POISSON FORMULA FOR RESONANCES IN EVEN DIMENSIONS.

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1. Introduction

We consider scattering by an abstract compactly supported perturbation in $\mathbb{R}^n$. To include the traditional cases of potential, obstacle and metric scattering without going into their particular nature we adopt the "black box" formalism developed jointly with Sjöstrand [23]. It is quite likely that one could extend the results presented here to the case of non-compactly supported perturbation as well – see [21] for a natural generalization of "black box" perturbations.

We review now the basic assumptions. We work with a complex Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)) , \tag{1.1}$$

and with an operator

$$\begin{aligned} &P : \mathcal{H} \longrightarrow \mathcal{H} , \ \text{ self-adjoint with a domain } \mathcal{D} \subset \mathcal{H} \\ &\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0, R_0)) \\ &\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} P = -\Delta|_{\mathbb{R}^n \setminus B(0,R_0)} , \end{aligned} \tag{1.2}$$

which satisfies

$$\exists\, k \text{ such that } \ \mathbf{1}_{B(0,R_0)}(P + i)^{-k} \ \text{ is of trace class } , \tag{1.3}$$

$$P \geq -C , \quad C \geq 0 . \tag{1.4}$$

These assumptions guarantee that the resolvent $R(\lambda) = (P-\lambda^2)^{-1}$ continues meromorphically as an operator $\mathcal{H}_{\text{comp}} \to \mathcal{D}_{\text{loc}}$ from $\operatorname{Im} \lambda < 0$, $\lambda^2 \notin \sigma_{\text{pp}}(P)$, to $\mathbb{C}$ when n is odd and to Λ, the logarithmic plane, when n is even. The poles of this meromorphic continuation are called *resonances*. At $\lambda \neq 0$ all reasonable definitions of multiplicity agree. We can for instance say that the multiplicity of a pole at $\lambda \neq 0$ is given by the rank of the polar part of $R(\zeta)$ near λ – see [17]. The situation is more subtle at 0 and rather than go into a detailed discussion we will take the multiplicity required by the trace formula – see [28],[30],[35] for the discussion of the resolvent near 0.

If $U(t)$ is the wave group for the operator P and $U_0(t)$ is the free wave group we consider the natural *wave trace*:

$$u(t) = \operatorname{tr} U(t) - U_0(t) ,$$

which is an even distribution in $t \in \mathbb{R}$. The notation used here is somewhat informal since U and U_0 act on different spaces – see [25]. The correct definition is given by

$$u(t) \stackrel{\text{def}}{=} \operatorname{tr} \left(U(t) - \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} U_0(t) \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} \right) + \operatorname{tr} \mathbb{1}_{B(0,R_0)} U_0(t) \mathbb{1}_{B(0,R_0)} . \tag{1.5}$$

In odd dimensions the following Poisson formula was established in increasing degrees of generality by Bardos-Guillot-Ralston [1], Melrose [14],[15] and Sjöstrand-Zworski [25]:

$$\begin{aligned} t^{n+1} u(t) &= t^{n+1} \sum_{\lambda \in \mathbb{C}} m(\lambda) e^{i\lambda|t|} , \\ m(\lambda) &= \text{ multiplicity of } \lambda \text{ as a resonance of } P , \end{aligned} \tag{1.6}$$

in the sense of distributions on $\mathbb{R}$. The observation that we only need to multiply by t^{n+1} was made in [35]. The formula also holds exactly for super-exponentially decaying perturbations as was pointed out by Sá Barreto-Zworski [19].

We note that for $t > 0$ the trace formula is equivalent to

$$\widehat{u\phi}(\lambda) = \sum_{\zeta \in \mathbb{C}} m(\zeta) \hat{\phi}(\lambda - \zeta) \,, \quad \phi \in \mathcal{C}^\infty_{\rm c}((0, \infty)) \,. \tag{1.7}$$

The original proofs of (1.6) were based on Lax-Phillips theory [12] and in particular on the strong Huyghens principle. The extension to the case of hyperbolic surfaces by Guillopé-Zworski [8] provided a proof which does not require the strong Huyghens principle and is also applicable in the euclidean case [35]. It is based on the Birman-Krein formula and "global minimum modulus" estimates on the scattering determinant. That was followed by a local trace formula of Sjöstrand [21] the proof of which did not involve any scattering theory but also used some "local minimum modulus" estimates for determinants of some holomorphic matrices. Sjöstrand's formula specialized to the even dimensional compactly supported case gives the following weaker version of (1.7):

$$\widehat{u\phi}(\lambda) = \sum_{\zeta \in \lambda \Omega} m(\zeta) \hat{\phi}(\lambda - \zeta) + \mathcal{O}(\langle \lambda \rangle^{-\infty}) \,, \quad \Omega = [1/2, 3/2] + i[0, 1/2] \,, \quad \phi \in \mathcal{C}^\infty_{\rm c}((0, \infty)) \,. \tag{1.8}$$

We remark however that the semi-classical local formula of [21] is much stronger than (1.8).

By using a "local minimum modulus" theorem in the argument of [8], [35] we can strengthen (1.8) to obtain a global formula. This extension was motivated by a question asked by Vodev (see Sect.3).

Theorem 1. *Let P be an operator sastisfying the assumptions* (1.1)-(1.4) *and let $u(t)$ be its normalized wave trace given by* (1.5). *Let Λ_ρ be an open conic neighbourhood of the real axis as shown in Fig.1, $\sigma(\lambda)$ the scattering phase of P and let $\psi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}; [0, 1])$ be equal to* 1 *near* 0. *Then*

$$\begin{gathered} u(t) = \sum_{\lambda \in \Lambda_\rho} m(\lambda) e^{i\lambda|t|} + \sum_{\substack{\lambda^2 \in \sigma_{\rm pp}(P) \cap (-\infty, 0) \\ \operatorname{Im} \lambda < 0}} m(\lambda) e^{i\lambda|t|} + m(0) + 2\int_0^\infty \psi(\lambda) \frac{d\sigma}{d\lambda}(\lambda) \cos t\lambda d\lambda + v_{\rho,\psi}(t) \,, \quad t \neq 0 \,, \\ v_{\rho,\psi} \in \mathcal{C}^\infty(\mathbb{R} \setminus \{0\}) \,, \quad \partial_t^k v_{\rho,\psi} = \mathcal{O}(t^{-N}) \,, \ \forall\, k, N \,, \quad |t| \longrightarrow \infty \,. \end{gathered} \tag{1.9}$$

The scattering phase, $\sigma(\lambda)$, is a standard object in scattering theory – see [17] and reference given there for background information and [2] for the discussion of the "black box" case. Here it is normalized so that the Birman-Krein formula holds – see (2.8) below.

2. Proof of the trace formula

To prove Theorem 1 we identify the subset of Λ shown in Fig.1 with $\mathbb{C} \setminus e^{i\pi/2}\overline{\mathbb{R}}_+$, where $e^{i\pi/2}\mathbb{R}$ is the cut. Then the resonances are symmetric with respect to the cut and they coincide with the poles of the scattering determinant $s(\lambda) = \det S(\lambda)$. The unitarity of $S(\lambda)$ for $\lambda > 0$ implies the usual relation $S(\lambda)^{-1} = S(\bar{\lambda})^*$ for $\operatorname{Re} \lambda > 0$. Hence $s(\lambda)^{-1} = \overline{s(\bar{\lambda})}$. To simplify the discussion[1] we will consider the scattering matrix in $\Lambda_\rho \cap \{\operatorname{Re} \lambda > 0\}$ only and *define* $s(\lambda)$ in $\Lambda_\rho \cap \operatorname{Re} \lambda < 0$ so that the scattering phase defined by $\sigma'(\lambda) = (i/2\pi) s'(\lambda)/s(\lambda)$ is even in λ – see (2.8).

The assumption (1.3) guarantees the existence of m such that for $\rho > 0$

$$\sum_{\substack{\lambda \in \Lambda_\rho \\ |\lambda| \leq r}} m(\lambda) \leq C_\rho r^{m+\epsilon} \,, \quad \forall\, \epsilon > 0 \,, \tag{2.1}$$

see [35]. This is deduced from the polynomial bounds of Vodev [30],[31] which for $\rho < \pi/2$ (all that is needed here) follow also from the earlier estimates of Sjöstrand-Zworski [23].

[1] so that we do not need to consider global analytic properties of $S(\lambda)$

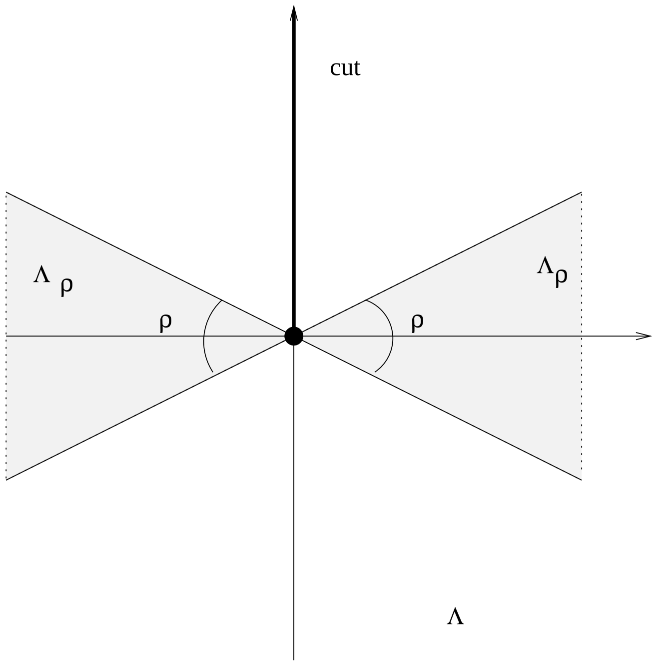

FIGURE 1. Conic neighbourhoods of the real axis on the logarithmic plane.

We now put

$$P_\rho(\lambda) = \prod_{\zeta \in \Lambda_\rho \setminus \mathbb{R}} E\left(\lambda/\zeta, m\right)^{m(\zeta)} \,, \quad E(z,p) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}} \,, \tag{2.2}$$

where the bound (2.1) guarantees the convergence of the Weierstrass product.

The poles of $s(\lambda)$ and $s(\lambda)^{-1}$ in Λ_ρ coincide (with multiplicities) with the zeros of $P_\rho(\lambda)$ and $P_\rho(-\lambda)$ respectively. Hence we can write

$$s(\lambda) = e^{g_\rho(\lambda)} \frac{P_\rho(-\lambda)}{P_\rho(\lambda)} \,, \quad \lambda \in \Lambda_\rho \cap \{\operatorname{Re}\lambda > 0\} \,, \tag{2.3}$$

where g_ρ is holomorphic in $\Lambda_\rho \cap \{\operatorname{Re}\lambda > 0\}$. We now extend $g_\rho(\lambda)$, and consequently $s(\lambda)$, to Λ_ρ by setting

$$g_\rho(-\lambda) = -g_\rho(\lambda) \,.$$

That clearly implies that $s'(\lambda)/s(\lambda) = s'(-\lambda)/s(-\lambda)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ and further analysis shows that this identity holds through $\lambda = 0$ – see Sect.3.

We want to estimate the function g_ρ. For that we need to estimate $s(\lambda)$ away from its poles and that is done exactly as in [8],[35] (see also [18]). We write

$$\begin{gathered} S(\lambda) = Id + A(\lambda) \,, \quad A(\lambda) = C_n \lambda^{n-2} \mathbb{E}^{\phi_1}(-\lambda)(I + K(\lambda, \lambda_0))^{-1}[\Delta, \chi]^t \mathbb{E}^{\phi_2}(\lambda) \,, \\ \mathbb{E}^\rho : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{S}^{n-1}) \,, \quad \mathbb{E}^\rho(\theta, x) = e^{i\lambda\langle x, \theta\rangle} \rho(x) \,, \; \rho \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n) \,, \end{gathered} \tag{2.4}$$

and where $K(\lambda, \lambda_0)$ is the operator constructed in Sect.3 of [23]:

$$\begin{gathered} K(\lambda, \lambda_0) = [\Delta, \chi_0] R_0(\lambda)(1-\chi_1)\chi_4 - [\Delta, \chi_2] R(\lambda_0)\chi_4 + \chi_2(\lambda_0^2 - \lambda^2) R(\lambda_0)\chi_4 \,, \\ \chi_i \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n) \,, \; \chi_0 \equiv 1 \text{ near } B(0, R_0) \,, \; \chi_i \equiv 1 \text{ near supp } \chi_{i-1} \,, \\ R(\lambda_0) = (P - \lambda_0^2)^{-1} \,, \; \operatorname{Im}\lambda_0 \ll 0 \,, \quad R_0(\lambda) = (-\Delta - \lambda^2)^{-1} \,. \end{gathered} \tag{2.5}$$

To estimate $s(\lambda)$ we will first estimate $\|(I+K(\lambda,\lambda_0))^{-1}\|$ and that is based on the inequality

$$\|(I+K(\lambda,\lambda_0))^{-1}\| \leq \frac{\det(I+|K(\lambda,\lambda_0)|^{m+1})}{|\det(I+K(\lambda,\lambda_0))^{m+1})|}, \tag{2.6}$$

from [4], Theorem 5.1, Chap.V.

Exactly as in [35], where we followed [29],[33], we see that for $\lambda \in \Lambda_\rho$

$$|\det(I+K(\lambda,\lambda_0)^{m+1})| \leq Ce^{C|\lambda|^{m+\epsilon}}.$$

Using the lower modulus theorem of H. Cartan[2] – see for instance [13], Theorem 4, Sect.11.3 – we obtain a lower bound:

$$|\det(I+K(\lambda,\lambda_0)^{m+1})| \geq Ce^{-Cr^{m+\epsilon}/\eta}, \quad \lambda \in D(r,\rho r/C) \setminus \bigcup_j D(\lambda_j, r_j), \quad \sum_j r_j \leq \eta r,$$

uniformly as $r \to \infty$. From this and (2.6) we obtain, as in [35],

$$|s(\lambda)| \leq Ce^{C(r^{m+\epsilon}/\eta)^n}, \quad \lambda \in D(r,\rho r/C) \setminus \bigcup_j D(\lambda_j, r_j), \quad \sum_j r_j \leq \eta r.$$

If we take $\eta \ll 1$ then for every r there exists $r/2C \leq k(r) \leq r/C$ such that the circle $|\lambda - r| = k(r)$ does not intersect any of the excluded discs. Then using the standard estimates for Weierstrass products and the maximum principle we see that

$$|\exp g_\rho(\lambda)||P_\rho(-\lambda)| \leq Ce^{|\lambda|^{(m+\epsilon)n}}, \ \lambda \in \Lambda_{\rho'}, \ 0 < \rho' \ll \rho.$$

We then conclude (as in the proof of Cartan's theorem or yet easier as in the proof of Hadamard's factorization theorem[3]) that

$$|\partial_\lambda^k g_\rho(\lambda)| \leq C|\lambda|^{(m+\epsilon)n-k}, \quad \lambda \in \Lambda_{\rho''}, \quad 0 < \rho'' < \rho', \tag{2.7}$$

where the symbolic property followed from Cauchy's inequalities.

We can now prove the Poisson formula. As in [8],[35] the starting point is the Birman-Krein formula:

$$\begin{gathered} u(t) = \widehat{\sigma'}(t) + \sum_{\lambda^2 \in \sigma_{\rm pp}(P)\setminus\{0\}} 2\cos(t\lambda) + m(0), \\ \sigma(\lambda) \stackrel{\rm def}{=} \frac{i}{2\pi}\log s(\lambda) \text{ for } \lambda > 0, \quad \sigma(\lambda) = -\sigma(-\lambda) \text{ for } \lambda < 0, \end{gathered} \tag{2.8}$$

where the Fourier transform is of course taken in the sense of distributions. For the "black box" perturbation the proof was given by Christiansen in Sect.1 of [2] but it is classical for all well known scattering problems. We note that our definition of $s(\lambda)$ implies that we can set

$$\sigma'(\lambda) = \frac{i}{2\pi}\frac{s'(\lambda)}{s(\lambda)}, \quad \forall\, \lambda \in \mathbb{R} \setminus \{0\}.$$

Let ψ be as in Theorem 1 and even. We can then write

$$\sigma'(\lambda) = \frac{i}{2\pi}\frac{s'(\lambda)}{s(\lambda)} = \frac{i}{2\pi}\left((1-\psi(\lambda))g_\rho'(\lambda) + \frac{d}{d\lambda}\left(\log P_\rho(-\lambda) - \log P_\rho(\lambda)\right) + \psi(\lambda)\frac{s'(\lambda)}{s(\lambda)}\right) + f_{\rho,\psi}(\lambda),$$

where $f_{\rho,\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$.

The argument of [8],[35] now easily gives

$$t^{2p}u(t) = t^{2p}\left(\sum_{\lambda\in\Lambda_\rho} e^{i\lambda|t|}m(\lambda) + \sum_{\substack{\lambda^2\in\sigma_{\rm pp}(P)\cap(-\infty,0)\\ \operatorname{Im}\lambda<0}} e^{i\lambda|t|} + m(0) + \widehat{\psi\sigma'}(t) + \widehat{(1-\psi)g_\rho}(t) + \hat{f}_{\rho,\psi}(t)\right),$$

[2] We remark that a much cruder estimate would suffice here but it is nice to quote the optimal result which is useful elsewhere in the theory of resonances – see [18] and [26], Sect.8.

[3] We can use for instance Carathéodory's inequality – see [27].

for p large enough. Since g_ρ is a symbol on $\mathbb{R}$,

$$v_{\rho,\psi}(t) = \frac{i}{2\pi}\widehat{(1-\psi)g_\rho}(t) + \hat{f}_{\rho,\psi}(t)\,,$$

has the properties stated in Theorem 1 and this completes its proof.

We remark here that *a posteriori* the bound on g_ρ on the real axis has to be much better than the bound provided by the estimate (2.7): we know the strength of the singularity of u at $t=0$ and the bound on the number of resonances gives an estimate on the strength of the singularity of the exponential sum. Hence for elliptic perturbations where $m=n$ we obtain

$$|\partial_\lambda^k g_\rho(\lambda)| \le C_{k,\epsilon}(1+|\lambda|)^{n+\epsilon-k}\,, \quad \forall \epsilon > 0\,.$$

3. Review of applications

The basic application of the trace formula is in obtaining lower bounds on the number of resonances from the singularities of the wave trace. The basic Tauberian lemma was given in Sjöstrand-Zworski [24] and it was applied there to problems in odd dimensions. One of the applications of the local trace formula of Sjöstrand [21] was the extension to even dimension – see Theoreom 10.1 there. That becomes even clearer when we use the global formula (1.9). One of the interesting consequencies is based on the trace formula of Guillemin-Melrose [5]:

Theorem 2. *Let P be the Dirichlet or Neumann Laplacian on a connected exterior domain $\mathbb{R}^n \setminus \mathcal{O}$ where $\mathcal{O}$ has a smooth boundary. Suppose that the there exists a non-degenerate closed transversally reflected trajectory of the broken geodesic flow of $\mathbb{R}^n \setminus \mathcal{O}$ such that no essentially different closed trajectory has the same period. Then for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$\sum \{m(\lambda) \;:\; |\lambda| \le r + C_\epsilon\,,\ |\operatorname{Im}\lambda| < \epsilon \log|\lambda|\} \ge r/C_\epsilon\,.$$

For more applications we refer to [24] and [21], Sect.10.

The new trace formula allows also an easy extension of some of the results of Ikawa on the distribution of resonances for several convex obstacles to even dimensions. That is particularly interesting in dimension two where most of the numerical studies were conducted – see for instance [3] for the discussion of symbolic dynamics. We remark that it is rather clear that the results of [21] would suffice for this purpose but the global formula makes the applications even more apparent. As an example we give the modification of the result of [9]:

Theorem 3. *Let P be the Neumann Laplacian on $\mathbb{R}^n \setminus \mathcal{O}$, $n \ge 2$, and let us assume that $\mathcal{O} = \bigcup_{j=1}^N \mathcal{O}_j$ where $\mathcal{O}_j$ are mutually disjoint strictly convex obstacles with smooth boundaries satisfying the following condition:*

$$\text{convex hull } (\mathcal{O}_k \cup \mathcal{O}_l) \cap \mathcal{O}_m = \emptyset \quad \forall\, k \ne m \ne l\,.$$

Then there exists $\alpha > 0$ for which

$$\sum \{m(\lambda) \;:\; |\operatorname{Im}\lambda| < \alpha\} = \infty\,.$$

The next theorem answers a question asked by Vodev [32] and does not seem to follow from the local trace formula:

Theorem 4. *Let P satisfy the assumptions (1.1)-(1.3) with $n \ge 4$ even. In addition let us assume that 0 is not an eigenvalue or a zero-resonance of P. Then for any $\gamma > 0$ and k*

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(u(t) - \sum_{\operatorname{Im}\lambda \le \gamma \log|\lambda|} m(\lambda) e^{i\lambda|t|} \right) \right| \le C_k t^{-n+2-k}\,, \quad t > t_k > (n+k)/\gamma\,. \tag{3.1}$$

We remark that in odd dimensions the polynomial bound on the number of resonances and the global formula (1.6) imply that the right hand side of (3.1) can be replaced by $\mathcal{O}(\exp(-\alpha t))$, $\alpha > 0$. It is quite clear from (1.9) that to establish Theorem 4 we need to understand the behaviour of $\sigma'(\lambda)$ as $\lambda \to 0+$. A finer analysis based for instance on [10],[11], should show that the estimate above is optimal and that in fact there exists an asymptotic expansion as $t \to \infty$. Also, we did not attempt to study the more involved two dimensional case.

Proof. We start by observing that

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(\sum_{\operatorname{Im}\lambda \leq \rho|\lambda|} m(\lambda) e^{i\lambda|t|} - \sum_{\operatorname{Im}\lambda \leq \gamma \log|\lambda|} m(\lambda) e^{i\lambda|t|} \right) \right| \leq C_k e^{-\alpha_k t}\,,$$
$$\alpha_k > 0\,, \quad t \geq t_k > (n+k)/\gamma\,,$$

which follows from the bound (2.1). In fact, the derivative of the difference can be estimated by

$$e^{-\alpha_k' t} + \int_A^\infty x^k x^{-\gamma t} dN(x) \leq e^{-\alpha_k' t} + CA^{-\gamma(t-(n+k)/\gamma)} + Ct\int_A^\infty x^{-\gamma(t-(n+k)/\gamma)-1} dx$$
$$\leq C_{k,\epsilon} e^{-\alpha_k t}\,, \quad t > (n+k)/\gamma + \epsilon\,.$$

Thus we need to study the behaviour of

$$\frac{s'(\lambda)}{s(\lambda)} = \operatorname{tr}\, S(\lambda)^* S'(\lambda) = \operatorname{tr}\, (I + A(\lambda)^*) A'(\lambda)\,, \tag{3.2}$$

as $\lambda \to 0+$, where we used the notation of (2.4).

The assumption that P has no resonance at zero implies that the cut-off resolvent, $\chi R(\lambda)\chi$, $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$, $\chi \equiv 1$ near $B(0, R_0)$, is holomorphic in $(\lambda, \lambda^{n-2}\log\lambda)$ for $|\lambda| < \epsilon$. To see that we recall that the free resolvent, $R_0(\lambda)$, is of the form $R_0'(\lambda) + \lambda^{n-2}\log\lambda M(\lambda)$ where $R_0'(\lambda)$ and $M(\lambda)$ are entire – see Sect.1 of [17]. Following [23], Sect.3, we write $\chi R(\lambda)\chi = \chi(Q_0(\lambda)\chi + Q_1(\lambda_0)\chi)(I + K(\lambda, \lambda_0))^{-1}$ where Q_0 is a cut-off free resolvent and K is given by (2.5). Analytic Fredholm theory shows that when $\chi R(\lambda)\chi$ is bounded, it is a holomorphic function of λ and $\lambda^{n-2}\log\lambda$ – see [28], [30] for more details.

To study (3.2) we need a different representation of $A(\lambda)$. If we recall the definition from Sect.3 of [35], $A(\lambda)$ comes from the radiation pattern of $R(\lambda)(-[\Delta, \chi]e^{i\langle \bullet, \omega\rangle})$. To obtain a formula similar to (2.4) but involving $R(\lambda)$ rather than $(I + K(\lambda, \lambda_0))^{-1}$ we take χ_j as in (2.5) and write

$$(1-\chi_2)R(\lambda)\chi_1 = R_0(\lambda)(-\Delta - \lambda^2)(1-\chi_2)R(\lambda)\chi_1 = R_0(\lambda)(-[\Delta, \chi_2]R(\lambda)\chi_1)\,,$$

since $(1-\chi_2)(-\Delta - \lambda^2) = (1-\chi_2)(P - \lambda^2)$ and $(1-\chi_2)\chi_1 = 0$. This shows that

$$A(\lambda) = c_n \lambda^{n-2} \mathbb{E}^{\phi_1}(-\lambda)[\Delta, \chi_2] R(\lambda) [\Delta, \chi]^t \mathbb{E}^{\phi_2}(\lambda)\,,$$

and the assumption on $R(\lambda)$ implies that $\lambda^{2-n}A(\lambda)$ is holomorphic in λ and $\lambda^{n-2}\log\lambda$ for $|\lambda| < \epsilon$. From this and (3.2) it follows that

$$\sigma'(\lambda) = \lambda^{n-3} f(\lambda, \lambda^{n-2}\log\lambda)\,, \quad \lambda > 0\,, \tag{3.3}$$

with f smooth near 0. We then easily check that

$$\int_0^\infty \lambda^{n-3} f(\lambda, \lambda^{n-2}\log\lambda)\psi(\lambda)\cos t\lambda d\lambda = (-1)^{\frac{n}{2}}(n-3)! f(0,0)\, t^{-n+2} + \mathcal{O}(t^{-n+1})\,, \quad t \to \infty\,,$$

which completes the proof as the estimates for derivatives clearly hold as well. □

Finally we compare this result with the estimates on the heat trace. As a consequence of well known estimates on heat kernels, Sá Barreto-Zworski [20] showed that when $P = -\Delta + V$ and P has no resonances with $\operatorname{Im}\lambda \leq 0$ (that is no eigenvalues and no zero resonance)

$$\operatorname{tr}\, (e^{-tP} - e^{t\Delta}) = \mathcal{O}(t^{-\frac{n}{2}+1})\,, \quad t > 0\,.$$

Werner Müller pointed out to the author that for the behaviour as $t \to \infty$ it is more natural to study $\sigma'(\lambda)$ near $\lambda = 0$ using the heat version of the Birman-Krein formula:

$$\operatorname{tr}\,(e^{-tP} - e^{t\Delta}) = \int_0^\infty e^{-t\lambda^2}\sigma'(\lambda)d\lambda + \sum_{\mu_j \in \sigma_{\rm pp}(P)} e^{-t\mu_j}\,, \quad t > 0\,, \tag{3.4}$$

where to make sense of the trace we used the convention employed in the definition of u, (1.5). Hence under the assumptions of Theorem 4 but for *any* $n \geq 3$ we obtain from its proof

$$\operatorname{tr}\,(e^{-tP} - e^{t\Delta}) = \sum_{\substack{\mu_j \in \sigma_{\rm pp}(P) \\ \mu_j < 0}} e^{-t\mu_j} + \frac{1}{2}\Gamma\left(\frac{n}{2} - 1\right) f(0,0)\, t^{-\frac{n}{2}+1} + \mathcal{O}(t^{-\frac{n}{2}+\frac{1}{2}})\,, \tag{3.5}$$

where f is as in (3.3). In odd dimension it is a function of one variable, λ, only.

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References

[1] C. Bardos, J.-C. Guillot and J.V. Ralston. *La relation de Poisson pour l'équation des ondes dans un ouvert non borné,* Commun. Partial Differ. Equations **7** (1982), 905–958.

[2] T. Christiansen. *Spectral asymptotics for general compactly supported perturbations of the Laplacian on* $\mathbb{R}^n$, to appear in Comm. P.D.E.

[3] P. Cvitanović and B. Eckhardt. *Symmetry decomposition of chaotic dynamics,* Nonlinearity **6** (1993), 277–311.

[4] I. C. Gohberg and M. G. Krein. *Introduction to the theory of linear nonselfadjoint operators*, Translations of mathematical monographs **18**, American Mathematical Society, Providence, 1969.

[5] V. Guillemin and R.B. Melrose. *The Poisson summation formula for for manifolds with with boundaries.* Adv. in Math. **32** (1979), 204-232.

[6] L. Guillopé. *Asymptotique de la phase de diffusion pour l'opérateur de Schrödinger avec potentiel*, C.R. Acad. Sci., Paris, Sér.I **293**(1981), 601-603.

[7] L. Guillopé and M. Zworski. *Upper bounds on the number of resonances for non-compact Riemann surfaces*, J. Funct. Anal. **129** (1995), 364–389.

[8] L. Guillopé and M. Zworski. *Scattering asymptotics for Riemann surfaces*, Ann. of Math. **145** (1997), 597-660.

[9] M. Ikawa. *On the existence of poles for several convex bodies,* Proc. Japan. Acad. **64** (1988), 91-93.

[10] A. Jensen. *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in* $L^2(\mathbf{R}^m)$, $m \geq 5$, Duke Math. J. **47**, (1980), 57-80.

[11] A. Jensen and T. Kato. *Spectral properties of Schrödinger operators and time-decay of the wave functions,* Duke Math. J. **46** (1979), 583-610.

[12] P. Lax and R. Phillips. *The time delay operator and a related trace formula.* in *Topics in Functional Analysis.* Advances in Math. Suppl. Studies **3** (1978), 197–295.

[13] B.Ya. Levin, *Lectures on Entire Functions,* Translations of Mathematical Monographs, American Mathematical Society, Providence, 1996.

[14] R.B. Melrose. *Scattering theory and the trace of the wave group*, J. Func. Anal. **45** (1982), 429–440.

[15] R.B. Melrose. *Polynomial bounds on the number of scattering poles*, J. Funct. Anal. **53** (1983), 287–303.

[16] R.B. Melrose. *Polynomial bounds on the distribution of poles in scattering by an obstacle*, Journées "Équations aux Dérivées partielles", Saint-Jean de Monts, 1984.

[17] R.B. Melrose. *Geometric scattering theory*, Cambridge University Press, Cambridge, New York, Melbourne, 1995.

[18] A. Sá Barreto and S.-H. Tang. *Existence of resonances in even dimensional potential scattering,* preprint, 1998.

[19] A. Sá Barreto and M. Zworski. *Existence of resonances in three dimensions*, Comm. Math. Phys. **173**(2) (1995), 401–415.

[20] A. Sá Barreto and M. Zworski. *Existence of resonances in potential scattering*, Comm. Pure Appl. Math. **49**(1996), 1271-1280.

[21] J. Sjöstrand. *A trace formula and review of some estimates for resonances*, in *Microlocal analysis and spectral theory* (Lucca, 1996), 377–437, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 490, Kluwer Acad. Publ., Dordrecht, 1997.

[22] J. Sjöstrand. *A trace formula for resonances and application to semi-classical Schrödinger operators*, Séminaire EDP, École Polytechnique, Novembre, 1996.

[23] J. Sjöstrand and M. Zworski. *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc. **4** (1991), 729–769.

[24] J. Sjöstrand and M. Zworski. *Lower bounds on the number of scattering poles*, Comm. P.D.E. **18** (1993), 847-857.
[25] J. Sjöstrand and M. Zworski. *Lower bounds on the number of scattering poles II*, J. Funct. Anal. **123** (1994), 336–367.
[26] J. Sjöstrand and M. Zworski. *Asymptotic distribution of resonances for convex obstacles.* preprint, 1998.
[27] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, 1939
[28] B. Vainberg. *Asymptotic methods in equations of mathematical physics,* Gordon and Breach, 1988.
[29] G. Vodev. *Sharp polynomial bounds on the number of scattering poles for perturbations of the Laplacian*, Comm. Math. Phys. **146** (1992), 39–49.
[30] G. Vodev. *Sharp bounds on the number of scattering poles in even-dimensional spaces,* Duke Math. J. **74**(1994), 1-17.
[31] G. Vodev. *Sharp bounds on the number of scattering poles in the two dimensional case,* Math. Nachr. **170** (1994), 287-297.
[32] G. Vodev. *Private communication,* 1998.
[33] M. Zworski. *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. **59** (1989), 311–323.
[34] M. Zworski. *Counting scattering poles.*, Proceedings of the Taniguchi International Workshop *Spectral and scattering theory*, M. Ikawa Ed., Marcel Dekker, New York, Basel, Hong Kong, 1994.
[35] M. Zworski. *Poisson formulæ for resonances,* Séminaire E.D.P. 1996-1997, École Polytechnique, XIII-1-XIII-12.

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