EXACT CONTROL FOR SCHRÖDINGER EQUATION ON TORUS FROM SMALL BALLS

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Abstract. For standard torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we prove observability for free Schrödinger equation from a ball of radius ε with explicit dependence of the observability constant on ε.

1. Introduction

We will follow some methods of Bourgain-Burq-Zworski [BBZ13][BuZw12] and Jin [Jin18] to prove a quantitative version of observability result for the Schrödinger equation on the 2-dimensional standard torus.

Theorem 1 (Semiclassical Observability Estimate)**.** *Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and $\Omega_\varepsilon = B(0,\varepsilon) \subset \mathbb{T}^2$. Then for any $\delta > 0$, there exists a numerical constant C such that for sufficiently small constant $\varepsilon > 0$ and $0 < h < h_0 = \varepsilon^{16+\delta}$,*

$$\|u\|_{L^2(\mathbb{T}^2)} \leq C\varepsilon^{-4}\|u\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{-2}h^{-2}\|(-h^2\Delta - 1)u\|_{L^2(\mathbb{T}^2)}.$$

From Theorem 1 we deduce the classical version

Theorem 2. *On the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ we have*

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq C_{\Omega_\varepsilon} \int_0^{\frac{1}{2\pi}} \|e^{it\Delta}u_0\|^2_{L^2(\Omega_\varepsilon)} dt$$

for $\Omega_\varepsilon = B(0,\varepsilon)$ and $C_{\Omega_\varepsilon} = \exp\exp\left(\frac{C\log\varepsilon^{-1}}{\log\log\varepsilon^{-1}}\right)$ with some constant C independent of ε.

The bad constant comes from the low frequency parts. For high frequency parts, we have a better estimate with constant ε^{-8}, see Theorem 3 for details.

1.1. Historical Remark.

Lebeau [Le92] first obtained the control for Schrödinger equation under the following geometric control condition

There exists $T > 0$ such that every geodesic of length T intersects Ω.

In general, the geometric control condition is not necessary. The observability estimate in the case of flat tori was obtained by Jaffard [Jaf90] and Haraux [Ha89] in dimension two and by Komornik [Ko92] in higher dimensions. In dimension two, Burq-Zworski [BuZw12] extended the result to Schrödinger equation with smooth potential. Bourgain-Burq-Zworski [BBZ13] further extended it to the case of L^2 potential. In higher dimensions, the result was obtained by Anantharaman-Macià [AnMa14] with some class of potentials including continuous ones.

For integrable systems, Macià-Rivière [MaRi17] studied observability for Zoll manifold. Anantharaman-Léautaud-Macià [ALM16] studied observability for the unit disk.

For compact negatively curved surfaces, the observability by any nonempty open set was proved by Dyatlov-Jin-Nonnenmacher [DJN19].

None of the above results provides an exact constant for torus. However, the observability estimate is proved for any $T > 0$. We expect that our exact constant is valid for any $T > 0$ but are not able to prove it for some technical reasons.

Theorem 1 gives a lower bound on quantum limits on the standard torus. A better bound can be provided by the explicit description of the quantum limits by Jakobson [Jak97].

1.2. **Outline of the proof.** Microlocal method relate the issues of observability and control to classical dynamics. The point of our proof is to keep track of different directions of the geodesic flow, which is possible as that flow is completely integrable.

In §2 we prove a quantitative estimate for one-dimensional case, which implies a control result for regions of the form $(-\varepsilon, \varepsilon) \times \mathbb{T}$. Then we present a brief review of semiclassical analysis in §3. Specifically we recall Egorov's theorem in the form we need (see Lemma 6, Proposition 8). In §4, we use combinatorial arguments to separate rational and irrational directions. In §5, we prove semiclassical observability estimates (Theorem 1) with rational and irrational directions treated separately: for rational directions, the estimate follows from the projection to one-dimensional case studied in §2. That is inspired by the methods of [BuZw12] but here the estimates are quantitative. In §6, we first derive the high frequency observability estimate from the semiclassical results of §5. Finally, the low frequency estimate is obtained using Nazarov-Turán lemma and a classical estimate coming from analytic number theory.

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2. Estimate in dimension one

To prove the semiclassical observability estimate (Theorem 1), we need to first show the following observability estimate for a strip. The technique is to first prove explicitly for one-dimensional case and then the strip case follows easily.

Proposition 1. *Let $\omega_\varepsilon = (-\varepsilon, \varepsilon) \times [0,1] \subset \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, then for any $u \in H^2(\mathbb{T}^2)$ and $h > 0$,*

$$\|u\|^2_{L^2(\mathbb{T}^2)} \leq C\varepsilon^{-3}\|u\|^2_{L^2(\omega_\varepsilon)} + 4h^{-4}\|(-h^2\Delta - 1)u\|^2_{L^2(\mathbb{T}^2)}.$$

Proof. Step 1

We follow the method in [BuZw05] to prove an estimate in dimension one. Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, for any $v \in H^2(\mathbb{T}^1)$ and $z \in \mathbb{R}$, we claim

$$\|v\|^2_{L^2(\mathbb{T}^1)} \leq C\varepsilon^{-3}\|v\|^2_{L^2((-\varepsilon,\varepsilon))} + 4h^{-4}\|(-h^2\partial_x^2 - z)v\|^2_{L^2(\mathbb{T}^1)}. \tag{2.1}$$

Denote $f = (-h^2\partial_x^2 - z)v$, we separate the proof into two cases:
Case 1: $z \leq 0$. In this case we have

$$\begin{aligned} h^2\|\partial_x v\|^2_{L^2(\mathbb{T}^1)} &\leq \int_{\mathbb{T}^1}((-h^2\partial_x^2 - z)\bar{v})v dx \\ &\leq \|f\|_{L^2(\mathbb{T}^1)}\|v\|_{L^2(\mathbb{T}^1)}. \end{aligned}$$

By integrating

$$\begin{aligned} |v(x)| &\leq \left|\int_t^x \partial_x v(y)dy\right| + |v(t)| \\ &\leq \|\partial_x v\|_{L^2(\mathbb{T}^1)} + |v(t)|, \end{aligned}$$

we get

$$\begin{aligned} \|v\|^2_{L^2(\mathbb{T}^1)} &\leq 2\|\partial_x v\|^2_{L^2(\mathbb{T}^1)} + \varepsilon^{-1}\|v\|^2_{L^2((-\varepsilon,\varepsilon))} \\ &\leq \frac{2}{h^2}\|v\|_{L^2(\mathbb{T}^1)}\|f\|_{L^2(\mathbb{T}^1)} + \varepsilon^{-1}\|v\|^2_{L^2((-\varepsilon,\varepsilon))} \\ &\leq \frac{1}{2}\|v\|^2_{L^2(\mathbb{T}^1)} + \frac{2}{h^4}\|f\|^2_{L^2(\mathbb{T}^1)} + \varepsilon^{-1}\|v\|^2_{L^2((-\varepsilon,\varepsilon))}. \end{aligned}$$

So

$$\|v\|^2_{L^2(\mathbb{T}^1)} \leq \frac{4}{h^4}\|f\|^2_{L^2(\mathbb{T}^1)} + 2\varepsilon^{-1}\|v\|^2_{L^2((-\varepsilon,\varepsilon))}.$$

Case 2: $z > 0$
First choose $\chi \in C_0^\infty(\mathbb{T}^1)$ such that $\chi = 0$ on $B(0, \frac{\varepsilon}{3})$ and $\chi = 1$ on $\mathbb{T}^1 \setminus B(0, \frac{\varepsilon}{2})$ with $|\chi^{(k)}(x)| \leq \frac{C_k}{\varepsilon^k}, \forall k \in \mathbb{N}$. We then have

$$(-h^2\partial_x^2 - z)(\chi v) = h^2\partial_x^2\chi v - 2h^2\partial_x(\partial_x\chi v) + \chi f = \tilde{f}.$$

The solution of this ODE is

$$\chi(x)v(x) = -\frac{1}{h\sqrt{z}}\int_0^x \sin\frac{\sqrt{z}(x-t)}{h}\tilde{f}(t)dt.$$

For each term we have

$$\left|-\frac{1}{h\sqrt{z}}\int_0^x \sin\frac{\sqrt{z}(x-t)}{h}h^2\partial_x^2\chi v(t)dt\right| \leq \|\partial_x^2\chi v\|_{L^1((-\varepsilon,\varepsilon))} \leq \frac{C}{\varepsilon^{\frac{3}{2}}}\|v\|_{L^2((-\varepsilon,\varepsilon))},$$

$$\begin{aligned}\left|-\frac{1}{h\sqrt{z}}\int_0^x \sin\frac{\sqrt{z}(x-t)}{h}h^2\partial_t(\partial_x\chi v)(t)dt\right| &= \left|-\frac{1}{h\sqrt{z}}\int_0^x \partial_t\left(\sin\frac{\sqrt{z}(x-t)}{h}\right)h^2\partial_x\chi v(t)dt\right| \\ &\leq \|\partial_x\chi v(t)\|_{L^1((-\varepsilon,\varepsilon))} \\ &\leq \frac{C}{\varepsilon^{\frac{1}{2}}}\|v\|_{L^2((-\varepsilon,\varepsilon))},\end{aligned}$$

$$\left|-\frac{1}{h\sqrt{z}}\int_0^x \sin\frac{\sqrt{z}(x-t)}{h}\chi f(t)dt\right| \leq \frac{1}{h^2}\|f\|_{L^1(\mathbb{T}^1)}.$$

Put them together we get

$$\begin{aligned}\|v\|_{L^2(\mathbb{T}^1)} &\leq \|\chi v\|_{L^2(\mathbb{T}^1)} + \|(1-\chi)v\|_{L^2(\mathbb{T}^1)} \\ &\leq \frac{C}{\varepsilon^{\frac{3}{2}}}\|v\|_{L^2((-\varepsilon,\varepsilon))} + \frac{C}{\varepsilon^{\frac{1}{2}}}\|v\|_{L^2((-\varepsilon,\varepsilon))} + \frac{1}{h^2}\|f\|_{L^1(\mathbb{T}^1)} + \|v\|_{L^2((-\varepsilon,\varepsilon))} \\ &\leq \frac{C}{\varepsilon^{\frac{3}{2}}}\|v\|_{L^2((-\varepsilon,\varepsilon))} + \frac{1}{h^2}\|f\|_{L^2(\mathbb{T}^1)}.\end{aligned}$$

Step 2

Let $g = (-h^2\Delta - 1)u$, we prove the 2-dimensional estimate by Fourier expansion in y.

Decompose $u = \sum_{k\in\mathbb{Z}} u_k(x)e_k(y)$, and $g = \sum_{k\in\mathbb{Z}} g_k(x)e_k(y)$ where $e_k(y) = e^{2k\pi iy}$, then

$$(-h^2\partial_x^2 + (2k\pi)^2h^2 - 1)u_k = g_k.$$

The proof follows from the one-dimensional estimate (2.1)

$$\|u_k\|^2_{L^2(\mathbb{T}^1)} \leq C\varepsilon^{-3}\|u_k\|^2_{L^2(B(0,\varepsilon))} + 4h^{-4}\|g_k\|^2_{L^2(\mathbb{T}^1)}.$$

□

Finally we point out that this is the best possible estimate by looking at

$$u(x,y) = \sin(2\pi x)e^{2\pi ik(h)y} \quad \text{or} \quad u(x,y) = \chi(x)e^{2\pi il(h)y}$$

where $4\pi^2h^2(k(h)^2+1) = 1$, $4\pi^2h^2l(h)^2 = 1$, and $\chi(x)$ is supported outside $(-\varepsilon,\varepsilon)$.

3. Semiclassical preliminaries

In this section we recall some semiclasscial preliminaries we will use. The general reference is [Zw12]. The results for the torus are also discussed in Anantharaman-Macià [AnMa14]. Throughout this section, we take $\mathbb{T}^n = \mathbb{R}^n/(L\mathbb{Z})^n$ for some $L \geq 1$.

3.1. L^2 boundedness of pseudo-differential operators.

We will recall several properties related to L^2 boundedness of pseudo-differential operators. First we recall the definition of Weyl quantization.

Definition 2. *Let $a(x,\xi) \in S^m(T^*\mathbb{T}^n)$, the Weyl quantization is defined as*

$$\mathrm{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a\left(\frac{x+y}{2},\xi\right)e^{i(x-y)\xi/h}u(y)dyd\xi$$

for $u \in \mathcal{D}'(\mathbb{T}^n)$, where a is regarded as a periodic function on $\mathbb{R}^n \times \mathbb{R}^n$. $\mathrm{Op}_h^w(a)$ is called an m-th order pseudo-differential operator.

0-th order pseudodifferential operators are bounded on $L^2(\mathbb{T}^n)$. In fact, we have

Lemma 3. *If $a \in S^0(T^*\mathbb{T}^n)$, then $\mathrm{Op}_h^w(a) : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ is bounded with*

$$\|\mathrm{Op}_h^w(a)\| \leq C \sum_{|\alpha|\leq Kn} h^{\frac{|\alpha|}{2}}\|\partial^\alpha a\|_{L^\infty}$$

for some universal constant K.

Proof. The proof follows from the proof of [Zw12, Theorem 4.23, Theorem 5.5]. □

Since we will need to estimate L^2 bound for remainders in composition formula, we prove an estimate for the composition formula of pseudo-differential operators.

Lemma 4. *Let $A(D) = \frac{1}{2}\langle QD, D\rangle$ with Q a real nonsingular symmetric matrix. Suppose $a \in C^\infty(\mathbb{R}^n)$, then for every $N \in \mathbb{N}$, there exists a constant C depending only on the dimension n, such that*

$$\sum_{|\alpha|\leq N} h^{\frac{|\alpha|}{2}}\|\partial^\alpha e^{ihA(D)}a\|_{L^\infty} \leq C \sum_{|\alpha|\leq N+n+1} h^{\frac{|\alpha|}{2}}\|\partial^\alpha a\|_{L^\infty}.$$

Proof. We just need to prove for $N=0$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function near 0 (i.e. $\chi(x) = 1$ in a neighbourhood of 0), then

$$\begin{aligned} e^{ihA(D)}a &= \frac{C}{h^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{\frac{i\phi(w)}{h}}a(z-w)dw \\ &= \frac{C}{h^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{\frac{i\phi(w)}{h}}\chi\left(\frac{w}{\sqrt{h}}\right)a(z-w)dw + \frac{C}{h^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{\frac{i\phi(w)}{h}}\left(1-\chi\left(\frac{w}{\sqrt{h}}\right)\right)a(z-w)dw \\ &= A_1 + A_2 \end{aligned}$$

where $\phi(w) = -\frac{1}{2}\langle Q^{-1}w, w\rangle$.

We have $|A_1| \le C\left|\int_{\mathbb{R}^n} e^{i\phi(w)}\chi(w)a(z-\sqrt{h}w)dw\right| \le C\|a\|_{L^\infty}$. On the other hand, applying $L = \frac{\langle \partial\phi, D\rangle}{|\partial\phi|^2}$ gives

$$\begin{aligned}
|A_2| &\le C\left|\int_{\mathbb{R}^n} e^{i\phi(w)}(1-\chi(w))a(z-\sqrt{h}w)dw\right| \\
&= C\left|\int_{\mathbb{R}^n} (L^{n+1}e^{i\phi(w)})(1-\chi(w))a(z-\sqrt{h}w)dw\right| \\
&= C\left|\int_{\mathbb{R}^n} e^{i\phi(w)}(L^T)^{n+1}((1-\chi(w))a(z-\sqrt{h}w))dw\right| \\
&\le C\sum_{|\alpha|\le n+1} h^{\frac{|\alpha|}{2}}\|\partial^\alpha a\|_{L^\infty}.
\end{aligned}$$

The estimates for A_1 and A_2 together give the desired estimate. ☐

Now in general, we have

$$\begin{aligned}
e^{ihA(D)}a &= \sum_{k=0}^{N}\frac{i^k h^k}{k!}A(D)^k a + \frac{i^{N+1}h^{N+1}}{N!}\int_0^1 (1-t)^N e^{ithA(D)}A(D)^{N+1}a dt \\
&= \sum_{k=0}^{N}\frac{i^k h^k}{k!}A(D)^k a + O_N(h^{N+1})\sum_{|\alpha|\le n+1} h^{\frac{|\alpha|}{2}}\|\partial^\alpha A(D)^{N+1}a\|_{L^\infty}.
\end{aligned}$$

This provides the following corollary.

Corollary 5. *Let $a, b \in S^0(T^*\mathbb{T}^n)$, then there exists a universal constant K such that*

$$\|\operatorname{Op}_h^w(a)\operatorname{Op}_h^w(b) - \operatorname{Op}_h^w(ab)\| \le Ch\sum_{|\alpha|\le Kn} h^{\frac{|\alpha|}{2}}\|\partial^\alpha\sigma(D)(a\otimes b)\|_{L^\infty}$$

where $\sigma(x,\xi,y,\eta) = \langle\xi, y\rangle - \langle x,\eta\rangle$ is the standard symplectic product on $T^*\mathbb{R}^{2n}$.

Proof. By composition formula for pseudo-differential operators [Zw12, Theorem 4.11],

$$\operatorname{Op}_h^w(a)\operatorname{Op}_h^w(b) = \operatorname{Op}_h^w(a\sharp b)$$

where

$$a\sharp b(x,\xi) = e^{ihA(D)}\,(a(x,\xi)b(y,\eta))|_{y=x,\eta=\xi}$$

and $A(D) = \frac{1}{2}\sigma(D_x, D_\xi, D_y, D_\eta)$. By the previous discussion after Lemma 4,

$$a\sharp b(x,\xi) = a(x,\xi)b(x,\xi) + O(h)\sum_{|\alpha|\le 4n+1} h^{\frac{|\alpha|}{2}}\|\partial^\alpha A(D)(a\otimes b)\|_{L^\infty}$$

with all the derivatives satisfying similar estimates. Then by Lemma 3 we get the desired result.

□

3.2. **Propagation of singularities.** We will study the quantitative version of propagation of singularities of Schrödinger equation. First we recall an important lemma which relates Schrödinger equation with the geodesic flow on the torus.

Lemma 6 (Egorov theorem)**.** *Let $a \in S^0(T^*\mathbb{T}^n)$, $v_h(t) = e^{-iht\Delta}$ be a unitary operator, and $\phi_t(x,\xi) = (x+2t\xi, \xi)$ be the corresponding Hamiltonian flow. Then*

$$v_h(t)\operatorname{Op}_h^w(a)v_h(-t) = \operatorname{Op}_h^w(a\circ\phi_t).$$

Proof. We recall the identity for Weyl quantization following e.g. by an explicit computation from [Zw12, Theorem 4.6]

$$[-h^2\Delta, \operatorname{Op}_h^w(a)] = -ih\operatorname{Op}_h^w(\{|\xi|^2, a\}).$$

Then let $A(t) = v_h(-t)\operatorname{Op}_h^w(a\circ\phi_t)v_h(t)$, we get

$$\begin{aligned}
\partial_t A(t) &= v_h(-t)(-\frac{i}{h}[-h^2\Delta, \operatorname{Op}_h^w(a\circ\phi_t)] + \operatorname{Op}_h^w(2\xi\cdot\partial_x a\circ\phi_t))v_h(t)\\
&= v_h(-t)(-\operatorname{Op}_h^w(\{|\xi|^2, a\circ\phi_t\}) + \operatorname{Op}_h^w(2\xi\cdot\partial_x a\circ\phi_t))v_h(t)\\
&= 0.
\end{aligned}$$

Hence

$$A(t) = A(0) = \operatorname{Op}_h^w(a)$$

or

$$v_h(t)\operatorname{Op}_h^w(a)v_h(-t) = \operatorname{Op}_h^w(a\circ\phi_t).$$

□

In addition, we have (see [DyJin18, Lemma 4.2])

Lemma 7.

$$\|e^{it(-h^2\Delta-1)/h}u - u\|_{L^2} \le \frac{|t|}{h}\|(-h^2\Delta-1)u\|_{L^2}$$

Proof. It is obvious from

$$\partial_t e^{it(-h^2\Delta-1)/h}u = \frac{i}{h}e^{it(-h^2\Delta-1)/h}(-h^2\Delta-1)u.$$

□

Combining Lemma 6 and Lemma 7, we have

$$\|\operatorname{Op}_h^w(a\circ\phi_t)u\| \le \|\operatorname{Op}_h^w(a)u\| + \frac{|t|}{h}\|\operatorname{Op}_h^w(a)\|\|(-h^2\Delta-1)u\|. \tag{3.1}$$

Now we can prove a quantitative version of propagation of singularities.

Proposition 8. *Let $a \in C_0^\infty(T^*\mathbb{T}^n;[0,1])$ and $b \in S^0(T^*\mathbb{T}^n;[0,1])$. If there exist $t_1,\cdots,t_M \in (0,t)$ such that for any $p \in \operatorname{supp}(a)$, there exists j such that $\phi_{t_j}(p) \in \{b = 1\}$, then for any $u \in H^2(\mathbb{T}^n)$, we have*

$$\|\operatorname{Op}_h^w(a)u\|_{L^2}^2 \le C_{a,b,1}\|\operatorname{Op}_h^w(b)u\|_{L^2}^2 + C_{a,b,2}\frac{|t|^2}{h^2}\|(-h^2\Delta-1)u\|_{L^2}^2 + C_{a,b,3}h\|u\|_{L^2}^2$$

where $C_{a,b,1} = C\|a\otimes b\|_{C^{Kn}_{h;M,t}}M$, $C_{a,b,2} = CM\|a\otimes b\|_{C^{Kn}_{h;M,t}}\|b\|^2_{C_h^{Kn}}$ and $C_{a,b,3} = C\|a \otimes b\|_{C^{Kn}_{h:M,t;2}}\|b\|_{C_h^{Kn}}$.

Here we use the notations: $\|f\|_{C_h^k} = \sum_{|\alpha|\le k} h^{\frac{|\alpha|}{2}}\|\partial^\alpha f\|_{L^\infty}$,

$$\|a\otimes b\|_{C^k_{h;s,t}} = \sum_{j\le k} h^{\frac{j}{2}} \sum_{l_0+l_1+\cdots+l_m=j} s^m t^{l_1+\cdots+l_m}\|a\|_{C^{l_0}}\|b\|_{C^{l_1}}\cdots\|b\|_{C^{l_m}},$$

$$\|a\otimes b\|_{C^k_{h;s,t;2}} = \sum_{j\le k} h^{\frac{j}{2}} \sum_{l_0+l_1+\cdots+l_m=j+2} s^m t^{l_1+\cdots+l_m}\|a\|_{C^{l_0}}\|b\|_{C^{l_1}}\cdots\|b\|_{C^{l_m}}.$$

Proof. Let $\chi = \sum_j |b\circ\phi_{t_j}|^2 \ge 1$ on $\operatorname{supp}(a)$. Let $q = \frac{|a|^2}{\chi}$, then by Lemma 3 and Corollary 5

$$\begin{aligned}
\langle \operatorname{Op}_h^w(|a|^2)u,u\rangle &\le \sum_j \langle \operatorname{Op}_h^w(\overline{\phi_{t_j}^* b})\operatorname{Op}_h^w(q)\operatorname{Op}_h^w(\phi_{t_j}^*b)u,u\rangle \\
&\quad + Ch\Big(\sum_j \|\sigma(D)(\phi_{t_j}^*b\otimes q)\|_{C_h^{Kn}}\|b\|_{C_h^{Kn}} + \|\sigma(D)(q\overline{\phi_{t_j}^*b}\otimes\phi_{t_j}^*b)\|_{C_h^{Kn}}\Big)\|u\|^2 \\
&\le C\|q\|_{C_h^{Kn}}\sum_j \|\operatorname{Op}_h^w(\phi_{t_j}^*b)u\|^2 + Ch\|a\otimes b\|_{C^{Kn}_{h:M,t;2}}\|b\|_{C_h^{Kn}}\|u\|^2 \\
&\le C\|a\otimes b\|_{C^{Kn}_{h;M,t}}\sum_j \|\operatorname{Op}_h^w(\phi_{t_j}^*b)u\|^2 + Ch\|a\otimes b\|_{C^{Kn}_{h:M,t;2}}\|b\|_{C_h^{Kn}}\|u\|^2.
\end{aligned}$$

By (3.1),

$$\begin{aligned}
\|\operatorname{Op}_h^w(\phi_{t_j}^*b)u\| - \|\operatorname{Op}_h^w(b)u\| &\le \frac{|t|}{h}\|\operatorname{Op}_h^w(b)\|\|(-h^2\Delta-1)u\| \\
&\le C\frac{|t|}{h}\|b\|_{C_h^{Kn}}\|(-h^2\Delta-1)u\|.
\end{aligned}$$

So

$$\begin{aligned}
\|\operatorname{Op}_h^w(a)u\|^2 &= \langle \operatorname{Op}_h^w(|a|^2)u,u\rangle \\
&\le C\|a\otimes b\|_{C^{Kn}_{h;M,t}}M\|\operatorname{Op}_h^w(b)u\|^2 + Ch\|a\otimes b\|_{C^{Kn}_{h:M,t;2}}\|b\|_{C_h^{Kn}}\|u\|^2 \\
&\quad + CM\frac{t^2}{h^2}\|a\otimes b\|_{C^{Kn}_{h;M,t}}\|b\|^2_{C_h^{Kn}}\|(-h^2\Delta-1)u\|^2.
\end{aligned}$$

□

4. Rational and irrational directions

To prove Theorem 1, the main point is to deduce the high frequency estimate by considering the geodesic flow on the torus. However, the dynamics on the torus does not satisfy the geometric control condition. So we divide the discussion into two cases-rational and irrational as follows.

Take $\Omega_\varepsilon = B(0,\varepsilon) \subset \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\phi_t(x,\xi) = (x + 2t\xi, \xi)$ be the geodesic flow. We will always assume $\varepsilon > 0$ is sufficiently small. We give the following definition

Definition 9. *A vector $\xi \in \mathbb{R}^2 \backslash \{0\}$ is called a direction. Two directions $\xi, \xi' \in \mathbb{R}^2_\xi \backslash \{0\}$ are equivalent iff $\xi = \lambda\xi'$ for some $\lambda \in \mathbb{R}^+$. We denote $\xi \sim \xi'$ for equivalent directions. If $\eta \sim (a,b) \in \mathbb{Z}^2 \setminus \{0\}$, then it is called rational. For rational directions, define*

$$L_\eta = \sqrt{a^2 + b^2}$$

to be the length of the primitive geodesic in direction η, where $\gcd(a,b) = 1$. If a rational direction η satisfies

$$L_\eta^2 = a^2 + b^2 < \frac{32}{\varepsilon^2},$$

then we call η an ε-rational direction.

Proposition 10. *Let $\xi \in \mathbb{R}^2 \setminus \{0\}$ be a direction of length 1 (i.e. $|\xi| = 1$) and $\varepsilon > 0$ be a small constant. If there exists constant $C > 0$ such that*

$$|\arg \xi - \arg \eta| \geq \frac{\varepsilon}{C L_\eta} \tag{4.1}$$

for any ε-rational direction η, then there exists $C' = C'(C)$ such that for any $x \in \mathbb{T}^2$, there exists $t \in [0, C'\varepsilon^{-1}]$ such that $x + t\xi \in B(0, \frac{\varepsilon}{3})$.

Proof. Assume $\xi \sim (1,\alpha)$ with $0 < \alpha < 1$ and $C > 12$ without loss of generality. Define $\|x\|_1 := \min_{m\in\mathbb{Z}} |x - m|$, consider $\{n\alpha \mod 1 : 1 \leq n \leq \frac{3C}{\varepsilon}\}$, by Pigeonhole Principle there exist $\frac{3C}{\varepsilon} \geq n' > n'' \geq 1$ such that

$$\|n'\alpha - n''\alpha\|_1 \leq \frac{\varepsilon}{2C}.$$

Let $n = n' - n''$, then $n \in [1, \frac{3C}{\varepsilon} - 1]$ and there exists $m \in \mathbb{Z}$ such that

$$|n\alpha - m| \leq \frac{\varepsilon}{2C}.$$

We have $0 \leq m \leq n$, assume $\gcd(n, m) = 1$ without loss of generality. Now let $\eta = \frac{(n,m)}{\sqrt{n^2+m^2}}$, then

$$\begin{aligned}\frac{\varepsilon}{2C} &\geq |n\alpha - m| \\ &= |(n, m) \times (1, \alpha)| \\ &= L_\eta \sqrt{1+\alpha^2} |\sin(\arg \xi - \arg \eta)| \\ &\geq \frac{2}{\pi} L_\eta |\arg \xi - \arg \eta|.\end{aligned}$$

So $|\arg \xi - \arg \eta| \leq \frac{\pi\varepsilon}{4CL_\eta}$, which means that η is not ε-rational by condition (4.1) (i.e. $\frac{4\sqrt{2}}{\varepsilon} \leq L_\eta \leq \frac{3\sqrt{2}C}{\varepsilon}$). The intersection of the closed trajectory $\gamma = \{(tn, tm) : t \in [0, 1]\} \subset \mathbb{T}^2$ with the circle $\{x_1 = 0\}$ is given by $\{(0, \frac{k}{n}) : 0 \leq k < n\}$. Thus each ball of radius $r > \frac{1}{2n}$ has to intersect γ. Since $n \geq \frac{L_\eta}{\sqrt{2}} \geq \frac{4}{\varepsilon}$, there exists $t \in [0, L_\eta]$, $x + t\eta \in B(0, \frac{\varepsilon}{7})$ for any $x \in \mathbb{T}^2$.

Now

$$|(x + t\xi) - (x + t\eta)| \leq L_\eta |\xi - \eta| \leq L_\eta |\arg \xi - \arg \eta| < \frac{\varepsilon}{C} < \frac{\varepsilon}{12}.$$

So $x + t\xi \in B(0, \frac{\varepsilon}{3})$, i.e. $C' = 3\sqrt{2}\max(C, 12)$ would work. □

In the following section, we will prove Theorem 1 by considering rational and irrational directions separately. First we note that for $\psi \in C_0^\infty(\mathbb{R}; [0, 1])$ such that $\psi(x) = 1$ on $[-K, K]$, we have

$$\|(1 - \psi)(-\Delta - h^{-2})u\|_{L^2(\mathbb{T}^2)} \leq \frac{1}{K} \|(-\Delta - h^{-2})u\|_{L^2(\mathbb{T}^2)}.$$

So we only need to consider the case when the frequency is close to h^{-1}. We choose a cutoff function $a \in C_0^\infty(N_\varepsilon; [0, 1])$ such that $a = 1$ on $N_{\frac{\varepsilon}{2}}$ where $N_\varepsilon = \{1 - \varepsilon^3 < |\xi| < 1 + \varepsilon^3\}$ and $|\partial^\alpha a| \leq C_\alpha \varepsilon^{-3|\alpha|}$. Furthermore, we make a partition of unity

$$a(\xi) = a_{\mathrm{irr}}(\xi) + \sum_\eta a_\eta(\xi)^2$$

requiring the following conditions, where the sum is over all ε-rational directions η.

- For any ε-rational direction η, there exists a_η such that $a_\eta^2 = a$ on $\{\xi \in N_\varepsilon : |\arg \xi - \arg \eta| < \frac{\varepsilon}{25L_\eta}\}$ and $a_\eta = 0$ outside $\{\xi \in N_\varepsilon : |\arg \xi - \arg \eta| < \frac{\varepsilon}{24L_\eta}\}$. In addition,

  $$\|a_\eta\|_{C^k} \leq C_k \max\left(\left(\frac{\varepsilon}{L_\eta}\right)^k, \varepsilon^{-3k}\right) \leq C_k \varepsilon^{-3k}.$$

  These a_η's are called rational cutoff functions. Their number is $O(\varepsilon^{-2})$.

- Define $a_{\mathrm{irr}} := a - \sum_{\eta} a_\eta^2$ to be the irrational part. It also satisfies

$$\|a_{\mathrm{irr}}\|_{C^k} \le C_k \varepsilon^{-3k}.$$

We claim that any two rational a_η and $a_{\eta'}$ have disjoint support. If there exists $\xi \in \operatorname{supp}(a_\eta) \cap \operatorname{supp}(a_{\eta'})$, assume $L_\eta \le L_{\eta'}$ and $\eta \sim (1, \frac{q}{p})$ and $\eta' \sim (1, \frac{q'}{p'})$ such that $0 < q < p$ and $0 < q' < p'$ without loss of generality, then

$$\frac{1}{pp'} \le \left| \frac{q}{p} - \frac{q'}{p'} \right| < 2|\arg \eta - \arg \eta'| \le \frac{2\varepsilon}{12 L_\eta} \le \frac{\varepsilon}{6p}.$$

Therefore, $\frac{1}{pp'} < \frac{\varepsilon}{6p}$, which means that $p' > \frac{6}{\varepsilon}$, contradictory to that η' is ε-rational. See Figure 1 for a picture of the rational cutoff functions.

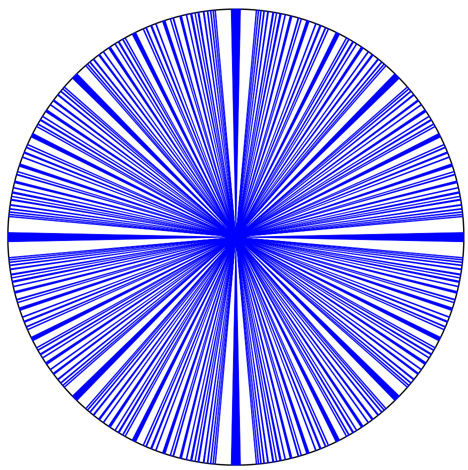

Figure 1. The blue set contains the union of the supports of the rational cutoff functions a_η

5. Proof of semi-classical observability

In this section we give the proof of Theorem 1 by treating the irrational and rational case separately. First we deal with the irrational case.

Proposition 11. *Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $h = O(\varepsilon^{8+\delta})$ for some $\delta > 0$, then for sufficiently small $\varepsilon > 0$, we have the following estimate*

$$\| \operatorname{Op}_h^w(a_{\mathrm{irr}}) u\|^2_{L^2(\mathbb{T}^2)} \le C\varepsilon^{-2} \|u\|^2_{L^2(\Omega_\varepsilon)} + C\varepsilon^{-4} h^{-2} \|(-h^2\Delta - 1)u\|^2_{L^2(\mathbb{T}^2)} + C\varepsilon^{-8} h \|u\|^2_{L^2(\mathbb{T}^2)}. \tag{5.1}$$

Proof. We choose a cutoff function $\chi \in C_0^\infty(\Omega_\varepsilon)$ with $\|\chi\|_{C^k} \le C_k \varepsilon^{-k}$ and $\chi = 1$ on $B(0, \frac{2\varepsilon}{3})$. On $\operatorname{supp}(a_{\mathrm{irr}})$ the assumption of Proposition 10 is satisfied for $C = 25$, so there exists C' such that for any $p \in \operatorname{supp}(a_{\mathrm{irr}})$, there exists $t \in [0, C'\varepsilon^{-1}]$ such that $\phi_t(p) \in T^*B(0, \frac{\varepsilon}{3})$. Let $t_j = j\frac{\varepsilon}{12}$, $j = 1, \cdots, M = \lceil 12C'\varepsilon^{-2} \rceil$, then $\phi_{t_j}(p) \in T^*B(0, \frac{2\varepsilon}{3}) \subset \{\chi = 1\}$ for some $j \in [1, M]$. By Proposition 8 we have

$$\|\operatorname{Op}_h^w(a_{\text{irr}})u\|_{L^2}^2 \leq C_{a_{\text{irr}},\chi,1}\|\operatorname{Op}_h^w(\chi)u\|_{L^2}^2 + C_{a_{\text{irr}},\chi,2}\frac{|t|^2}{h^2}\|(-h^2\Delta - 1)u\|_{L^2}^2 + C_{a_{\text{irr}},\chi,3}h\|u\|_{L^2}^2$$

with $M = O(\varepsilon^{-2})$ and $t = O(\varepsilon^{-1})$, so

$$\begin{aligned}
C_{a_{\text{irr}},\chi,1} &\leq C\varepsilon^{-2}(1 + h^{\frac{1}{2}}\varepsilon^{-4})^{2K}, \\
C_{a_{\text{irr}},\chi,2} &\leq C\varepsilon^{-2}(1 + h^{\frac{1}{2}}\varepsilon^{-4})^{2K}(1 + h^{\frac{1}{2}}\varepsilon^{-1})^{4K}, \\
C_{a_{\text{irr}},\chi,3} &\leq C\varepsilon^{-8}(1 + h^{\frac{1}{2}}\varepsilon^{-4})^{2K}(1 + h^{\frac{1}{2}}\varepsilon^{-1})^{2K}.
\end{aligned}$$

Therefore, let $h = O(\varepsilon^{8+\delta})$, then

$$\|\operatorname{Op}_h^w(a_{\text{irr}})u\|_{L^2(\mathbb{T}^2)}^2 \leq C\varepsilon^{-2}\|u\|_{L^2(\Omega_\varepsilon)}^2 + C\varepsilon^{-4}h^{-2}\|(-h^2\Delta - 1)u\|_{L^2(\mathbb{T}^2)}^2 + C\varepsilon^{-8}h\|u\|_{L^2(\mathbb{T}^2)}^2.$$

□

Then we deal with the rational case.

Proposition 12. *Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $h = O(\varepsilon^{8+\delta})$ for some $\delta > 0$, then for sufficiently small $\varepsilon > 0$ and an ε-rational direction η, we have the following estimate*

$$\begin{aligned}
\|\operatorname{Op}_h^w(a_\eta)u\|_{L^2(\mathbb{T}^2)}^2 \leq{}& C\varepsilon^{-6}\|u\|_{L^2(\Omega_\varepsilon)}^2 + C\varepsilon^{-4}h^{-4}\|\operatorname{Op}_h^w(a_\eta)(-h^2\Delta - 1)u\|_{L^2(\mathbb{T}^2)}^2 \\
&+ Ch^{-2}\varepsilon^{-10}\|(-h^2\Delta - 1)u\|_{L^2(\mathbb{T}^2)}^2 + C\varepsilon^{-14}h\|u\|_{L^2(\mathbb{T}^2)}^2.
\end{aligned} \tag{5.2}$$

Proof. For ε-rational direction $\eta \sim (a, b) \in \mathbb{Z}^2 \setminus \{0\}$ ($\gcd(a, b) = 1$), let $L = L_\eta$ be the new period and a_η be the corresponding ε-rational cutoff function ($\|a_\eta\|_{C^k} \leq C_k\varepsilon^{-3k}$). Cover $\mathbb{T}^2$ with a larger square with edges in direction η and $\eta^\perp$. The square has area $a^2 + b^2$ and induces a torus $\tilde{\mathbb{T}}^2 = \tilde{\mathbb{T}}^1_\eta \times \tilde{\mathbb{T}}^1_{\eta^\perp}$. We extend the function u to the larger torus periodically. Let $b_\eta \in C^\infty(\tilde{\mathbb{T}}^1_{\eta^\perp})$ such that $\operatorname{supp}(b_\eta) \subset (-\frac{\varepsilon}{3}, \frac{\varepsilon}{3})$ and $b_\eta = 1$ on $(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$ with $\|b_\eta\|_{C^k} \leq C_k\varepsilon^{-k}$. See Figure 2 for the covering and cutoff.

By Proposition 1 we have

$$\begin{aligned}
\|\operatorname{Op}_h^w(a_\eta)u\|_{L^2(\tilde{\mathbb{T}}^2)}^2 \leq{}& C\left(\frac{L}{\varepsilon}\right)^3\|\operatorname{Op}_h^w(b_\eta)\operatorname{Op}_h^w(a_\eta)u\|_{L^2(\tilde{\mathbb{T}}^2)}^2 \\
&+ 4L^4h^{-4}\|\operatorname{Op}_h^w(a_\eta)(-h^2\Delta - 1)u\|_{L^2(\tilde{\mathbb{T}}^2)}^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\|(\operatorname{Op}_h^w(a_\eta b_\eta) - \operatorname{Op}_h^w(b_\eta)\operatorname{Op}_h^w(a_\eta))u\|_{L^2(\tilde{\mathbb{T}}^2)} \\
&\leq Ch\sum_{|\alpha| \leq 2K} h^{\frac{|\alpha|}{2}}\|\partial^\alpha\sigma(D)(b_\eta \otimes a_\eta)\|_{L^\infty}\|u\|_{L^2(\tilde{\mathbb{T}}^2)} \\
&\leq Ch\varepsilon^{-4}(1 + h^{\frac{1}{2}}\varepsilon^{-3})^{2K}\|u\|_{L^2(\tilde{\mathbb{T}}^2)}.
\end{aligned}$$

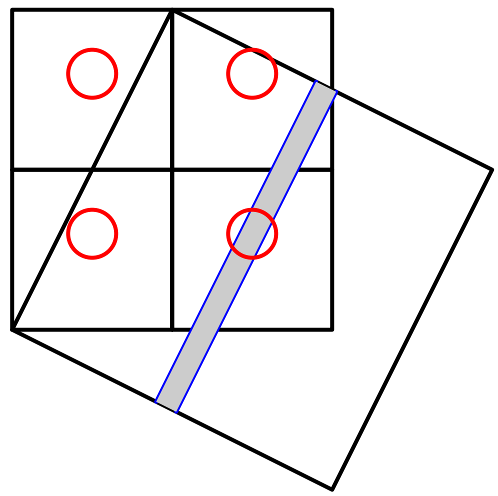

FIGURE 2. Cover the standard torus with a larger one. The cutoff function b_η restricts the consideration into a small strip coming through the small ball.

We then choose a cutoff function $\chi \in C_0^\infty(\Omega_\varepsilon)$ with $\|\chi\|_{C^k} \le C_k\varepsilon^{-k}$ and $\chi = 1$ on $B(0, \frac{2\varepsilon}{3})$. Notice that for any $p \in \operatorname{supp}(a_\eta b_\eta)$, there exists $t = O(L) \le C'\varepsilon^{-1}$ such that $\phi_t(p) \in T^*B(0, \frac{9\varepsilon}{24})$. Let $t_j = j\frac{\varepsilon}{12}$, $j = 1, \cdots, M = \lceil 12C'\varepsilon^{-2}\rceil$, then $\phi_{t_j}(p) \in T^*B(0, \frac{2\varepsilon}{3}) \subset \{\chi = 1\}$ for some $j \in [1, M]$. By Proposition 8,

$$\begin{aligned}\|\operatorname{Op}_h^w(a_\eta b_\eta)u\|_{L^2}^2 \le C_{a_\eta b_\eta,\chi,1}\|\operatorname{Op}_h^w(\chi)u\|_{L^2}^2 \\ + C_{a_\eta b_\eta,\chi,2}\frac{|t|^2}{h^2}\|(-h^2\Delta - 1)u\|_{L^2}^2 + C_{a_\eta b_\eta,\chi,3}h\|u\|_{L^2}^2\end{aligned}$$

with $M = O(\varepsilon^{-2})$ and $t = O(\varepsilon^{-1})$, so

$$\begin{aligned}C_{a_\eta b_\eta,\chi,1} &\le C\varepsilon^{-2}(1 + h^{\frac{1}{2}}\varepsilon^{-4})^{2K},\\ C_{a_\eta b_\eta,\chi,2} &\le C\varepsilon^{-2}(1 + h^{\frac{1}{2}}\varepsilon^{-4})^{2K}(1 + h^{\frac{1}{2}}\varepsilon^{-1})^{4K},\\ C_{a_\eta b_\eta,\chi,3} &\le C\varepsilon^{-8}(1 + h^{\frac{1}{2}}\varepsilon^{-4})^{2K}(1 + h^{\frac{1}{2}}\varepsilon^{-1})^{2K}.\end{aligned}$$

So for $h = O(\varepsilon^{8+\delta})$,

$$\begin{aligned}\|\operatorname{Op}_h^w(a_\eta b_\eta)u\|_{L^2(\tilde{\mathbb{T}}^2)}^2 \le C\varepsilon^{-2}\|\operatorname{Op}_h^w(\chi)u\|_{L^2(\tilde{\mathbb{T}}^2)}^2 \\ + C\varepsilon^{-4}h^{-2}\|(-h^2\Delta - 1)u\|_{L^2(\tilde{\mathbb{T}}^2)}^2 + \varepsilon^{-8}h\|u\|_{L^2(\tilde{\mathbb{T}}^2)}^2\end{aligned}$$

and

$$\|\operatorname{Op}_h^w(b_\eta)\operatorname{Op}_h^w(a_\eta)u\|_{L^2(\tilde{\mathbb{T}}^2)}^2 \le 2\|\operatorname{Op}_h^w(a_\eta b_\eta)u\|_{L^2(\tilde{\mathbb{T}}^2)}^2 + \varepsilon^{-8}h^2\|u\|_{L^2(\tilde{\mathbb{T}}^2)}^2.$$

Put them together

$$\begin{aligned}\|\operatorname{Op}_h^w(a_\eta)u\|^2_{L^2(\tilde{\mathbb{T}}^2)} &\le C\left(\frac{L}{\varepsilon}\right)^3\|\operatorname{Op}_h^w(b_\eta)\operatorname{Op}_h^w(a_\eta)u\|^2_{L^2(\tilde{\mathbb{T}}^2)}\\ &\quad+\frac{4L^4}{h^4}\|\operatorname{Op}_h^w(a_\eta)(-h^2\Delta-1)u\|^2_{L^2(\tilde{\mathbb{T}}^2)}\\ &\le C\varepsilon^{-5}L^3\|u\|^2_{L^2(\Omega_\varepsilon)}+C\frac{\varepsilon^{-10}}{h^2}\|(-h^2\Delta-1)u\|^2_{L^2(\tilde{\mathbb{T}}^2)}\\ &\quad+\frac{C\varepsilon^{-4}}{h^4}\|\operatorname{Op}_h^w(a_\eta)(-h^2\Delta-1)u\|^2_{L^2(\tilde{\mathbb{T}}^2)}+C\varepsilon^{-14}h\|u\|^2_{L^2(\tilde{\mathbb{T}}^2)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\operatorname{Op}_h^w(a_\eta)u\|^2_{L^2(\mathbb{T}^2)} &\le C\varepsilon^{-6}\|u\|^2_{L^2(\Omega_\varepsilon)}+C\varepsilon^{-4}h^{-4}\|\operatorname{Op}_h^w(a_\eta)(-h^2\Delta-1)u\|^2_{L^2(\mathbb{T}^2)}\\ &\quad+Ch^{-2}\varepsilon^{-10}\|(-h^2\Delta-1)u\|^2_{L^2(\mathbb{T}^2)}+C\varepsilon^{-14}h\|u\|^2_{L^2(\mathbb{T}^2)}.\end{aligned}$$

∎

We combine the irrational estimate (5.1) and rational estimate (5.2) to get

Proof of Theorem 1. The number of ε-rational directions is $O(\varepsilon^{-2})$. For $h=O(\varepsilon^{16+\delta})$ we have

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{T}^2)} &\le \langle\operatorname{Op}_h^w(a)u,u\rangle+\langle\operatorname{Op}_h^w(1-a)u,u\rangle\\ &\le \|\operatorname{Op}_h^w(a_{\mathrm{irr}})u\|_{L^2(\mathbb{T}^2)}\|u\|_{L^2(\mathbb{T}^2)}+\sum_\eta\|\operatorname{Op}_h^w(a_\eta)u\|^2_{L^2(\mathbb{T}^2)}\\ &\quad+C\varepsilon^{-6}\|(-h^2\Delta-1)u\|^2_{L^2(\mathbb{T}^2)}\\ &\le C(\|\operatorname{Op}_h^w(a_{\mathrm{irr}})u\|^2_{L^2(\mathbb{T}^2)}+\sum_\eta\|\operatorname{Op}_h^w(a_\eta)u\|^2_{L^2(\mathbb{T}^2)}+C\varepsilon^{-6}\|(-h^2\Delta-1)u\|^2_{L^2(\mathbb{T}^2)})\\ &\le C\varepsilon^{-8}\|u\|^2_{L^2(\Omega_\varepsilon)}+C\frac{\varepsilon^{-4}}{h^4}\|(-h^2\Delta-1)u\|^2_{L^2(\mathbb{T}^2)}.\end{aligned}$$

This ends the proof of Theorem 1. ∎

6. Classical observability

In this section, we deduce the classical observability estimate from the semiclassical estimate on $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$. Let $\{\varphi_{\lambda,k}=e^{2\pi i(px+qy)}\}$ be eigenfunctions of $-\Delta$ with respect to eigenvalue λ^2 (i.e. $\lambda^2=4\pi^2(p^2+q^2)$) such that $\{\varphi_{\lambda,k}\}$ forms an orthonormal basis of $L^2(\mathbb{T}^2)$. Let $\Pi_N=\sum\limits_{\lambda=N}\langle u,\varphi_{\lambda,k}\rangle\varphi_{\lambda,k}$ and $\Pi_{\le N}=\sum\limits_{\lambda\le N}\Pi_\lambda$. Similarly, $\Pi_{>N}=\sum\limits_{\lambda>N}\Pi_\lambda$.

6.1. **High frequency estimate.** We first prove the high frequency estimate.

Theorem 3. *Let $\Omega_\varepsilon = B(0,\varepsilon) \subset \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\Pi = \Pi_{>h_0^{-1}}$ where $h_0 = \varepsilon^{16+\delta}$ for $\delta > 0$ and sufficiently small $\varepsilon > 0$, then we have*

$$\|\Pi u_0\|^2_{L^2(\mathbb{T}^2)} \le C\varepsilon^{-8} \int_0^{\frac{1}{2\pi}} \|e^{it\Delta}\Pi u_0\|^2_{L^2(\Omega_\varepsilon)} dt.$$

Proof. Let $u_0 = \Pi u_0$, $\chi \in C_0^\infty((0,1))$, $u = e^{it\Delta}u_0$ and $v = \chi_T(t)u$ where $\chi_T(t) = \chi(\frac{t}{T})$. We argue similarly to [BBZ13, Proposition 3.1],

$$(i\partial_t - \Delta)v = \frac{i}{T}\chi_T'(t)u.$$

Take Fourier transform in t,

$$(-\tau - \Delta)\hat{v} = \frac{i}{T}(\widehat{\chi_T'(t)u}).$$

For $\tau > h_0^{-2}$, apply Theorem 1, we have

$$\|\hat{v}\|_{L^2(\mathbb{T}^2)} \le \frac{C\varepsilon^{-2}}{T}\|(\widehat{\chi_T'(t)u})\|_{L^2(\mathbb{T}^2)} + C\varepsilon^{-4}\|\hat{v}\|_{L^2(\Omega_\varepsilon)}.$$

This is obviously true for $\tau \le h_0^{-2}$. So

$$\|\chi_T(t)u\|_{L^2(\mathbb{T}^2\times\mathbb{R})} \le \frac{C\varepsilon^{-2}}{T}\|\chi_T'(t)u\|_{L^2(\mathbb{T}^2\times\mathbb{R})} + C\varepsilon^{-4}\|\chi_T(t)u\|_{L^2(\Omega_\varepsilon\times\mathbb{R})},$$

$$\|\chi\|\|u_0\|_{L^2(\mathbb{T}^2)} \le \frac{C\varepsilon^{-2}}{T}\|\chi'\|\|u_0\|_{L^2(\mathbb{T}^2)} + C\varepsilon^{-4}\|\chi(t)u(tT,x)\|_{L^2(\Omega_\varepsilon\times\mathbb{R})}.$$

Take appropriate $T = O(\varepsilon^{-2})$, we have

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \le C\frac{\varepsilon^{-8}}{T}\|u\|^2_{L^2(\Omega_\varepsilon\times(0,T))}.$$

Because $e^{it\Delta}u_0$ has period $\frac{1}{2\pi}$, we have

$$\frac{1}{T}\|u\|^2_{L^2(\Omega_\varepsilon\times(0,T))} \le C\int_0^{\frac{1}{2\pi}} \|e^{it\Delta}\Pi u_0\|^2_{L^2(\Omega_\varepsilon)} dt.$$

This ends the proof. ∎

6.2. **Low frequency estimate.** Then we estimate the low frequency part. By [HaWr60, Theorem 317], we have $\operatorname{rank}\Pi_\lambda \le e^{C\log\lambda/\log\log\lambda}$. Now we want to determine the constant in the following estimate

$$\|\Pi_\lambda u_0\|^2_{L^2(\mathbb{T}^2)} \le C(\varepsilon,\lambda)\|\Pi_\lambda u_0\|^2_{L^2(\Omega)}.$$

We use the following Nazarov-Turán lemma [Na00]

Lemma 13 (Nazarov-Turán). *Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, $E \subset \mathbb{T}^1$ be a measurable subset, and $p(x) = \sum_{k=1}^{n} c_k e^{2\pi ikx}$ be a trigonometric polynomial in n characters, then exists numerical constant C such that*

$$\|p\|_{L^2(\mathbb{T}^1)} \leq \left(\frac{C}{|E|}\right)^{n-\frac{1}{2}} \|p\|_{L^2(E)}.$$

Corollary 14. *Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\lambda \leq h_0^{-1} = \varepsilon^{-17}$, there exists constant C such that*

$$\|\Pi_\lambda u_0\|^2_{L^2(\mathbb{T}^2)} \leq e^{e^{C\log\varepsilon^{-1}/\log\log\varepsilon^{-1}}} \|\Pi_\lambda u_0\|^2_{L^2(\Omega_\varepsilon)}.$$

Proof. Let $u_0 = \Pi_\lambda u_0 = \sum_{k=1}^{n} c_k \varphi_{\lambda,k}$ where $n \leq \operatorname{rank}\Pi_\lambda \leq e^{C\log\lambda/\log\log\lambda}$. We first fix y and apply Nazarov-Turán lemma to get

$$\|u_0\|^2_{L^2(\mathbb{T}^1_x\times\{y\})} \leq (C\varepsilon^{-1})^{2n} \|u_0\|^2_{L^2([-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]\times\{y\})}.$$

By integrating it on y, we get

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq (C\varepsilon^{-1})^{2n} \|u_0\|^2_{L^2([-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]\times\mathbb{T}^1_y)}.$$

Similarly, we have

$$\|u_0\|^2_{L^2([-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]\times\mathbb{T}^1_y)} \leq (C\varepsilon^{-1})^{2n} \|u_0\|^2_{L^2([-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]\times[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}])}.$$

In conclusion, we get

$$\begin{aligned}
\|u_0\|^2_{L^2(\mathbb{T}^2)} &\leq (C\varepsilon^{-1})^{4n} \|u_0\|^2_{L^2([-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]^2)} \\
&\leq (C\varepsilon^{-1})^{4n} \|u_0\|^2_{L^2(\Omega_\varepsilon)} \\
&\leq e^{e^{C\log\varepsilon^{-17}/\log\log\varepsilon^{-17} + \log\log(C\varepsilon^{-1})}} \|u_0\|^2_{L^2(\Omega_\varepsilon)} \\
&\leq e^{e^{C\log\varepsilon^{-1}/\log\log\varepsilon^{-1}}} \|u_0\|^2_{L^2(\Omega_\varepsilon)}.
\end{aligned}$$

□

Now for any eigenvalues $\lambda \neq \mu \in \operatorname{Spec}(\sqrt{-\Delta})$,

$$\int_0^{\frac{1}{2\pi}} \langle e^{it\Delta}\Pi_\lambda u_0, e^{it\Delta}\Pi_\mu u_0\rangle_{L^2(\Omega_\varepsilon)} dt = 0.$$

This orthogonality gives

$$\|\Pi_{\leq h_0^{-1}} u_0\|^2_{L^2(\mathbb{T}^2)} \leq e^{e^{C\log\varepsilon^{-1}/\log\log\varepsilon^{-1}}} \int_0^{\frac{1}{2\pi}} \|e^{it\Delta}\Pi_{\leq h_0^{-1}} u_0\|^2_{L^2(\Omega_\varepsilon)} dt.$$

Combine this with Theorem 3 we get

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq e^{e^{C\log\varepsilon^{-1}/\log\log\varepsilon^{-1}}} \int_0^{\frac{1}{2\pi}} \|e^{it\Delta} u_0\|^2_{L^2(\Omega_\varepsilon)} dt.$$

This ends the proof of Theorem 2.

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