LOCALISED DAVIES GENERATORS FOR UNBOUNDED OPERATORS

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Abstract. A classical Davies generator provides a Lindbladian for which the Gibbs state is stationary. Its construction involves precise knowledge of the Bohr spectrum or equivalently state evolution for all times. Recently Chen, Kastoryano and Gilyen [CKG23] proposed a construction involving localisation in time and carried out in the case of finite dimensional Hilbert spaces. The resulting generators are called quantum Gibbs samplers as the corresponding Lindblad is expected to settle to the Gibbs state. In this paper, we show that the construction also works for classes of unbounded operators, including pseudodifferential operators used in the study of classical/quantum correspondence in Lindblad evolution. We also show that the jump operators in that construction are pseudodifferential and in particular, pseudo-local. That involves a novel version of Egorov's theorem.

1. Introduction

For a quantum system at a non-zero temperature $1/\beta$ the role of a ground state of a self-adjoint Hamiltonian, P, is taken by the Gibbs state $\rho_\beta := e^{-\beta P}/\mathrm{tr} e^{-\beta P}$. A quantum expectation value of an observable A (an operator on a Hilbert space) at temperature $1/\beta$ is then defined as $\langle A\rangle_\beta := \mathrm{tr} A\rho_\beta$ and its efficient evaluation is of interest in different situations (see the references below). One possible approach for this evaluation is designing an interaction with an open system. That can be modelled by a Lindblad evolution generated by a super-operator $\mathcal{L}$ – see (1.7) below for an example and [GaZw25] for an introduction from a PDE perspective – for which the Gibbs state is an equilibrium state. In that case,

$$\langle A\rangle_\beta = \lim_{t\to+\infty} \mathrm{tr} A\, e^{t\mathcal{L}} B, \quad \mathrm{tr} B = 1.$$

A necessary condition for this to hold is

$$\mathcal{L}\rho_\beta = 0. \tag{1.1}$$

If we have (1.1), one can expect that the dissipative term in $\mathcal{L}$ produces convergence to the Gibbs state. Here we only address finding $\mathcal{L}$ so that (1.1) holds. Our setting is that of differential and pseudo-differential operators as Hamiltonians.

In various works the importance of locality of the coherent part of the Lindbladian (see (1.8)) and of the jump operators (see $\mathcal{D}_f$ in (1.7)) is stressed. From the PDE point of view strict locality of linear operators acting on smooth functions on $\mathbb{R}^n$ ($\operatorname{supp} Pu \subset \operatorname{supp} u$) holds only for differential operators. Their generalisation, pseudodifferential operators (see (1.19)), are *pseudolocal* in the sense that if

$$Au(x) = \int_{\mathbb{R}^n} K_A(x,y)u(y)dy,$$

(an informal expression which should be understood in the sense of distributions – see [Hö03, §5.2]) then $K_A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta)$, $\Delta := \{(x,x) : x \in \mathbb{R}^n\}$, and

$$\forall\, N\, \exists\, C_N \ \ |K_A(x,y)| \leq C_N|x-y|^{-N}, \qquad |x-y| \geq 1. \tag{1.2}$$

This is one possible relevance of our showing pseudodifferential nature of jump operators. The proof of that involves a novel version of Egorov's theorem on the classical/quantum correspondence – see Theorem 5 for the statement and [GaZw25] for an introduction to the classical/quantum correspondence in the context of Lindbladians.

1.1. Localised generators.

A now classical construction of $\mathcal{L}$ satisfying (1.1) was given by Davies [Da74],[Da76] and we review it in the case of matrices in §2.1 following [BrPe02, §3.3]. It is constructed using an essentially arbitrary family of operators, $\mathcal{A}$, but it involves their evolution under e^{-itP} for all times – see §2.2.

Here we are motivated by a recent paper by Chen–Kastoryano–Gilyén [CKG23] where a construction involving localisation in time was proposed and carried out in the case of finite dimensional Hilbert spaces. (See also Ramkumar–Soleimanifar [RaSo24] and Ding–Li–Lin [DLL25] for further developments and references). We show that under certain conditions (natural in the case of differential and pseudo-differential operators) the construction works in the case of unbounded operators. A different approach to the infinite dimensional case has also been recently proposed by Becker–Rouzé–Salzmann [BRS26]. We refer to that paper and to [CKG23] for references and background in the context of quantum information. The motivating unbounded example is given by,

$$\begin{gathered} P = -\Delta + V(x), \quad V \in C^\infty(\mathbb{R}^n, \mathbb{R}), \\ \partial^\alpha V(x) = \mathcal{O}(1), \quad |\alpha| \geq 2, \quad V(x) \geq |x|^2/C - C, \end{gathered} \tag{1.3}$$

and (in the construction of the Lindbladian below)

$$A = \langle a_1, \partial_x\rangle + \langle a_2, x\rangle, \quad a_j \in \mathbb{C}^n. \tag{1.4}$$

To describe the construction in [CKG23, Theorem I.1], we recall the *operator Fourier transform*: for a self-adjoint operator P and for A (in a class described below) we define

$$\hat{A}_f(\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iPt} A e^{-iPt} e^{-i\omega t} f(t)dt, \tag{1.5}$$

where, following [CKG23], we will (mostly) take

$$f(t) = f_\sigma(t) := \sigma^{\frac{1}{2}}\pi^{\frac{1}{4}}e^{-t^2\sigma^2/2}, \quad \hat{f}_\sigma(\tau) = f_{1/\sigma}(\tau), \quad \|f_\sigma\|_{L^2} = 1, \tag{1.6}$$

where $\hat{g}(\tau) := (1/\sqrt{2\pi})\int g(t)e^{-it\tau}dt$. For a self-adjoint Hamiltonian P and a set of operators (with properties specified below), $\mathcal{A}$, we define the following Lindbladian

$$\begin{gathered}\mathcal{L}_f T := -i[\beta^{-1}P + B, T] + \mathcal{D}_f(T) \\ \mathcal{D}_f(T) := \int_{\mathbb{R}} \gamma(\omega) \sum_{A\in\mathcal{A}} \left(\hat{A}_f(\omega)T\hat{A}_f(\omega)^* - \tfrac{1}{2}\left\{\hat{A}_f(\omega)^*\hat{A}_f(\omega), T\right\}\right) d\omega\end{gathered} \tag{1.7}$$

The additional *coherent term* in Lindblad evolution is defined as

$$B := \sum_{A\in\mathcal{A}} \int_{\mathbb{R}} b_1(t)e^{-iPt}\left(\int_{\mathbb{R}} e^{iPs}A^*e^{-2iPs}Ae^{isP}b_2(s)ds\right)e^{iPt}dt, \tag{1.8}$$

where b_1, b_2 satisfy

$$b_1, b_2 \in L^1(\mathbb{R};\mathbb{R}). \tag{1.9}$$

As we indicate in the case of matrices (see 2.2), the Davies generator is the delocalisation limit, $\sigma \to 0$ (the Gaussian becomes flat) and hence we refer to the [CKG23] generators as *localised Davies generators*. They are more commonly known as *quantum Gibbs samplers* but that implicitly assumes convergence to the Gibbs state in Lindblad evolution which is a question not addressed here.

1.2. **Statements of the results.**

We now specify our assumptions on P and on $\mathcal{A}$. We first consider an abstract setting and then show how additional structure gives results for pseudodifferential operators.

For $\mathcal{H}$, a separable Hilbert space,

$$P : \mathcal{H} \to \mathcal{H} \ \text{ is an unbounded, self-adjoint operator with } P \geq 1. \tag{1.10}$$

(Since adding a constant to P does not change any of the objects above, for our purposes, this is equivalent to P being bounded from below, a natural condition for considering e^{-P}.) We can define (using the notation X' for the dual of X, with duality defined via the inner product on $\mathcal{H}$)

$$\mathscr{D}^s := \mathcal{D}(P^s), \quad \mathscr{D}^s = P^{-s}\mathcal{H}, \quad s \geq 0, \quad \mathscr{D}^s = (\mathscr{D}^{-s})', \quad s \leq 0. \tag{1.11}$$

We now assume that there exists $k \in \mathbb{R}$ such that $A : \mathscr{D}^k \to \mathcal{H}$ and $A^* : \mathscr{D}^k \to \mathcal{H}$. Here at first $A^* : \mathcal{H} \to \mathscr{D}^{-k}$ and we require this additional mapping property. More precisely, we assume that

$$\sum_{A\in\mathcal{A}} \|A\|^2_{\mathscr{D}^k\to\mathcal{H}} + \|A^*\|^2_{\mathscr{D}^k\to\mathcal{H}} < \infty \tag{1.12}$$

It is then immediate that for $\gamma \in L^1(\mathbb{R};[0,\infty))$, $f, b_j \in L^1(\mathbb{R})$, $\mathcal{L}_f$ defined in (1.7) has the following mapping property:

$$\mathcal{L}_f : \mathcal{L}(\mathscr{D}^{-k}, \mathscr{D}^k) \to \mathcal{L}(\mathscr{D}^k, \mathscr{D}^{-k}). \tag{1.13}$$

When we specialise to Gaussian f's and assume in addition that $\mathcal{A}$ is closed under taking the adjoints we need a *balance* condition on γ's guaranteeing that we can find B of the form (2.11) such that $\exp(-P)$ is stationary for (1.7). The standard *Kubo–Martin–Schwinger detailed balanced* condition (see Benoist at al [B*25] for a recent abstract investigation of balanced conditions) on γ, needed in the Davies generator (2.5), is the limiting case as $\sigma \to 0$, that is, when the Gaussian in the sampling (1.5) becomes flat (see §2.2). The balance condition (1.14) appeared already in [RaSo24, Lemma 7.1] and the expressions for b_j's in B are essentially the same as in [CKG23].

Theorem 1. *Assume* (1.10), (1.12), $f = f_\sigma$ *in* (1.6), $\mathcal{A}$ *is closed under taking adjoints, and*

$$\gamma(\omega) = e^{-\omega/2}\varphi\left(\omega + \tfrac14\sigma^2\right), \quad \varphi(-\omega) = \varphi(\omega) \in e^{-|\omega|/2}L^1(\mathbb{R}). \tag{1.14}$$

If B is defined by (1.8) *with*

$$\hat b_1(\tau) = \frac{1}{4\sigma\sqrt{\pi}i}e^{-\tau^2/4\sigma^2}\tanh\Big(\frac{\tau}{4}\Big), \quad \hat b_2(\tau) = e^{-\tau/2}\int_{\mathbb{R}}\gamma(\omega)\, e^{-(\omega+\tau/2)^2/\sigma^2}\,d\omega, \tag{1.15}$$

(see (2.16) *for $b_j(t)$'s) then, in the notation of* (1.7), $\mathcal{L}_f(e^{-P}) = 0$.

An important issue now is to see if $\mathcal{L}_f$ generates a contraction on the trace class. For that we make the following assumption: there is $\delta > 0$ such that

$$\sum_{A\in\mathcal{A}} \left(\|AP^{-\frac12}\|^2_{\mathcal{H}\to\mathcal{H}} + \|[P,A]P^{-\frac12}\|^2_{\mathcal{H}\to\mathcal{H}} + \|[P,[P,A]]P^{-1+\delta}\|^2_{\mathcal{H}\to\mathcal{H}}\right) < \infty, \tag{1.16}$$

where the operators are initially defined as operators $\mathscr{D}^N \to \mathscr{D}^{-2}$, for sufficiently large N. When the operators in $\mathcal{A}$ are bounded, it is enough to assume

$$\sum_{A\in\mathcal{A}} \|A\|^2 < \infty. \tag{1.17}$$

The key now is that (1.16) implies that $\mathcal{L}_f$ in (1.7) satisfies the Davies condition [Da77]:

$$Y := i(\beta^{-1}P + B) - \tfrac12 \sum_{A\in\mathcal{A}} \int_{\mathbb{R}} \gamma(\omega)\hat A_f(\omega)^* \hat A_f(\omega)d\omega \tag{1.18}$$

is the infinitesimal generator of a strongly continuous one parameter contraction semigroup on $\mathcal{H}$. This may be of independent interest and it gives

Theorem 2. *Suppose that $\mathcal{A}$ satisfies* (1.16), P *satisfies* (1.10) *and* $\gamma \in L^1(\mathbb{R};[0,\infty))$. *Then for $f \in L^1$, $\mathcal{L}_f$ is a well defined Lindbladian, in the sense that $e^{t\mathcal{L}_f}$ is a contraction on the space of trace class operators $\mathcal{L}_1(\mathcal{H})$.*

1.3. **Differential and pseudodifferential operators.** When we assume more structure, an analogue of Theorem 2 is valid for pseudodifferential operators, even though (1.16) may be violated. The operators we consider are in the class used in recent works on classical/quantum correspondence in Lindblad evolution by Hernández–Ranard–Riedel [HRR25] and the authors [GaZw25] (see also Li [Li25] and Smith [Sm26] for more recent progress on the PDE study of Lindblad evolution). For harmonic oscillators (which are a very special case of our operators) questions related to Lindbladians with stationary Gibbs states were addressed by Cipriani–Fagnola–Lindsay [CFM00] and, for localised generators, by Šmid et al [S*25], where very precise results about return to equilibrium were provided.

To state the result we recall the Weyl quantization. Let $a \in C^\infty(\mathbb{R}^{2n})$ (a classical observable, that is, a function on phase space) satisfy $|\partial^\alpha a| \leq (1+|x|+|\xi|)^M$ for some M and all α. For $u \in \mathscr{S}(\mathbb{R}^n)$ (the class of functions with rapidly decaying derivatives [Zw12, §3.1]) the action of the Weyl quantisation of a on u is given by

$$a^{\mathrm{w}}(x,D)u = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} a\left(\tfrac{1}{2}(x+y),\xi\right) u(y) e^{i\langle x-y,\xi\rangle} dy d\xi. \tag{1.19}$$

We use the notation

$$\rho = (x,\xi) \in \mathbb{R}^{2n}, \quad \langle \rho \rangle = (1+|x|^2+|\xi|^2)^{\frac{1}{2}}.$$

Theorem 3. *Suppose that* $p \in C^\infty(\mathbb{R}^{2n})$ *satisfies*

$$|\partial^\alpha p(\rho)| \leq C_\alpha, \quad |\alpha| \geq 2, \quad p(\rho) \geq \langle\rho\rangle^2/C_0 - C_0, \tag{1.20}$$

$P = p^{\mathrm{w}}(x,D)$ *and* $\mathcal{A} = \{a^{\mathrm{w}}(x,D) : a \in \mathcal{A}_{\mathrm{cl}} \subset C^\infty(\mathbb{R}^{2n})\}$ *where,*

$$\sum_{a\in\mathcal{A}_{\mathrm{cl}}} \sum_{1\leq|\alpha|\leq N_0} |\partial^\alpha a(\rho)|^2 \leq C, \quad \overline{\mathcal{A}}_{\mathrm{cl}} = \mathcal{A}_{\mathrm{cl}}, \tag{1.21}$$

where N_0 *is a (large) constant depending on the dimension. Then the assumptions of Theorem 1 are satisfied and the conclusions of Theorem 2 hold.*

In the case of P and A's of Theorem 3, we have the following result about pseudodifferential structure of the jump operators $A_f(\omega)$ in (1.7). In view of (1.2) this shows that they are pseudolocal.

Theorem 4. *Suppose that* $P = p^{\mathrm{w}}(x,D)$, $A = a^{\mathrm{w}}(x,D)$ *where*

$$|\partial^\alpha p(\rho)| \leq C_\alpha, \quad |\alpha| \geq 2, \quad |\partial^\alpha a(\rho)| \leq C_\alpha, \quad |\alpha| \geq 1, \quad \rho \in \mathbb{R}^n \times \mathbb{R}^n,$$

and p *is real valued. Then, for* f *satisfying* $|f(t)| \leq C_N e^{-N|t|}$, *for all* N, *and in the notation of* (1.5),

$$A_f(\omega) = a_f(\omega)^{\mathrm{w}}(x,D),$$

where $|\partial^\alpha a_f(\omega,\rho)| \leq C_\alpha$, $|\alpha| \geq 1$, *uniformly in* ω.

Even with the choices of f as in Theorem 2, we cannot exclude the possibility that B is not pseudodifferential if only (1.20) is assumed. That is due to the fact that $b_1(t)$ is not super-exponentially decaying in that case. It is possible that under stronger assumptions, for instance (1.3), the exponential decay of $b_1(t)$ is sufficient to guarantee pseudodifferential structure of B.

We conclude this introduction with some examples.

1.3.1. *Examples for Theorem 1.* The assumptions in Theorem 1 are very weak and apply to large classes of examples. Here is one formulated using order functions [Zw12, §4.4,§8.2] used for definitions and composition rules for pseudodifferential operators. We say that $m : \mathbb{R}^{2n} \to [0,\infty)$ is an order function if there exist C, N such that for $X, Y \in \mathbb{R}^{2n}$, $m(X) \leq (1+|X-Y|)^N m(Y)$. Then a classical observable $a \in C^\infty(\mathbb{R}^{2n})$ (a function on the phase space) is a symbol associated to m, $a \in S(m)$, if $|\partial^\alpha a| \leq C_\alpha m$ for all $\alpha \in \mathbb{N}^n$ (here $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots \alpha_n$. For $u \in \mathscr{S}(\mathbb{R}^n)$ (the class of functions with rapidly decaying derivatives [Zw12, §3.1]) a quantisation of $a \in S(m)$, is given by (1.19).

Suppose now that $m \geq 1$,

$$P = p^{\mathrm{w}}(x,D), \quad p \in S(m), \quad p \geq m/C - C, \tag{1.22}$$

We assume that P is self-adjoint with the domain given by the generalised Sobolev space $H(m)$ (see [Zw12, 8.2]; we consider the case of $h = 1$). We also assume that $P \geq 1$ which implies that $P^{-1} \in S(m^{-1})$ (see the Appendix). We then have $\mathscr{D}^s = H(m^s)$. The condition (1.12) holds provided that

$$\mathcal{A} = \{a^{\mathrm{w}}(x,D) : a \in \mathcal{A}_{\mathrm{cl}}\}, \quad \sum_{a \in \mathcal{A}_{\mathrm{cl}}} \sum_{|\alpha| \leq N_0} \sup_{\mathbb{R}^{2n}} |m^{-k} \partial^\alpha a|^2 < \infty, \quad \overline{\mathcal{A}}_{\mathrm{cl}} = \mathcal{A}_{\mathrm{cl}}, \tag{1.23}$$

where N_0 is some (large) fixed constant depending only on n.

We now list some concrete examples which fit into this framework: the condition $P \geq 1$ can be obtained by a adding a constant to P. One was already given in Theorem 3.

Example 1. Consider $\mathcal{A}_0 \subset C^\infty(\mathbb{R}^n; \mathbb{C}^{n+1})$, and

$$P = -\Delta + V(x), \quad |\partial^\alpha V(x)| \leq C_\alpha,$$
$$\mathcal{A} = \{\langle a_1(x), \partial_x \rangle + a_2(x) : (a_1, a_2) \in \mathcal{A}_0\}, \quad \sum_{a \in \mathcal{A}_0} \sum_{|\alpha| \leq N_0} \sup_{\mathbb{R}^n} |\partial^\alpha a|^2 < \infty,$$

and that, (to have the set closed under complex conjugation),

$$(a_1, a_2) \in \mathcal{A}_0 \implies (-\bar{a}_1, -\operatorname{div} \bar{a}_1 + \bar{a}_2) \in \mathcal{A}_0.$$

In this case, $m = (1+|\xi|)^2$ and $H(m^s) = H^{2s}(\mathbb{R}^n)$ (the usual Sobolev space) and (1.23) holds with $k = \frac{1}{2}$. Note that in this case e^{-P} is *not* of trace class.

Example 2. Suppose (M, g) is a compact Riemannian manifold and $P = -\Delta_g + V(x)$, $V \in C^\infty(M, \mathbb{R})$, and $\mathcal{A}$ a family of vector fields with coefficients bounded in $C^{N_0}(M)$. This could be generalised of P being any self-adjoint elliptic operator with a non-negative principal symbol of order m and $\mathcal{A}$ a family of psedodifferential operators of order $k/2$ with boundedness condition similar to (1.23) – see [Zw12, §14.2] for operators on manifolds.

1.3.2. *Examples for Theorem 2.* The assumption (1.16) is not valid for operators in Theorem 3. We can however consider some cases in which it is valid and to which Theorem 3 does not apply.

The first example generalises (1.3) slightly:

Example 3. Consider $P = -\Delta + V(x)$ satisfying

$$|\partial^\alpha V(x)| \leq \begin{cases} C(1 + |x|) & |\alpha| = 1, \\ C_\alpha (1 + |x|)^{1-2\delta}, & |\alpha| \geq 2, \end{cases} \quad V(x) \geq |x|^2/C - C,$$

(for instance $V(x) = x^2 + (1 + x^2)^{\frac{1}{2}(1-2\delta)} \cos x$, $x \in \mathbb{R}$) with

$$\mathcal{A} = \{\langle a_1, \partial_x \rangle + \langle a_2, x \rangle : (a_1, a_2) \in \mathcal{A}_0 \subset \mathbb{C}^{2n}\}, \quad \sum_{a \in \mathcal{A}_0} |a|^2 < \infty.$$

Example 4. Consider the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ and an elliptic self-adjoint (on $L^2(\mathbb{R}^n, dx)$) second order operator

$$P = - \sum_{1 \leq i,j \leq n} \partial_{x_i} p_{ij}(x) \partial_{x_j} + V(x), \quad p_{ij} \in C^\infty(\mathbb{T}^n), \quad V \in C^\infty(\mathbb{T}^n),$$
$$p_{ij}(x) = \overline{p_{ji}(x)}, \quad \sum_{i,j} p_{ij}(x) \xi_i \xi_j \geq |\xi|^2/C, \quad \xi \in \mathbb{R}^n.$$

Then the following family works:

$$\mathcal{A} = \{\langle a, \partial_x \rangle + b(x) : a \in \mathcal{A}_0 \subset \mathbb{C}^n, \ b \in \mathcal{B} \subset C^\infty(\mathbb{T}^n)\},$$
$$\sum_{a \in \mathcal{A}_0} |a|^2 + \sum_{\mathcal{B}} \sum_{|\alpha| \leq N_0} \sup |\partial^\alpha b| < \infty.$$

We should note that the operators $u \mapsto bu$ do not satisfy (1.16) but they are bounded operators and hence can be handled directly.

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2. The matrix case revisited

We start by presenting a different perspective on the case of

$$P = P^* \in \mathcal{L}(\mathcal{H}), \quad \mathcal{A} \subset \mathcal{L}(\mathcal{H}), \quad \dim \mathcal{H} < \infty. \tag{2.1}$$

The functional equation which determines the properties of γ and the functions b_j in (1.8) will be the same in the infinite dimensional case but it can be presented in a straightforward way. We start with the description of the Davies generator [Da74],[Da76] as presented in [BrPe02]. We then show how the Davies generator arises as the limit of the localised generator. This is followed by analysis of the localised version from [CKG23].

2.1. The Davies generator.

Following [BrPe02, §3.3] (see also [Te13, equation (4)]), we introduce the Davies generator. This generator provides a Lindbladian for which the Gibbs state, $\rho = e^{-P}/\mathrm{tr} e^{-P}$, is stationary. It is constructed as follows. Define

$$A_\nu := \sum_{E_2 - E_1 = \nu} \mathbb{1}_{E_2}(P) A \, \mathbb{1}_{E_1}(P), \quad [P, A_\nu] = \nu A_\nu, \quad e^P A_\nu e^{-P} = e^\nu A_\nu. \tag{2.2}$$

The index ν varies along the Bohr spectrum of P:

$$\mathcal{B}(P) := \{E_2 - E_1 : E_j \in \mathrm{Spec}(P)\}.$$

A version of the Davies generator is now defined for a family $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ as

$$\mathcal{D}(T) := \sum_{A \in \mathcal{A}} \sum_{\nu \in \mathcal{B}(P)} \gamma(\nu) \left(A_\nu T A_\nu^* - \tfrac{1}{2} (A_\nu^* A_\nu T + T A_\nu^* A_\nu) \right). \tag{2.3}$$

We now have

Proposition 1. *Suppose that $\mathcal{D}$ is given by (2.3), $\mathcal{A}$ is closed under taking adjoints, and*

$$\sum_{A \in \mathcal{A}} \|A\|^2 < \infty. \tag{2.4}$$

If γ satisfies the following balance condition

$$\gamma(-\omega) = e^\omega \gamma(\omega) \iff \gamma(\omega) = e^{-\frac{1}{2}\omega} \varphi(\omega), \quad \varphi(\omega) = \varphi(-\omega), \tag{2.5}$$

then

$$\mathcal{D}(e^{-P}) = 0.$$

In particular e^{-P} *is stationary for the Lindbladian* $\mathcal{L}T := -i[\beta^{-1}P, T] + \mathcal{D}(T)$*.*

Proof. From (2.2) we obtain

$$A_\nu e^{-P} A_\nu^* = e^\nu A_\nu A_\nu^* e^{-P}, \quad e^{-P} A_\nu^* A_\nu = A_\nu^* A_\nu e^{-P},$$

which means that we need to check that

$$\sum_{A \in \mathcal{A}} \sum_{\nu \in \mathcal{B}(P)} \left(\gamma(\nu) e^\nu A_\nu A_\nu^* - \gamma(\nu) A_\nu^* A_\nu\right) = 0.$$

We now observe that $A_{-\nu} = (A^*)_\nu^*$ so that the balance condition (2.5) shows that the sum is equal to

$$\sum_{A \in \mathcal{A}} \sum_{\nu \in \mathcal{B}(P)} \gamma(\nu) \left(A_{-\nu} A_{-\nu}^* - A_\nu^* A_\nu\right) = \sum_{A \in \mathcal{A}} \sum_{\nu \in \mathcal{B}(P)} \gamma(\nu) \left((A^*)_\nu^* (A^*)_\nu - A_\nu^* A_\nu\right).$$

Since $\mathcal{A}$ is assumed to be invariant under taking adjoints the sum vanishes. □

2.2. **Davies generator as a limit of localised generators.** The following proposition relates the Davies generator from §2.1 to the Lindbladian described in §1 under the assumption that $\dim \mathcal{H} < \infty$ is fixed. Since we present this only to relate the two generators we do not consider uniformity in the dimension. We make a somewhat strong assumption on γ – it is satisfied by Gaussians considered in [CKG23].

Proposition 2. *Suppose that* (2.4) *holds and that*

$$\gamma(\omega) \in C^\infty(\mathbb{R}), \quad \widehat{e^{-\omega}\gamma(\omega)} \in L^1(\mathbb{R}). \tag{2.6}$$

For $\mathcal{L}_f$ *and* $\mathcal{D}$ *defined in* (1.7) *(with* f_σ *and* b_j*'s in* (1.6) *and* (1.15)*) and* (2.3) *respectively,*

$$\lim_{\sigma \to 0} \|\mathcal{L}_{f_\sigma}(T) - \mathcal{D}(T)\|_{\mathcal{L}_p(\mathcal{H})} = 0, \quad T \in \mathcal{L}_p(\mathcal{H}), \quad 1 \le p \le \infty, \tag{2.7}$$

where $\mathcal{L}_p(\mathcal{H})$ *denotes the* p*–Schatten class (see* [Ka80, Chapter 10, §1.3]*).*

Proof. In the case of matrices (since $\mathcal{B}(P)$ is finite), definitions (1.5) and (2.2) give

$$\widehat{A}_{f_\sigma}(\omega) = \sum_\nu A_\nu \hat{f}_\sigma(\omega - \nu),$$

and hence

$$\mathcal{D}_{f_\sigma}(T) - \mathcal{D}(T) = \sum_{A \in \mathcal{A}} \sum_{\nu, \nu'} \left(A_\nu T A_{\nu'}^* - \tfrac{1}{2}\left\{A_\nu^* A_{\nu'}, T\right\}\right) \left(G_\sigma(\nu, \nu') - \gamma(\nu)\delta_{\nu\nu'}\right).$$

The formula (1.6) gives

$$\begin{aligned} G_\sigma(\nu,\nu') &:= \int_{\mathbb{R}} \gamma(\omega) f_{1/\sigma}(\omega-\nu) f_{1/\sigma}(\omega-\nu') d\omega \\ &= e^{-(\nu-\nu')^2/4\sigma^2} \frac{\sqrt{\pi}}{\sigma} \int_{\mathbb{R}} \gamma(\omega) e^{-\sigma^{-2}(\omega-\frac{1}{2}(\nu+\nu'))^2} d\omega \to \gamma(\nu)\delta_{\nu\nu'}. \end{aligned}$$

Since $\|A_\nu T A_{\nu'}^*\|_{\mathcal{L}_p} \leq \|A\|^2 \|T\|_{\mathcal{L}_p}$, with similar estimates for other terms, the finiteness of the sum over ν and ν' shows that for $T \in \mathcal{L}_p$, $\mathcal{D}_{f_\sigma}(T) \to \mathcal{D}(T)$ in $\mathcal{L}_p$ as $\sigma \to 0$.

It remains to show that $i[B,T] \to 0$ as $\sigma \to 0$ and we do it by showing that $\|B\| \to 0$. For that we go to the inverse Fourier transforms of the b_j's appearing in (1.8) and given in (2.16). We write b_1 as follows

$$b_1(t) = \tfrac{1}{8}\pi^{\frac{1}{2}} \int_0^\infty \frac{e^{-\sigma^2(t+s)^2} - e^{-\sigma^2(t-s)^2}}{\sinh(2\pi s)} ds.$$

Since $(t+s)^2 \geq (t-s)^2$ for $t \geq 0$ with the opposite inequality for $t \leq 0$,

$$\begin{aligned} \|b_1\|_{L^1} &= \tfrac{1}{4}\pi^{\frac{1}{2}} \int_0^\infty \int_0^\infty \frac{e^{-\sigma^2(t-s)^2} - e^{-\sigma^2(t+s)^2}}{\sinh(2\pi s)} ds dt = \tfrac{1}{4}\pi^{\frac{1}{2}} \int_0^\infty \int_{-s}^s \frac{e^{-\sigma^2 u^2}}{\sinh(2\pi s)} du ds \\ &\leq \tfrac{1}{2}\pi^{\frac{1}{2}} \int_0^\infty \frac{s}{\sinh(2\pi s)} ds = \tfrac{1}{32}\pi^{\frac{1}{2}}. \end{aligned}$$

For b_2 we use the assumption (2.6) noting that $\hat{\gamma}(t-i)$ in (2.16) is the Fourier transform of $e^{-\omega}\gamma(\omega)$. Hence, $\|b_2\|_1 \leq \sqrt{\pi}\sigma e^{\frac{1}{4}\sigma^2} \|\hat{\gamma}(\bullet - i)\|_1 = \mathcal{O}(\sigma)$, $\sigma \to 0$. This and (2.4) show that $\|B\| \to 0$ as $\sigma \to 0$, completing the proof. □

2.3. **A localised version.** We now consider (1.5) in the case of matrices. For that we define

$$\begin{aligned} \hat{A}(\omega) &:= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itP} A e^{-itP} e^{-it\omega} dt \\ &= \sqrt{2\pi} \sum_{\nu \in B(P)} A_\nu \delta(\nu - \omega), \qquad \hat{A} \in \mathscr{S}'(\mathbb{R}; \mathcal{L}(\mathcal{H})), \end{aligned}$$

where $\mathcal{H}$ is the finite dimensional Hilbert space on which the operators act.

The operator in (1.5) becomes (specifying the dependence on f in the notation)

$$\hat{A}_f(\omega) = \frac{1}{\sqrt{2\pi}} \hat{A} * \hat{f}(\omega), \quad \hat{f}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\omega} dt.$$

In view of (2.2),

$$e^{-P}\hat{A}(\omega) = e^{-\omega}\hat{A}(\omega)e^{-P}, \quad \hat{A}(\omega)^* = \widehat{A^*}(-\omega). \tag{2.8}$$

We now write $\mathcal{D}_f(e^{-P}) = \frac{1}{2\pi}\sum_j I_j$, where, assume that $\hat{f}$ is real valued, and that $\mathcal{A}$ is invariant under taking adjoints,

$$\begin{aligned} I_1 &:= \sum_{A\in\mathcal{A}} \int_{\mathbb{R}^3} \gamma(\omega)\hat{A}(\tau)e^{-P}\widehat{A^*}(-\tau')\hat{f}(\omega-\tau)\hat{f}(\omega-\tau')d\tau d\tau' d\omega \\ &= \int_{\mathbb{R}^2} \sum_{A\in\mathcal{A}} \hat{A}(\tau)\widehat{A^*}(-\tau')e^{-P}\left(\int_{\mathbb{R}} \gamma(\omega)e^{\tau'}\hat{f}(\omega-\tau)\hat{f}(\omega-\tau')d\omega\right) d\tau d\tau' \\ &= \int_{\mathbb{R}^2} \sum_{A\in\mathcal{A}} \hat{A}(\tau)^*\hat{A}(\tau')e^{-P}\left(\int_{\mathbb{R}} \gamma(\omega)e^{-\tau'}\hat{f}(\omega+\tau)\hat{f}(\omega+\tau')d\omega\right) d\tau d\tau', \end{aligned}$$

$$\begin{aligned} I_2 &:= -\sum_{A\in\mathcal{A}} \tfrac{1}{2}\int_{\mathbb{R}^3} \gamma(\omega)\widehat{A^*}(-\tau)\hat{A}(\tau')e^{-P}\hat{f}(\omega-\tau)\hat{f}(\omega-\tau')d\omega d\tau d\tau' \\ &= \sum_{A\in\mathcal{A}} \int_{\mathbb{R}^2} \hat{A}(\tau)^*\hat{A}(\tau')e^{-P}\left(-\tfrac{1}{2}\int_{\mathbb{R}} \gamma(\omega)\hat{f}(\omega-\tau)\hat{f}(\omega-\tau')d\omega\right) d\tau d\tau', \end{aligned}$$

and

$$\begin{aligned} I_3 &:= -\sum_{A\in\mathcal{A}} \tfrac{1}{2}\int_{\mathbb{R}^3} \gamma(\omega)e^{-P}\widehat{A^*}(\tau)\hat{A}(-\tau')\hat{f}(\omega+\tau)\hat{f}(\omega+\tau')d\omega d\tau d\tau' \\ &= \sum_{A\in\mathcal{A}} \int_{\mathbb{R}^2} \widehat{A^*}(\tau)\hat{A}(-\tau')e^{-P}\left(-\tfrac{1}{2}\int_{\mathbb{R}} \gamma(\omega)e^{\tau'-\tau}\hat{f}(\omega+\tau)\hat{f}(\omega+\tau')d\omega\right) d\tau d\tau' \\ &= \sum_{A\in\mathcal{A}} \int_{\mathbb{R}^2} \hat{A}(\tau)^*\hat{A}(\tau')e^{-P}\left(-\tfrac{1}{2}\int_{\mathbb{R}} \gamma(\omega)e^{\tau-\tau'}\hat{f}(\omega-\tau)\hat{f}(\omega-\tau')d\omega\right) d\tau d\tau'. \end{aligned}$$

If we define

$$F(\tau,\tau') := \int_{\mathbb{R}} \gamma(\omega)\left(e^{-\tau'}\hat{f}(\omega+\tau)\hat{f}(\omega+\tau') - \tfrac{1}{2}(1+e^{\tau-\tau'})\hat{f}(\omega-\tau)\hat{f}(\omega-\tau')\right) d\omega, \tag{2.9}$$

the formulas for I_j's show that

$$\mathcal{D}_f(e^{-P}) = \frac{1}{2\pi}\sum_{A\in\mathcal{A}}\int_{\mathbb{R}^2} F(\tau,\tau')\hat{A}(\tau)^*\hat{A}(\tau')e^{-P}d\tau d\tau'. \tag{2.10}$$

We now want to find a self-adjoint operator on $\mathcal{H}$, B, such that, for ω and b with suitable properties,

$$\mathcal{D}_f(e^{-P}) = i[B, e^{-P}], \quad B = \frac{1}{2\pi}\sum_{A\in\mathcal{A}}\int_{\mathbb{R}^2} b(\tau,\tau')\hat{A}(\tau)^*\hat{A}(\tau')d\tau d\tau'. \tag{2.11}$$

Since

$$B^* = \frac{1}{2\pi}\sum_{A\in\mathcal{A}}\int_{\mathbb{R}^2} \overline{b(\tau,\tau')}\hat{A}(\tau')^*\hat{A}(\tau)d\tau d\tau',$$

we have $B = B^*$ (for any choice of $\mathcal{A}$) if

$$\overline{b(\tau,\tau')} = b(\tau',\tau). \tag{2.12}$$

We now compute the commutator using (2.8) and

$$e^{-P}\hat{A}(\tau)^* = e^{-P}\widehat{A^*}(-\tau) = e^{\tau}\widehat{A^*}(-\tau)e^{-P} = e^{\tau}\hat{A}(\tau)^*e^{-P},$$

to obtain

$$\begin{aligned}[B, e^{-P}] &= \frac{1}{2\pi}\sum_{A\in\mathcal{A}}\int_{\mathbb{R}^2} b(\tau,\tau')(\hat{A}(\tau)^*\hat{A}(\tau')e^{-P} - e^{-P}\hat{A}(\tau)^*\hat{A}(\tau'))d\tau d\tau' \\ &= \frac{1}{2\pi}\sum_{A\in\mathcal{A}}\int_{\mathbb{R}^2} b(\tau,\tau')(1-e^{\tau-\tau'})\hat{A}(\tau)^*\hat{A}(\tau')e^{-P}d\tau d\tau'.\end{aligned}$$

In the notation of (2.10) and (2.11) this leads to the following functional equation:

$$F(\tau,\tau') = i(1-e^{\tau-\tau'})b(\tau,\tau'). \tag{2.13}$$

It turns out that for f given by the Gaussian centred at 0, a condition on the function γ is sufficient for obtaining a solution – see §2.4.

Before solving the functional equation, we relate the form of B in (2.11) with that from [CKG23] presented in (1.8).

Lemma 3. *The operator B in (2.11) is given by the formula (1.8) if*

$$b(\tau,\tau') = 2\pi\hat{b}_1(\tau'-\tau)\hat{b}_2(\tau+\tau'). \tag{2.14}$$

Proof. Make the following change of variables in (2.11), $w = s-t$, $v = -s-t$, so that the integral becomes

$$\tfrac{1}{2}\int_{\mathbb{R}^2} b_1(-\tfrac{1}{2}(v+w))b_2(\tfrac{1}{2}(w-v))e^{iwP}A^*e^{-iwP}e^{ivP}Ae^{-ivP}dvdw.$$

To apply Plancherel's theorem we calculate

$$\begin{aligned}\tfrac{1}{2}\int b_1(-\tfrac{1}{2}(v+w))b_2(\tfrac{1}{2}(w-v))e^{-iw\tau-iv\tau'}dvdw &= \int b_1(t)b_2(s)e^{-i(s-t)\tau+i(s+t)\tau'}dtds \\ &= \hat{b}_1(-\tau-\tau')\hat{b}_2(\tau-\tau').\end{aligned}$$

Hence Plancherel's theorem shows that B given in (1.8) can be written as

$$\begin{aligned}B &= \sum_{A\in\mathcal{A}}\int_{\mathbb{R}^2}\hat{b}_1(-\tau-\tau')b_2(\tau-\tau')\widehat{A^*}(-\tau)\hat{A}(-\tau')d\tau d\tau' \\ &= \sum_{A\in\mathcal{A}}\int_{\mathbb{R}^2}\hat{b}_1(\tau'-\tau)b_2(\tau+\tau')\widehat{A}(\tau)^*A(\tau')d\tau d\tau',\end{aligned}$$

which gives (2.14). □

2.4. **Solving functional equation (2.13) in the Gaussian case.** A necessary and sufficient condition for the existence of a function $b(\tau,\tau')$ such that (2.13) holds is $F(\tau,\tau)=0$ for all $\tau\in\mathbb{R}$. Equivalently,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \gamma(\omega)\Big(e^{-\tau}\hat f(\omega+\tau)^2 - \hat f(\omega-\tau)^2\Big)\,d\omega \\ &= \int_{\mathbb{R}} \hat f(\eta)^2\Big(e^{-\tau}\gamma(\eta-\tau) - \gamma(\eta+\tau)\Big)\,d\eta \qquad (2.15)\\ &= \sigma^{-1}\sqrt{\pi}\int_{\mathbb{R}}\Big(e^{-\tau}\gamma(\eta-\tau)-\gamma(\eta+\tau)\Big)e^{-\eta^2/\sigma^2}\,d\eta, \end{aligned}$$

where in the last equality we used $f=f_\sigma$ from (1.6), so that $\hat f_\sigma(\eta)^2=\sigma^{-1}\sqrt{\pi}\,e^{-\eta^2/\sigma^2}$. If we define $G_\sigma(x):=\sigma^{-1}\sqrt{\pi}\,e^{-x^2/\sigma^2}$, then condition (2.15) may be written as

$$(\gamma * G_\sigma)(\tau) = e^{-\tau}(\gamma * G_\sigma)(-\tau) = (e^{\bullet}[\gamma * G_\sigma](\bullet))(-\tau).$$

Passing to the Fourier transform side, and assuming the required analyticity of $\hat\gamma$, we obtain

$$\hat\gamma(t)\widehat{G}_\sigma(t) = \hat\gamma(-t+i)\widehat{G}_\sigma(-t+i).$$

Since

$$\widehat{G}_\sigma(s) = 2^{-1/2}e^{-\sigma^2 s^2/4},$$

this is equivalent to

$$\hat\gamma(t) = e^{i\sigma^2 t/2}e^{\sigma^2/4}\,\hat\gamma(-t+i).$$

Equivalently, if we define

$$\hat\varphi(t) := e^{-i\sigma^2 t/4}\hat\gamma\big(t+\tfrac12 i\big)$$

then

$$\hat\varphi(t) = \hat\varphi(-t).$$

Taking inverse Fourier transforms, we obtain the following description of all γ for which (2.15) holds:

$$\gamma(\omega) = e^{-\omega/2}\,\varphi\big(\omega+\tfrac14\sigma^2\big)\,, \qquad \varphi(\omega)=\varphi(-\omega).$$

To obtain the expressions for b_1 and b_2 in (2.14), it is convenient to introduce $\xi := \tau-\tau'$, $\zeta:=\tau+\tau'$, and

$$A(\zeta) := \int_{\mathbb{R}} \gamma(\omega)e^{-\omega^2/\sigma^2}e^{-\zeta\omega/\sigma^2}\,d\omega.$$

A direct computation then gives

$$F(\tau,\tau') = \sigma^{-1}\sqrt{\pi}\,e^{-(\zeta^2+\xi^2)/4\sigma^2}\left(e^{-(\zeta-\xi)/2}A(\zeta) - \tfrac12(1+e^{\xi})A(-\zeta)\right).$$

Under the divisibility condition $F(\tau,\tau)=0$, equivalently

$$A(-\zeta) = e^{-\zeta/2}A(\zeta),$$

this becomes

$$F(\tau,\tau') = \sigma^{-1}\sqrt{\pi}\, i(1-e^{\xi})\, \beta_1(\xi)\, \beta_2(\zeta),$$

where

$$\beta_1(\xi) = \frac{1}{2i} e^{-\xi^2/4\sigma^2} \tanh\left(\frac{\xi}{4}\right), \qquad \beta_2(\zeta) = e^{-\zeta/2} \int_{\mathbb{R}} \gamma(\omega)\, e^{-(\omega+\zeta/2)^2/\sigma^2}\, d\omega.$$

In particular, we may write

$$b(\tau,\tau') = 2\pi\, \hat{b}_1(\xi)\hat{b}_2(\zeta),$$

with

$$\hat{b}_1(\xi) = \frac{1}{4\sigma\sqrt{\pi}\, i}\, e^{-\xi^2/4\sigma^2} \tanh\left(\frac{\xi}{4}\right), \qquad \hat{b}_2(\zeta) = e^{-\zeta/2} \int_{\mathbb{R}} \gamma(\omega)\, e^{-(\omega+\zeta/2)^2/\sigma^2}\, d\omega.$$

This gives Theorem 1 in the case of matrices. Here are the expressions b_j's after taking the inverse Fourier transform (less clean in the case of b_1):

$$b_1(t) = -\tfrac{1}{8}\pi^{-\frac{1}{2}}(e^{-\sigma^2\bullet^2} * \sinh(2\pi\bullet)^{-1})(t), \quad b_2(t) = 2\sqrt{\pi}\,\sigma\, e^{-\frac{1}{4}\sigma^2(2t-i)^2}\, \hat{\gamma}(2t-i), \tag{2.16}$$

where $\sinh(2\pi x)^{-1}$ is considered the distribution defined by taking the principal value at 0: $\sinh(2\pi\bullet)^{-1}(\varphi) := \lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \sinh(2\pi t)^{-1}\varphi(t)dt$, $\varphi \in \mathscr{S}(\mathbb{R})$.

3. UNBOUNDED OPERATORS

The assumptions for Theorem 1 are very general as illustrated in §1.3.1. The proof is also straightforward since we assume that the operators A act on spaces defined using P – see (1.11) and (1.12). We also note that for those spaces $e^{-P} : \mathscr{D}^{-N} \to \mathscr{D}^{N}$ for any N.

Proof of Theorem 1. The strategy is to mimic the proof in §2.3 considering $A(t) := e^{itP}Ae^{-itP}$ as an operator-valued tempered distribution. Then $\hat{A}(\omega)$ can be considered as the Fourier transform of $A(t)$ in the distributional sense (see [Zw12, §3.2] for a brief introduction), and for $f \in \mathscr{S}(\mathbb{R})$ $A_f(\omega) = \hat{A} * f(\omega)$.

More precisely, using (1.12)

$$A(t) \in L^\infty(\mathbb{R}_t; \mathcal{L}(\mathcal{H}, \mathscr{D}^{-k})) \subset \mathscr{S}'(\mathbb{R}; \mathcal{L}(\mathcal{H}, \mathscr{D}^{-k})),$$

and, in the sense of the distributions $\hat{A}(g) := A(\hat{g})$, $\hat{A}(\omega) \in \mathscr{S}'(\mathbb{R}_\omega; \mathcal{L}(\mathcal{H}, \mathscr{D}^{-k}))$. Here $A(\hat{g})$ and $\hat{A}(g)$ denotes distributional pairing, informally written as

$$A(\hat{g}) = \int_{\mathbb{R}} A(t)\hat{g}(t)dt = \int_{\mathbb{R}} \hat{A}(\omega)g(\omega)d\omega = \hat{A}(g), \quad \hat{g}(t) := \frac{1}{\sqrt{2\pi}} \int g(\omega)e^{-it\omega}d\omega. \tag{3.1}$$

We also have the corresponding statements for A^*.

The key fact now is the distributional analogue of (2.8):

Lemma 4. *Suppose that* $g\in\mathscr{S}(\mathbb{R})$*,* $\hat g$ *is holomorphic in* $\mathbb{R}+i(-1-\varepsilon,\varepsilon)$*,* $\varepsilon>0$ *and*

$$|\hat g(t-i\mu)|\le C\langle t\rangle^{-2},\quad \mu\in[0,1]. \tag{3.2}$$

Then

$$e^{-P}\hat A(g)=\hat A(e^{-\bullet}g(\bullet))e^{-P}\in\mathcal{L}(\mathcal{H}) \tag{3.3}$$

Proof. We proceed using the definition (3.1) and contour deformation, justified by the holomorphy of $\hat g$ and the fact that $e^{i\zeta P}:\mathscr{D}^s\to\mathscr{D}^s$ for $\operatorname{Im}\zeta\ge 0$ (recall that $P\ge 1$):

$$e^{-P}\hat A(g)=e^{-P}\int e^{itP}Ae^{-itP}\hat g(t)dt=\int_{\Gamma_R}e^{izP}Ae^{-i(z-i)P}\hat g(z-i)dz,$$

and the contour is given by $\Gamma_R=\Gamma_R^+ + \Gamma_R^- + \gamma_R^+ + \gamma_R^- + I_R$, where $I_R=[-R,R]$ with the positive orientation, and $\Gamma_R^\pm=\pm[R,\infty)+i$, $\gamma_R^\pm=\pm R+i[0,1]$, with matching orientations. Condition (3.2) then shows that

$$\int_{\Gamma_R^\pm}e^{izP}Ae^{-i(z-i)P}\hat g(z-i)dz,\int_{\gamma_R^\pm}e^{izP}Ae^{-i(z-i)P}\hat g(z-i)dz\to 0,\quad R\to\infty,$$

implying that

$$e^{-P}\hat A(g)=\int_{\mathbb{R}}e^{itP}Ae^{-itP}\hat g(t-i)dte^{-P}.$$

Since $\hat g(\bullet-i)=\widehat{e^{-\bullet}g(\bullet)}$, this gives (3.3). □

With the lemma in place, the distributional point of view allows us to carry the calculations in §2.3 in the case of f given by the Gaussians (1.6). □

To prove Theorem 2 via Proposition 5 below we start with some preliminaries about mapping properties of A and A^*. The assumption (1.16) shows that the following operators are continuous:

$$A:\mathscr{D}^{\frac12}\to\mathcal{H},\quad A^*:\mathcal{H}\to\mathscr{D}^{-\frac12}. \tag{3.4}$$

We also not that the assumption that $[P,A]P^{-\frac12}:\mathcal{H}\to\mathcal{H}$ gives

$$\begin{aligned}\|A\|_{\mathscr{D}^{-\frac12}\to\mathscr{D}^{-1}}&=\|A^*\|_{\mathscr{D}^1\to\mathscr{D}^{\frac12}}\\&=\|P^{\frac12}A^*P^{-1}\|_{\mathcal{H}\to\mathcal{H}}\le\|P^{-\frac12}A^*\|_{\mathcal{H}\to\mathcal{H}}+\|P^{\frac12}[P^{-1},A^*]\|_{\mathcal{H}\to\mathcal{H}}\\&=\|P^{-\frac12}A^*\|_{\mathcal{H}\to\mathcal{H}}+\|P^{-\frac12}[P,A^*]P^{-1}\|_{\mathcal{H}\to\mathcal{H}}\\&\le\|AP^{-\frac12}\|_{\mathcal{H}\to\mathcal{H}}+\|[P,A]P^{-\frac12}\|_{\mathcal{H}\to\mathcal{H}}.\end{aligned} \tag{3.5}$$

since $\|P^{-1}\|_{\mathcal{H}\to\mathcal{H}}\le 1$. Complex interpolation then implies that

$$\|A\|_{\mathscr{H}\to\mathscr{D}^{-\frac12}}\le\|A\|^{1/2}_{\mathscr{D}^{-1/2}\to\mathscr{D}^{-1}}\|A\|^{1/2}_{\mathscr{D}^{1/2}\to\mathcal{H}}\le C\|AP^{-\frac12}\|_{\to\mathcal{H}}+\|[P,A]P^{-\frac12}\|_{\mathcal{H}\to\mathcal{H}}.$$

We also have

$$\|A\|_{\mathscr{D}^{\frac32}\to\mathscr{D}^1}=\|PAP^{-\frac32}\|_{\mathcal{H}\to\mathcal{H}}\le\|AP^{-\frac12}\|_{\mathcal{H}\to\mathcal{H}}+\|[P,A]P^{-\frac12}\|_{\mathcal{H}\to\mathcal{H}} \tag{3.6}$$

and hence, using interpolation again $A : \mathscr{D} \to \mathcal{H}$. Consequently, we also have the following uniform continuity statement:

$$e^{itP}A^*e^{i(s-t)P}Ae^{-isP} : \mathcal{H} \to \mathscr{D}^{-1}, \quad e^{itP}A^*e^{i(s-t)P}Ae^{-isP} : \mathscr{D}^1 \to \mathcal{H}. \tag{3.7}$$

To prove Theorem 2 we recall (1.18):

$$Y := i(\beta^{-1}P + B) - \tfrac{1}{2}\sum_{A\in\mathcal{A}}\int \gamma(\omega)\hat{A}_f(\omega)^*\hat{A}_f(\omega)d\omega \tag{3.8}$$

where B is defined in (1.8). In view of the results of [Da77], Theorem 2 follows from

Proposition 5. *Suppose that* $\gamma, f, b_j \in L^1(\mathbb{R})$, $j = 1, 2$, $\gamma \geq 0$, *and that for some* $\delta > 0$,

$$\sum_{A\in\mathcal{A}}\left(\|AP^{-\frac{1}{2}}\|^2_{\mathcal{H}\to\mathcal{H}} + \|[P,A]P^{-\frac{1}{2}}\|^2_{\mathcal{H}\to\mathcal{H}} + \|[P,[P,A]]P^{-1+\delta}\|^2_{\mathcal{H}\to\mathcal{H}}\right) < \infty, \tag{3.9}$$

Then Y *in* (3.8) *is the generator of a contraction semigroup on* $\mathcal{H}$.

Proof of Proposition 5. As in the proof of Lemma 3, we can write B and the last term on the right hand side of (3.8) as

$$R := \int_{\mathbb{R}^2} e^{itP}A^*e^{i(s-t)P}Ae^{-isP}F(t,s)dtds, \tag{3.10}$$

with

$$F(t,s) = \tfrac{1}{2}b_1(-\tfrac{1}{2}(t+s))b_2(\tfrac{1}{2}(t-s)) \in L^1(\mathbb{R}^2),$$

and

$$F(t,s) = \int_{\mathbb{R}} \gamma(\omega)e^{i(t-s)\omega}f(t)f(s)d\omega \in L^1(\mathbb{R}^2),$$

respectively (we assumed $\gamma, f, b_j \in L^1$). From (3.7) we see that $R : \mathcal{H} \to \mathscr{D}^{-1}$ is continuous and hence so is Y.

To apply the Hille-Yosida theorem (see [Ka80, Chapter 9, §1.2]), we need to show that Y with the domain

$$\mathcal{D}(Y) := \{u \in \mathcal{H} : Yu \in \mathcal{H}\}.$$

(in view of the the discussion above, we interpret $Yu \in \mathscr{D}^{-1} \supset \mathcal{H}$) is closed and prove the estimate

$$\|(Y-\lambda)^{-1}\|_{L^2\to L^2} \leq \lambda^{-1}, \qquad \lambda > 0. \tag{3.11}$$

To show that Y is closed we first consider $Y_0 = Y|_{\mathscr{D}^1}$ and show that $\overline{Y}_0 = Y$. For that, suppose $u_n \in \mathscr{D}^1$ with $u_n \overset{\mathcal{H}}{\to} u$, $Y_0u_n \overset{\mathcal{H}}{\to} w$. Since this implies that $Y_0u_n \overset{\mathscr{D}^{-1}}{\to} w$, and Y is continuous from $\mathcal{H} \to \mathscr{D}^{-1}$, we have $w = Yu$ (where the right hand side is understood to be in $\mathscr{D}^{-1}$. In particular, $\overline{Y}_0 \subset Y$.

To show equality, let $u \in \mathcal{D}(Y)$. We will show that

$$\forall\, u \in \mathcal{D}(Y)\, \exists\, u_\varepsilon \in \mathscr{D}^1 \qquad u_\varepsilon \xrightarrow{\mathcal{H}} u, \quad Y_0 u_\varepsilon \xrightarrow{\mathcal{H}} Yu, \quad \varepsilon \to 0. \tag{3.12}$$

In other words, $Y \subset \overline{Y}_0$, and hence $Y = \overline{Y}_0$ is closed.

To prove this we need the following Lemma:

Lemma 6. *Suppose that* $R : \mathscr{S} \to \mathscr{S}$ *satisfies*

$$\| P^{-\frac{1}{2}} [P, R] P^{-\frac{1}{2}} \|_{\mathcal{H} \to \mathcal{H}} + \| P^{-1+\delta} [P, [P, R]] P^{-1+\delta} \|_{\mathcal{H} \to \mathcal{H}} < \infty, \tag{3.13}$$

for some $\delta > 0$. *Then, for all* $\chi_0, \chi_1 \in C_c^\infty(\mathbb{R})$ *with* $\chi_1|_{[-1,1]} \equiv 1$ *and* $\operatorname{supp}(1 - \chi_0) \cap \operatorname{supp} \chi_1 = \emptyset$, *there is* $C > 0$ *such that for* $0 < \varepsilon < 1$ *there* A_ε *and* B_ε *such that*

$$[\chi_0(\varepsilon P), R] = A_\varepsilon (1 - \chi_1(\varepsilon P)) + \varepsilon^{2\delta} B_\varepsilon, \qquad \|B_\varepsilon\|_{\mathcal{H} \to \mathcal{H}} + \|A_\varepsilon\|_{\mathcal{H} \to \mathcal{H}} \le C. \tag{3.14}$$

Proof. We use the Helffer–Sjöstrand formula (see [DiSj99, Theorem 8.1]):

$$\chi_j(\varepsilon P) = \pi^{-1} \int_{\mathbb{C}} \bar{\partial}_z \tilde{\chi}_j(z) (\varepsilon P - z)^{-1} dm(z), \tag{3.15}$$

where $dm(z)$ is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$ and $\tilde{\chi}_j \in C_c^\infty(\mathbb{C})$ satisfies $\tilde{\chi}_j|_{\mathbb{R}} = \chi_j$, $\bar{\partial}_z \tilde{\chi}_h(z) = (|\operatorname{Im} z|^\infty)$. (That means that $\tilde{\chi}_j$ is an *almost analytic* extension of χ_j.) We can assume that $\widetilde{\chi}_0 = 1$ on the support of $\widetilde{\chi}_1$. (See the construction in the beginning of [DiSj99, Chapter 8].) We then have the decomposition in (3.14) with

$$\begin{aligned} A_\varepsilon &= [\chi_0(\varepsilon P), R] = \pi^{-1} \varepsilon \int_{\mathbb{C}} \bar{\partial}_z \tilde{\chi}_0(z) (\varepsilon P - z)^{-1} [R, P] (\varepsilon P - z)^{-1} dm(z), \\ B_\varepsilon &= \varepsilon^{-2\delta} [\chi_0(\varepsilon P), R] \chi_1(\varepsilon P). \end{aligned}$$

To estimate $\|A_\varepsilon\|$ we use the spectral theorem to see that for $z \notin \mathbb{R}$,

$$\| (\varepsilon P - z)^{-1} P^{1-\delta} \|_{\mathcal{H} \to \mathcal{H}} \le \sup_{x \in [1, \infty)} \frac{x^{1-\delta}}{|\varepsilon x - z|} \le \frac{|z|^{1-\delta}}{\varepsilon^{1-\delta} |\operatorname{Im} z|}. \tag{3.16}$$

This with $\delta = \frac{1}{2}$ and the first bound in (3.13) show that $\|A_\varepsilon\|_{\mathcal{H} \to \mathcal{H}}$ is uniformly bounded.

To estimate the norm of B_ε, let $\chi_m \in C_c^\infty(\mathbb{R})$ satisfy

$$\operatorname{supp}(1 - \chi_m) \cap \operatorname{supp} \chi_1 = \emptyset, \quad \operatorname{supp} \chi_m \cap \operatorname{supp}(1 - \chi_0) = \emptyset.$$

Then,

$$\begin{aligned} B_\varepsilon &= \varepsilon^{-2\delta} [\chi_0(\varepsilon P), R] \chi_1(\varepsilon P) = -\varepsilon^{-2\delta} (1 - \chi_0(\varepsilon P)) R \chi_m(\varepsilon P) \chi_1(\varepsilon P) \\ &= -\varepsilon^{-2\delta} (1 - \chi_0(\varepsilon P)) [[R, \chi_m(\varepsilon P)], \chi_1(\varepsilon P)]. \end{aligned}$$

Hence, using the Helffer Sjöstrand formula,

$$\begin{aligned} B_\varepsilon &= \pi^{-2} \varepsilon^{2-2\delta} (1 - \chi_0(\varepsilon P)) \int_{\mathbb{C}^2} \bar{\partial}_z \tilde{\chi}(z) \bar{\partial}_w \tilde{\chi}_m(w) Q_\varepsilon(z, w) dm(z) dm(w), \\ Q_\varepsilon(z, w) &:= (\varepsilon P - w)^{-1} (\varepsilon P - z)^{-1} [[R, P], P] (\varepsilon P - z)^{-1} (\varepsilon P - w)^{-1}. \end{aligned}$$

Using (3.16) with $\delta = 0$ to estimate $(\varepsilon P - w)^{-1} : \mathcal{H} \to \mathcal{H}$, (3.16) as stated to estimate $P^{1-\delta}(\varepsilon P - z)^{-1} : \mathcal{H} \to \mathcal{H}$, and the second hypothesis in (3.13), we see that $\|B_\varepsilon\|_{\mathcal{H}\to\mathcal{H}} \leq C$. ☐

To apply the lemma we need the following:

Lemma 7. *Suppose that (3.9) holds. Then for all $F \in L^1(\mathbb{R}^2)$, (3.13) holds for operators, R, of the form*

$$R = \sum_{A\in\mathcal{A}} \int_{\mathbb{R}^2} e^{itP} A^* e^{i(s-t)P} A e^{-isP} F(t,s)\,dtds$$

Proof. Since $P^{1-\delta}P^{-1+\delta'} : \mathcal{H} \to \mathcal{H}$ is bounded when $\delta' < \delta$, we may assume without loss of generality that $\delta \geq \frac{1}{2}$.

To prove the lemma, we compute

$$[P,R] = \int_{\mathbb{R}^2} e^{itP}\big([P,A^*]e^{i(s-t)P}A + A^*e^{i(s-t)P}[P,A]\big)e^{-isP}F(t,s)\,dtds.$$

Hence, the bounds

$$\sum_{A\in\mathcal{A}} \|[P,A]P^{-\frac{1}{2}}\|^2 + \|AP^{-\frac{1}{2}}\|^2 < \infty$$

together with the fact that P is self-adjoint imply the first estimate in (3.13).

Next,

$$[P,[P,R]] = \int_{\mathbb{R}^2} e^{itP}Q(t,s)e^{-isP}F(t,s)\,dtds$$

$$Q(t,s) := [P,[P,A^*]]e^{i(s-t)P}A + 2[P,A^*]e^{i(s-t)P}[P,A] + A^*e^{i(s-t)P}[P,[P,A]].$$

Hence, the bounds

$$\sum_{A\in\mathcal{A}} \|[P,A]P^{-\frac{1}{2}}\|^2 + \|AP^{-\frac{1}{2}}\|^2 + \|[P,[P,A]]P^{-1+\delta}\|^2 < \infty$$

imply the second estimate in (3.13). ☐

By Lemma 7, (3.13) holds for Y. Using the notation of Lemma 6, we put $u_\varepsilon = \chi_0(\varepsilon P)u$. Then, using (3.14) and the assumptions in (3.12),

$$\begin{aligned} Yu_\varepsilon &= Yu + (1-\chi_0(\varepsilon P))Yu + A_\varepsilon(1-\chi_1(\varepsilon P))Yu + \varepsilon^{2\delta}B_\varepsilon Yu \\ &= Yu + o(1)_{\mathcal{H}} + \mathcal{O}(\varepsilon^{2\delta}) \xrightarrow{\mathcal{H}} 0, \quad \varepsilon \to 0. \end{aligned}$$

(For any $v \in \mathcal{H}$, $(1-\chi_j(\varepsilon P))v \overset{\mathcal{H}}{\to} 0$ as $\varepsilon \to 0$.) This proves (3.12) and completes the argument for the closedness of Y.

We now need to prove (3.11). Observe that for $u \in \mathscr{D}^1$, the estimate (3.5) shows that $A_f(\omega) \in \mathscr{D}^{\frac{1}{2}} \subset \mathcal{H}$ and hence,

$$\mathrm{Re}\langle (Y-\lambda)u, u\rangle = -\lambda \|u\|_{\mathcal{H}}^2 - \tfrac{1}{2}\int \gamma(\omega)\|A_f(\omega)u\|_{\mathcal{H}}^2 d\omega \leq -\lambda\|u\|_{\mathcal{H}}^2.$$

For $u \in \mathcal{D}(Y)$ and with u_ε as in (3.12). Then,

$$\mathrm{Re}\langle (Y-\lambda)u, u\rangle = \lim_{\varepsilon\to 0}\mathrm{Re}\langle (Y-\lambda)u_\varepsilon, u_\varepsilon\rangle \leq -\lambda\|u_\varepsilon\|_{\mathcal{H}}^2 \to -\lambda\|u\|_{\mathcal{H}}^2.$$

Hence (using the same argument for Y^* with its maximal domain), we have

$$\lambda\|u\|_{\mathcal{H}} \leq \|(Y-\lambda)u\|_{\mathcal{H}}, u \in \mathcal{D}(Y), \qquad \lambda\|u\|_{\mathcal{H}} \leq \|(Y^*-\lambda)u\|_{\mathcal{H}}, u \in \mathcal{D}(Y^*), \tag{3.17}$$

Since (3.17) implies $Y-\lambda$ is injective, and provides the estimate (3.11) if the inverse exists, it remains only to show that Y is surjective. For this, suppose that $v \in \mathcal{H}$ such that

$$\langle (Y-\lambda)u, v\rangle = 0, \qquad \forall u \in \mathscr{D}^1 \subset \mathcal{D}(Y).$$

Then, in $\mathscr{D}^{-1}$, $(Y^*-\lambda)v = 0$, and hence $v \in \mathcal{D}(Y^*)$ and (3.17) implies that $v = 0$.

We conclude that the inverse exists and (3.17) shows that (3.11) holds, completing the proof of the proposition. □

4. PROOF OF THEOREM 3

In this section we show how the proof of Proposition 5 can be modified to apply to pseudodifferential operators quantising observables satisfying (1.20) and (1.21). That will prove Theorem 3.

We define $m = (1+|x|^2+|\xi|^2)^{\frac{1}{2}}$ (a slightly different convention than in §1.3.1 and the Appendix) and recall the following definitions:

$$\Psi(m^r) = \{a^{\mathrm{w}}(x,D) : a \in S(m^r)\}, \quad \Psi_{(k)} = \{a^{\mathrm{w}}(x,D) : a \in S_{(k)}\}, \quad k = 1,2,$$

where

$$a \in S(m^r) \Longleftrightarrow \partial^\alpha a = \mathcal{O}(m^r), \quad |\alpha| \geq 0,$$
$$a \in S_{(k)} \Longleftrightarrow \partial^\alpha a = \mathcal{O}(1), \quad |\alpha| \geq k, \quad k = 0,1,2.$$

We note that $S_{(k)} \subset S(m^k)$. Also all results are valid with only a finite (but large depending on the dimension) number of derivatives needed.

We want to investigate the structure of $\chi(\varepsilon P)$ for $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R})$, and

$$P = p(x,D), \quad p \in S_{(2)}, \quad p \geq cm^2. \tag{4.1}$$

One can show that P with the domain $H(m^2)$ (see [Zw12, §8.3] for definitions; in this case it is particularly simple) is self-adjoint (see [GaZw25, Proposition A.2]). We assume in addition that $P \geq 1$ which can always be achieved by adding a constant to P. We have mapping properties

$$P^s : H(m^r) \to H(m^{r-2s}), \quad A : H(m^r) \to H(m^{r-k}), \ A \in \Psi(m^k), \quad r, s \in \mathbb{R}. \tag{4.2}$$

The assumptions (4.1) and the fact that P is invertible implies that $P^{-1} \in \Psi(m^{-2})$ (see the Appendix).

To follow the same strategy as in §3 it suffices the prove the following analogue of Lemmas 6 and 7:

Proposition 8. *Suppose that $R_1, R_2 \in \Psi_{(1)}$, $t \in \mathbb{R}$. Then, for*

$$\chi, \chi_0 \in C_c^\infty(\mathbb{R}), \quad \operatorname{supp} \chi_0 \cap \operatorname{supp}(1-\chi) = \emptyset,$$

$$[\chi(\varepsilon P), R_1 e^{itP} R_2] = A_\varepsilon(1 - \chi_0(\varepsilon P)) + \varepsilon B_\varepsilon, \quad \|B_\varepsilon\|_{L^2 \to L^2} + \|A_\varepsilon\|_{L^2 \to L^2} \leq C, \tag{4.3}$$

where C is independent of ε.

The proof is based on the following

Lemma 9. *Suppose that $\chi \in C_{\rm c}^\infty(\mathbb{R})$ and P satisfies the assumptions above. Then*

$$\chi(\varepsilon p^{\rm w}(x, D)) = (\varepsilon p)^* \chi^{\rm w}(x, D) + \varepsilon q_\varepsilon^{\rm w}(x, D), \quad q_\varepsilon \in S(m^{-2}), \tag{4.4}$$

uniformly in ε, that with $|\partial^\alpha q_\varepsilon| \leq C_\alpha m^{-2}$, with constants independent of ε.

Remark. We obtain a stronger result about q_ε but we do not stress it as (4.4) is sufficient for our purposes.

Proof. In what follows we assume that $|z| \leq C$. We first show that for for all $M \geq 0$ there is $N > 0$ such that for $\{\ell_j\}_{j=1}^M$ linear,

$$\| \operatorname{ad}_{\ell_1^{\rm w}} \operatorname{ad}_{\ell_2^{\rm w}} \dots \operatorname{ad}_{\ell_M^{\rm w}} (\varepsilon P - z)^{-1} \|_{L^2 \to L^2} \leq C_M |\operatorname{Im} z|^{-N_M} \begin{cases} \varepsilon & M \geq 2 \\ \varepsilon^{\frac{1}{2}M} & 0 \leq M \leq 1, \end{cases} \tag{4.5}$$

Hence for $\operatorname{Im} z \neq 0$, Beals's Lemma, or rather [Zw12, Theorem 8.1],

$$(\varepsilon P - z)^{-1} = a_\varepsilon^{\rm w}(z, x, D), \quad |\partial^\alpha a_\varepsilon| \leq C |\operatorname{Im} z|^{-N_\alpha} \varepsilon^{\min(1, |\alpha|/2)}. \tag{4.6}$$

(Here and elsewhere N_α denotes a constant depending on α which may be different in different estimates.)

To prove (4.5), we first observe that $\operatorname{ad}_{\ell_1^{\rm w}} P \in \Psi_{(1)}$. Since

$$\operatorname{ad}_{\ell^{\rm w}} (\varepsilon P - z)^{-1} = -\varepsilon (\varepsilon P - z)^{-1} (\operatorname{ad}_{\ell^{\rm w}} P) P^{-\frac{1}{2}} P^{\frac{1}{2}} (\varepsilon P - z)^{-1},$$

and

$$\|(\operatorname{ad}_{\ell^{\rm w}} P) P^{-\frac{1}{2}}\|_{L^2 \to L^2}^2 = \|(\operatorname{ad}_{\ell^{\rm w}} P) P^{-1} (\operatorname{ad}_{\ell^{\rm w}} P)^*\|_{L^2 \to L^2} \leq C,$$

(since $P^{-1} \in \Psi(m^{-2})$ and $\operatorname{ad}_{\ell^{\mathrm{w}}} P \in \Psi_{(1)} \subset \Psi(m)$) and $\|(\varepsilon P - z)^{-1} P^{-\frac{1}{2}}\| \leq C\varepsilon^{\frac{1}{2}} |\operatorname{Im} z|^{-1}$, we have

$$\| \operatorname{ad}_{\ell^{\mathrm{w}}} (\varepsilon P - z)^{-1} \|_{L^2 \to L^2} \leq C\varepsilon^{\frac{1}{2}} |\operatorname{Im} z|^{-2}. \tag{4.7}$$

For $M \geq 2$ we argue as in the proof of [DiSj99, Proposition 8.6] but for our class of operators. For that we note that

$$\operatorname{ad}_{\ell_1^{\mathrm{w}}} \dots \operatorname{ad}_{\ell_M^{\mathrm{w}}} (\varepsilon P - z)^{-1}$$

is a sum of terms of the form

$$\varepsilon^L (\varepsilon P - z)^{-1} (\operatorname{ad}_{\ell_1^{\mathrm{w}}} \dots \operatorname{ad}_{\ell_{j_1}^{\mathrm{w}}} P)(\varepsilon P - z)^{-1} \dots (\operatorname{ad}_{\ell_{j_{L-1}+1}^{\mathrm{w}}} \dots \operatorname{ad}_{\ell_M^{\mathrm{w}}} P)(\varepsilon P - z)^{-1},$$

where $1 \leq L \leq M$ and and $1 \leq j_1 < j_2 \dots < j_{L-1} < M$. The bound is then seen from considering the two extreme cases:

$$\varepsilon^M (\varepsilon P - z)^{-1} (\operatorname{ad}_{\ell_1^{\mathrm{w}}} P)(\varepsilon P - z)^{-1} (\operatorname{ad}_{\ell_2^{\mathrm{w}}} P)(\varepsilon P - z)^{-1} \cdots (\operatorname{ad}_{\ell_M^{\mathrm{w}}} P)(\varepsilon P - z)^{-1}$$

and

$$\varepsilon (\varepsilon P - z)^{-1} (\operatorname{ad}_{\ell_1^{\mathrm{w}}} \cdots \operatorname{ad}_{\ell_M^{\mathrm{w}}} P)(\varepsilon P - z)^{-1}.$$

In the first case we proceed as in the proof of (4.7) to obtain the bound $\varepsilon^{M/2} |\operatorname{Im} z|^{-M-1} = \mathcal{O}(\varepsilon) |\operatorname{Im} z|^{-M-1}$. In the second we use the fact that for $M \geq 2$, $\operatorname{ad}_{\ell_1^{\mathrm{w}}} \cdots \operatorname{ad}_{\ell_M^{\mathrm{w}}} P \in \Psi_{(0)}$, so the bound becomes $\mathcal{O}(\varepsilon) |\operatorname{Im} z|^{-2}$.

We write

$$[(\varepsilon p - z)^{-1}]^{\mathrm{w}} (\varepsilon P - z) = I + \varepsilon E(z),$$

where $E(z) = e^{\mathrm{w}}(z, x, D)$ with

$$e(z, x, \xi) = \int_0^1 (1 - t) e^{itA(D)} (iA(D))^2 (\varepsilon^{-1} (\varepsilon p(\rho_1) - z)(\varepsilon p(\rho_2) - z)^{-1})|_{\rho_1 = \rho_2 = (x, \xi)}$$

and $A(D) := -\frac{1}{2} \sigma(D_{\rho_1}, D_{\rho_2})$. The terms to which $e^{itA(D)}$ is applied are, schematically, of the form

$$D^2 p(\rho_1) \left(\frac{\varepsilon D^2 p(\rho_2)}{(\varepsilon p(\rho_2) - z)^2} + \frac{\varepsilon^2 D p(\rho_2) D p(\rho_2)}{(\varepsilon p(\rho_2) - z)^3} \right).$$

Using that $p \geq cm^2$, for $|z| < 1$ and $\varepsilon > 0$ small enough, we have

$$|(\varepsilon p(\rho_2) - z)|^{-1} \leq |\operatorname{Im} z|^{-1} \min\left(1, 2\varepsilon^{-1} c^{-1} m(\rho_2)^{-2}\right).$$

Thus, for $|\operatorname{Im} z| > 0$, we have

$$\partial^\alpha_{\rho_1, \rho_2} (iA(D))^2 (\varepsilon^{-1} (\varepsilon p(\rho_1) - z)(\varepsilon p(\rho_2) - z)^{-1}) = \mathcal{O}(|\operatorname{Im} z|^{-N_\alpha} m(\rho_2)^{-2}).$$

Since in the estimates $e^{itA(D)} : S(m_1(\rho_1) m_2(\rho_2)) \to S(m_1(\rho_1) m_2(\rho_2))$ only finitely many derivatives are used, we conclude that

$$\partial^\alpha e(z, x, \xi) \leq C |\operatorname{Im} z|^{-N_\alpha} m(x, \xi)^{-2}.$$

Combing this with (4.6) we obtain

$$(\varepsilon P - z)^{-1} = [(\varepsilon p(x, \xi) - z)^{-1}]^{\mathrm{w}} + \varepsilon R(z),$$

where $R(z) = E(z)(\varepsilon P - z)^{-1} = r^{\rm w}(z, x, D)$ and

$$\partial^\alpha r(z, x, \xi) = \mathcal{O}(|\operatorname{Im} z|^{-N_\alpha} m(x, \xi)^{-2}).$$

We now apply the Helffer–Sjöstrand formula (3.15) (see the proof of Lemma 6) which gives (4.4). □

Proof of Proposition 8. We put

$$A_\varepsilon := [\chi(\varepsilon P), R_1 e^{itP} R_2], \quad B_\varepsilon = \varepsilon^{-1}[\chi(\varepsilon P), R_1 e^{itP} R_2]\chi_0(\varepsilon P),$$

so that

$$\begin{aligned} A_\varepsilon &= [\chi(\varepsilon P), R_1] e^{itP} R_2 + R_1 e^{itP}[\chi(\varepsilon P), R_2] \\ &= [\chi(\varepsilon P), R_1] P^{\frac12} e^{itP} P^{-\frac12} R_2 + R_1 P^{-\frac12} e^{itP} P^{\frac12}[\chi(\varepsilon P), R_2] \end{aligned}$$

In view of (4.2) and (4.4), to show that $A_\varepsilon : L^2 \to L^2$ is uniformly bounded, it is enough to show that

$$\|[(\varepsilon p)^*\chi^{\rm w}, R^*] P [(\varepsilon p)^*\chi^{\rm w}, R]\|_{L^2\to L^2} \le C, \quad R \in \Psi_{(1)} \tag{4.8}$$

with C independent of ε. Since $m\partial^\alpha(\chi(\varepsilon p)) = \mathcal{O}(\varepsilon p)(\partial^\alpha\chi)(\varepsilon p)\mathcal{O}(1)$ for $|\alpha| = 1$, this follows from the composition formula as in [GaZw25, Proposition A.1].

To analyse B_ε, let $\chi_1 \in C^\infty_{\rm c}(\mathbb{R})$ with $\operatorname{supp}\chi_0 \cap \operatorname{supp}(1-\chi_1) = \emptyset$ and $\operatorname{supp}\chi_1 \cap \operatorname{supp}(1-\chi) = \emptyset$. Then,

$$\begin{aligned} B_\varepsilon &= \varepsilon^{-1}[\chi(\varepsilon P), R_1] e^{itP} R_2 \chi_1(\varepsilon P)\chi_0(\varepsilon P) + \varepsilon^{-1} R_1 e^{itP}[\chi(\varepsilon P), R_2]\chi_0(\varepsilon P) \\ &= \varepsilon^{-1}[\chi(\varepsilon P), R_1] e^{itP}[R_2, \chi_1(\varepsilon P)]\chi_0(\varepsilon P) \\ &\quad + \varepsilon^{-1}[\chi(\varepsilon P), R_1]\chi_1(\varepsilon P) P^{\frac12} e^{itP} P^{-\frac12} R_2 \chi_0(\varepsilon P) \\ &\quad + \varepsilon^{-1} R_1 P^{-\frac12} e^{itP} P^{\frac12}[\chi(\varepsilon P), R_2]\chi_0(\varepsilon P) \\ &=: I + II + III. \end{aligned} \tag{4.9}$$

To prove the lemma, we show that I, II, and III are $O(1)_{L^2\to L^2}$.

We start by estimating I. To do this, we claim that

$$\|[\chi(\varepsilon P), R]\|_{L^2\to L^2} = \mathcal{O}(\varepsilon^{\frac12}). \tag{4.10}$$

The form of I then implies $I = O(1)_{L^2\to L^2}$. To prove (4.10), we use (4.2) and (4.4) to replace the the cut-off operator with $\big((\varepsilon p)^*\chi\big)^{\rm w}(x, D)$ so that it is enough to show

$$\|[(\varepsilon p)^*\chi^{\rm w}, R]\|_{L^2\to L^2} = \mathcal{O}(\varepsilon^{\frac12}). \tag{4.11}$$

This follows by arguing as in the proof of (4.8).

Next, to estimate II, we claim that

$$\|\varepsilon^{-1}[\chi(\varepsilon P), R_1]\chi_1(\varepsilon P) P^{\frac12}\|_{L^2\to L^2} = O(1). \tag{4.12}$$

Since $\|P^{-\frac12} R_2\|^2 = \|R_2^* P^{-1} R_2\| = O(1)_{L^2\to L^2}$, this implies $II = O(1)_{L^2\to L^2}$.

To prove (4.12) we put $\tilde{\chi}_1(x) := x\chi_1(x) \in C^\infty_{\rm c}(\mathbb{R})$ and observe that

$$\begin{aligned}
&\|\varepsilon^{-1}[\chi(\varepsilon P), R_1]\chi_1(\varepsilon P)P^{\frac{1}{2}}\|^2_{L^2\to L^2}\\
&= \|\varepsilon^{-2}[\chi(\varepsilon P), R_1]\chi_1(\varepsilon P)P\chi_1(\varepsilon P)[\chi(\varepsilon P), R_1]\|_{L^2\to L^2}\\
&= \|\varepsilon^{-3}[\chi(\varepsilon P), R_1]\tilde{\chi}_1(\varepsilon P)\chi_1(\varepsilon P)[\chi(\varepsilon P), R_1]\|_{L^2\to L^2}\\
&\le \varepsilon^{-3}\|[\chi(\varepsilon P), R_1]\tilde{\chi}_1(\varepsilon P)\|_{L^2\to L^2}\|[\chi(\varepsilon P), R_1]\chi_1(\varepsilon P)\|_{L^2\to L^2}.
\end{aligned} \tag{4.13}$$

Now, we claim that for $\psi \in C^\infty_c$ with $\operatorname{supp}\psi \cap \operatorname{supp}(1-\chi) = \emptyset$,

$$\|[\chi(\varepsilon P), R_1]\psi(\varepsilon P)\|_{L^2\to L^2} = O(\varepsilon^{3/2}). \tag{4.14}$$

The estimate (4.12) then follows by using (4.15) with $\psi = \tilde{\chi}_1$ and $\psi = \chi_1$ in (4.13).

Tp prove (4.15) we again use (4.2) and (4.4) to replace the the cut-off operator with $\big((\varepsilon p)^*\chi\big)^{\rm w}(x,D)$. That is, we need to show

$$\|[(\varepsilon p)^*\chi^{\rm w}, R_1](\varepsilon p)^*\psi^{\rm w}\|_{L^2\to L^2} = O(\varepsilon^{3/2}). \tag{4.15}$$

The symbolic calculus (see [GaZw25, Proposition A.1]) gives

$$[(\varepsilon p)^*\chi^{\rm w}, R_1] = \varepsilon^{1/2}\big(\varepsilon^{1/2}\{p, r_1\}\chi'(\varepsilon p)\big)^{\rm w} + \varepsilon b_1^{\rm w}, \tag{4.16}$$

where $b_1 \in S_{(0)}$. Since $\{p, r_1\} \in S(m)$, and $p \ge m^2$,

$$\varepsilon^{1/2}\{p, r_1\}p\chi'(\varepsilon p) \in S_{(0)}.$$

Therefore, using [GaZw25, Proposition A.1] again and the fact that $\operatorname{supp}\chi' \cap \operatorname{supp}\psi = \emptyset$ in the first equality

$$\varepsilon^{1/2}\big(\varepsilon^{1/2}\{p, r_1\}\chi'(\varepsilon p)\big)^{\rm w}(\varepsilon p)^*\psi^{\rm w} = \varepsilon^{3/2}b_2^{\rm w}, \qquad \varepsilon b_1^{\rm w}(\varepsilon p)^*\psi^{\rm w} = \varepsilon^{3/2}b_3^{\rm w}, \tag{4.17}$$

where $b_2, b_3 \in S_{(0)}$. Putting (4.16) and (4.17) together implies (4.15) and hence completes the proof of the bound on II.

Finally, to estimate III we observe that

$$\|P^{\frac{1}{2}}[\chi(\varepsilon P), R_2]\chi_0(\varepsilon P)\|^2_{L^2\to L^2} = \|\chi_0(\varepsilon P)[\chi(\varepsilon P), R_2]P[\chi(\varepsilon P), R_2]\chi_0(\varepsilon P)\|_{L^2\to L^2}$$

and

$$\begin{aligned}
&\chi_0(\varepsilon P)[\chi(\varepsilon P), R_2]P[\chi(\varepsilon P), R_2]\chi_0(\varepsilon P)\\
&= \chi_0(\varepsilon P)[\chi(\varepsilon P), R_2][\chi(\varepsilon P), R_2]P\chi_0(\varepsilon P) + [\chi_0(\varepsilon P), R_2][P,[\chi(\varepsilon P), R_2]]\chi_0(\varepsilon P)\\
&= \chi_0(\varepsilon P)[\chi(\varepsilon P), R_2][\chi(\varepsilon P), R_2]P\chi_0(\varepsilon P) - [\chi_0(\varepsilon P), R_2][\chi(\varepsilon P), R_3]\chi_0(\varepsilon P),
\end{aligned}$$

where $R_3 := [R_2, P] \in \Psi_{(1)}$. Arguing as in the proof of the estimate on II, we obtain

$$\|[\chi(\varepsilon P), R_2]P\chi_0(\varepsilon P)\|_{L^2\to L^2} = O(\varepsilon^{1/2}), \quad \|\chi_0(\varepsilon P)[\chi(\varepsilon P), R_2]\|_{L^2\to L^2} = O(\varepsilon^{3/2}),$$
$$\|[\chi(\varepsilon P), R_3]\chi_0(\varepsilon P)\|_{L^2\to L^2} = O(\varepsilon^{3/2}).$$

Therefore, using (4.10) (or more precisely its analog with χ replaced by χ_0) completes the proof that $III = O(1)_{L^2\to L^2}$ and hence of the lemma. ☐

5. A NEW EGOROV THEOREM AND THE PROOF OF THEOREM 4

In this section, we give a version of Egorov's theorem which holds in our context. For standard versions and references see [Zw12, Chapter 11] and for some recent advances [Gu24] and [Pr24].

Recalling that $a \in S_{(k)}$ if $\partial^\alpha a = \mathcal{O}(1)$ for $|\alpha| \geq k$, we assume that $p \in S_{(2)}$ is real valued and define $P := p^{\mathrm{w}}$. We will show that for $b \in S_{(1)}$, and $t \in \mathbb{R}$, $e^{itP} b^{\mathrm{w}} e^{-itP}$ is a pseudodifferential operator and obtain quantitative control on the derivatives of its symbol. More precisely, we prove the following:

Theorem 5. *Suppose that $b \in S_{(1)}$. Then for all $k \geq 0$ there are $C_k > 0$ such that for all $t \in \mathbb{R}$ there is $r_t \in S_{(0)}$ satisfying*

$$e^{itP} b^{\mathrm{w}} e^{-itP} = (b \circ \varphi_t)^{\mathrm{w}} + r_t^{\mathrm{w}}$$

and

$$\|r_t\|_{C^k} \leq C_k e^{C_k |t|}.$$

Remark 1. *It is possible to prove a version of Theorem 5 for $b \in S_{(\ell)}$ for any $\ell \geq 0$, using the calculus in $\Psi_{(\ell)}$ to reduce to the case of $S_{(0)}$. However, since we do not need it here, we do not pursue this generalisation.*

Theorem 5 allows us to conclude that the jump operators $\hat{A}_f(\omega)$ iare pseudodifferential when f is super-exponentially decaying and $A = a^{\mathrm{w}}$ with $a \in S_{(1)}$. The structure of the coherent term is more complicated as $b_1(t)$ is *not* super-exponentially decreasing.

To prove Theorem 5, we start with an estimate on the flow for $p \in S_{(2)}$. For this, we follow [Zw12, Lemma 11.1].

Lemma 10. *Let $p \in S_{(2)}$ be real valued and $\varphi_t := \exp(tH_p)$. There exist $\Lambda > 0$ and $C_\alpha > 0$, such that for all $b \in S_{(1)}$ and all $t \in \mathbb{R}$,*

$$|\partial^\alpha (b \circ \varphi_t)| \leq C_\alpha e^{\Lambda |\alpha| |t|} \sum_{1 \leq |\beta| \leq |\alpha|} \|\partial^\beta b\|_{L^\infty}, \quad |\alpha| \geq 1, \quad \alpha \in \mathbb{N}^{2d}.$$

Proof. The lemma follows from showing that

$$|\partial^\alpha \varphi_t(x,\xi)| \leq C_\alpha e^{\Lambda |\alpha| |t|}, \qquad |\alpha| \geq 1. \tag{5.1}$$

The flow $\rho(t) = \varphi_t((x_0, \xi_0))$ is defined by

$$\dot{\rho}(t) = H_p(\rho(t)), \qquad \rho(0) = (x_0, \xi_0). \tag{5.2}$$

Differentiating (5.2) once, we obtain for $|\alpha| = 1$,

$$\frac{d}{dt}(\partial^\alpha \varphi_t) = \partial H_p(\rho(t)) \partial^\alpha \varphi_t.$$

Therefore, since $\partial H_p \in L^\infty$, we obtain

$$|\partial^\alpha \varphi_t| \leq e^{\|\partial H_p\|_\infty |t|}, \qquad |\alpha| = 1,$$

which gives (5.1) with $\Lambda \geq \|\partial H_p\|_\infty$.

Let $\ell \geq 2$ and suppose by induction that (5.1) holds for all $1 \leq |\alpha| \leq \ell - 1$. Then for $|\alpha| = \ell$, differentiating the equation (5.2), we obtain

$$\frac{d}{dt}(\partial^\alpha \varphi_t) = \partial H_p(\rho(t))\partial^\alpha \varphi_t + \gamma(t), \tag{5.3}$$

where $\gamma(t)$ is a sum of terms of the form

$$g\, \partial^{\alpha_1}\varphi_t \ldots \partial^{\alpha_k}\varphi_t,$$

where $\alpha_1 + \cdots + \alpha_k = \alpha$ and $0 < |\alpha_j| < |\alpha| = \ell$ and g is bounded. The induction hypothesis implies

$$|\gamma(t)| \leq Ce^{\Lambda|\alpha||t|},$$

and using this in (5.3) gives

$$\frac{d}{dt}|\partial^\alpha \varphi_t|^2 \leq |\partial^\alpha \varphi_t|^2 \|\partial H_p\|_\infty + Ce^{\Lambda|\alpha||t|}|\partial^\alpha \varphi_t|.$$

This implies (5.1) after taking $\Lambda > \|\partial H_p\|_\infty$ and an application of Grönwall's inequality. □

Next, we give a preliminary decomposition of the conjugated operator.

Lemma 11. *There is $N > 0$ such that for all $a \in S_{(1)}$ with*

$$A_t := e^{itP}a^{\mathrm{w}}e^{-itP}, \tag{5.4}$$

and all $t \in \mathbb{R}$, there is $q_t \in S_{(0)}$ satisfying

$$A_t = (a \circ \varphi_t)^{\mathrm{w}} - Q(a,t), \qquad Q(a;t) := i\int_0^t e^{i(t-s)P}q_s^{\mathrm{w}}e^{i(s-t)P}ds, \tag{5.5}$$

and

$$|\partial^\alpha q_t| \leq C_\alpha e^{\Lambda(|\alpha|+N)|t|} \sum_{1\leq|\beta|\leq|\alpha|+N} \|\partial^\beta a\|_\infty. \tag{5.6}$$

Proof. Observe that

$$\begin{aligned} D_t\big(e^{-itP}(a\circ\varphi_t)^{\mathrm{w}}e^{itP}\big) &= e^{-itP}\big(-[P,(a\circ\varphi_t)^{\mathrm{w}}] - i(H_p(a\circ\varphi_t))^{\mathrm{w}}\big)e^{itP} \\ &=: e^{-itP}q_t^{\mathrm{w}}e^{itP}. \end{aligned} \tag{5.7}$$

This definition of q_t, [GaZw25, Lemma A.1] and Lemma 10 show that there is $N > 0$ such that for all α,

$$|\partial^\alpha q_t| \leq C_\alpha \sum_{2\leq|\beta|\leq|\alpha|+N} \|\partial^\beta(a\circ\varphi_t)\|_\infty \leq C_\alpha e^{\Lambda(|\alpha|+N)|t|} \sum_{1\leq|\beta|\leq|\alpha|+N} \|\partial^\beta a\|_\infty.$$

In particular, $q_t \in S_{(0)}$. Integrating (5.7) in t gives

$$\begin{aligned} e^{itP}a^{\mathrm{w}}e^{-itP} - (a \circ \varphi_t)^{\mathrm{w}} &= -i \int_0^t D_s\big(e^{i(t-s)P}(a\circ\varphi_s)^{\mathrm{w}}e^{i(s-t)P}\big)ds \\ &= -i\int_0^t e^{i(t-s)P}q_s^{\mathrm{w}}e^{i(s-t)P}ds \end{aligned}$$

which is (5.5). □

Before proceeding to the full case of $S_{(1)}$, we prove the Egorov theorem for $S_{(0)}$.

Lemma 12. *Let $p \in S_{(2)}$ be real valued. Then, for $k \geq 0$, there are $C_k, N_k > 0$ such that for all $t \in \mathbb{R}$ and $a \in S_{(0)}$, there is $a_t \in S_{(0)}$ satisfying*

$$A_t := e^{itP}a^{\mathrm{w}}e^{-itP} = a_t^{\mathrm{w}}, \qquad \|a_t\|_{C^k} \leq C_k e^{C_k|t|}\|a\|_{C^{N_k}}.$$

Proof. For this, we employ Beals's characterisation of pseudodifferential operators [Zw12, Theorem 8.12]. In particular, it is enough to show that for any $M \geq 0$ there is $N_M > 0$ such that for any fixed linear functions $L_j(x,\xi)$, $j = 1,\dots M$, with $\|\nabla L_j\| \leq 1$, there are $C_M > 0$ such that

$$\|\operatorname{ad}_{L_1^{\mathrm{w}}}\dots\operatorname{ad}_{L_M^{\mathrm{w}}} A_t\|_{L^2\to L^2} \leq C_M e^{C_M|t|}\|a\|_{C^{N_M}}. \tag{5.8}$$

Step 1: Boundedness of commutators with operators of linear growth.

In the notation of (5.5), and with $B_t := e^{itP}b^{\mathrm{w}}e^{-itP}$, $b \in S_{(1)}$,

$$e^{-itP}[b^{\mathrm{w}}, A_t]e^{itP} = [B_{-t}, a^{\mathrm{w}}] = [(b\circ\varphi_{-t})^{\mathrm{w}} - Q(b;-t), a^{\mathrm{w}}], \tag{5.9}$$

where

$$Q(b,-t) := i\int_0^{-t} e^{-i(t+s)P}q_s^{\mathrm{w}}e^{i(s+t)P}ds$$

and q_s satisfies (5.6) with a replaced by b. Then unitarity of e^{itP} and [Zw12, Theorem 4.23] show that there is $N > 0$ such that

$$\|Q(b;-t)\|_{L^2\to L^2} \leq C\sum_{0\leq|\alpha|\leq N}\int_{\min(0,-t)}^{\max(0,-t)}\|\partial^\alpha q_s\|_\infty \leq C|t|e^{2N\Lambda|t|}\sum_{1\leq|\beta|\leq 2N}\|\partial^\beta b\|_\infty$$

and hence (absorbing the $|t|$ into the exponential factor) we conclude that for N_0 large enough,

$$\|Q(b;-t)\|_{L^2\to L^2} \leq Ce^{\Lambda M_0|t|}\sum_{1\leq|\beta|\leq N_0}\|\partial^\beta b\|_\infty. \tag{5.10}$$

Lemma 10, [GaZw25, Lemma A.1], and [Zw12, Theorem 4.23], combined with (5.9) and (5.10) then imply that there are $M_0, N_0 > 0$ such that for $b \in S_{(1)}$,

$$\|[b^{\mathrm{w}}, A_t]\|_{L^2\to L^2} \leq Ce^{\Lambda M_0|t|}\sum_{1\leq|\beta|\leq N_0}\|\partial^\beta b\|_\infty\sum_{0\leq|\beta|\leq N_0}\|\partial^\beta a\|_\infty. \tag{5.11}$$

Step 2: Commutators with error terms.

In the inductive process, we also require the estimate: for $b_1, b \in S_{(1)}$,

$$\|[(b_1 \circ \varphi_{-t})^{\mathrm{w}}, Q(b;-t)]\|_{L^2\to L^2} \le Ce^{3\Lambda M_0|t|} \sum_{1\le|\beta|\le N_0} \|\partial^\beta b_1\|_\infty \sum_{1\le|\beta|\le N_0} \|\partial^\beta b\|_\infty \tag{5.12}$$

To prove (5.12), observe that for $\tilde{b}_1 \in S_{(1)}$,

$$[Q(b;-t), \tilde{b}_1^{\mathrm{w}}] = -i\int_{-t}^0 [e^{i(-t-s)P} q_s^{\mathrm{w}} e^{i(s+t)P}, \tilde{b}_1^{\mathrm{w}}]ds.$$

Now, by (5.6) and (5.11) (with a replaced by q_s and t by $t+s$)

$$\begin{aligned}
&\|[e^{i(-t-s)P} q_s^{\mathrm{w}} e^{i(s+t)P}, \tilde{b}_1^{\mathrm{w}}]\|_{L^2\to L^2}\\
&\le Ce^{M_0\Lambda|t+s|} \sum_{1\le|\beta|\le N_0} \|\partial^\beta \tilde{b}_1\|_\infty \sum_{0\le|\alpha|\le N_0} \|\partial^\alpha q_s\|_\infty\\
&\le Ce^{\Lambda(M_0|t+s|+(N_0+N)|s|)} \sum_{1\le|\beta|\le N_0} \|\partial^\beta \tilde{b}_1\|_\infty \sum_{1\le|\beta|\le N_0+N} \|\partial^\beta b\|_\infty\\
&\le Ce^{2\Lambda M_0|t|} \sum_{1\le|\beta|\le N_0} \|\partial^\beta \tilde{b}_1\|_\infty \sum_{1\le|\beta|\le N_0+N} \|\partial^\beta b\|_\infty,
\end{aligned}$$

where the last line follows if M_0 is large enough.

Now, setting $\tilde{b}_1 = b_1 \circ \varphi_{-t}$, using Lemma 10, enlarging N_0, choosing M_0 sufficiently large, and absorbing the factor $|t|$ into the exponential, we obtain (5.12).

Step 3: Iterated commutators.

We claim that for all $L \ge 1$, there are $C_L > 0$, N_L such that for all $\{b_i\}_{i=1}^L \subset S_{(1)}$, and $b \in S_{(1)}$, all $1 \le j \le L$, and all $a \in S_{(0)}$ ($A_t := e^{itP} a^{\mathrm{w}} e^{-itP}$),

$$\|\operatorname{ad}_{b_j^{\mathrm{w}}} \dots \operatorname{ad}_{b_1^{\mathrm{w}}} A_t\|_{L^2\to L^2} \le C_L e^{C_L|t|} \sum_{0\le|\alpha|\le N_L} \|\partial^\alpha a\|_\infty \prod_{\ell=1}^{j} \sum_{1\le|\beta|\le N_L} \|\partial^\beta b_\ell\|_\infty \tag{5.13}$$

and for $t \in \mathbb{R}$, and $0 \le j \le L$,

$$\begin{aligned}
&\|\operatorname{ad}_{(b_j\circ\varphi_{-t})^{\mathrm{w}}} \dots \operatorname{ad}_{(b_1\circ\varphi_{-t})^{\mathrm{w}}} Q(b;-t)\|_{L^2\to L^2}\\
&\qquad \le C_L e^{C_L|t|} \sum_{1\le|\alpha|\le N_L} \|\partial^\alpha b\|_\infty \prod_{\ell=1}^{j} \sum_{1\le|\beta|\le N_L} \|\partial^\beta b_\ell\|_\infty.
\end{aligned} \tag{5.14}$$

Since (5.13) implies (5.8) the proof of the lemma will be complete once that inequality is established.

For $L = 1$ we have proved (5.13) and (5.14) in steps 1 and 2 respectively. Therefore, we suppose by induction that (5.13) and (5.14) hold for some $L \ge 1$. Let $\{b_i\}_{i=1}^{L+1} \in S_{(1)}$, set

$$B_{i,1} := (b_i \circ \varphi_{-t})^{\mathrm{w}}, \qquad B_{i,0} := -Q(b_i;-t),$$

and consider

$$\begin{aligned} e^{-itP}(\operatorname{ad}_{b^{\mathrm{w}}_{L+1}} \dots \operatorname{ad}_{b^{\mathrm{w}}_1} A_t)e^{itP} &= \operatorname{ad}_{B_{L+1,1}+B_{L+1,0}} \dots \operatorname{ad}_{B_{1,1}+B_{1,0}} a^{\mathrm{w}} \\ &= \sum_{\vec{\sigma}\in\{0,1\}^{L+1}} \operatorname{ad}_{B_{L+1,\vec{\sigma}_{L+1}}} \dots \operatorname{ad}_{B_{1,\vec{\sigma}_1}} a^{\mathrm{w}}, \end{aligned} \tag{5.15}$$

where we used (5.9) to obtain the first equality. The second equality follows from expanding $\operatorname{ad}_{B_{j,1}+B_{j,0}} = \operatorname{ad}_{B_{j,1}} + \operatorname{ad}_{B_{j,0}}$.

The Jacobi identity, $\operatorname{ad}_{A_1} \operatorname{ad}_{A_2} = \operatorname{ad}_{A_2} \operatorname{ad}_{A_1} + \operatorname{ad}_{[A_1,A_2]}$, allows us to rewrite this so that all the terms with $\sigma_j = 1$ are on the right:

$$\operatorname{ad}_{B_{L+1,\vec{\sigma}_{L+1}}} \dots \operatorname{ad}_{B_{1,\vec{\sigma}_1}} a^{\mathrm{w}} = \sum_{\gamma\in\Gamma_{\vec{\sigma}}} c_\gamma \operatorname{ad}_{\widetilde{Q}_{\gamma,1}} \dots \operatorname{ad}_{\widetilde{Q}_{\gamma,R_\gamma}} \operatorname{ad}_{B_{j_{\gamma,1},1}} \dots \operatorname{ad}_{B_{j_{\gamma,J_\gamma},1}} a^{\mathrm{w}}, \tag{5.16}$$

where for a fixed $\vec{\sigma} \in \{0,1\}^{L+1}$, $\Gamma_{\vec{\sigma}}$ is the set of terms obtained after this reordering, with γ labelling its elements, $c_\gamma \in \{1,-1\}$, and

$$\widetilde{Q}_{\gamma,j} := \operatorname{ad}_{B_{\ell_{\gamma,j,1},1}} \operatorname{ad}_{B_{\ell_{\gamma,j,2},1}} \dots \operatorname{ad}_{B_{\ell_{\gamma,j,M_{\gamma,j}},1}} Q(b_{i_{\gamma,j}};-t).$$

Here $0 \le M_{\gamma,j}$ (with the convention that when $M_{\gamma,j} = 0$, $\widetilde{Q}_{\gamma,j} = Q(b_{i_{\gamma,j}};-t)$ and

$$R_\gamma + \sum_{j=1}^{R_\gamma} M_{\gamma,j} + J_\gamma = L+1.$$

Each b_i appears exactly once in each summand in (5.16).

To estimate the terms in (5.16), we first see that [GaZw25, Lemma A.1], [Zw12, Theorem 4.23], and Lemma 10 imply there is N_0 such that

$$\begin{aligned} \| \operatorname{ad}_{B_{j_{\gamma,1},1}} \dots \operatorname{ad}_{B_{j_{\gamma,J_\gamma},1}} a^{\mathrm{w}} \|_{L^2\to L^2} &\le \sum_{|\beta|\le J_\gamma+N_0} \|\partial^\beta a\|_\infty \prod_{\ell=1}^{J_\gamma} \sum_{1\le|\beta|\le N_0+J_\gamma} \|\partial^\beta b_{j_{\gamma,\ell}} \circ \varphi_{-t}\|_\infty \\ &\le C e^{J_\gamma(N_0+J_\gamma)\Lambda|t|} \sum_{|\beta|\le J_\gamma+N_0} \|\partial^\beta a\|_\infty \prod_{\ell=1}^{J_\gamma} \sum_{1\le|\beta|\le N_0+J_\gamma} \|\partial^\beta b_{j_{\gamma,\ell}}\|_\infty. \end{aligned}$$

Then, the inductive hypothesis (5.14) implies that

$$\begin{aligned} \| \operatorname{ad}_{\widetilde{Q}_{\gamma,j}} A\|_{L^2\to L^2} &\le 2\|\widetilde{Q}_{\gamma,j}\|_{L^2\to L^2}\|A\|_{L^2\to L^2} \\ &\le 2C_L e^{C_L|t|}\|A\|_{L^2\to L^2} \sum_{1\le|\alpha|\le N_L} \|\partial^\alpha b_{i_{\gamma,j}}\|_\infty \prod_{m=1}^{M_{\gamma,j}} \sum_{1\le|\beta|\le N_L} \|\partial^\beta b_{\ell_{\gamma,j,m}}\|_\infty. \end{aligned}$$

Hence using the unitarity of e^{itP}, there are $\tilde{C}_{L+1}, \tilde{N}_{L+1} > 0$ such that

$$\begin{aligned}\| \operatorname{ad}_{b_{L+1}^{\mathrm{w}}} \dots \operatorname{ad}_{b_1^{\mathrm{w}}} A_t\|_{L^2\to L^2}& \\ \le \tilde{C}_{L+1} e^{\tilde{C}_{L+1}|t|} \sum_{|\beta|\le \tilde{N}_{L+1}} \|\partial^\beta a\|_\infty \prod_{\ell=1}^{L+1} \sum_{1\le|\beta|\le \tilde{N}_{L+1}} \|\partial^\beta b_\ell\|_\infty.& \qquad (5.17)\end{aligned}$$

In particular (5.13) holds with L replaced by $L+1$.

To prove (5.14) with L replaced by $L+1$, we write

$$\begin{aligned}&\| \operatorname{ad}_{(b_{L+1}\circ\varphi_{-t})^{\mathrm{w}}} \dots \operatorname{ad}_{(b_1\circ\varphi_{-t})^{\mathrm{w}}} Q(b;-t)\|_{L^2\to L^2} \\ &\le \int_{\min(0,-t)}^{\max(-t,0)} \| \operatorname{ad}_{(b_{L+1}\circ\varphi_{-t})^{\mathrm{w}}} \dots \operatorname{ad}_{(b_1\circ\varphi_{-t})^{\mathrm{w}}} e^{i(-t-s)P} q_s^{\mathrm{w}} e^{i(s+t)P}\|_{L^2\to L^2} ds\end{aligned}$$

where q_s satisfies (5.6). Hence, applying (5.17) with $a = q_s$, and b_j replaced by $b_j\circ\varphi_{-t}$, we have

$$\begin{aligned}&\| \operatorname{ad}_{(b_{L+1}\circ\varphi_{-t})^{\mathrm{w}}} \dots \operatorname{ad}_{(b_1\circ\varphi_{-t})^{\mathrm{w}}} Q(b;-t)\|_{L^2\to L^2} \\ &\le \int_{\min(0,-t)}^{\max(-t,0)} \tilde{C}_{L+1} e^{\tilde{C}_{L+1}|s+t|} \sum_{0\le|\alpha|\le \tilde{N}_{L+1}} \|\partial^\alpha q_s\|_\infty \prod_{\ell=1}^{L+1} \sum_{1\le|\beta|\le \tilde{N}_{L+1}} \|\partial^\beta b_\ell\circ\varphi_{-t}\|_\infty \\ &\le C_{L+1} e^{C_{L+1}|t|} \sum_{1\le|\alpha|\le \tilde{N}_{L+1}+N} \|\partial^\alpha b\|_\infty \prod_{\ell=1}^{L+1} \sum_{1\le|\beta|\le \tilde{N}_{L+1}} \|\partial^\beta b_\ell\|_\infty,\end{aligned}$$

which completes the proof of (5.14) with L replaced by $L+1$.

Taking $N_{L+1} \ge \max(\tilde{N}_{L+1}+N, N_L, N_0+L+1)$ and taking $C_{L+1} \ge \tilde{C}_{L+1}$, (5.17) implies (5.13) with L replaced by $L+1$. Thus we have shown that (5.13) and (5.14) hold for all $L \ge 1$. □

We now complete the proof of the Egorov theorem by reducing the proof for symbols in $S_{(1)}$ to that of $S_{(0)}$.

Proof of theorem 5. Observe that by Lemma 11

$$e^{itP} b^{\mathrm{w}} e^{-itP} = (b\circ\varphi_t)^{\mathrm{w}} - i\int_0^t e^{i(t-s)P} q_s^{\mathrm{w}} e^{i(s-t)P} ds,$$

where q_s satisfies (5.6). By Lemma 12, there is $q_{t,s}$ such that

$$e^{i(t-s)P} q_s^{\mathrm{w}} e^{i(s-t)P} = q_{t,s}^{\mathrm{w}},$$

and

$$\begin{aligned}\|q_{t,s}\|_{C^k} \le C_k e^{C_k|t-s|} \|q_s\|_{C^{N_k}} &\le C_k e^{C_k|t-s| + \Lambda(N_k+N)|s|} \sum_{1\le|\alpha|\le N_k+N} \|\partial^\alpha b\|_\infty \\ &\le \tilde{C}_k e^{\tilde{C}_k(|t-s|+|s|)}.\end{aligned}$$

Hence we get

$$-i\int_0^t e^{i(t-s)P} q_s^{\rm w} e^{i(s-t)P} ds = r_t^{\rm w}, \qquad r_t := -i\int_0^t q_{t,s} ds$$

where $\|r_t\|_{C^k} \leq |t|\tilde{C}_k e^{\tilde{C}_k|t|}$. This completes the proof. $\square$

Proof of Theorem 4 is now immediate: we apply Theorem 5 and Lemma 10.

APPENDIX

We first consider the general assumptions in §1.3.1. We first use the results of [Zw12, §8.2] to see that there exists

$$G = e^{g^{\rm w}(x,D)}, \quad G^s := e^{sg^{\rm w}(x,D)} = e_s^{\rm w}(x,D), \quad e_s \in S(m^s), \quad s \in \mathbb{R}. \tag{A.1}$$

From this we conclude that $PG^{-1} = r(x,D)$, $r \in S(1)$, and that $r(x,D)^{-1} = GP^{-1} : L^2 \to L^2$ exists (since the domain of P was assumed to be $H(m)$, $P^{-1} : L^2 \to H(m)$ and $G : H(m) \to L^2$). Beals's Lemma (see [Zw12, Theorem 8.3]) shows that $r(x,D)^{-1} = \tilde{r}(x,D)$, $\tilde{r} \in S(1)$, and hence,

$$P^{-1} = G^{-1}\tilde{r}(x,D) = q(x,D), \quad q \in S(m^{-1}). \tag{A.2}$$

Then the assumptions in Theorem 1 follow from [Zw12, Theorems 4.23 and 8.12].

In Example 4, we can compute $c^{\rm w}(x,D) := [P,A]$, $c_1^{\rm w}(x,D) := [P,[P,A]]$ explicitly, and show that $c \in S(m^{\frac{1}{2}})$ and $c_1 \in S(m^{1-\delta})$, $m := (1+|x|+|\xi|)^2$.

Finally we comment on the condition under which e^{-P} is of trace class. In the general case of a self-adjoint P satisfying (1.22) (with the domain $H(m)$), $P \geq 1$, we use (A.2) to conclude that $P^{-\ell} = q_\ell^{\rm w}(x,D)$, $q_\ell \in S(m^{-\ell})$. We then recall from [Zw12, (C.3.6)] that if $m^{-\ell} \in L^2(\mathbb{R}^{2n}, dxd\xi)$ then $P^{-\ell}$, and consequently, $\exp(-P)$ is of trace class.

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