SPECTRAL ASYMPTOTICS FOR KINETIC BROWNIAN MOTION ON RIEMANNIAN MANIFOLDS

QIUYU REN AND ZHONGKAI TAO

Abstract. We prove the convergence of the spectrum of the generator of the kinetic Brownian motion to the spectrum of the base Laplacian for closed Riemannian manifolds. This generalizes recent work of Kolb–Weich–Wolf [KWW22] on constant curvature surfaces and of Ren–Tao [RT22] on locally symmetric spaces. As an application, we prove a conjecture of Baudoin–Tardif [BT18] on the optimal convergence rate to the equilibrium.

1. Introduction

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$ and $SM = \{(x, v) \in TM : |v|_g = 1\}$ be the unit tangent bundle. For any $p \in M$, the fiber S_pM is a standard sphere, and we denote by Δ_{S_pM} the (positive) spherical Laplacian on S_pM. We then define the vertical Laplacian Δ_V on SM by $(\Delta_V f)|_{S_pM} := \Delta_{S_pM}(f|_{S_pM})$ for every $p \in M$. Let X be the generator of the geodesic flow on SM. From these two operators we construct the generator of the *kinetic Brownian motion* on SM (see below for motivation) as

$$P_\gamma := -\gamma X + c_n \gamma^2 \Delta_V, \qquad c_n = \frac{1}{n(n-1)}, \quad \gamma > 0. \tag{1.1}$$

We are interested in the spectrum of the operator $P_\gamma : D(P_\gamma) = \{u \in L^2(SM) : P_\gamma u \in L^2\} \to L^2(SM)$, which we denote by $\sigma(P_\gamma)$. The operator P_γ is hypoelliptic, hence it has discrete spectrum with finite multiplicities (see e.g. [KWW22, Proposition 2.1]). The main result of this paper is

Theorem 1. *Let Δ_M be the (positive) Laplace–Beltrami operator on M. Then,*

$$\sigma(P_\gamma) \cap U \to \sigma(\Delta_M) \cap U, \quad \gamma \to \infty \tag{1.2}$$

uniformly on any bounded open set $U \Subset \mathbb{C}$, with the agreement of multiplicities. Moreover, for any $s \in \mathbb{R}$,

$$\|(P_\gamma - \lambda)^{-1} - (\Delta_M - \lambda)^{-1}\|_{H^s \to H^{s+1/4}} \to 0, \quad \gamma \to \infty \tag{1.3}$$

uniformly for $\lambda \in U \Subset \mathbb{C} \setminus \sigma(\Delta_M)$.

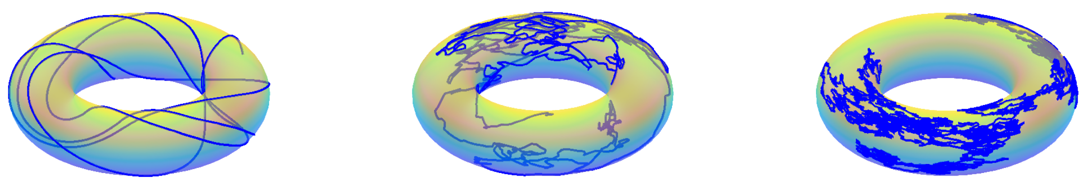

FIGURE 1. Simulation of kinetic Brownian motion for $\gamma = 10^{-2}$, 10, 10^4 (click for movies) on $S(\mathbb{R}^2/2\pi\mathbb{Z}^2)$ projected to the base $\mathbb{R}^2/2\pi\mathbb{Z}^2$.

The convergence in (1.3) is in fact quantitative, and we show in (2.18) that the left hand side of (1.3) is $\mathcal{O}(\gamma^{-1/10})$. We have not, at this stage, attempted to find the optimal rate of convergence or optimal regularity improvement. This allows us to keep the proof relatively short.

As an application, we prove a conjecture of Baudoin–Tardif [BT18] who proposed an optimal convergence rate to the equilibrium:

Theorem 2. *Suppose in addition to Theorem 1 that M is connected and the spectrum of Δ_M is given by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, then for any $0 < \beta < \lambda_1$, there is $\gamma_0 > 0$ such that for any $\gamma > \gamma_0$, there exists $C_\gamma > 0$ such that*

$$\left\| e^{-tP_\gamma} u - \frac{1}{\mathrm{Vol}_g(SM)} \int_{SM} u \,\mathrm{dVol}_g \right\|_{L^2} \leq C_\gamma e^{-\beta t} \|u\|_{L^2}, \quad t > 0.$$

A more precise asymptotic expansion is given in Theorem 3.

The operator P_γ is the generator of a stochastic process called kinetic Brownian motion. It is a form of a Langevin equation where Brownian motion occurs only in the fiber variables. It was studied by several authors, including Franchi–Le Jan [FL07], Grothaus–Stilgenbauer [GS13], Angst–Bailleul–Tardif [ABT15] and Li [Li16]. In particular, [ABT15] and [Li16] proved that the kinetic Brownian motion interpolates between the geodesic flow and Brownian motion on the base manifold. Figure 1 is a simulation of the kinetic Brownian motion on the flat torus projected to the base. One can see that when γ is small, the flow behaves like the geodesic flow; but as γ grows, it gets closer to the Brownian motion on the base manifold.

Bismut [Bis05] introduced another, more functorial, family of hypoelliptic operators interpolating between the generator of the geodesic flow and the Laplacian on the base. His hypoelliptic Laplacian, L_γ, is defined for forms on the cotangent bundle T^*M. Bismut–Lebeau [BL08] proved that L_γ converges to Δ_M in a certain strong sense for arbitrary closed manifolds. Bismut [Bis11] also studied the limit $\gamma \to 0$ for a related hypoelliptic Laplacian on locally symmetric spaces and obtained formulas for orbit integrals. One motivation came from Fried's conjecture [Fri86; Fri95], which relates special values of dynamical zeta functions to the Reidemeister torsion – see Shen [She21] for a recent survey. The proof of Fried's conjecture in the locally symmetric case by Shen [She17] used methods of [Bis11].

Our Theorem 1 should consequently be compared with [BL08, Theorem 17.21.5]. We use a Grushin problem similar to that in [BL08, §17.2] but we do not need more sophisticated aspects of semiclassical analysis used there. This is despite the fact that P_γ is less functorial and hence does not enjoy special properties as in [BL08, Chapter 16] related to the harmonic oscillator structure in the fibers which are crucial in [BL08, Chapter 17].

One possible reason for the difficulties in proving Fried's conjecture using hypoelliptic Laplacian is that its properties as $\gamma \to 0$ for general negatively curved manifolds are not clear. On the other hand, if we think of P_γ as an analogue of hypoelliptic Laplacian on SM, Drouot [Dro17] proved that, uniformly on compact sets,

$$\sigma(X + \gamma\Delta_V) \to \mathrm{Res}(X), \quad \gamma \to 0$$

for negatively curved manifolds, where $\mathrm{Res}(X)$ is the set of Pollicott–Ruelle resonances. They are defined as the spectrum of X on certain anisotropic Sobolev spaces and (at least in principle) appear in expansions of correlations.

Concerning the limiting properties of P_γ as $\gamma \to \infty$, the first breakthrough was achieved by Kolb–Weich–Wolf [KWW19; KWW22], who proved a weaker version of convergence for constant curvature surfaces. This approach was generalized in Ren–Tao [RT22] to the case of locally symmetric spaces. We should stress that [RT22] provides a strong convergence as stated in (1.2), while [KWW22] only proved convergence in each Casimir eigenspace. The new ingredient in [RT22] is a careful study of the localization of eigenfunctions in the Fourier space. In this paper, we implement the strategy to general Riemannian manifolds and prove similar localization results. This leads to the proof of Theorem 1.

The connection between kinetic Brownian motion and Fried's conjecture is still mysterious. We propose it as an open question.

Question. *How is P_γ related to the Reidemeister torsion?*

A positive answer to this question would give us a new way to understand Fried's conjecture.

The paper is organized as follows. We prove Theorem 1 in §2. This is done by introducing a finite rank semi-positive operator Q_A such that $P_\gamma - \lambda + Q_A$ is invertible. In this way, the problem is transformed to the study of the spectrum of the finite rank operator $(P_\gamma - \lambda + Q_A)^{-1} Q_A$, and can be solved using a Grushin problem as in [RT22, Section 3.4]. The invertibility of $P_\gamma - \lambda + Q_A$ is proved in Lemma 2.4 in a quantitative form, and is the key technical result of the paper. Roughly speaking, we study the decomposition into spherical harmonics in the fiber variables and prove that eigenfunctions of P_γ are localized to 0-th spherical harmonics. Then we use projection to first order spherical harmonics to conclude eigenfunctions of P_γ are also localized in the horizontal direction, hence completely localized in the Fourier space. Thus the potential Q_A gives the invertibility of $P_\gamma - \lambda + Q_A$. The improvement of regularity is a corollary of a uniform hypoelliptic estimate (see Proposition 2.5) following [Rad69; Koh73; Hör07]. As an application, we prove Theorem 2 in §3. This is done by writing e^{-tP_γ} as the inverse Mellin transform of the resolvent, and then deforming the contour. Our method only provides information on compact sets (or on vertical strips). In the faraway region, we use a result in Eckmann–Hairer [EH03], which is based on earlier work of Hérau–Nier [HN04], to obtain a spectral free region near infinity.

Acknowledgement. We would like to thank Alexis Drouot for sharing with us his notes on kinetic Brownian motion which suggested the Grushin problem used here and in [RT22], and for helpful discussions. ZT would also like to thank Maciej Zworski for many helpful discussions and for his encouragement. ZT was partially supported by National Science Foundation under the grant DMS-1901462 and by Simons Targeted Grant Award No. 896630.

2. Convergence of spectrum

In this section, we prove Theorem 1. We will first recall some important properties of Δ_V and X studied in Ren–Tao [RT22]. Then we prove the key invertibility lemmas: Lemma 2.4 and Lemma 2.7, and use them to conclude Theorem 1.

2.1. Decomposition of $H^s(SM)$.

The result in this section is basically the same as [RT22, Section 3.2], except we use more general H^s spaces defined as $H^s(SM) = \{u \in \mathcal{D}'(SM) : (1+\Delta)^{s/2} u \in L^2\}$. Here we have three different (positive) Laplacians: the total Laplacian Δ, the horizonal Laplacian Δ_H and the vertical Laplacian Δ_V.

- The total Laplacian Δ is the Laplace–Beltrami operator associated to the Sasaki metric on SM.
- The vertical Laplacian Δ_V is defined as $(\Delta_V f)|_{S_pM} := \Delta_{S_pM}(f|_{S_pM})$ for every $p \in M$.
- The horizontal Laplacian Δ_H is defined as $\Delta_H = \Delta - \Delta_V$.

We recall from [BB82, Theorem 1.5] that $\Delta|_{L^2(M)} = \Delta_H|_{L^2(M)} = \Delta_M$ and $\Delta, \Delta_H, \Delta_V$ commute with each other.

Let $s \in \mathbb{R}$. The total Laplacian Δ is a self-adjoint operator on $H^s(SM)$ with discrete spectrum. Since Δ_V commutes with the total Laplacian Δ on SM, we can do spectral decomposition on each eigenspace of Δ. Thus we get the following orthogonal decomposition:

$$H^s(SM) = \bigoplus_k V_k^s \tag{2.1}$$

where $V_k^s = \{u \in H^s(SM) : \Delta_V u = k(k+n-2)u\}$ is the k-th eigenspace of Δ_V.

Let $\Pi_k^s : H^s(SM) \to V_k^s$ denote the orthogonal projection with the abbreviated notation $\Pi = \Pi_0^s : H^s(SM) \to V_0^s$ and $\Pi^\perp = \mathrm{id} - \Pi : H^s(SM) \to V_{>0}^s$. The difficulty is that the geodesic vector field X does not commute with Δ_V, but it satisfies the following properties from [RT22, Lemma 3.2]. We include the proofs for completeness.

Lemma 2.1.
- *X is anti-self-adjoint with respect to the natural $L^2(SM)$ norm defined via the metric;*
- *X sends V_k into $V_{k+1} \oplus V_{k-1}$ with the convention that $V_{-1} = 0$;*
- *$n\Pi X^2 \Pi = -\Delta_M$.*

Proof.
- The fact X is anti-self-adjoint on L^2 follows from the fact that $\exp(tX)$ is volume-preserving. This is essentially Liouville's theorem that geodesic flow preserves the volume.
- This is done by a computation in local coordinates. We choose normal coordinates $\{x^i\}$ at $p \in M$, so that $g_{ij}(p) = \delta_{ij}$ and $\partial_k g_{ij}(p) = 0$. Then at p, $X = \sum v^j \partial_{x^j}$ where v^j's are the induced coordinates on TM. The claim follows from the fact that multiplying spherical harmonics of degree k by linear functionals gives a combination of spherical harmonics in degree $k-1$ and $k+1$.
- Again we compute in normal coordinates $\{x^i\}$ and the induced coordinates $\{v^i\}$ near the fiber over $p \in M$. The geodesic flow is given by

  $$X = \sum v^i \partial_{x^i} - \sum v^i v^j \Gamma_{ij}^k \partial_{v^k} = \sum (v^i \partial_{x^i} + \mathcal{O}(x)\partial_{v^i}),$$

  and $X^2 = \sum v^i v^j \partial_{x^i}\partial_{x^j} + \sum \mathcal{O}(1)\partial_{v^k} + \mathcal{O}(x)$. Since $\partial_{v^k}\Pi = 0$, it follows that at p, $\Pi X^2 \Pi = \sum \Pi(v^i v^j)\partial_{x^i}\partial_{x^j}$. Here $\Pi(v^i v^j)$ is the L^2 orthogonal projection of $v^i v^j$ to constants. If $i \neq j$, then the projection is zero. If $i = j$, then the projection is given by

  $$\frac{1}{\mathrm{Vol}(S^{n-1})}\int_{S^{n-1}} (v^j)^2 d\sigma(v) = \frac{1}{n\,\mathrm{Vol}(S^{n-1})}\int_{S^{n-1}} \sum_{i=1}^n (v^j)^2 d\sigma(v) = \frac{1}{n}.$$

Thus $\Pi X^2\Pi = \frac{1}{n}\sum_{i=1}^n \partial_{x^i}^2 = -\frac{1}{n}\Delta_M.$ $\square$

2.2. **Invertibility lemmas.** In this section, we prove the crucial invertibility lemmas: Lemma 2.4 and Lemma 2.7. In order to keep track of the dependence on the parameters, we use $A \lesssim_s B$ to mean $A \le C(s)B$ with the implicit constant $C(s)$ depending on s. Similarly, $A \ll_s B$ means we choose $A \le c(s)B$ for some sufficiently small $c(s) > 0$ depending on s. Since everything will depend on the dimension n and regularity s, we will often omit s, n in the dependence to keep the notations simple.

We start by recalling the hypoelliptic estimate essentially from [Smi20, Theorem 6.3].

Lemma 2.2. *For any $\gamma > 0, s \in \mathbb{R}, N \in \mathbb{N}$, $u \in C^\infty(SM)$ we have*

$$\|Xu\|_{H^s} + \|\Delta_V u\|_{H^s} + \|u\|_{H^{s+2/3}} \le C_{\gamma,s,N}(\|P_\gamma u\|_{H^s} + \|u\|_{H^{-N}}). \tag{2.2}$$

Since Δ commutes with Δ_V, it is easy to see $C^\infty(SM)$ is dense in $D^s(P_\gamma) = \{u \in H^s(SM) : P_\gamma u \in H^s(SM)\}$. Thus (2.2) works for any $u \in D^s(P_\gamma)$. In particular, $(P_\gamma - \lambda)u \in C^\infty(SM)$ implies $u \in C^\infty(SM)$. A basic accretive estimate shows $P_\gamma : D^s(P_\gamma) \to H^s(SM)$ is a Fredholm operator with index 0.

Lemma 2.3. *For $\operatorname{Re}\lambda < 0$, $P_\gamma - \lambda$ is invertible on L^2. For $\operatorname{Re}\lambda < 0$ sufficiently negative (depending on s and γ), $P_\gamma - \lambda$ is invertible on H^s.*

Proof. We will only prove the claim for L^2. The proof for H^s is similar. First we recall $\operatorname{Re}(Xu, u)_{L^2} = 0$ since X is anti-self-adjoint. Thus

$$\operatorname{Re}((P_\gamma - \lambda)u, u) = c_n\gamma^2(\Delta_V u, u) - \operatorname{Re}\lambda\|u\|^2 \ge -\operatorname{Re}\lambda\|u\|^2.$$

For $\operatorname{Re}\lambda < 0$, this shows $P_\gamma - \lambda : D(P_\gamma) \to L^2$ is injective and the image is closed. We claim it is also surjective. If there is $v \in L^2(SM)$ such that $((P_\gamma - \lambda)u, v) = 0$, then distributionally

$$(P_\gamma^* - \bar{\lambda})v = 0.$$

By hypoellipticity, $v \in C^\infty(SM)$. However,

$$0 = \operatorname{Re}((P_\gamma^* - \bar{\lambda})v, v) = \gamma^2(\Delta_V v, v) - \operatorname{Re}\lambda\|v\|^2 \ge -\operatorname{Re}\lambda\|v\|^2$$

implies $v = 0$. So P_γ must be surjective and thus invertible. $\square$

When $\operatorname{Re}\lambda \ge 0$, it is possible that $P_\gamma - \lambda$ is not invertible. The following lemma essentially says that any such eigenfunction must be localized to finite frequency. In order to implement the heuristics, for $A > 0$ we introduce $Q_A = A^2\Pi\mathbb{1}_{(\Delta_M \le A^2)}\Pi : H^s(SM) \to H^s(SM)$. This is a finite rank smoothing operator localized to finite frequencies. Here $\mathbb{1}_{(\lambda \le A^2)}$ is the characteristic function of the set $\{\lambda \leq A^2\}$ and $\mathbb{1}_{(\Delta_M \leq A^2)}$ is the spectral projection to eigenspaces of Δ_M with eigenvalue $\leq A^2$, defined using functional calculus of self-adjoint operators.

Lemma 2.4. *For any $C_0 > 0$, $s \in \mathbb{R}$, there exists $C_1 = C_1(C_0, n) > 0$ such that for any $\gamma > C_1$, $A > C_1$ and $|\lambda| \leq C_0$, the operator*

$$P_\gamma - \lambda + Q_A : D^s(P_\gamma) = \{u \in H^s(SM) : P_\gamma u \in H^s\} \to H^s(SM)$$

is invertible. For $\gamma > A \gg_{C_0,n,s} 1$, the inverse has the bound

$$\|(P_\gamma - \lambda + Q_A)^{-1}\|_{H^s \to H^s} \lesssim_{C_0,n,s} A^{-1}. \tag{2.3}$$

Proof. Since $P_\gamma - \lambda + Q_A$ is hypoelliptic and Fredholm of index 0, we only need to prove it has no kernel. Suppose by contradiction that for some $u \in H^s(SM) \setminus \{0\}$,

$$(P_\gamma - \lambda + Q_A)u = 0. \tag{2.4}$$

Then $u \in C^\infty$ by hypoellipticity. Suppose $\|u\|_{L^2} = 1$ and denote $u_k = \Pi_k u$. Pairing with u gives

$$c_n \gamma^2 (\Delta_V u, u) - \gamma \operatorname{Re}(Xu, u) - \operatorname{Re} \lambda \|u\|_{L^2}^2 + (Q_A u_0, u_0) = 0. \tag{2.5}$$

Since $\operatorname{Re}(Xu, u)_{L^2} = 0$, we get

$$\|\Pi^\perp u\|_{L^2} \lesssim_{C_0} \gamma^{-1}.$$

Similaly pairing (2.4) with $(\Delta_H + 1)u$ gives

$$c_n \gamma^2 (\Delta_V (\Delta_H + 1)u, u) - \gamma \operatorname{Re}(Xu, (\Delta_H + 1)u) - \operatorname{Re} \lambda \|u\|_{H^1}^2 + (Q_A u_0, (\Delta_H + 1)u_0) = 0.$$

Moreover,

$$2 \operatorname{Re}(Xu, (\Delta_H + 1)u) = ([\Delta_H, X](\Pi^\perp u + u_0), \Pi^\perp u + u_0) \lesssim \|\Pi^\perp u\|_{H^1}^2 + \|\Pi^\perp u\|_{H^1} \|u_0\|_{H^1}.$$

Note $\|\Pi^\perp u\|_{H^1} \|u_0\|_{H^1} \leq \epsilon_n \gamma \|\Pi^\perp u\|_{H^1}^2 + \gamma^{-1} \epsilon_n^{-1} \|u_0\|_{H^1}^2$, we conclude

$$\|\Pi^\perp u\|_{H^1} \lesssim_{C_0} \gamma^{-1} \|u_0\|_{H^1}.$$

We come back to (2.4). Projecting it to V_1 gives

$$\frac{1}{n} \gamma^2 u_1 - \gamma \Pi_1 X(u_0 + u_2) = \lambda u_1.$$

Recall $\|u_1\|_{L^2} \lesssim_{C_0} \gamma^{-1}$, and

$$\|Xu_0\|_{L^2}^2 = -(\Pi X^2 \Pi u_0, u_0) = \frac{1}{n} (\Delta_M u_0, u_0) \gtrsim \|u_0\|_{H^1}^2 - \|u_0\|_{L^2}^2,$$
$$\|Xu_2\|_{L^2}^2 \lesssim \|u_2\|_{H^1}^2 \lesssim_{C_0} \gamma^{-1} \|u_0\|_{H^1}^2.$$

We conclude $\|u_0\|_{H^1} \lesssim_{C_0} 1$. The key observation is that the constant in this estimate is independent of A. On the other hand, (2.5) also implies $\|A\mathbb{1}_{(\Delta\leq A^2)}u_0\|_{L^2} \lesssim_{C_0} 1$. Taking $A \gg_{C_0} 1$ gives a contradiction. This shows the invertibility of $P_\gamma - \lambda + Q_A$.

In order to get the bound for the inverse, let $f \in C^\infty$ and $u \in C^\infty$ such that

$$(P_\gamma - \lambda + Q_A)u = f. \tag{2.6}$$

Pairing with u in H^s gives

$$c_n\gamma^2(\Delta_V u, u)_{H^s} - \gamma \operatorname{Re}(Xu, u)_{H^s} - \operatorname{Re}\lambda\|u\|_{H^s}^2 + (Q_A u, u)_{H^s} = (f, u)_{H^s}. \tag{2.7}$$

Since

$$\begin{aligned}\operatorname{Re}(Xu, u)_{H^s} = \operatorname{Re}(Xu, (1+\Delta)^s u)_{L^2} &= \frac{1}{2}([(1+\Delta)^s, X]u, u)_{L^2} \\ &\lesssim \|\Pi^\perp u\|_{H^s}^2 + \|\Pi^\perp u\|_{H^s}\|u_0\|_{H^s},\end{aligned}$$

we conclude as before

$$\|\Pi^\perp u\|_{H^s} \lesssim_{C_0} \gamma^{-1}(\|u_0\|_{H^s} + \|f\|_{H^s}), \quad \|A\mathbb{1}_{(\Delta\leq A^2)}u_0\|_{H^s} \lesssim_{C_0} \|u_0\|_{H^s} + \|f\|_{H^s}.$$

Now we look at the Π_1 component of (2.6), i.e.

$$\frac{1}{n}\gamma^2 u_1 - \gamma\Pi_1 X(u_0 + u_2) = \lambda u_1 + f_1.$$

We conclude

$$\begin{aligned}\|u_0\|_{H^s} &\lesssim \|Xu_0\|_{H^{s-1}} + \|u_0\|_{H^{s-1}} \\ &\lesssim_{C_0} \gamma\|u_1\|_{H^{s-1}} + \gamma^{-1}\|f\|_{H^{s-1}} + \|u_2\|_{H^s} + \|u_0\|_{H^{s-1}} \\ &\lesssim_{C_0} \|u_0\|_{H^{s-1}} + \|f\|_{H^{s-1}} + \gamma^{-1}\|f\|_{H^s}.\end{aligned} \tag{2.8}$$

Now we divide into two cases.

- If $f = \mathbb{1}_{(\Delta>\gamma^2)}f$ is in high frequency, then (2.8) implies that

$$\begin{aligned}\|u_0\|_{H^s} &\lesssim_{C_0} \|\mathbb{1}_{(\Delta\leq A^2)}u_0\|_{H^{s-1}} + \|\mathbb{1}_{(\Delta>A^2)}u_0\|_{H^{s-1}} + \gamma^{-1}\|f\|_{H^s} \\ &\lesssim_{C_0} A^{-1}(\|u_0\|_{H^{s-1}} + \|f\|_{H^{s-1}}) + A^{-1}\|u_0\|_{H^s} + \gamma^{-1}\|f\|_{H^s} \\ &\lesssim A^{-1}\|u_0\|_{H^s} + A^{-1}\|f\|_{H^s}.\end{aligned}$$

- If $f = \mathbb{1}_{(\Delta\leq\gamma^2)}f$ is in low frequency, then (2.8) with s replaced by $s+1$ gives

$$\begin{aligned}\|u_0\|_{H^{s+1}} &\lesssim_{C_0} \|u_0\|_{H^s} + \|f\|_{H^s} + \gamma^{-1}\|f\|_{H^{s+1}} \\ &\lesssim \|u_0\|_{H^s} + \|f\|_{H^s}.\end{aligned}$$

  On the other hand, we have

$$\|u_0\|_{H^{s+1}} \geq \|\mathbb{1}_{(\Delta>A^2)}u_0\|_{H^{s+1}} \geq A\|\mathbb{1}_{(\Delta>A^2)}u_0\|_{H^s},$$

thus

$$\begin{aligned}\|u_0\|_{H^s} &\leq \|\mathbb{1}_{(\Delta\leq A^2)}u_0\|_{H^s} + \|\mathbb{1}_{(\Delta>A^2)}u_0\|_{H^s}\\ &\lesssim_{C_0} A^{-1}(\|u_0\|_{H^s} + \|f\|_{H^s}).\end{aligned}$$

In both cases we have $\|u_0\|_{H^s} \lesssim_{C_0} A^{-1}\|f\|_{H^s}$ and we conclude

$$\|u\|_{H^s} \lesssim_{C_0} \|u_0\|_{H^s} + \gamma^{-1}\|f\|_{H^s} \lesssim_{C_0} A^{-1}\|f\|_{H^s}. \qquad \square$$

In order to get the improvement of regularity in (1.3), we prove a uniform hypoelliptic estimate following [Rad69; Koh73] and [Hör07, Theorem 22.2.1].

Proposition 2.5. *For $A, B > 1$, $\gamma > A + B^2$, $y \in \mathbb{R}$, there exists $C = C(n, s)$ independent of A, B, γ, y such that*

$$\|u\|_{H^{s+1/4}} \leq CB^{-1}\|(P_\gamma + Q_A)u\|_{H^s} + CB\|u\|_{H^s}. \tag{2.9}$$

$$\|u\|_{H^{s+1/8}} \leq CB^{-1}\|(P_\gamma - iy)u\|_{H^s} + CB\|u\|_{H^s}. \tag{2.10}$$

Proof. It suffices to compute locally. We will only give the proof of (2.9), but (2.10) is proved in the same way using the fact that for a local basis X_i of vertical vector fields, the vector fields

$$X_i, [X_i, X], [X_i, [X_i, X]], i = 1, 2, \cdots, n-1$$

generate all directions.

In order to get (2.9), since $X, X_i, [X_i, X], i = 1, 2, \cdots, n-1$ generate all directions, it suffices to bound $\|X_i u\|_{H^s}$, $\|Xu\|_{H^{s-1/2}}$ and $\|[X_i, X]u\|_{H^{s-3/4}}$ by the right hand side of (2.9). First, we have

$$\begin{aligned}(Q_A u, u)_{H^s} + \gamma^2(\Delta_V u, u)_{H^s} &\lesssim \operatorname{Re}((P_\gamma + Q_A)u, u)_{H^s} + C\|u\|^2_{H^s}\\ &\lesssim B^{-2}\|(P_\gamma + Q_A)u\|^2_{H^s} + B^2\|u\|^2_{H^s}.\end{aligned} \tag{2.11}$$

We will abbreviate pseudodifferential operators of order k by Ψ^k to simplify the notation. For $\|Xu\|_{H^{s-1/2}}$, we have

$$\|Xu\|^2_{H^{s-1/2}} = (Xu, \Psi^0 u)_{H^s} = \gamma^{-1}((c_n\gamma^2\Delta_V + Q_A - (P_\gamma + Q_A))u, \Psi^0 u)_{H^s}.$$

The first term is estimated as

$$\begin{aligned}\gamma^{-1}(\gamma^2\Delta_V u, \Psi^0 u)_{H^s} &= \gamma^{-1}(\gamma\nabla^V u, \Psi^0\gamma\nabla^V u)_{H^s} + (\gamma\nabla^V u, \Psi^0 u)_{H^s}\\ &\lesssim B^{-2}\|(P_\gamma + Q_A)u\|^2_{H^s} + B^2\|u\|^2_{H^s}.\end{aligned}$$

The second term is estimated as

$$\gamma^{-1}(Q_A u, \Psi^0 u)_{H^s} \lesssim \gamma^{-1}\|Q_A u\|_{H^s}\|u\|_{H^s} \lesssim B^{-2}\|(P_\gamma + Q_A)u\|^2_{H^s} + B^2\|u\|^2_{H^s}.$$

The third term is estimated as

$$\gamma^{-1}((P_\gamma+Q_A)u,\Psi^0 u)_{H^s}\lesssim \gamma^{-1}\|(P_\gamma+Q_A)u\|_{H^s}\|u\|_{H^s}\leq \gamma^{-1}\|(P_\gamma+Q_A)u\|_{H^s}^2+\gamma^{-1}\|u\|_{H^s}^2.$$

Thus we conclude

$$\|Xu\|_{H^{s-1/2}}\lesssim B^{-1}\|(P_\gamma+Q_A)u\|_{H^s}+B\|u\|_{H^s}. \tag{2.12}$$

For $\|[X_i,X]u\|_{H^{s-3/4}}$, we have

$$\begin{aligned}
&\|[X_i,X]u\|_{H^{s-3/4}}^2\\
&=([X_i,X]u,\Psi^{-1/2}u)_{H^s}\\
&=(X_iXu,\Psi^{-1/2}u)_{H^s}-(XX_iu,\Psi^{-1/2}u)_{H^s}\\
&=-(Xu,\Psi^{-1/2}X_iu)_{H^s}+(Xu,\Psi^{-1/2}u)_{H^s}+(X_iu,\Psi^{-1/2}Xu)_{H^s}-(X_iu,\Psi^{-1/2}u)_{H^s}\\
&=\operatorname{Re}(Xu,\Psi^{-1/2}X_iu)_{H^s}+\operatorname{Re}(Xu,\Psi^{-1/2}u)_{H^s}+\operatorname{Re}(X_iu,\Psi^{-1/2}u)_{H^s}.
\end{aligned}$$

The last term is estimated as

$$\operatorname{Re}(X_iu,\Psi^{-1/2}u)_{H^s}\lesssim \gamma^{-1}\|(P_\gamma+Q_A)u\|_{H^s}^2+\gamma^{-1}\|u\|_{H^s}^2,$$

The second term is estimated as

$$\begin{aligned}
&\operatorname{Re}(Xu,\Psi^{-1/2}u)_{H^s}\\
&=\gamma^{-1}\operatorname{Re}((c_n\gamma^2\Delta_V+Q_A-(P_\gamma+Q_A))u,\Psi^{-1/2}u)_{H^s}\\
&\lesssim \gamma^{-1}\|\gamma\nabla^V u\|_{H^s}\|\gamma\nabla^V\Psi^{-1/2}u\|_{H^s}+\gamma^{-1}(Q_Au,\Psi^{-1/2}u)_{H^s}+\gamma^{-1}\|(P_\gamma+Q_A)u\|_{H^s}^2+\gamma^{-1}\|u\|_{H^s}^2\\
&\lesssim B^{-2}\|(P_\gamma+Q_A)u\|_{H^s}^2+B^2\|u\|_{H^s}^2.
\end{aligned}$$

The first term is estimated as

$$\begin{aligned}
&\operatorname{Re}(Xu,\Psi^{-1/2}X_iu)_{H^s}=\gamma^{-1}\operatorname{Re}((c_n\gamma^2\Delta_V+Q_A-(P_\gamma+Q_A))u,\Psi^{-1/2}X_iu)_{H^s}\\
&\lesssim \gamma^{-1}(\gamma^2\Delta_Vu,u)_{H^s}^{1/2}(\gamma^2\Delta_V\Psi^{-1/2}X_iu,\Psi^{-1/2}X_iu)_{H^s}^{1/2}\\
&+\gamma^{-1}(Q_Au,\Psi^{-1/2}X_iu)_{H^s}+\gamma^{-1}\|(P_\gamma+Q_A)u\|_{H^s}\|X_iu\|_{H^s}\\
&\lesssim B^{-2}\|(P_\gamma+Q_A)u\|_{H^s}^2+B^2\|u\|_{H^s}^2
\end{aligned}$$

where we used $(\Delta_V u, v)_{H^s} \le (\Delta_V u, u)_{H^s}^{1/2} (\Delta_V v, v)_{H^s}^{1/2}$ and

$$
\begin{aligned}
&(\gamma^2 \Delta_V \Psi^{-1/2} X_i u, \Psi^{-1/2} X_i u)_{H^s} \\
&\lesssim \operatorname{Re}(P_\gamma \Psi^{-1/2} X_i u, \Psi^{-1/2} X_i u)_{H^s} + C\|\Psi^{-1/2} X_i u\|_{H^s}^2 \\
&\lesssim \operatorname{Re}(\Psi^{-1/2} X_i P_\gamma u, \Psi^{-1/2} X_i u)_{H^s} + \operatorname{Re}((\gamma^2 \Psi^{1/2} \nabla^V + \gamma \Psi^{1/2}) u, \Psi^{-1/2} X_i u)_{H^s} + C\|\Psi^{-1/2} X_i u\|_{H^s}^2 \\
&\lesssim B^{-2} \|P_\gamma u\|_{H^s}^2 + B^2 \|u\|_{H^s}^2 \\
&\lesssim B^{-2} \|(P_\gamma + Q_A) u\|_{H^s}^2 + B^2 \|u\|_{H^s}^2 + \|Q_A u\|_{H^s}^2.
\end{aligned}
$$

Thus we conclude

$$
\|[X_i, X] u\|_{H^{s-3/4}} \lesssim B^{-1} \|(P_\gamma + Q_A) u\|_{H^s} + B\|u\|_{H^s}. \tag{2.13}
$$

Combining (2.11), (2.12), (2.13), we conclude (2.9). □

As a corollary, we can improve the regularity in (2.3).

Corollary 2.6. *In Lemma 2.4, we have*

$$
\|(P_\gamma - \lambda + Q_A)^{-1}\|_{H^s \to H^{s+1/4}} \lesssim_{C_0, n, s} A^{-1/2}. \tag{2.14}
$$

Proof. We take $B = A^{1/2}$ in (2.9), then

$$
\begin{aligned}
\|u\|_{H^{s+1/4}} &\lesssim_{C_0} A^{-1/2} \|(P_\gamma - \lambda + Q_A) u\|_{H^s} + A^{1/2} \|u\|_{H^s} \\
&\lesssim_{C_0} A^{-1/2} \|(P_\gamma - \lambda + Q_A) u\|_{H^s}.
\end{aligned}
$$
□

We will also need the following invertibility lemma. We use the semiclassical notation $h = \gamma^{-1}$ and $\tilde{P}_h = c_n \Delta_V - hX$. Note that $P_\gamma = \gamma^2 \tilde{P}_h$.

Lemma 2.7. *Let $s \in \mathbb{R}$, $|\lambda| \le C_0$, there exists $h_0 = h_0(C_0, n, s) > 0$ such that for $0 < h < h_0$, the operator*

$$
\Pi^\perp (\tilde{P}_h - h^2 \lambda) \Pi^\perp : \{u \in V_{>0}^s : \Pi^\perp \tilde{P}_h u \in H^s\} \to V_{>0}^s
$$

is invertible. The inverse has norm

$$
\|(\Pi^\perp (\tilde{P}_h - h^2 \lambda) \Pi^\perp)^{-1}\|_{H^s \to H^s} \lesssim_{C_0, n, s} 1.
$$

Proof. For $u \in V_{>0}^s$, $h \ll_{C_0} 1$,

$$
\operatorname{Re}((\tilde{P}_h - h^2 \lambda) u, u)_{H^s} = c_n (\Delta_V u, u)_{H^s} - h \operatorname{Re}(Xu, u)_{H^s} - h^2 \operatorname{Re} \lambda \|u\|_{H^s}^2 \gtrsim_{C_0} \|u\|_{H^s}^2.
$$

So $\Pi^\perp (\tilde{P}_h - h^2 \lambda) \Pi^\perp$ is injective and has closed image. Suppose it is not surjective, then there exists a nonzero $v \in V_{>0}^s$ such that

$$
(\Pi^\perp (\tilde{P}_h - h^2 \lambda) u, v)_{H^s} = 0, \quad \forall u \in C^\infty(SM) \cap V_{>0}.
$$

Thus $\Pi^\perp(\tilde{P}_h^* - h^2\bar{\lambda})v = 0$ where the adjoint is taken in H^s. Let $\chi \in C_0^\infty(\mathbb{R};[0,1])$ be a cutoff function such that $\chi = 1$ near 0 and $v_\epsilon = \chi(\epsilon^2\Delta)v$, then

$$\begin{aligned}\|v_\epsilon\|_{H^s} &\lesssim_{C_0} \|\Pi^\perp(\tilde{P}_h^* - h^2\bar{\lambda})v_\epsilon\|_{H^s}\\ &\lesssim_{C_0} h\|[X^*, \chi(\epsilon^2\Delta)]v\|_{H^s}\\ &\lesssim_{C_0} h\|v\|_{H^s}.\end{aligned}$$

Let $h \ll_{C_0} 1$ and $\epsilon \ll 1$, we conclude $v = 0$, a contradiction. Thus $\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp$ is also surjective and thus invertible. □

2.3. **Spectral convergence.** In this section we prove the convergence of the spectrum in Theorem 1 by a Grushin problem following [RT22].

Let $i_0 : V_0^s \to H^s(SM)$ be the inclusion. Intuitively, we want to consider the following Grushin problem for $P_\gamma - \lambda + Q_A$.

$$\begin{pmatrix} P_\gamma - \lambda + Q_A & \gamma i_0 \\ \gamma\Pi & 0 \end{pmatrix} : D^s(P_\gamma) \oplus V_0^s \to H^s(SM) \oplus V_0^s.$$

However, it is not clear what the correct space is to set up the Grushin problem. Instead we will just directly write down a formula (2.17) that works distributionally. Using same methods in [RT22], we can solve the equations

$$\begin{cases} (P_\gamma - \lambda + Q_A)u + \gamma u_- = v, & (u, u_-) \in \mathcal{D}'(SM) \oplus \mathcal{D}'(M), \\ \gamma\Pi u = v_+, & (v, v_+) \in \mathcal{D}'(SM) \oplus \mathcal{D}'(M). \end{cases} \tag{2.15}$$

The solution we get is

$$\begin{cases} u &= (\Pi^\perp(P_\gamma - \lambda)\Pi^\perp)^{-1}\Pi^\perp(v + Xv_+) + \gamma^{-1}v_+, \\ u_- &= \gamma^{-1}\Pi v + \gamma^{-2}(\lambda - Q_A)v_+ + \Pi X(\Pi^\perp(P_\gamma - \lambda)\Pi^\perp)^{-1}\Pi^\perp(v + Xv_+). \end{cases} \tag{2.16}$$

Now we write (at least formally)

$$(P_\gamma - \lambda + Q_A)^{-1} = E - E_+E_{-+}^{-1}E_- \tag{2.17}$$

where

$$E = (\Pi^\perp(P_\gamma - \lambda)\Pi^\perp)^{-1}\Pi^\perp, \qquad E_+ = (\Pi^\perp(P_\gamma - \lambda)\Pi^\perp)^{-1}\Pi^\perp X + \gamma^{-1},$$

$$E_- = \gamma^{-1}\Pi + \Pi X(\Pi^\perp(P_\gamma - \lambda)\Pi^\perp)^{-1}\Pi^\perp, \quad E_{-+} = \gamma^{-2}(\lambda + \Pi X(\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp)^{-1}X\Pi - Q_A).$$

We need to justify that E_{-+} is invertible, and the inverse has a good control. So let us look at the equation $\gamma^2 E_{-+}u = f$. Let $v = (\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp)^{-1}X\Pi u$, we have

$$\lambda u + \Pi X v - Q_A u = f, \qquad \Pi^\perp(\tilde{P}_h - h^2\lambda)v = Xu.$$

Thus $\gamma^2 E_{-+}u = f$ is equivalent to

$$(P_\gamma - \lambda + Q_A)(u + hv) = -f.$$

By Lemma 2.4, $P_\gamma - \lambda + Q_A$ is invertible, we conclude E_{-+} is also invertible. So the formula (2.17) makes sense distributionally for $\gamma > A \gg_{C_0,n,s} 1$ depending on the Sobolev regularity s.

In order to apply (2.17), we write

$$P_\gamma - \lambda = P_\gamma - \lambda + Q_A - Q_A = (P_\gamma - \lambda + Q_A)(I - (P_\gamma - \lambda + Q_A)^{-1}Q_A).$$

We claim

Proposition 2.8. *For $|\lambda| \leq C_0$, $\gamma > A \gg_{C_0,n,s,N} 1$, we have*

$$\begin{aligned}
&\|(P_\gamma - \lambda + Q_A)^{-1}Q_A - (\Delta_M - \lambda + Q_A)^{-1}Q_A\|_{H^s \to H^{s+N}} \lesssim_{C_0,n,s,N} A^{N+2}\gamma^{-1}, \\
&\|Q_A(P_\gamma - \lambda + Q_A)^{-1} - Q_A(\Delta_M - \lambda + Q_A)^{-1}\|_{H^s \to H^{s+N}} \lesssim_{C_0,n,s,N} A^{N+2}\gamma^{-1},
\end{aligned}$$

for any $s \in \mathbb{R}, N \geq 0$.

Proof. We will only prove the first one, but the second one is proved exactly the same way.

Note by (2.17), $(P_\gamma - \lambda + Q_A)^{-1}Q_A = -\gamma^{-1}E_+E_{-+}^{-1}Q_A$. We first prove a bound for $\gamma^{-2}E_{-+}^{-1}Q_A$ in $H^s \to H^{s+N}$ for any $N \geq 0$. Let $\gamma^2 E_{-+}u = Q_A f$, then for $v = (\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp)^{-1}X\Pi u$ we have

$$(P_\gamma - \lambda + Q_A)(u + hv) = -Q_A f.$$

By Lemma 2.4, we conclude

$$\|u\|_{H^{s+N}} \lesssim_{C_0} A^{-1}\|Q_A f\|_{H^{s+N}} \lesssim_{C_0} A^{N+1}\|f\|_{H^s}.$$

Now we can estimate the difference

$$\begin{aligned}
&(P_\gamma - \lambda + Q_A)^{-1}Q_A - (\Delta_M - \lambda + Q_A)^{-1}Q_A = -\gamma^{-1}E_+E_{-+}^{-1}Q_A - (\Delta_M - \lambda + Q_A)^{-1}Q_A \\
&= (-\gamma^{-2}E_{-+}^{-1} - (\Delta_M - \lambda + Q_A)^{-1})Q_A - \gamma^{-3}(\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp)^{-1}\Pi^\perp X E_{-+}^{-1}Q_A.
\end{aligned}$$

For the second term,

$$\begin{aligned}
\|\gamma^{-3}(\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp)^{-1}\Pi^\perp X E_{-+}^{-1}Q_A\|_{H^s \to H^{s+N}} &\lesssim_{C_0} \|\gamma^{-3}E_{-+}^{-1}Q_A\|_{H^s \to H^{s+N+1}} \\
&\lesssim_{C_0} A^{N+2}\gamma^{-1}.
\end{aligned}$$

For the first term,

$$\begin{aligned}
&(-\gamma^{-2}E_{-+}^{-1} - (\Delta_M - \lambda + Q_A)^{-1})Q_A \\
=&(\Delta_M - \lambda + Q_A)^{-1}(-\Delta_M - \Pi X(\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp)^{-1}X\Pi)\gamma^{-2}E_{-+}^{-1}Q_A \\
=&(\Delta_M - \lambda + Q_A)^{-1}(\Pi X((c_n\Pi^\perp\Delta_V\Pi^\perp)^{-1} - (\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp)^{-1})X\Pi)\gamma^{-2}E_{-+}^{-1}Q_A \\
=&(\Delta_M - \lambda + Q_A)^{-1}(\Pi X((c_n\Pi^\perp\Delta_V\Pi^\perp)^{-1}\Pi^\perp(-hX - h^2\lambda)\Pi^\perp \\
&\qquad\qquad(\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp)^{-1})X\Pi)\gamma^{-2}E_{-+}^{-1}Q_A.
\end{aligned}$$

Note

$$\|\gamma^{-2}E_{-+}^{-1}Q_A\|_{H^s\to H^{s+N+3}} \lesssim_{C_0} A^{N+4}, \|X\|_{H^{s+N+3}\to H^{s+N+2}} \lesssim 1,$$

$$\|(\Pi^\perp(\tilde{P}_h - h^2\lambda)\Pi^\perp)^{-1}\|_{H^{s+N+2}\to H^{s+N+2}} \lesssim_{C_0} 1, \|hX + h^2\lambda\|_{H^{s+N+2}\to H^{s+N+1}} \lesssim_{C_0} h,$$

$$\|X(c_n\Pi^\perp\Delta_V\Pi^\perp)^{-1}\|_{H^{s+N+1}\to H^{s+N}} \lesssim 1, \|(\Delta_M - \lambda + Q_A)^{-1}\|_{H^{s+N}\to H^{s+N}} \lesssim_{C_0} A^{-2}.$$

We conclude

$$\|(P_\gamma - \lambda + Q_A)^{-1}Q_A - (\Delta_M - \lambda + Q_A)^{-1}Q_A\|_{H^s\to H^{s+N}} \lesssim_{C_0} A^{N+2}\gamma^{-1}. \qquad \square$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We first show the spectrum convergence (1.2). It is direct to check $I - (P_\gamma - \lambda + Q_A)^{-1}Q_A$ is invertible on $L^2(SM)$ if and only if it is invertible on $D(P_\gamma)$. So for $U \Subset \mathbb{C}$,

$$\begin{aligned}
\sigma(P_\gamma) \cap U &= \{\lambda \in U : I - (P_\gamma - \lambda + Q_A)^{-1}Q_A \text{ is not invertible on } L^2(SM)\} \\
&= \text{ zeros of } \det(I - (P_\gamma - \lambda + Q_A)^{-1}Q_A) \text{ in } U.
\end{aligned}$$

By Proposition 2.8, for fixed A, the determinant $\det(I - (P_\gamma - \lambda + Q_A)^{-1}Q_A)$ convergences to $\det(I - (\Delta_M - \lambda + Q_A)^{-1}Q_A)$ locally uniformly as $\gamma \to \infty$. So the zeros also converge to zeros of $\det(I - (\Delta_M - \lambda + Q_A)^{-1}Q_A)$, which are exactly eigenvalues of Δ_M.

Now we prove the resolvent convergence (1.3). We will choose $A = \gamma^{1/5} \to \infty$ below. Let $u \in H^s$ with $u_{\text{low}} = \mathbb{1}_{(\Delta_M \le 10A^2)}\Pi u$ and $u_{\text{high}} = u - u_{\text{low}}$, we have

$$\|(\Delta_M - \lambda)^{-1}u_{\text{high}}\|_{H^{s+1/4}} \lesssim_{C_0} A^{-7/4}\|u_{\text{high}}\|_{H^s}.$$

We note

$$\|(I - Q_A(\Delta_M - \lambda + Q_A)^{-1})^{-1}\|_{H^s\to H^s} = \|I - Q_A(\Delta_M - \lambda)^{-1}\|_{H^s\to H^s} \lesssim_{C_U} 1 + A^2$$

where C_U^{-1} is the distance between $\sigma(\Delta_M)$ and U. For $A^2\gamma^{-1} \ll_U A^{-2}$, we have

$$\|Q_A(P_\gamma - \lambda + Q_A)^{-1} - Q_A(\Delta_M - \lambda + Q_A)^{-1}\|_{H^s\to H^s} \ll \|(I - Q_A(\Delta_M - \lambda + Q_A)^{-1})^{-1}\|_{H^s\to H^s}^{-1}$$

so that

$$\|(I - Q_A(P_\gamma - \lambda + Q_A)^{-1})^{-1}\|_{H^s \to H^s} \lesssim_U A^2.$$

Moreover,

$$\begin{aligned}
&\|((I - Q_A(P_\gamma - \lambda + Q_A)^{-1})^{-1} - (I - Q_A(\Delta_M - \lambda + Q_A)^{-1})^{-1})v\|_{H^s} \\
&\leq \|(I - Q_A(P_\gamma - \lambda + Q_A)^{-1})^{-1}(Q_A(P_\gamma - \lambda + Q_A)^{-1} - Q_A(\Delta_M - \lambda + Q_A)^{-1}) \\
&(I - Q_A(\Delta_M - \lambda + Q_A)^{-1})^{-1}v\|_{H^s} \\
&\lesssim_U A^4\gamma^{-1}\|(I - Q_A(\Delta_M - \lambda + Q_A)^{-1})^{-1}v\|_{H^s}.
\end{aligned}$$

Using (2.14), we conclude

$$\begin{aligned}
\|(P_\gamma - \lambda)^{-1}u_{\text{high}}\|_{H^{s+1/4}} &= \|(P_\gamma - \lambda + Q_A)^{-1}(I - Q_A(P_\gamma - \lambda + Q_A)^{-1})^{-1}u_{\text{high}}\|_{H^{s+1/4}} \\
&\lesssim_{C_0} A^{-1/2}\|(I - Q_A(P_\gamma - \lambda + Q_A)^{-1})^{-1}u_{\text{high}}\|_{H^s} \\
&\lesssim_U A^{-1/2}\|(I - Q_A(\Delta_M - \lambda + Q_A)^{-1})^{-1}u_{\text{high}}\|_{H^s} \\
&= A^{-1/2}\|u_{\text{high}}\|_{H^s}.
\end{aligned}$$

In the last step we use the fact $Q_A u_{\text{high}} = 0$. Now we are left with the finite dimensional part u_{low} and by Proposition 2.8 we have

$$\begin{aligned}
&\|((P_\gamma - \lambda)^{-1} - (\Delta_M - \lambda)^{-1})u_{\text{low}}\|_{H^{s+1/4}} \\
&= \|((I - (P_\gamma - \lambda + Q_A)^{-1}Q_A)^{-1}(P_\gamma - \lambda + Q_A)^{-1} \\
&- (I - (\Delta_M - \lambda + Q_A)^{-1}Q_A)^{-1}(\Delta_M - \lambda + Q_A)^{-1})\mathbb{1}_{(\Delta_M \leq 10A^2)}\Pi u_{\text{low}}\|_{H^{s+1/4}} \\
&\leq \|(I - (P_\gamma - \lambda + Q_A)^{-1}Q_A)^{-1}((P_\gamma - \lambda + Q_A)^{-1} - (\Delta_M - \lambda + Q_A)^{-1})u_{\text{low}}\|_{H^{s+1/4}} + \\
&\|((I - (P_\gamma - \lambda + Q_A)^{-1}Q_A)^{-1} - (I - (\Delta_M - \lambda + Q_A)^{-1}Q_A)^{-1})(\Delta_M - \lambda + Q_A)^{-1}u_{\text{low}}\|_{H^{s+1/4}} \\
&\lesssim_U A^{2+1/4}\gamma^{-1}\|u_{\text{low}}\|_{H^s} + A^{4+1/4}\gamma^{-1}\|u_{\text{low}}\|_{H^s}.
\end{aligned}$$

We conclude

$$\|(P_\gamma - \lambda)^{-1} - (\Delta_M - \lambda)^{-1}\|_{H^s \to H^{s+1/4}} \lesssim_U A^{-1/2} + A^{4+1/4}\gamma^{-1} \lesssim \gamma^{-1/10}. \tag{2.18}$$

This finishes the proof of (1.3). □

3. Convergence to equilibrium

In this section we give the proof of Theorem 2. In fact, we will prove the following more general Theorem 3. If we take β in Theorem 3 to be smaller than the first eigenvalue of Δ_M, then there is only a single term coming from the zero eigenvalue of P_γ in the expansion, and this gives Theorem 2.

Theorem 3. *For any $\beta, \epsilon > 0$ there exists $\gamma_0 > 0$ such that for $\gamma > \gamma_0$, $\sigma(P_\gamma) \cap \{\operatorname{Re}\lambda \le \beta\} = \{\lambda_0 = 0, \lambda_1, \cdots, \lambda_r\}$ is finite, and there exists $C_\gamma > 0$ such that*

$$\left\| e^{-tP_\gamma} u - \sum_{j=0}^{r} \sum_{l=0}^{m_j - 1} \frac{(-t)^l}{l!} e^{-t\lambda_j} (P_\gamma - \lambda_j)^l \Pi_{\lambda_j} u \right\|_{L^2} \le C_\gamma e^{-\beta t} \|u\|_{L^2}, \quad t \ge 1,$$

where Π_{λ_j} is the spectral projector to the generalized eigenspace of P_γ with eigenvalue λ_j. Moreover, for each j there is $\lambda_j^0 \in \sigma(\Delta_M)$ such that $|\lambda_j - \lambda_j^0| < \epsilon$.

Proof. First we claim there are only finitely many eigenvalues of P_γ in the region $\{\operatorname{Re}\lambda \le \beta\}$, and they all satisfy $|\operatorname{Im}\lambda| \lesssim \beta$. We prove by contradiction again. Suppose λ is an eigenvalue of P_γ such that $\operatorname{Re}\lambda \le \beta$, then there exists $u \in C^\infty(SM)$ such that

$$P_\gamma u = c_n \gamma^2 \Delta_V u - \gamma X u = \lambda u. \tag{3.1}$$

As in the proof of Lemma 2.4, we have

$$\|\Pi^\perp u\|_{H^s} \lesssim \sqrt{\beta}\gamma^{-1} \|u_0\|_{H^s}, \quad s = 0, 1.$$

Projecting the equation (3.1) to V_0 gives

$$-\gamma \Pi_0 X u_1 = \lambda u_0.$$

Thus $|\lambda| \|u_0\|_{L^2} \lesssim \gamma \|u_1\|_{H^1} \lesssim \sqrt{\beta} \|u_0\|_{H^1}$. Projecting the equation (3.1) to V_1 gives

$$\frac{1}{n}\gamma^2 u_1 - \gamma \Pi_1 X(u_0 + u_2) = \lambda u_1$$

which gives as before $\|u_0\|_{H^1} \lesssim \sqrt{\beta}(1 + \gamma^{-2}|\lambda|)\|u_0\|_{L^2}$. We conclude

$$|\lambda| \lesssim \beta(1 + \gamma^{-2}|\lambda|).$$

Taking $\gamma^2 \gg \beta$, we conclude $|\lambda| \lesssim \beta$. Along with Theorem 1, this shows $|\lambda_j - \lambda_j^0| < \epsilon$ for some $\lambda_j^0 \in \sigma(\Delta_M)$ once $\gamma > \gamma_0$ is taken large enough.

Now we consider the Laplace transform of e^{-tP_γ}:

$$\int_0^\infty e^{\lambda t} e^{-tP_\gamma} dt = (P_\gamma - \lambda)^{-1}, \quad \operatorname{Re}\lambda < 0.$$

We can then express e^{-tP_γ} as the inverse Laplace transform

$$e^{-tP_\gamma} = \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} (P_\gamma - \lambda)^{-1} e^{-\lambda t} d\lambda.$$

We deform the contour from $\operatorname{Re}\lambda = -1$ to ρ and conclude

$$e^{-tP_\gamma} = \sum_{j=0}^{r} \operatorname{Res}_{\lambda=\lambda_j}((\lambda - P_\gamma)^{-1} e^{-\lambda t}) + \frac{1}{2\pi i} \int_\rho (P_\gamma - \lambda)^{-1} e^{-\lambda t} d\lambda \tag{3.2}$$

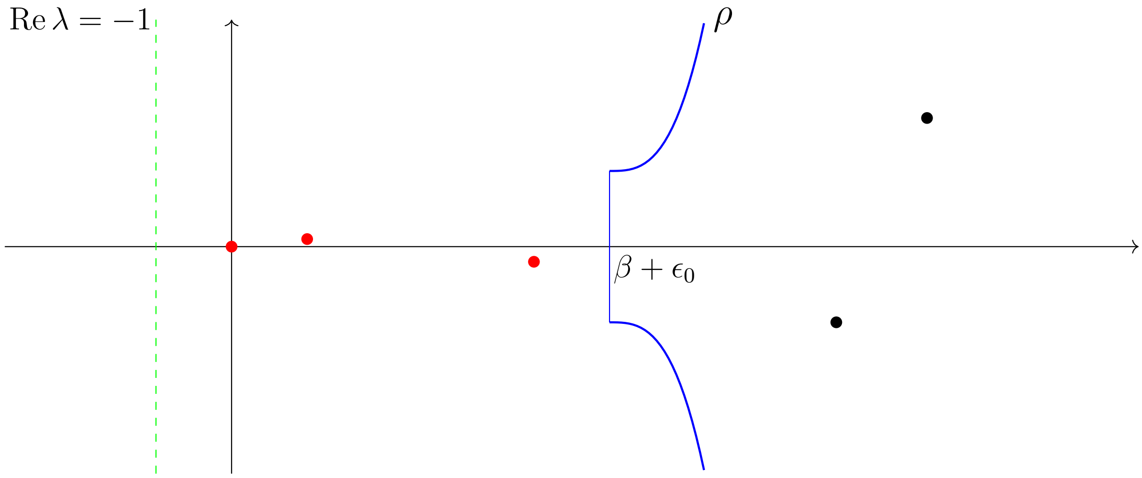

FIGURE 2. Contour deformation

where ρ is given by $\{\operatorname{Re}\lambda = \beta + \epsilon_0, |\operatorname{Im}\lambda| \leq A_\gamma\}$ and $\{|\operatorname{Im}\lambda| = C_\gamma(\operatorname{Re}\lambda - \beta - \epsilon_0)^{16} + A_\gamma, \operatorname{Re}\lambda > \beta + \epsilon_0\}$. See Figure 2 for a picture of the contours. In order to conclude the proof we need the following Lemma 3.1 from Eckmann–Hairer [EH03, Theorem 4.1, 4.3].

Lemma 3.1. *There exists $C > 0$ (independent of γ) such that P_γ does not have spectrum in $\{|\operatorname{Im}\lambda| \geq C(\operatorname{Re}\lambda + \gamma^{1/4})^{16} + 1, \operatorname{Re}\lambda > 0\}$. Moreover, we have for such λ*

$$\|(P_\gamma - \lambda)^{-1}\|_{L^2 \to L^2} \lesssim 1.$$

Proof. The lemma follows from the uniform hypoelliptic estimate (2.10)

$$\|u\|_{H^{1/8}} \leq C(B^{-1}\|(P_\gamma - iy)u\|_{L^2} + B\|u\|_{L^2}), \qquad \forall y \in \mathbb{R}$$

with constant $C > 0$ independent of y and γ. Taking $B = \gamma^{1/8}$, we get (using [HN04, [Proposition B.1])

$$\begin{aligned}
\frac{1}{4}|\lambda + 1|^{1/8}\|u\|_{L^2}^2 &\leq (((P_\gamma + 1)^*(P_\gamma + 1))^{1/16}u, u)_{L^2} + \|(P_\gamma - \lambda)u\|_{L^2}^2 \\
&\lesssim \gamma^{1/4}\|u\|_{H^{1/8}}^2 + \|(P_\gamma - \lambda)u\|_{L^2}^2 \\
&\lesssim (\gamma^{1/4} + \operatorname{Re}\lambda)^2\|u\|_{L^2}^2 + \|(P_\gamma - \lambda)u\|_{L^2}^2.
\end{aligned}$$

Thus for $|\lambda + 1| \geq C_1(\gamma^{1/4} + \operatorname{Re}\lambda)^{16} + 1$ we conclude

$$\|u\|_{L^2} \lesssim \|(P_\gamma - \lambda)u\|_{L^2}. \qquad \square$$

Theorem 3 then follows from (3.2) where the residues are given by

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_j}((\lambda - P_\gamma)^{-1}e^{-\lambda t}) &= \mathrm{Res}_{\lambda=\lambda_j}\left(\sum_{l=0}^{m_j-1}(P_\gamma-\lambda_j)^l\Pi_{\lambda_j}(\lambda-\lambda_j)^{-l-1}e^{-\lambda t}\right)\\ &= \sum_{l=0}^{m_j-1}\frac{(-t)^l}{l!}e^{-\lambda_j t}(P_\gamma-\lambda_j)^l\Pi_{\lambda_j}\end{aligned}$$

and the remainder is estimated as

$$\left\|\int_\rho (P_\gamma-\lambda)^{-1}e^{-\lambda t}d\lambda\right\|_{L^2\to L^2} \lesssim_\gamma \int_\rho e^{-\operatorname{Re}\lambda t}d|\lambda| \lesssim_\gamma e^{-\beta t}. \qquad \square$$

References

[ABT15] J. Angst, I. Bailleul, and C. Tardif, *Kinetic Brownian motion on Riemannian manifolds*, Electronic Journal of Probability **20** (2015), 1–40.

[BB82] L. B. Bergery and J.-P. Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibres*, Illinois Journal of Mathematics **26** (1982), no. 2, 181–200.

[Bis05] J.-M. Bismut, *The hypoelliptic Laplacian on the cotangent bundle*, Journal of the American Mathematical Society **18** (2005), no. 2, 379–476.

[Bis11] J.-M. Bismut, *Hypoelliptic Laplacian and Orbital Integrals (AM-177)*, vol. 177, Princeton University Press, 2011.

[BL08] J.-M. Bismut and G. Lebeau, *The Hypoelliptic Laplacian and Ray–Singer Metrics.(AM-167)*, vol. 167, Princeton University Press, 2008.

[BT18] F. Baudoin and C. Tardif, *Hypocoercive estimates on foliations and velocity spherical Brownian motion*, Kinetic and Related Models **11** (2018), no. 1, 1–23.

[Dro17] A. Drouot, *Stochastic stability of Pollicott–Ruelle resonances*, Communications in Mathematical Physics **356** (2017), no. 2, 357–396.

[EH03] J.-P. Eckmann and M. Hairer, *Spectral properties of hypoelliptic operators*, Communications in mathematical physics **235** (2003), no. 2, 233–253.

[FL07] J. Franchi and Y. Le Jan, *Relativistic diffusions and Schwarzschild geometry*, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences **60** (2007), no. 2, 187–251.

[Fri86] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Inventiones mathematicae **84** (1986), no. 3, 523–540.

[Fri95] D. Fried, *Meromorphic zeta functions for analytic flows*, Communications in mathematical physics **174** (1995), no. 1, 161–190.

[GS13] M. Grothaus and P. Stilgenbauer, *Geometric Langevin equations on submanifolds and applications to the stochastic melt-spinning process of nonwovens and biology*, Stochastics and Dynamics **13** (2013), no. 04, Id/No 1350001.

[HN04] F. Hérau and F. Nier, *Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential*, Archive for Rational Mechanics and Analysis **171** (2004), no. 2, 151–218.

[Hör07] L. Hörmander, *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*, Springer Science & Business Media, 2007.

[Koh73] J. J. Kohn, *Pseudo-differential operators and hypoellipticity*, Partial Differential Equations, Proceedings of Symposia in Pure Mathematics **23** (1973), 61–69.

[KWW19] M. Kolb, T. Weich, and L. L. Wolf, *Spectral asymptotics for kinetic Brownian motion on hyperbolic surfaces*, arXiv preprint arXiv:1909.06183 (2019).

[KWW22] M. Kolb, T. Weich, and L. L. Wolf, *Spectral asymptotics for kinetic Brownian motion on surfaces of constant curvature*, Annales Henri Poincaré **23** (2022), no. 4, 1283–1296.

[Li16] X.-M. Li, *Random perturbation to the geodesic equation*, The Annals of Probability **44** (2016), no. 1, 544–566.

[Rad69] E. V. Radkevich, *On a theorem of L. Hormander*, Uspekhi Matematicheskikh Nauk **24** (1969), no. 2, 233–234.

[RT22] Q. Ren and Z. Tao, *Spectral asymptotics for kinetic Brownian motion on locally symmetric spaces*, arXiv preprint arXiv:2208.13111 (2022).

[She17] S. Shen, *Analytic torsion, dynamical zeta functions, and the Fried conjecture*, Analysis and PDE **11** (2017), no. 1, 1–74.

[She21] S. Shen, *Analytic torsion and dynamical flow: a survey on the Fried conjecture*, in: *Arithmetic L-Functions and Differential Geometric Methods*, Springer, 2021, 247–299.

[Smi20] H. F. Smith, *Parametrix for a semiclassical subelliptic operator*, Analysis and PDE **13** (2020), no. 8, 2375–2398.

Email address: `qiuyu_ren@berkeley.edu`

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA

Email address: `ztao@math.berkeley.edu`

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA