needs to be considered as we are in a neighbourhood of T^*N) we can restrict our attention to a neighbourhood of K in $S^*_{\epsilon_0}X$. Consequently

$$\begin{aligned}
&\{\alpha\in\Lambda_{tG_\theta}\cap\widetilde{T^*N}\ :\ |p\restriction_{\Lambda_{tG_\theta}}-\omega_0|^2\leq(r_0+\mu)^2\}\\
&\quad\subset\ \exp tH^{\mathrm{Im}\,\sigma}_{tG_\theta}\left(\{m\in S^*_{\epsilon_0}N_1\ :\ (r_0+t(1+\mathcal{O}(t))H_{g_\theta}G(m))^2\leq(r_0+\mu)^2,\ (\mathrm{Re}\,p_\theta-\mathrm{Re}\,\omega_0)^2\leq 2r_0\mu+\mu^2\}\right)\\
&\quad\subset\ \exp tH^{\mathrm{Im}\,\sigma}_{G\theta}\left(\left\{m\in S^*_{\epsilon_0}N_1\ :\ tH_{g_\theta}G\leq C\mu\,,\ |\mathrm{Re}\,p_\theta-\mathrm{Re}\,\omega_0|\leq C\mu^{\frac{1}{2}}\right\}\right)\,.
\end{aligned}$$

Since K is assumed to be of pure dimension (see (3.7) and the discussion following it), (6.3) gives the desired bound. ☐

Theorem of Sect.1 follows from Propositions 8.1 and 8.2 as in [28],[32],[29] and we briefly recall the argument. As in Sect.3 of [28] (see Lemma 3.1 there) for any $\epsilon > 0$ there exist finite rank operators $\Xi_{q,t}^{\mu,\epsilon}$ such that for $v \in H^2_{\Phi_{q,t}}(\Omega_q^\sharp)$

$$\|f_{q,t}^{\mu}(v - \Xi_{q,t}^{\mu,\epsilon} v)\|_{\Phi_{q,t}} \leq C\epsilon \|v\|_{\Phi_{q,t}}$$

$$\text{rank}\ \Xi_{q,t}^{\mu,\epsilon} \leq \min\{M \ : \ \text{supp}\ f_{q,t}^{\mu} \subset \bigcup_{j=1}^{M} B(w_j, \epsilon h^{\frac{1}{2}}),\ w_j \in \Omega_q^\sharp\}\,.$$

Here we use the special structure of the set in Lemma 8.4. Hence, by Proposition 8.2 we can arrange that

$$\text{rank}\ \Xi_{q,t}^{\mu,\epsilon} \leq C\mu^{\frac{5}{2}-\frac{m}{2}} \left(\epsilon h^{\frac{1}{2}}\right)^{-4}\,.$$

Consequently Proposition 8.1 can be rewritten as

$$|||(P_\theta - \omega_0)u|||^2_{tG_\theta} \geq \left((r_0+\mu)^2 - \mathcal{O}(h) - \mathcal{O}(\epsilon\mu)\right) |||u|||^2_{tG_\theta} - |||Q_\theta^{\mu,\epsilon} u|||^2_{tG_\theta}\,,$$

$$\text{rank}\ Q_\theta^{\mu,\epsilon} \leq C_\epsilon \mu^{\frac{5}{2}-\frac{m}{2}} h^{-2}\,.$$

For $Ch < \mu < \mu_0 \ll_t 1$ we can now apply the Weyl estimates for eigenvalues in terms of characteristic values (see Sect.7 of [32]) to obtain the bound $\mathcal{O}(\mu^{\frac{5}{2}-\frac{m}{2}} h^{-2})$ on the number of resonances of $P(h)$ in $|\operatorname{Re} z - 1| < \mu^{\frac{1}{2}}/C$, $-\operatorname{Im} z > \mu$.

When $\mu = Ch^{1-\alpha}$ then a covering argument first gives the bound $\mathcal{O}(h^{-\frac{1}{2}(1-\alpha)} h^{(1-\alpha)\left(\frac{5}{2}-\frac{m}{2}\right)} h^{-2}) = \mathcal{O}(h^{-2+(1-\alpha)\left(2-\frac{m}{2}\right)})$ on the number of resonances in

$$|\operatorname{Re} z - 1| \leq \frac{1}{2}\,, \quad -\operatorname{Im} z > Ch^{1-\alpha}\,,\ 0 \leq \alpha \leq 1\,.$$

Another covering argument (see Sect.7 of [32]) and the identity $m = 2 + 2\delta$ give the Theorem of Sect.1.

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