Teaching Geometry in Grade 8 and High School According to the Common Core Standards

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Preface

This is the companion article to Teaching Geometry According to the Common Core Standards.

The Common Core State Standards for Mathematics (CCSSM) have reorganized the geometry curriculum in grade 8 and high school. Because there are at present very few (if any) ready references for such a reorganization, this document is being offered as a stopgap measure.

In terms of the topics covered, there is hardly any difference between what is called for by the CCSSM and by the other curricula. The change occurs mainly in the internal (mathematical) reorganization and the change of (mathematical) focus. For example, transformations are usually taught as rote skills in middle school with no mathematical applications or relevance, and the concepts of congruence and similarity are *talked about* but never defined except in the case of polygons. By contrast, the CCSSM develop all these topics on the foundation of transformations, thereby giving them coherence and purposefulness. The "coherence" of the CCSSM has been much bandied about in recent discussions, but it is time to realize that the coherence of the CCSSM is not an educational slogan but a ***mathematical*** fact, and one of its manifestations is the coherence of the geometry curriculum embedded in the CCSSM. For the benefit of students' learning, this change is a welcome development. However, it is unfortunately the case that while these basic topics are routinely discussed in the mathematics literature, not much of this information can be found in the education literature except perhaps H. Wu, Pre-Algebra. The intentions of the CCSSM have thus become hidden for the time being. If the detailed account given in this document is at all successful, it will furnish a bridge across this mathematical chasm for the time being.

My specific targets are middle and high school mathematics teachers as well as the publishers of textbooks. I hope that teachers will find this account helpful in their preparations for the implementation of CCSSM by year 2014. If, in addition, their school district can offer professional development, then maybe they can make use of this document to articulate the kind of professional development they want. We are entering an era when teachers must take an active role in their own professional life. The CCSSM are charting a new course, and district administrators and professional developers have to work together with teachers to find their new bearings in the transitional period.

As for the publishers, my contact with them in the past fifteen years has made me aware that their claim of not having the needed resources to improve their books is indeed entirely legitimate. Our educational system has been broken for a long time and we have to find ways to forge a new beginning. At a time when the CCSSM are initiating a significant change in the teaching of geometry, it would be unconscionable—as in the days of the New Math—to once again ask for change without providing the necessary support for this change. It is hoped that this document will provide some temporary relief in the present absence of this support.

This document is essentially a compendium of selected topics from the lecture notes for the annual summer professional development institutes (MPDI) and upper division courses (Math 151–153) at Berkeley that I have given since 2006. I have been advocating this transformations-based approach to the teaching of middle school and high school geometry because, in terms of student learning, it is a more reasonable alternative to the existing ones (see the discussions on page 79 ff. and page 125 ff. for part of the reason). By a happy coincidence, the CCSSM agreed with this judgment. (The reference, *Wu, H., Lecture Notes for the 2009 Pre-Algebra Institute, September 15, 2009.* on page 92 of the CCSSM is the same as H. Wu, Pre-Algebra.) In any case, the detailed development of this approach to middle school and high school geometry, together with exercises, will be found in the following textbooks by the author: **From Pre-Algebra to Algebra** (for middle school teachers, to appear in late 2014), and **Mathematics of the Secondary School Curriculum** (a two volume set for high school teachers, to appear probably in late 2015).

It may also be mentioned that I expect to post detailed *student* lessons for grade 8 according to the CCSSM by the fall of 2014.

Acknowledgements. I wish to thank Wolfgang Buettner for his interesting contributions, Mark Saul for his willingness to read through the grade 8 portion and make suggestions, and Larry Francis for his usual excellent editorial assistance. To Angelo Segalla and Clinton Rempel, I owe an immense debt for numerous corrections.

Conventions

A turquoise box around a phrase or a sentence (such as H. Wu, Pre-Algebra) indicates an active link to an article online.

The standards on geometry are listed at the beginning of each grade in sans serif fonts.

GRADE 8

Geometry 8.G

Understand congruence and similarity using physical models, transparencies, or geometry software.

1. Verify experimentally the properties of rotations, reflections, and translations:
a. Lines are taken to lines, and line segments to line segments of the same length.
b. Angles are taken to angles of the same measure.
c. Parallel lines are taken to parallel lines.

2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

3. Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

4. Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

5. Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

Understand and apply the Pythagorean Theorem.

6. Explain a proof of the Pythagorean Theorem and its converse.

7. Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system. Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

9. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Goals of eighth grade geometry

1. An *intuitive* introduction of the concept of *congruence* using rotations, translations, and reflections, and their compositions (**page 8**)
2. An *intuitive* introduction of the concepts of *dilation* and *similarity* (**page 42**)
3. An informal argument for the angle-angle criterion (AA) of similar triangles (**page 57**)
4. Use of AA for similarity to prove the Pythagorean Theorem (**page 61**)
5. An informal argument that the angle sum of a triangle is 180 degrees (**page 66**)
6. Introduction of some basic volume formulas (**page 68**)

These six goals are intended to be achieved with an emphasis on the intuitive *geometric* content through the ample use of hands-on activities. They will prepare eighth graders to learn about the geometry of linear equations in beginning algebra. They are also needed to furnish eighth graders with a firm foundation for the more formal development of high school geometry.

1. Basic rigid motions and congruence

Overview

The main new ideas are the concepts of translations, reflections, rotations, and dilations in the plane. The first three—translations, reflections, rotations—are collectively referred to as the **basic rigid motions**, and they will be the subject of inquiry in this section. Dilation will be explained in the next.

Before proceeding further, we note that the basic rigid motions are quite subtle concepts whose precise definitions require a bit of preparation about more advanced topics such as transformations of the plane, the concept of transformations that are one-to-one and "onto", separation properties of lines in the plane, distance in the plane, and other concepts that are necessary for a more formal development. Such precision is neither necessary nor desirable in an introductory treatment in eighth grade. Rather,

> *it is the intuitive geometric content of the basic rigid motions that needs to be emphasized.*

In the high school course on geometry, more of this precision will be supplied in order to carry out the detailed mathematical reasoning for the proofs of theorems. For eighth grade, however, we should minimize the formalism and emphasize the geometric intuition instead. Fortunately, the availability of abundant teaching tools makes it easy to convey this intuitive content. In this document, we will rely exclusively on the use of transparencies as an aid to the explanation of basic rigid motions. This expository decision should be complemented by two remarks, however.

The first is to caution against the premature use of computer software for learning about basic rigid motions. While computer software will eventually be employed for the purpose of geometric explorations, it is strongly recommended, on the basis of professional judgment and available experience, that students begin the study of basic rigid motions with transparencies but *not* with computer software. Primitive objects such as transparencies have the advantage that students can easily achieve complete control over them without unforeseen software-related subtleties interfering with the learning process. Let students first be given an extended opportunity to gain the requisite geometric intuition through direct, tactile experiences *before* they approach the computer. A second remark is that if you believe some other manipulatives are more suitable for your own classroom needs and *you are certain that these manipulatives manage to convey the same message*, then feel free to use them.

In the following, a *basic rigid motion* will mean a translation, a reflection, or a rotation in the plane, with the precise definition to follow. In general, a basic rigid motion is a rule F so that, for each point P of the plane, F assigns a point $F(P)$ to P.[1] Before describing this rule separately for translations, reflections, and rotations, we first introduce a piece of terminology that will minimize the possible confusion with the language of "assigning $F(P)$ to P". We will sometimes say that

> *a basic rigid motion F is a rule that* ***moves*** *each point of the plane P to a point $F(P)$ in the plane.*

This terminology expresses the intuitive content of a rigid motion better than the original language of a "rule of assignment", and is in fact the reason behind the term "rigid motion".

Now all basic rigid motions share certain desirable properties, but rather than listing these properties right at the beginning, we first define these basic rigid motions, namely, *reflection* across a given line, *translation* along a vector, and *rotation* of a fixed degree around a point. Then we examine their effect on simple geometric figures in the plane. Through these examples, the properties in question will appear naturally.

Before proceeding further, it may be worthwhile to pursue the analogy of basic rigid motions with functions of one variable a bit further. If f is a function of the

[1]This should remind you of the definition of a function of one variable.

variable x, then $f(x)$ is a real number for each real number x. So f assigns to each point x on the *number line* the point $f(x)$ on the *number line*, just as a basic rigid motion F assigns to each point P in the *plane* the point $F(P)$ in the *plane*. Now, on the number line, one clearly cannot "rotate", but the analogs of translation and reflection do make sense. The function g so that $g(x) = x + 2$ for each x "moves" each point x of the number line to a point which is the same number of units, namely, 2 to the right of x regardless of what x may be. This will be seen to be the analog of a translation in the plane. Next, let the function h assign to each point x on the number line point its opposite $-x$ on the number line. Thus h "flips" x to the other side of 0, and this will be seen to the analog of a reflection across a line.

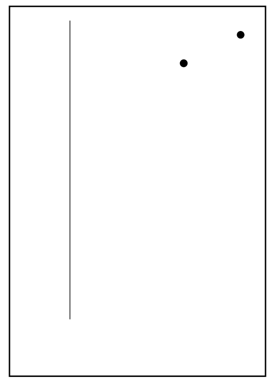
In general, if a rigid motion F is given, then its rule of assignment is usually not simple, and the effort needed to decode this description will likely distract a beginner's attention from the geometric content of F itself. Fortunately, the rule associated with each of the three basic rigid motions can be given visually with the help of transparencies. This is what we will describe next. The additional advantage of giving the definition in terms of transparencies is that it will effortlessly reveal the above-mentioned desirable properties that these three basic rigid motions have in common.

We proceed to describe how to move a transparency over a piece of paper to illustrate the three basic rigid motions, but it is well to note that, ***in the classroom, a face-to-face demonstration of the the manipulation of a transparency is far easier to understand than the clumsy verbal and graphical description given below.*** Please keep this fact in mind as you read.

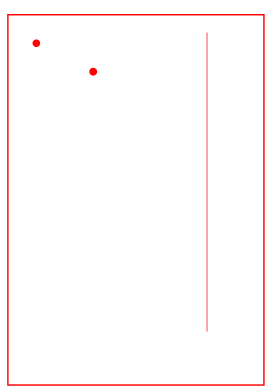
Preliminary definitions of basic rigid motions

We begin with the **reflection $\boldsymbol{R}$ across a given line $\boldsymbol{L}$**. The line L will be called the **line of reflection** of R. For definiteness, let us say L is a vertical line and let us say two arbitrary points in the plane are given. We now describe how R moves these points. Let the line L and the dots be drawn on a piece of paper in black, as in the picture below.[2] The black rectangle indicates the border of the paper.

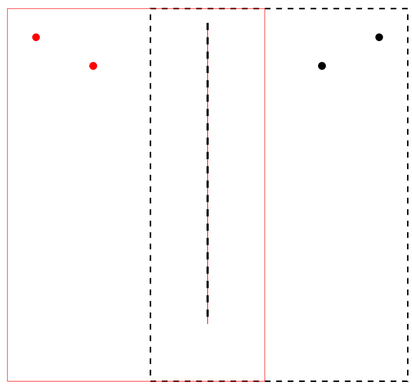
[2] If a teacher wants to do a demonstration for the whole class on an overhead projector, then on some overhead projectors, it may be necessary to do the drawing on a transparency to begin with.



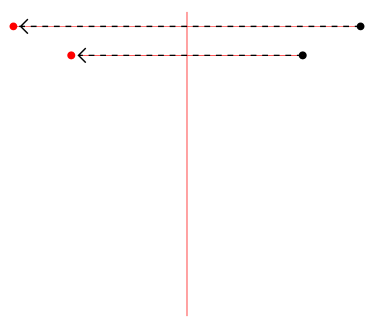
Note that this pictorial representation of the plane has severe limitations: *We are using a finite rectangle to represent the plane which is infinite in all directions, and a finite segment to represent a line which is infinite in both directions.* With this understood, trace the line and the points exactly on a transparency (of exactly the same size as the paper, of course) using a different color, say red. ***In all subsequent discussions of demonstrations with a transparency, the piece of paper containing the original drawing is understood to stay fixed and only the transparency is moved around.*** With this understood, now pick up the transparency and set it down again in alignment with the vertical line, *interchanging left and right*, while keeping every point on the red vertical line on top of the same point on the black vertical line. Clearly, the latter is achieved if the upper (respectively, lower) endpoint of the red vertical line is kept on top of the upper (respectively, lower) endpoint of the black vertical line. The position of the red figure of two dots and the red line on the transparency now represents how the original figure has been reflected.



Because we want to see the new position of the reflected figure relative to the original, we are going to show the red figure together with the black figure, and this calls for some explanation. In the classroom, the black figure will be just black, of course, but for the purpose of the verbal explanation in this article, we are going to replace the black lines by dashed black lines to suggest the correct psychological response, namely, that we are leaving the original figure behind in order to concentrate on the red figure on the transparency. Thus the dashed black figure represents where the red figure *used to be*, and is therefore just background information. (The red rectangle indicates the border of the transparency.)



In other words, if we now look at the plane itself, then the rule of assignment of the reflection R is to move the points in the plane represented by the black dots to the corresponding points in the plane represented by the red dots, but leave every point on the vertical line unchanged. The dashed arrows below are meant to suggest the assignment:



In partial symbolic notation:

$$R(\text{upper black dot}) = \text{upper red dot}, \qquad R(\text{lower black dot}) = \text{lower red dot}$$

The following animation by Sunil Koswatta is meant to go with this definition of a reflection:

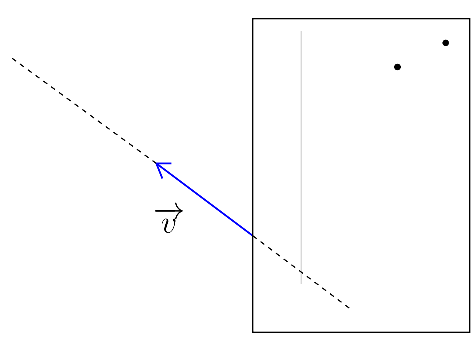
http://www.harpercollege.edu/˜skoswatt/RigidMotions/reflection.html

It goes without saying that R moves *every* point in the plane not lying on L to the "opposite side" of L, and the two points above are meant to merely suggest what happens in general. There is a reason why we used two points (dots) instead of one in this discussion of the reflection R. Of course, one point already suffices for the definition itself. However, the use of two points highlights an obvious property of R: the distance between the original two points (black dots) is equal to the distance between the two reflected points (red dots). This is because the transparency is rigid and cannot be distorted, so the distance between the red dots cannot differ from that between the black dots. We may formulate this fact in more precise language, as follows. If we take two points A and B in the plane and

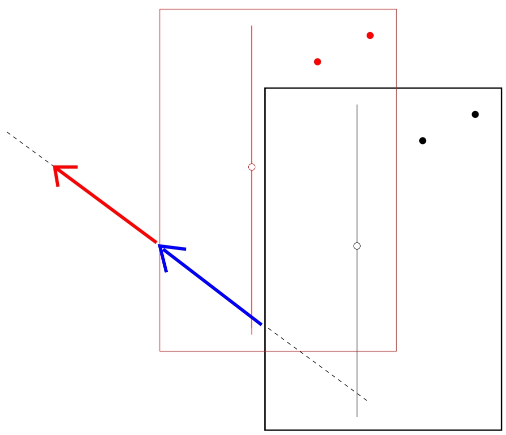
> *if R moves the points A and B to $R(A)$ and $R(B)$, respectively, then the distance between A and B is equal to the distance between $R(A)$ and $R(B)$.*

We refer to this property of R as the **distance-preserving** property.

Next, we define **translation along a given vector $\overrightarrow{v}$**. Let us continue with the same picture of a vertical line with two dots on a piece of paper, and we will describe how to translate this figure. A **vector** is just a segment together with the designation of one of its two endpoints as a **starting point**; the other endpoint will be referred to simply as **the endpoint** of the vector and will be pictorially distinguished by an arrowhead, as shown in the blue vector below.



We are going to define the translation T along the given blue vector $\overrightarrow{v}$. We copy the line and the dots and the vector on a transparency in red; in particular, the copy of $\overrightarrow{v}$ on the transparency will be referred to as the *red* vector. Let the line containing the blue vector be denoted by ℓ (this is the slant dashed line in the picture below). We now **slide the transparency along** $\boldsymbol{\overrightarrow{v}}$, in the sense that the red vector on the transparency glides along ℓ in the direction of $\overrightarrow{v}$ until the starting point of the red vector rests on the endpoint of the blue arrow, as shown.

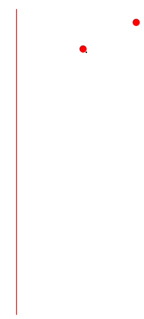


The whole red figure is seen to move "in the direction of $\overrightarrow{v}$ by the same distance". Then by definition, T moves the black dots to the red dots. Precisely, the rule of assignment of T moves the point *in the plane* represented by the upper (respectively, lower) black dot to the point in the plane represented by the upper (respectively, lower) red dot.

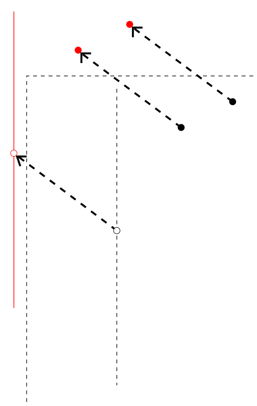
The following animation of essentially this translation by Sunil Koswatta would be helpful to a beginner:

http://www.harpercollege.edu/˜skoswatt/RigidMotions/translation.html

If we draw the translated figure (of the above vertical line and two black dots) by itself without reference to the original, it would be visually indistinguishable from the original:

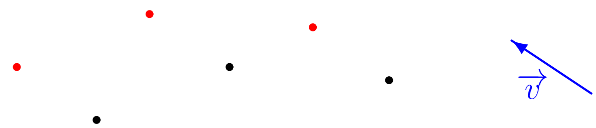


So we put in the black figure as background information to show where the red figure *used to be*. Then T moves the points represented by the black dots to the corresponding points represented by the red dots, and moves each point in the black vertical line to a point in the red vertical line. The dashed arrows are meant to suggest the assignment.



For exactly the same reason as in the case of a reflection, a translation is distance-preserving: if A and B are any two points in the plane and if T assigns the points $T(A)$ and $T(B)$ to A and B, respectively, then the distance between A and B is equal to the distance between $T(A)$ and $T(B)$.

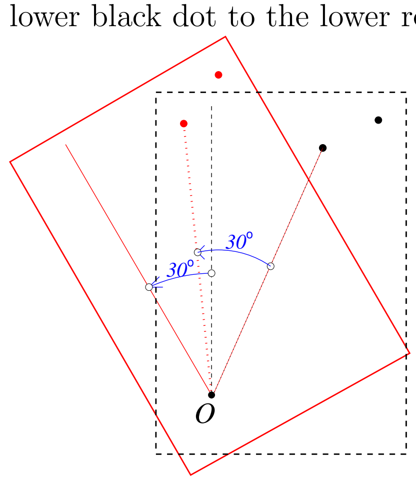
Intuitively, if T is the translation along $\overrightarrow{v}$, then no matter what the point A is, the vector with starting point A and endpoint $T(A)$ will have the same length and the same direction as $\overrightarrow{v}$. This fact will be proved in the high school course on geometry as it follows from one of the standard characterizations of a parallelogram. In the eighth grade, it suffices to verify such phenomena experimentally by measurements. Thus if each A and $T(A)$ are represented by a black dot and a red dot, respectively, then we have the following pictorial representation of the translation of three points.



(*Question:* If a segment joins two of the black dots, what will happen to all the points on the segment?)

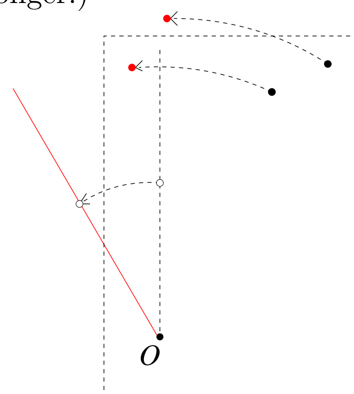
We make an observation about translations: If $\overrightarrow{0}$ is the **zero vector**, i.e., the vector with 0 length, which is a point, the translation along $\overrightarrow{0}$ then leaves every point unchanged. This is the **identity** basic rigid motion, usually denoted by I. Thus $I(P) = P$ for every point P.

Finally, we define a **rotation *Ro* around a given point *O* of a fixed degree**. The point O is called the **center of rotation** of Ro. For definiteness, let the center O of this rotation be the lower endpoint of the vertical line segment we have been using, and let the rotation be 30 degrees *counterclockwise* around this point (one could also do a clockwise rotation, see below). Again, we trace the vertical line segment and the two dots on a transparency in red. Then we pin the transparency down at the lower endpoint of the segment and (keeping the paper fixed, of course) rotate the transparency counterclockwise 30 degrees, i.e., so that the angle between the black segment and the red segment is 30 degrees. In the picture below, the rotated figure is superimposed on the original figure and, as usual, the red rectangle represents the border of the transparency. By definition, the rotation moves the upper black dot to the upper red dot, and the lower black dot to the lower red dot.



Observe that the angle formed by the ray from the center of rotation to a black dot and the ray from the center of rotation to the corresponding red dot is also 30 degrees.

We now draw the rotated figure as a geometric figure in the plane with the dashed black figure provided as background information to show where the red figure *used to be*. The dashed arcs indicate the rule of assignment by Ro. (Note that by the nature of a rotation around a given center, the farther a point is away from the center of rotation, the farther it gets rotated. This is why the dashed arc that is farther away from the center of rotation is longer.)



So far we have discussed rotations of positive degrees, and they are, by definition, the counter-clockwise rotations. A rotation of negative degree is defined exactly as above, except that the transparency is now rotated *clockwise*.

The following two animations by Sunil Koswatta show how a rotation of 35 degrees (respectively, -35 degree) rotates a geometric figure consisting of three points and an angle whose vertex is the center of the rotation:

http://www.harpercollege.edu/˜skoswatt/RigidMotions/rotateccw.html

http://www.harpercollege.edu/˜skoswatt/RigidMotions/rotatecw.html

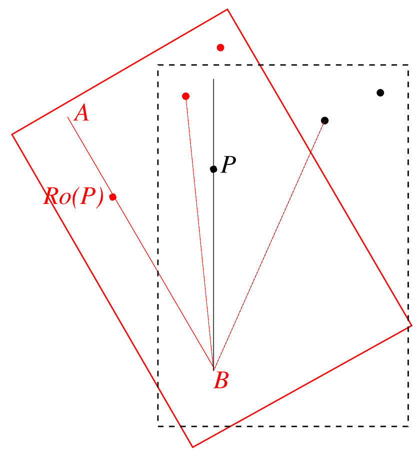
For the usual reasons, a rotation is distance-preserving. Note that a rotation of 0 degrees is also the identity basic rigid motion I.

Motions of entire geometric figures

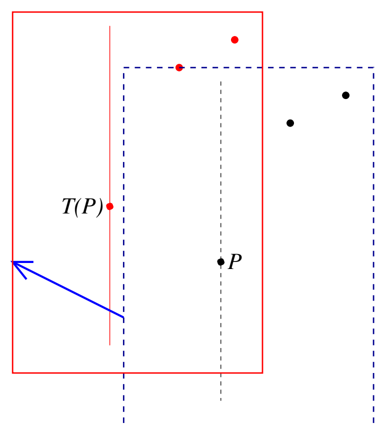
We now introduce some terminology to facilitate the ensuing discussion. Given a basic rigid motion F, it assigns a point—to be denoted by $F(P)$—to a given point P in the plane. We say $F(P)$ is the **image** of P under F, or that F **maps** P to $F(P)$. If you wonder about why we use the word "map" in this context, think about the drawing of the street map of a city, for example. Are we not *mapping* points in the streets one by one on a piece of paper?

We have given a description of how a reflection, a translation, or a rotation moves each point, but such information is not particularly illuminating because it does not reveal in a distinctive way what a reflection, translation, or rotation does, or how a reflection is different from a rotation. What we do next is to examine a bit how a basic rigid motion moves, not just a point, but a whole geometric figure, in the following sense. Given a geometric figure $\mathcal{S}$ in the plane, then a given point P in $\mathcal{S}$ is mapped by F to another point $F(P)$. Now focus entirely on $\mathcal{S}$ and observe what the total collection of *all* the points $F(P)$ looks like *when P is restricted to be a point of $\mathcal{S}$*. For understandable reasons, we denote such a collection by the symbol $\boldsymbol{F(\mathcal{S})}$ and call it the **image of $\boldsymbol{\mathcal{S}}$ by $\boldsymbol{F}$**. (We also say **$\boldsymbol{F}$ maps $\boldsymbol{\mathcal{S}}$ to $\boldsymbol{F(\mathcal{S})}$.**) For example, in the preceding picture of the 30-degree counterclockwise rotation Ro around the lower endpoint of the vertical segment, let the lower endpoint be denoted by B and let $\mathcal{S}$ denote this vertical segment.[3] Then $Ro(\mathcal{S})$ is the red segment AB, as shown. It makes a 30-degree angle with $\mathcal{S}$.

[3]Notice that in this instance, we are taking the picture literally and regard the segment for what it really is: a segment. By contrast, we have, up to this point, used this segment to represent the whole vertical line.

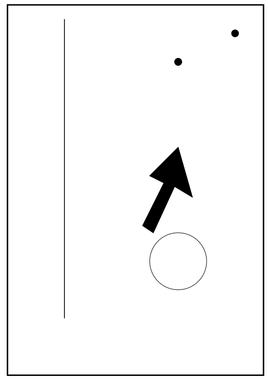


Now, let T be the translation along the blue vector that we encountered earlier on page 14, and if $\mathcal{S}$ continues to denote the same vertical segment, then $T(\mathcal{S})$ becomes the red segment which is now parallel to $\mathcal{S}$ rather than making a 30-degree angle with $\mathcal{S}$ at its lower endpoint.

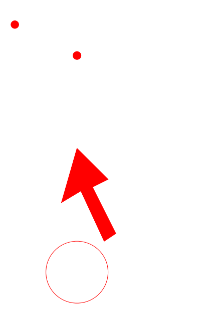


We see that by looking at the image of a *segment*, we obtain at a glance a fairly comprehensive understanding of the basic difference between a rotation and a translation, something that is not possible if we just look at the cut-and-dried descriptions of how these basic rigid motions move the points, *one point at a time*.

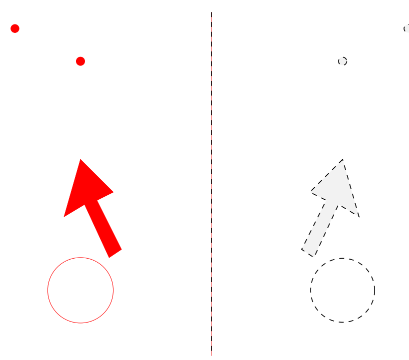
We proceed to create a geometric figure slightly more complex than a mere segment, one that will better reveal the effects of the three basic rigid motions. We add a non-vertical, solid arrow and a circle, as shown.



Make the drawing on the paper in black, as shown, and trace the whole figure on a transparency in red, as usual. Consider first the reflection across the vertical line. Then flipping the transparency across the vertical line exactly as before, we obtain the following reflected image of the figure all by itself.



We now superimpose it on the original black figure in order to get a sense of how the figure has been moved by the reflection. The black figure, drawn with dots, represents where the red figure *used to be*.

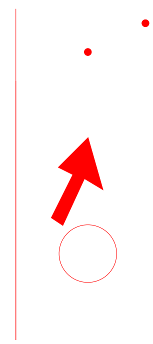


Observe that, under the reflection, the images of the two black dots are the two red dots, respectively. The image of the black arrow is the red arrow and the image of the black circle is the red circle. The image of the vertical line of reflection is of course the vertical line itself, and the image of every point on this line is the point itself. While the black arrow is on the right of the vertical line and points to the upper right, the red arrow to the left of the line of reflection points to the upper left. The latter is because the distance of the tip of the black arrow is further from the vertical line of reflection than the lower end of the black arrow. Since the flipping preserves the distance, the tip of the red arrow is likewise further from the vertical line than the lower end of the red arrow. But the red arrow being to the left of the vertical line, we see that the red arrow is now pointing to the upper left. Similarly, the black circle, being slightly to the right of the black arrow, is reflected to the red circle which is slightly to the left of the red arrow. The fact that a reflection across a vertical line switches left and right can be easily verified by looking at yourself in a mirror.

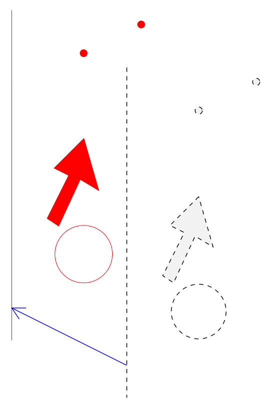
The reflection of a more complex figure allows us to make additional observations about reflections. A reflection **preserves lines, rays, and segments** in the sense that the image of a line (respectively, a ray, a segment) by a reflection is a line (respectively, a ray, a segment). After all, a line or a ray or a segment on the transparency, no matter where it is placed on a piece of paper, remains a line, a ray, or a segment after the reflection. For exactly the same reason, a reflection is not only distance-preserving, as we already know by now, but is also **degree-preserving** in the sense that the image of an angle is an angle of the same degree. For example, the angle at the tip of the black arrow is reflected to the angle at the tip of the red arrow, and the two angles must have the same degree because the red arrow is an exact copy of the black arrow.

By experimenting with reflections of different figures across different lines, students can obtain a robust intuitive understanding of what a reflection does to points in the plane.

Next, we look at the effect that a translation has on the same figure. Let us translate along the same blue vector $\vec{v}$ as before. We trace the figure on a transparency in red, and we slide the transparency along $\vec{v}$ (see page 14 for the definition). Then the red figure, which is the translated image of the original, now looks like this:

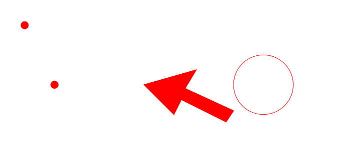


By itself, the translated image sits in the plane in such a way that it "looks exactly like the original". It makes more sense to show it against the background of where it used to be, so we place it alongside the original black figure, now drawn with dots. Keep in mind, however, that the black figure is merely background information.

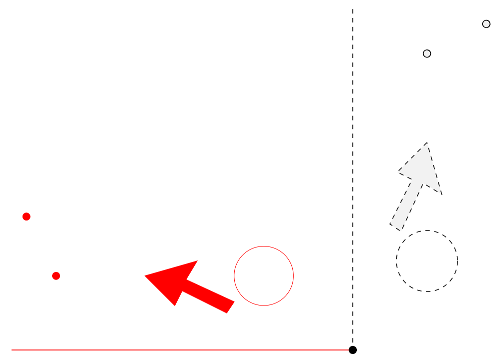


Naturally, *a translation preserves lines, rays, and segments, and is both distance- and degree-preserving.*

Finally, we rotate the same figure around the lower endpoint of the vertical segment, 90 degrees counterclockwise. This is realized by pinning the transparency at the lower endpoint of the vertical line segment and then rotating the transparency 90 degrees counterclockwise. Here is the rotated image:



Here is the superimposed image of the transparency on the paper; it gives a better sense of what this 90-degree rotation does to the plane:



Note that although we have used only rotations of 30 and 90 degrees for illustration, direct manipulations of a transparency make it easy to do rotations of *any* degree around any point. In a classroom, students should be encouraged to experiment with rotations of arbitrary degrees to deepen their intuitive grasp of what rotations are like.

As before, we observe that *rotations map lines, rays, and segments to lines, rays, and segments, and are distance- and degree- preserving* for exactly the same reason.

Assumptions on basic rigid motions

Let us summarize our findings thus far. Hands-on experiences, such as those above, predispose us to accept as true that the basic rigid motions (reflections, translations, and rotations) share three common "rigidity" properties:

1. They map lines to lines, rays to rays, and segments to segments.

2. They are **distance-preserving**, meaning that the distance between the images of two points is always equal to the distance between the original two points.

3. They are **degree-preserving**, meaning that the degree of the image of an angle is always equal to the degree of the original angle.

Notice that property 1 implies that a basic rigid motion maps angles to angles, and this is why in property 3 we can speak about "the degree of the image of an angle".

These are our **assumptions** about basic rigid motions, i.e., we will henceforth agree that every basic rigid motion has these properties.

We will also accept as true the fact that there are "plenty of" basic rigid motions, in the following sense:

R Given any line, there is always a reflection across that line.

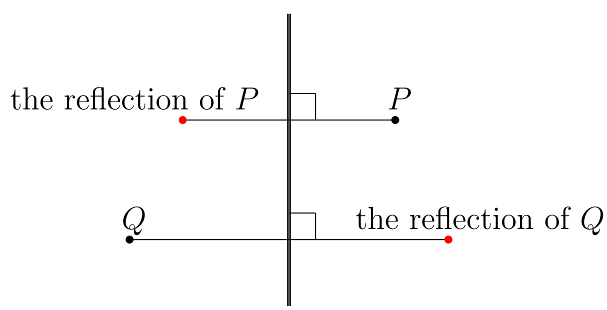
T Given any vector, there is always a translation along that vector.

Ro Given a point and a degree, there is always a rotation (clockwise or counterclockwise) of that degree around the point.

These too are part of our **assumptions** about basic rigid motions.

We now give a few more details, *for the teachers*, about the definitions of the basic rigid motions. Whether or not such details should be presented in an eighth grade classroom is the kind of decision only a teacher can make on a case-by-case basis.

The reflection across the line of reflection assigns to each point on the line of reflection the point itself, and to any point not on the line of reflection it assigns the point which is **symmetric to it with respect to the line of reflection**, in the sense that the line of reflection becomes the perpendicular bisector of the line segment joining the point to its reflected image.

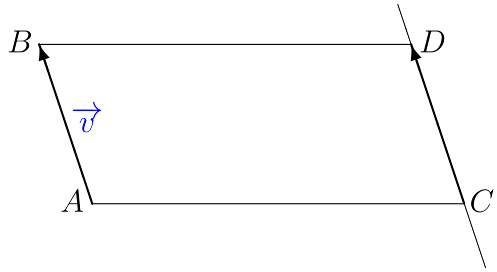


For a translation along a given vector $\overrightarrow{v}$, we describe the point D that is assigned to a given point C. First, assume that C does not lie on line L_{AB}. Let the starting point and endpoint of $\overrightarrow{v}$ be A and B, respectively. Then:

(1) Draw the line passing through C and parallel to line L_{AB}.

(2) Draw the line passing through B and parallel to the line L_{AC}.

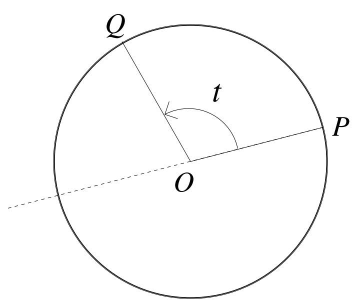
(3) Let the point of intersection of the lines in (1) and (2) be D. By definition, the translation assigns the point D to C.



If C lies on the same line as A and B, then D is obtained by going along this line from C in the same direction as A to B until the length of CD is equal to the length of AB.

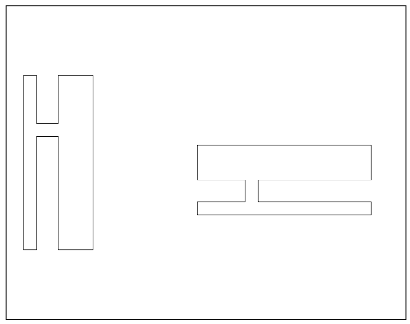
Because we will be discussing the length of a segment AB often, we will agree to use the symbol $\boldsymbol{|AB|}$ to denote this length. Be aware that there is no universal agreement on this particular notation. In any case, with this notation understood, we see that if the translation along a given vector $\overrightarrow{AB}$ moves C to D, then $|CD| = |AB|$.

Finally, we define the rotation Ro of t degrees around a given point O, where $-180 \leq t \leq 180$. The rotation could be counterclockwise or clockwise, depending on whether the degree of rotation is positive or negative, but for definiteness, we will deal with the counterclockwise case, i.e., $t > 0$. Let a point P be given. If $P = O$, then by definition, $Ro(O) = O$. If P is distinct from O, draw the circle with center O and radius $|OP|$ (see picture below). On this circle, go from P in a counterclockwise direction until we reach the point Q so that $|\angle QOP| = t^\circ$. Then by definition, Ro assigns Q to P.



Compositions of basic rigid motions

Having explained the meaning of the basic rigid motions, it is time to go to the next level and explain the concept of a **composition of basic rigid motions**, which means moving the points of the plane by use of two or more basic rigid motions *in succession*, one after the other. The need for composition is easily seen by considering an example. Suppose the following two identical H's are paced in the plane as shown. Is there a single basic rigid motion so that it maps one of these H's to the other?



Nothing obvious immediately comes to mind, so it would seem reasonable to do it in more than one step[4]: we will try to use two or more basic rigid motions applied in succession to map one H to the other.

There are many ways to accomplish this, but a key observation is that the upper right corner of the vertical H matches the upper left corner of the horizontal H. Therefore it makes sense to first bring these two corners together by a translation and then use an additional rotation to bring the vertical H to the horizontal H. So we translate along the blue vector (see picture below) going from the upper right corner of the *left* H to the upper left corner of the *right* H, and follow by rotating 90 degrees

[4]Actually there is a rotation that gets it done, but that is not obvious.

counterclockwise around the endpoint of the vector (designated by a black dot).

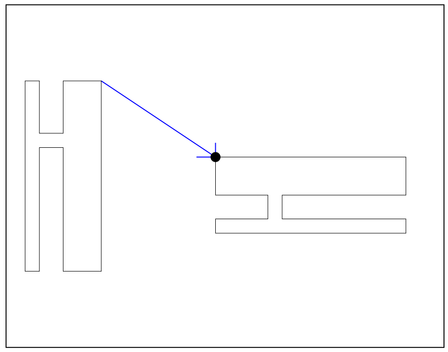


Figure 1

To visualize this translation, we use a transparency. Let the preceding figure of the two H's be drawn on a piece of paper in black. Trace the figure in red on a transparency and, while keeping the paper fixed, slide the transparency along the blue vector (see page 14 for the definition of "slide"); the translated figure in red is now shown together with the original figure in dashed lines. (Again, we use the dashed figure to remind ourselves of the original position of the figure.) Our focus is on the fact that the vertical black H has now been moved to the position of the vertical red H. In the process, of course the horizontal black H is also moved and the black dot is likewise moved, as shown below. But the key point is that, while a basic rigid motion necessarily moves every point in the plane, we have to remember why we use this particular basic rigid motion in the first place and maintain our concentration on the object of interest. In this case, it is the vertical black H and we will follow this particular figure assiduously.

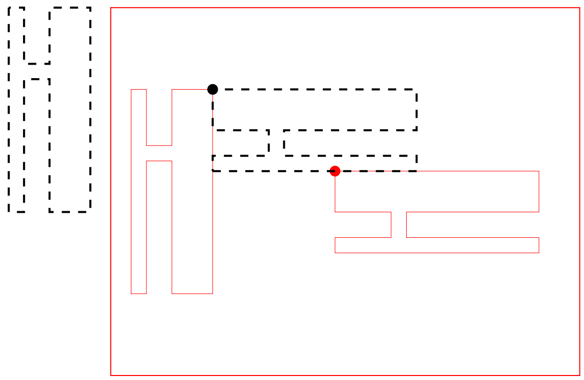


Figure 2

In order to move the (points in the) vertical red H to the position of the horizontal black H (indicated by the dashed lines), we will now apply the second basic rigid motion to the plane by rotating the plane 90 degrees counterclockwise around the black dot. This rotation maps every point in the plane to a different point (with the exception of the black dot, which remains fixed), but *we are only interested in what this rotation does to* (*the points in*) *the vertical red H.* We can find out by rotating the transparency around the *black* dot 90 degrees counterclockwise, as shown:

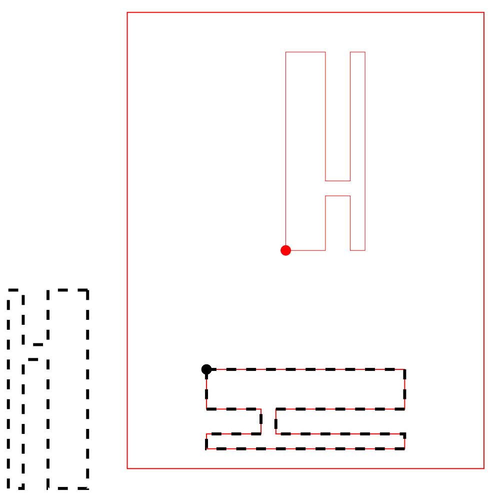
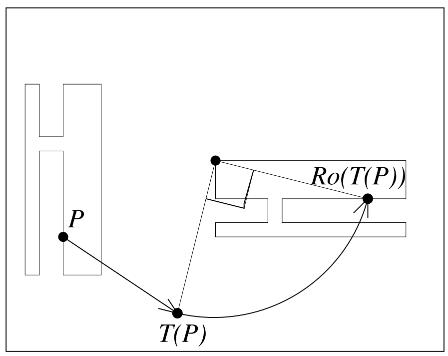


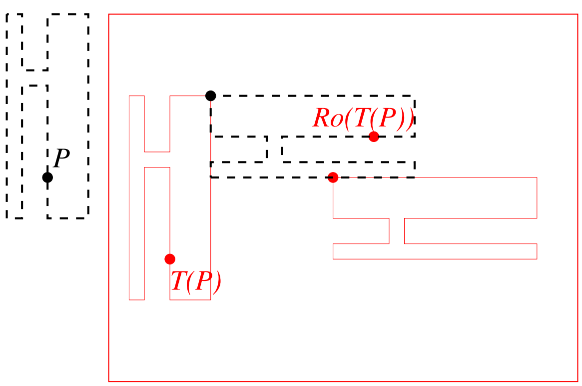
Figure 3

Now observe that the red vertical H coincides with the black dashed horizontal H, as shown in Figure 3. This is what we want.

Several additional comments may be helpful. The first is that we are only interested in what the rotation does to the points in the red vertical H in Figure 2. The fact that the rotation also moves the red *horizontal* H in Figure 2 to the red vertical H in Figure 3 was consequently ignored in the discussion because this fact was irrelevant to our concern. In addition, the situation may be better understood with the use of symbols. Let T denote the translation along the blue vector and Ro denote the 90-degree counterclockwise rotation of the plane around the black dot. Given a point P, then T moves it to another point $T(P)$ in the plane. (For this notation, see page 18.) Similarly, the point assigned by Ro to a given point Q will be denoted by $Ro(Q)$. The picture below shows how a typical point P of the vertical H in Figure 1 is moved to $T(P)$ by the translation, and then to $Ro(T(P))$ by the rotation.



Let us show the same points superimposed on Figure 2: If we follow the points $P \to T(P) \to Ro(T(P))$, then we get a new perspective on the way the transparency was moved twice to get from Figure 1 to Figure 2 and finally to Figure 3.

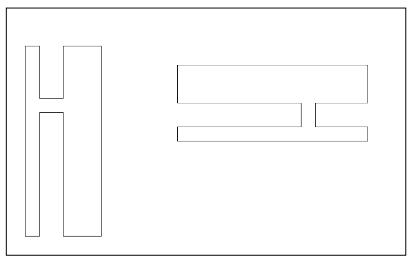


Finally, the rotation Ro is by definition around the black dot of Figure 1, which is also the black dot of Figure 2. This is why when we rotated the transparency in

Figure 2, we did so around that black dot, but *not* around the red dot regardless of the fact that the translation had moved the black dot to the red dot.

To summarize: (1) We apply the above translation to the plane, and (2) follow it with the preceding rotation. Then (3) the two basic rigid motions together move the left vertical black H of Figure 1 to coincide exactly with the right horizontal black H (in Figure 1).

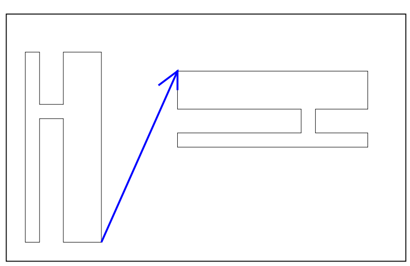
We give another example of composition of rigid motions. Suppose two identical H's have been placed in the plane as shown. What basic rigid motions can we use in succession to move the H on the left to coincide with the H on the right?



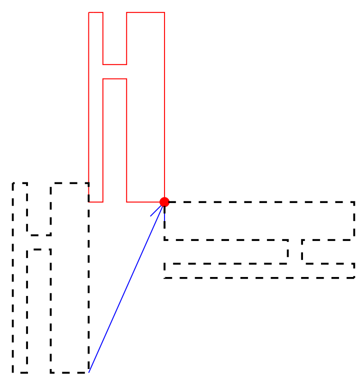
Before we describe a sequence of basic rigid motions to achieve this goal, we point out that Larry Francis has created an animation to accompany this description:

Composition of Rigid Motions (translation, rotation, and reflection)

First, we map the lower right corner of the left vertical H to the upper left corner of the right horizontal H. So we translate the plane along the blue vector, as shown:



Again we will demonstrate the effect of the translation with the help of a transparency, but this time we will do something a little different. We have made the point above that when we apply a basic rigid motion to the plane for a specific purpose, this motion will move *all* the points in the plane, including those that we may not be interested in. Assuming that this point has been well made, the succeeding figures will focus completely on the figure of interest and ignore everything else. Therefore this time around, we only trace the vertical H onto a transparency in red, and then we slide the transparency along the blue vector (see page 14 for the definition of *slide*). Then the black vertical H is shown to be moved to the position indicated by the red H of the transparency. The black dashed H's in the picture below serve to remind us where the original figure used to be. In particular, they remind us that our goal is to move the vertical red H to where the dashed horizontal H is.



We have to ***move the plane*** (i.e., the transparency) again by using basic rigid motions until the (points of the plane indicated by the) vertical red H on the transparency coincides with the black dashed horizontal H. If we rotate the plane (i.e., the transparency) around the endpoint of the blue vector (indicated by the red dot) 90 degrees clockwise, the red figure will assume the following position:

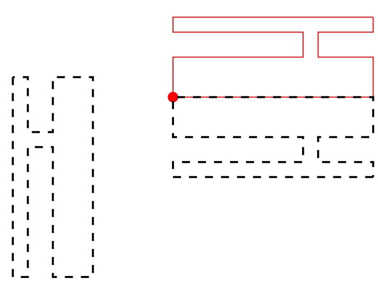
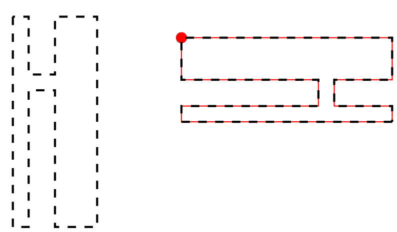


Figure 4

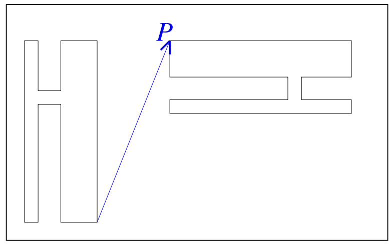
Let us be clear about what the red figure means: this is the position of the black H moved first by the translation along the blue vector, and then *followed by* the clockwise rotation of 90 degrees around the endpoint of the blue vector. Specifically, under the consecutive actions of these two rigid motions, the vertical black H is moved to the position of the horizontal red H in Figure 4. But even after applying these two rigid motions to the plane, we have not yet achieved the goal we set for ourselves, namely, to move the vertical black H to the black dashed horizontal H. We have only moved the vertical black H to the horizontal red H. At this point, it is clear that our goal will be achieved if can move the horizontal red H in Figure 4 to the dashed horizontal black H (of Figure 4). This can be done by reflecting the plane along the horizontal line that passes through the red dot (not drawn in Figure 4). *What does this reflection do to the red figure?* We can find out as follows. Trace this horizontal line on the transparency. Then lift the transparency, rotate it in space to interchange "above" and "below" with respect to the horizontal line that passes through the red dot, and then put it down on the paper so that the horizontal line on the transparency coincides with the horizontal line on the paper containing the red dot—and so that the red dot on the transparency coincides with the red dot on the paper. When we do that, the new position of the red H is now the following:



Thus the horizontal red H is now exactly where the horizontal black H used to be.

After applying three basic rigid motions in succession, we finally achieve our goal.

To summarize: In order to move the left vertical H of the following picture to coincide with the right horizontal H, we compose three basic rigid motions. We first translate along the blue vector (whose endpoint we now call P), then follow the translation by a 90-degree clockwise rotation around P, and then follow the rotation by a reflection of the plane across the horizontal line that contains P, as shown below.



We now give a more formal definition of composition. Let F and G be two basic rigid motions. Then the **composition $\boldsymbol{F \circ G}$**, or **$\boldsymbol{G}$ followed by $\boldsymbol{F}$** [5] is defined to be the rule that assigns to a given point P the following point: first G assigns the point $G(P)$ to P, and then F assigns the point $F(G(P)$ to $G(P)$, so *by definition*, the rule $F \circ G$ assigns this point $F(G(P))$ to P. As a shorthand, we abbreviated the long-winded description by writing symbolically:

$$\boldsymbol{(F \circ G)(P) \stackrel{\text{def}}{=} F(G(P))}$$

where the symbol "$\stackrel{\text{def}}{=}$" indicates that this is a definition.

Let us make sure that this definition makes sense. First of all, G moves P to the point $G(P)$ of the plane, so it makes sense for the rigid motion F to move the given point $G(P)$ to the point $F(\,G(P)\,)$. Thus the rule that assigns the point $F(G(P))$ to the point P does make sense.

Notice that $F \circ G$ so defined also satisfies properties 1–3 on page 23 shared by the basic rigid motions. Indeed, if we think back on our use of transparencies to define basic rigid motions, then it is clear that the image of a figure under $F \circ G$ is just a relocation of the same figure on the transparency to a different part of the plane, and therefore if the figure is a line, or a ray, or a segment, the image remains a line, or a ray, or a segment. For the same reason, distances and degrees are preserved by $F \circ G$. Of course one can also show this without reference to transparencies. Take a line ℓ, for example, and we will show that $F \circ G$ maps ℓ to a line (first property on page 23). Because G is a basic rigid motion, the *image* (see page 18) $G(\ell)$ of ℓ by G is a line. Now since $G(\ell)$ is a line, the image $F(\,G(\ell)\,)$ of $G(\ell)$ by F is also a line (F is after all a basic rigid motion). But by the definition of $F \circ G$, the image $(F \circ G)(\ell)$ of ℓ by $F \circ G$ is exactly $F(G(\ell))$. Hence $F \circ G$ maps the line ℓ to a line $F(G(\ell))$. In the

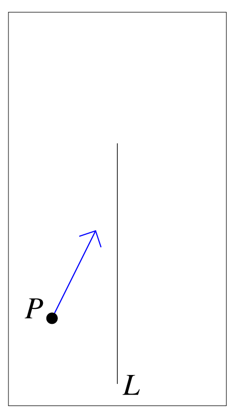
[5] G comes before F, as the following definition makes clear. It is unfortunate that the writing of $F \circ G$ gives the opposite impression when read from left to right.

same way, because a ray or a segment is part of a line, we see that $F \circ G$ maps rays to rays and segments to segments; it also preserves distance and degrees of angles. The conclusion: the composition $F \circ G$ enjoys the same properties 1–3 on page 23 shared by the basic rigid motions..

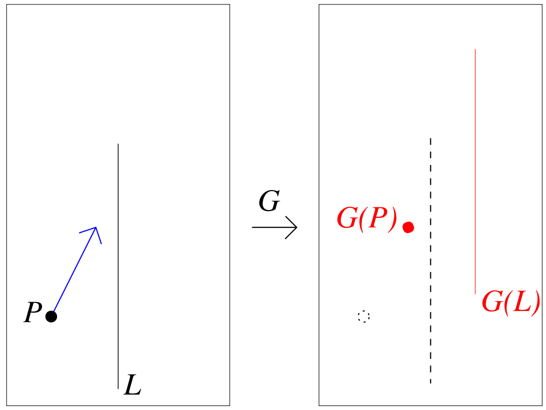
Let us analyze $F \circ G$ a little bit more. The understanding of the composite $F \circ G$ begins with an understanding of what it assigns to a *single* point P. Schematically, this assignment comes in two stages:

$$P \xrightarrow{\quad G \quad} G(P) \xrightarrow{\quad F \quad} F(G(P)) = (F \circ G)(P)$$

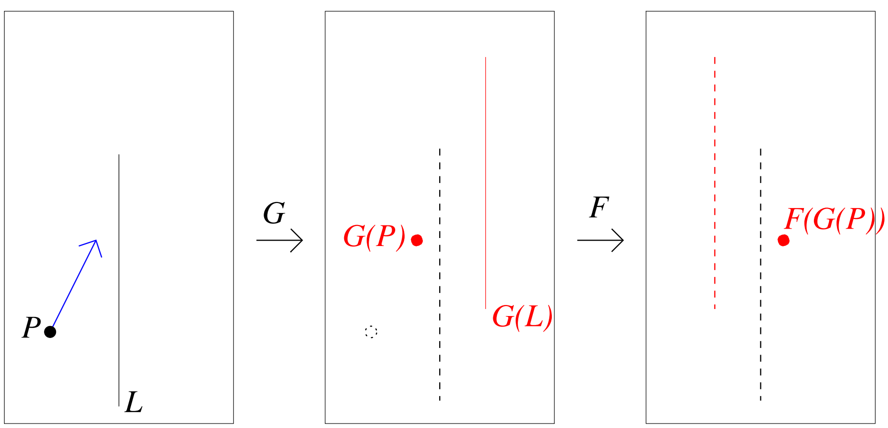
Thus G moves the point P, and then F moves the point $G(P)$. Of course F, being a basic rigid motion of the plane, moves every point of the plane, but for the purpose of finding out what $(F \circ G)(P)$ is, what matters is not what F does to P or any other point but what it does to the specific point $G(P)$. For example, let the plane be represented by the rectangle below, and let a point P and a line L be given. Let G be the translation along the blue vector and F be the reflection across L. We want $(F \circ G)(P)$.



First we apply G to the plane, and G moves P and L to new positions, indicated in red, while the original positions of P and L are indicated by dashed lines.



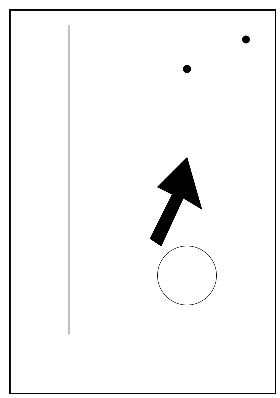
Now $F(G(P))$ is the point assigned to $G(P)$ by F. Here we must be careful about what F is: it is by the definition the reflection across L. Although G has moved L to the red vertical line in the middle rectangle (see the picture below), we must reflect $G(P)$ across L itself, *not* across the red line $G(L)$. In other words, if we reflect the red dot $G(P)$ across the dashed vertical line, we would get $F(G(P))$. (Notice that we make no mention of what $F(P)$ is or what the reflection of $G(P)$ across $G(L)$ is, because neither is relevant. Notice also that F moves the red line $G(L)$ in the middle rectangle to the *dashed* red line in the right rectangle, but this fact is also irrelevant in the determination of what $F(G(P))$ is.)



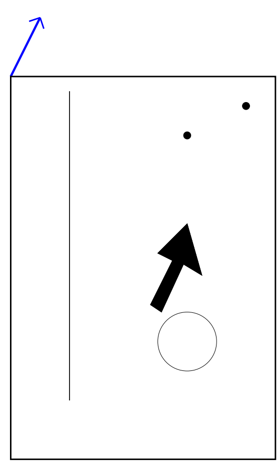
If we have a good idea of what $F \circ G$ does individually to a few points, then we can begin to look at what it does to a figure $\mathcal{S}$. Perhaps reviewing the two preceding examples at this point would be a good idea.

In summary: To find out which point $F \circ G$ assigns to a given point P, first we obtain $G(P)$ and then we focus on what F does to $G(P)$. In terms of transparencies, this observation corresponds to our insistence that, once we have moved the transparency according to the first basic rigid motion G, we are no longer concerned with applying the second rigid motion to the plane (i.e., the paper) itself but *only to the points of the transparency.*

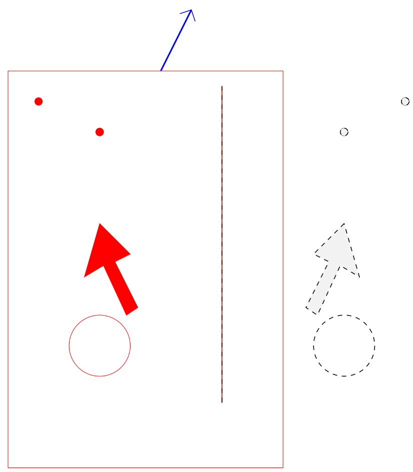
We give a final illustration of the composition of basic rigid motions. Let us go back to our first example of the line-dots-arrow-circle figure in the plane.



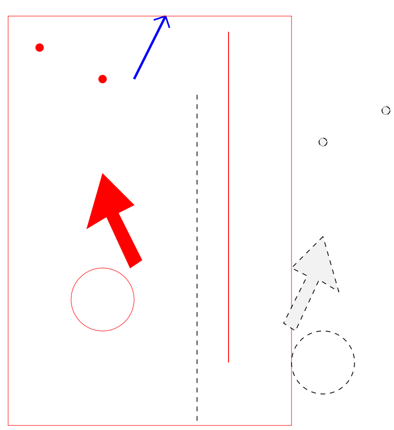
Suppose we want to see the combined effect on this geometric figure of the composition of the reflection across the vertical line followed by the translation along the blue vector as shown.



Trace the figure (note: the blue vector is not part of the figure) and the vertical line on a transparency in red. Recall that the reflection across the vertical line has been shown before: the figure is moved by flipping the transparency across the vertical line to the position in the plane indicated by the red figure.

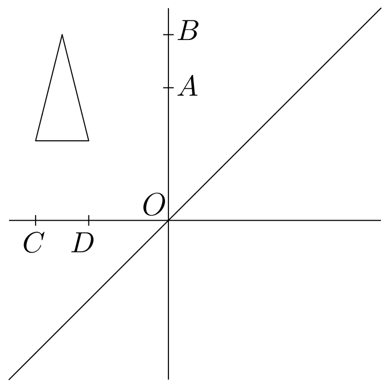


To follow the reflection by the translation, we now have to slide the (flipped) transparency along the blue vector. The red figure now looks like this:



The composition of these two basic rigid motions therefore moves the black figure to the red figure.

It is convenient to assess students' understanding of the circle of ideas surrounding the basic rigid motions by making use of coordinates. Consider for example the following figure, where O is the origin, the line is the "diagonal" that makes a 45-degree angle with the x-axis, the triangle is isosceles, the point $C = (-5, 0)$ is directly below the left vertex of the triangle, the horizontal line through B passes through the top vertex of the triangle, and $A = (0, 5)$.



Let R be the reflection across the "diagonal" line and let T be the translation along the vector $\overrightarrow{OA}$. Consider the two compositions $\varphi = T \circ R$, and $\psi = R \circ T$. If $\mathcal{S}$ denotes the isosceles triangle shown above, what is $\varphi(\mathcal{S})$ and what is $\psi(\mathcal{S})$? The two sets turn out to be not equal, and this is a good illustration of some of the subtleties of composition.

It is easy to make up many other problems of this genre. For example, if Ro is the clockwise rotation of 90 degrees around the origin O, how does $(T \circ Ro)(\mathcal{S})$ compare with $(Ro \circ T)(\mathcal{S})$? Again, they are different.

A more substantial application of the concept of composition is given in the next subsection.

The concept of congruence

There are good reasons why one should devote a generous amount of class time to the composition of basic rigid motions. We have seen that when we are given two identical figures placed in different parts of the plane, it sometimes takes more than one basic rigid motion to map one to the other. It takes a composition of several such motions. (You may paraphrase this by saying that mathematics has very few one-step problems.) A more fundamental reason is the fact that if a rule assigns a point to each point of the plane in a way that preserves distance, then it is equal to a composition of basic rigid motions. We will not need to use this fact and therefore will not prove it here, but it is a good idea to keep it in mind. Incidentally, this is why the basic rigid motions are "basic".

In general, we say two geometric figures are **congruent** if a composition of a finite number of basic rigid motions maps one to the other. We also call the composition of a finite number of basic rigid motions a **congruence**. From the definition, we see that a composition of congruences is also a congruence.

Our hands-on experience using transparencies shows that a basic rigid motion maps a geometric figure to a figure that is, intuitively, "the same size and the same shape." For this reason, two congruent figures are intuitively "the same size and same shape". However, this well-known phrase cannot be used as a definition of congruence because the concepts of "same size" and "same shape" are too imprecise to furnish a basis for logical reasoning. We repeat: the only correct definition of congruence between two-dimensional figures is that one can be obtained from the other by a composition of a finite number of rotations, reflections, and translations (see Standard 8.G 2).

Because basic rigid motions preserve lines, rays, segments, lengths and degrees, two congruent triangles necessarily have three pairs of equal sides and three pairs of equal angles. The converse is also true: two triangles with three pairs of equal sides and three pairs of equal angles are congruent. However, much stronger versions of the converse exist, and they are as useful as they are important. The overriding idea is that triangles are special, so instead of requiring *six* sets of conditions to guarantee triangle congruence (three for sides and three for angles), a judicious choice of *three* sets of conditions is sufficient. The following three theorems along this line are the best known. Their proofs can be given right now, but since there are more urgent things to do in eighth grade, they will be put off until the high school course on geometry.

> **SAS criterion for congruence.** If two triangles have a pair of equal angles (i.e., same degree) and corresponding sides of these angles in the triangles are pairwise equal (in length), then the two triangles are congruent.
>
> **ASA criterion for congruence.** If two triangles have two pairs of equal angles and the common side of the angles in one triangle is equal to the

corresponding side in the other triangle, then the triangles are congruent.

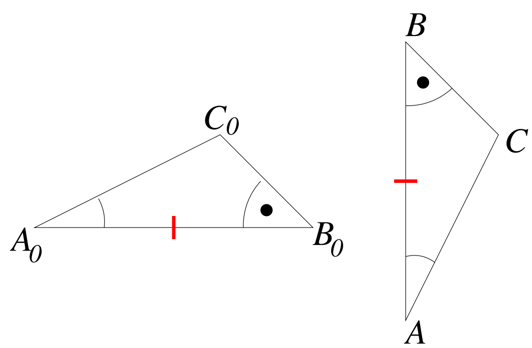
SSS criterion for congruence. If two triangles have three pairs of equal sides, then they are congruent.

At this stage, it suffices for students to verify these theorems experimentally by drawing pictures or by using a geometric software that allows for the precise drawing of triangles and the ability to move a geometric figure undistorted across the computer screen so that the congruence of triangles can be checked visually. Here we will offer an *informal* proof of ASA ("informal" means that, while the overall idea is correct, some details are missing) together with a link to an animation of the proof created by Larry Francis:

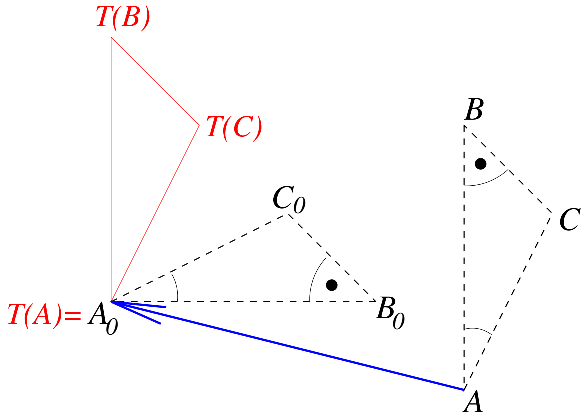
Angle-Side-Angle Congruence by Basic Rigid Motions

Thus we have two triangles ABC and $A_0B_0C_0$ so that $|\angle A| = |\angle A_0|$, $|AB| = |A_0B_0|$, and $|\angle B| = |\angle B_0|$. We have to produce a congruence (see page 39) F so that $\boldsymbol{F(\triangle ABC) = \triangle A_0B_0C_0}$, where the notation means:

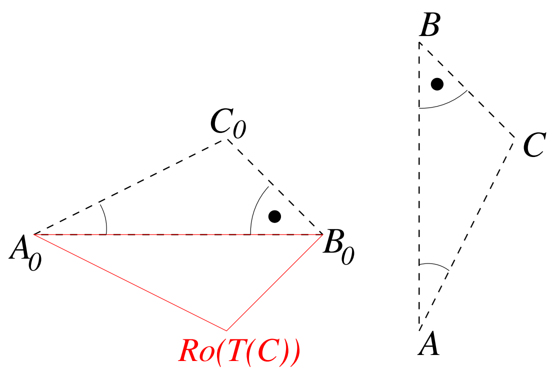
$$F(A) = A_0, \quad F(B) = B_0, \quad F(C) = C_0$$



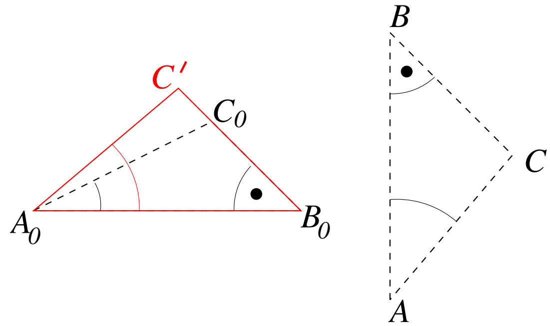
Step 1: Bring vertices A and A_0 together. If $A = A_0$ already, do nothing. If not, let T be the translation along the vector $\overrightarrow{AA_0}$. Then $T(\triangle ABC)$ is a triangle with one vertex in common with $\triangle A_0B_0C_0$. Visually this translation can be realized by the use of transparency, as follows. On a transparency, trace out $\triangle ABC$ in red and then slide $\triangle ABC$ along the vector $\overrightarrow{AA_0}$ (see picture below). Notice that we leave out $\triangle A_0B_0C_0$ from the transparency because what T does to it is not our concern at the moment. We also draw the original positions of $\triangle ABC$ and $\triangle A_0B_0C_0$ in dashed lines as a reminder of where things used to be.



Step 2: Bring the sides AB and A_0B_0 together. If the translated AB, $T(AB)$, already coincides with A_0B_0, do nothing. Otherwise, since $A = A_0$, a rotation Ro of a suitable degree around A_0 would bring the ray from A_0 to $T(B)$ to coincide with the ray from A_0 to B_0. Then because of the assumption that $|AB| = |A_0B_0|$, the same rotation would bring $T(B)$ to B_0.



Step 3: Bring vertices C and C_0 together. If the point $Ro(T(C))$ and the point C_0 are on opposite sides of the line joining A_0 to B_0, then the reflection R across this line would bring the point $Ro(T(C))$ to the same side of C_0. We may therefore assume that, after (possibly) a translation and a rotation and a reflection, the point C is brought to a point C' which lies on the same side as the point C_0 with respect to the line joining A_0 to B_0. See the following picture.



Now, we claim that, appearance to the contrary (as in the above picture), the ray from A_0 to C' must coincide with the ray from A_0 to C_0. This is because the basic rigid motions preserve degrees of angles (see page 23) and therefore $\angle C'A_0B_0$ is equal to $\angle A$, which is assumed to be equal to $\angle C_0A_0B_0$. Thus $|\angle C'A_0B_0| = |\angle C_0A_0B_0|$, and since C' and C_0 are on the same side of the line joining A_0 to B_0, the two sides A_0C_0 and A_0C' coincide *as rays*. Similarly, the ray from B_0 to C_0 coincides with the ray from B_0 to C'. But C' is the intersection of the ray from A_0 to C' and the ray from B_0 to C', while C_0 is the intersection of the ray from A_0 to C_0 and the ray from A_0 to C_0. Thus $C' = C_0$, which means after (possibly) a translation and a rotation and a reflection, A, B, and C are brought respectively to A_0, B_0, and C_0. We have proved the ASA criterion.

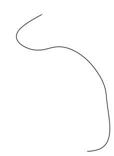
It remains to point out that Larry Francis has also created an animation for the proof of SAS:

Side-Angle-Side Congruence by basic rigid motions

2. Dilation and similarity

Dilations and the Fundamental Theorem of Similarity

So far we have dealt with rules of assignment in the plane that move points in a distance-preserving manner (see page 23 for the definition). Now we will confront an important class of such rules that definitely are *not* distance-preserving. Consider this question: Given a wiggly curve such as the following, how can we "double its size"?



One of the purposes of this section is to show how this can be done and, in the process, clarify what it means to "double the size" of a geometric figure.

The heart of the matter is how to devise a rule D that moves points of the plane in such a way that the distance between *every* two points is changed by a "scale factor" of r, where r is a positive number,i.e., $r > 0$. Let us make sure the the meaning of this statement is clear. Suppose we start with two points P and Q, and suppose D moves them to P' and Q', respectively. Then to say D changes distance by a scale factor of 5 means that the distance between P' and Q', which is taken to be the length $|P'Q'|$ of the segment $P'Q'$, is 5 times the distance $|PQ|$ between P and Q, no matter what P and Q may be. (We introduced the notation of $|PQ|$ on page 25.) In symbolic shorthand, this information is succinctly expressed as

$$|P'Q'| \;=\; 5\,|PQ| \quad \text{for all } P \text{ and for all } Q.$$

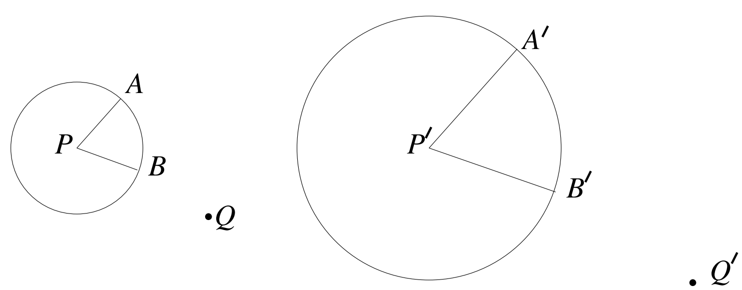
Similarly, to say D changes distance by a scale factor of $\frac{1}{3}$ means that the distance $|P'Q'|$ between P' and Q' is $\frac{1}{3}$ times the distance $|PQ|$ between P and Q, no matter what P and Q may be, i.e.,

$$|P'Q'| \;=\; \frac{1}{3}\,|PQ| \quad \text{for all } P \text{ and for all } Q.$$

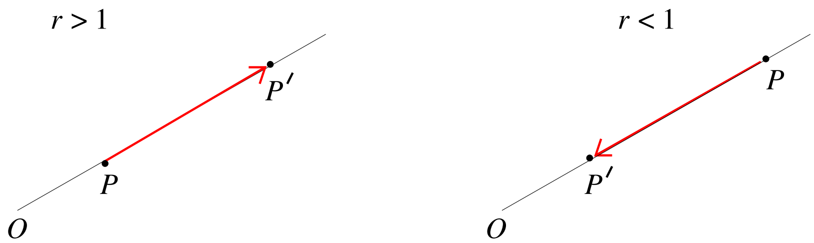
Thus such a D either magnifies or contracts, depending on whether the scale factor r is bigger than 1 (i.e., $r > 1$) or smaller than 1 (i.e., $r < 1$). Now we return to the initial question: how to get such a D? It would seem that this is impossible because we are asking for too much, in the following sense. Let us fix the scale factor to be 2 (let us say) and let us look at a point P and a circle $\mathcal{C}$ of radius 1 around P. As before, let $P' = D(P)$. If A is a point of $\mathcal{C}$ and $A' = D(P)$, then $|P'A'| = 2\,|PA| = 2$ so that A' lies on the circle of radius 2 around P'. Thus we would expect the image $\mathcal{C}' = D(\mathcal{C})$, to be the circle of radius 2 around P'. Now take any two points A and B on $\mathcal{C}$, so that $|PA| = |PB| = 1$. Let $A' = D(A)$ and $B' = D(B)$. If D does magnify distance by a factor of 2, then we would expect $|A'B'| = 2\,|AB|$, *no matter what A and B may be*. Now this does not seem likely because A' and B' are where they are in order to satisfy the requirement that $|P'A'| = 2\,|PA|$ and $|P'B'| = 2\,|PB|$, and there is no reason to expect that they would satisfy this additional requirement of $|A'B'| = 2\,|AB|$. To make matters worse, if Q is another point and $Q' = D(Q)$, then A', B' must also satisfy

$$|A'Q'| \;=\; 2\,|AQ| \quad \text{and} \quad |B'Q'| \;=\; 2\,|BQ|,$$

while at the same time $|P'Q'| = 2\,|PQ|$. How is this possible?



The answer to this question turns out to be both surprising and simple. Fix a point O and a scale factor r ($r > 0$), and we now describe D: D does not move O, i.e., $D(O) = O$, but moves any other point P in such a way that, if $r > 1$, D pushes P away from O along the ray R_{OP} by a factor of r (i.e., if $P' = D(P)$, we require $|OP'| = r\,|OP|$,) but if $r < 1$, D pulls P toward O along the ray R_{OP} by the same factor r.



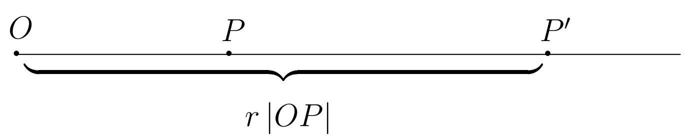
The surprising part is that such a simple definition of D actually meets all the requirements (see (i) on page 47).

We now make the formal definition.

Definition. *A* **dilation D with center O and scale factor r ($r > 0$)** is a rule that assigns to each point P of the plane a point $D(P)$ so that

(1) $D(O) = O$.

(2) *If $P \neq O$, the point $D(P)$, to be denoted more simply by P', is the point on the ray R_{OP} so that $|OP'| = r|OP|$.*



Here we restrict the scale factor to be a positive number. This is just a pedagogical decision. One could allow it to be negative too, but it is neither necessary nor (in our opinion) desirable.

Note that unless the scale factor r is equal to 1, a dilation is not a congruence. The easiest way to see this is to consider two points P and Q **in the same radial direction** of O, meaning that P and Q lie on the same ray issuing from O. Thus let P and Q be any two points on such a ray so that $|OQ| > |OP|$ and let $P' = D(P)$ and $Q' = D(Q)$.

We claim: $|P'Q'| = r\,|PQ|$. This is because

$$|P'Q'| = |OQ'| - |OP'| = r\,|OQ| - r\,|OP| = r\,(|OQ| - |OP|) = r\,|PQ|$$

This shows that the distances between points in the same radial direction are changed by the dilation D by a factor of r. Consequently, D is not distance-preserving (see page 23) if $r \neq 1$.

Now if points P and Q are not in the same radial direction of O, does the equality $|P'Q'| = r\,|PQ|$ continue to hold? The answer is affirmative. We first dispose of the special case where the segment PQ contains O.

Then by the above, we have $|OP'| = r\,|OP|$ and $|OQ'| = r\,|OQ|$, so that

$$|P'Q'| \;=\; |P'O| + |OQ'| \;=\; r\,(|PO| + |OQ|) \;=\; r\,|PQ|,$$

as desired.

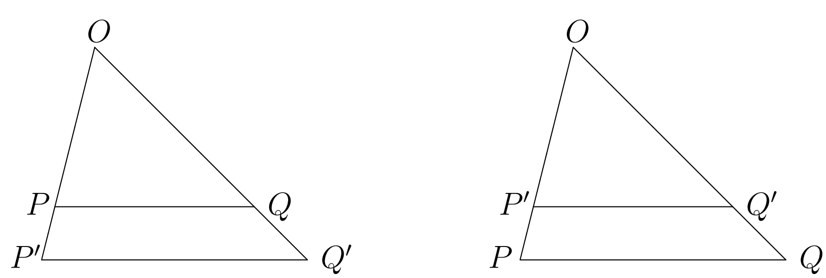
If PQ does not contain O, then the fact that the same result continues to hold is not obvious anymore. This is, in fact, the content of the following **Fundamental Theorem of Similarity**, usually abbreviated to **FTS**. In the statement of the theorem, we adopt a common abuse of notation: Let $\boldsymbol{L_{PQ}}$ (respectively, $L_{P'Q'}$) denote the line joining P and Q (resp., P' and Q'). Then instead of writing $L_{P'Q'} \parallel L_{PQ}$, we usually write:

$$P'Q' \parallel PQ$$

when there is no danger of confusion.

Theorem (FTS). *Let D be a dilation with center O and scale factor $r > 0$. Let P and Q be two points so that L_{PQ} does not contain O. If $D(P) = P'$ and $D(Q) = Q'$, then*

$$P'Q' \parallel PQ \quad \text{and} \quad |P'Q'| \;=\; r\,|PQ|$$



The case $r > 1$ The case $r < 1$

The eighth grade is not the right place to prove this theorem; a high school course will be able to handle such a proof better. What one can do in an eighth grade class is to verify simple cases of FTS by direct measurement to gain confidence in its validity. For example, one can start with $r = 2, 3, 4$, and then verify that (within the bounds of measurement errors), indeed, $|P'Q'| = 2|PQ|$, $|P'Q'| = 3|PQ|$, $|P'Q'| = 4|PQ|$, respectively. Then do the same with $r = \frac{1}{2}$, $r = \frac{2}{3}$, $r = \frac{3}{4}$, etc.

What we can do with FTS is to learn how to apply it to deduce the most basic properties of a dilation. This is what we do next.

Basic properties of dilations

There are four of them. First of all, notice that insofar as a dilation is a rule of assignment in the plane, we will simply take over the terminology associated with basic rigid notions such as *maps*, *image*, *composition*, etc. We will also fix the notation as in FTS, i.e., we have a dilation D with center at O and scale factor r. Then the discussion leading up to FTS about $|P'Q'|$ may now be summarized as follows:

> ***(i)*** *Let D be a dilation with scale factor r. Then the distance between the images $P' = D(P)$, $Q' = D(Q)$ of any two points P and Q is r times the distance between P and Q, i.e.,*
>
> $$|P'Q'| \;=\; r\,|PQ|.$$

We paraphrase (i) by saying that *a dilation with scale factor r changes distance by a factor of r*. A second property is this:

> ***(ii)*** *A dilation maps lines to lines, rays to rays, and segments to segments.*

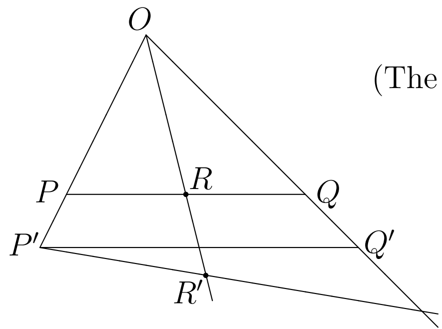
We will consider only the case of lines; the other two assertions about rays and segments are similar. So given a line L_{PQ}, we have to show that $D(L_{PQ})$ is a line. If L_{PQ} contains the center of dilation O, it is easy to see that $D(L_{PQ}) = L_{PQ}$. We will therefore assume that L_{PQ} does not contain O so that FTS becomes applicable. With $D(P) = P'$ and $D(Q) = Q'$ as usual, (ii) asserts that $D(L_{PQ}) = L_{P'Q'}$. In greater detail, this means

(1) if R is a point on the line L_{PQ}, then $D(R)$ lies on $L_{P'Q'}$, and

(2) conversely, every point R' on line $L_{P'Q'}$ is the image of some point R on L_{PQ}, i.e., $D(R) = R'$.

For eighth grade students, we suggest that it suffices for them to know the meaning of both statements and verify, by direct measurements, various special cases where the scale factor is a whole number or a simple fraction such as $\frac{3}{2}$ or $\frac{4}{5}$. For teachers, however, knowing the following simple argument would be essential.

Let us show that for a point R on the line L_{PQ}, the point $R' = D(R)$ lies on $L_{P'Q'}$.



(The case $r > 1$.)

To show that R' lies on $L_{P'Q'}$, it suffices to show that the line $L_{P'R'}$ and the line $L_{P'Q'}$ coincide. Now,

$D(P) = P'$ and $D(Q) = Q'$ imply $P'Q' \parallel PQ$, by FTS.

$D(P) = P'$ and $D(R) = R'$ imply $P'R' \parallel PR$, by FTS.

Thus we have $P'Q' \parallel PQ$, and $P'R' \parallel PQ$. So it seems that we have two different lines $L_{P'Q'}$ and $L_{P'R'}$ and both go through P' and both are parallel to L_{PQ}. We would like to say this is impossible unless the two lines coincide. To reach this conclusion, we have to bring to the fore the following basic assumption of plane geometry.

Parallel Postulate. Through a point A not lying on a line L passes one and only one line which is parallel to L.

According to the Parallel Postulate, since the two lines $P'R'$ and $P'Q'$ both pass through P' and are both parallel to PQ, they must be one and the same line. This is exactly what we want to prove.

The reasoning for the converse (i.e., every point R' on line $L_{P'Q'}$ is the image of some point R on L_{PQ}, i.e., $D(R) = R'$) is entirely similar if we look at the dilation D_1 with center O but with scale factor $\frac{1}{r}$ and observe that $P = D_1(P')$ and $Q = D_1(Q')$. So the same argument shows that $D_1(R')$ is a point of L_{PQ}. If we denote $D_1(R')$ by R, then this implies $D(R) = R'$. This shows $D(L_{PQ}) = L_{P'Q'}$.

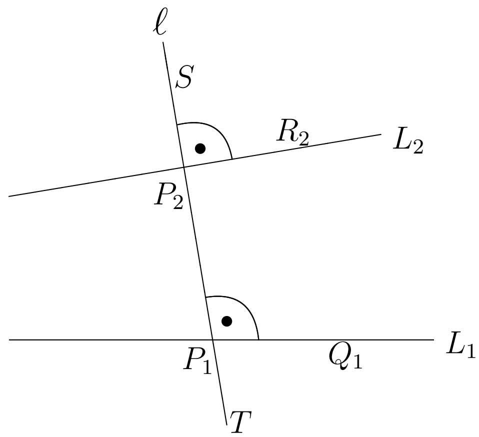
Now that we know a dilation D maps a line PQ to the line $P'Q'$, where P', Q' are the images of P, Q under D, FTS now implies:

> ***(iii)*** *A dilation maps a line not containing the center of dilation to a parallel line.*

A fourth basic property of dilation is the following.

> ***(iv)*** *A dilation preserves degrees of angles.*

We note first of all that (*iv*) makes sense because by (*ii*) above, a dilation maps rays to rays and therefore angles to angles. So it makes sense to ask for the degree of the image angle by a dilation. For eighth grade students, the following informal argument for (*iv*) would be enough and a rigorous proof can be postponed until a high school course. We begin with a fact about parallel lines. Let L_1 and L_2 be two distinct lines and let ℓ be a **transversal** of L_1 and L_2 in the sense that ℓ intersects both. Suppose ℓ meets L_1 and L_2 at P_1 and P_2, respectively. Then the angles $\angle SP_2R_2$ and $\angle P_2P_1Q_1$ in the picture below, with vertices at P_1 and P_2 and lying on the same side of the line ℓ, are called a pair of **corresponding angles** of the transversal ℓ with respect to L_1 and L_2. Similarly, $\angle R_2P_2P_1$ and $\angle Q_1P_1T$ are also corresponding angles of ℓ with respect to L_1 and L_2.

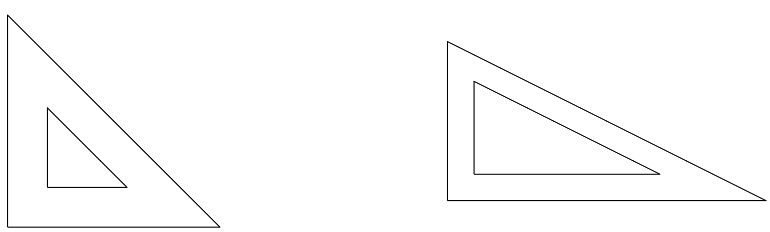


Then we have the following theorem.

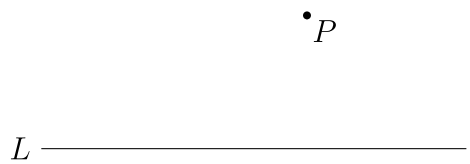
Theorem 1. (a) *Corresponding angles of a transversal with respect to parallel lines are equal.* (b) *Conversely, if a pair of corresponding angles of a transversal with respect to two lines are equal, then the two lines are parallel.*

We suggest that eighth graders simply verify special cases of this theorem by direct measurement. In fact, it would be very instructive to teach them about part (b) of this theorem by showing them the following efficient method of drawing a pair of parallel lines using plastic triangles.

There are two kinds of plastic triangles on the market, the 90-45-45 one and the 90-60-30 one, as shown.



Given a point P and a line L not containing P, we will show how to draw a line passing through P parallel to L.

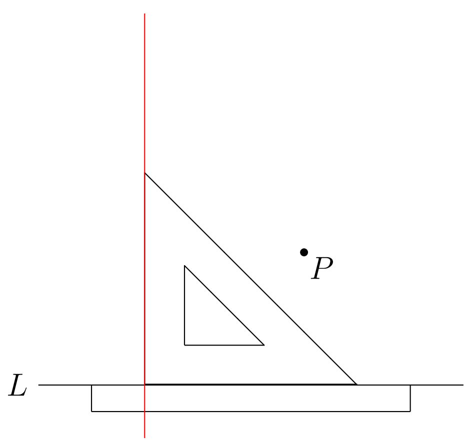


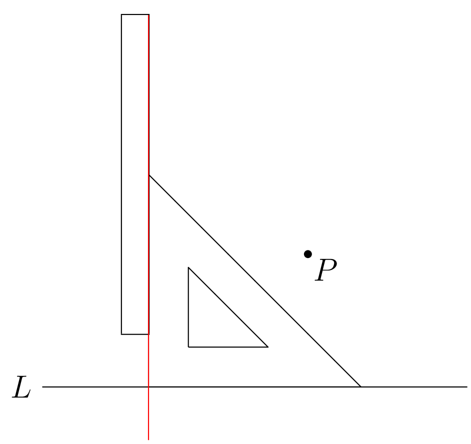
We can use either triangle for this purpose, but for definiteness, we will use the 90-45-45 triangle in the following discussion. The procedure is given in steps (1)–(5), and the simple explanation of the procedure as it is related to Theorem 1 will be given after step (5).

(1) Place a ruler along the line L. See the pictures below; the red vertical line is an imaginary line that will be explained later.

(2) Holding the ruler in place with one hand, put one leg of the triangle flush against the ruler as shown in the left picture below.

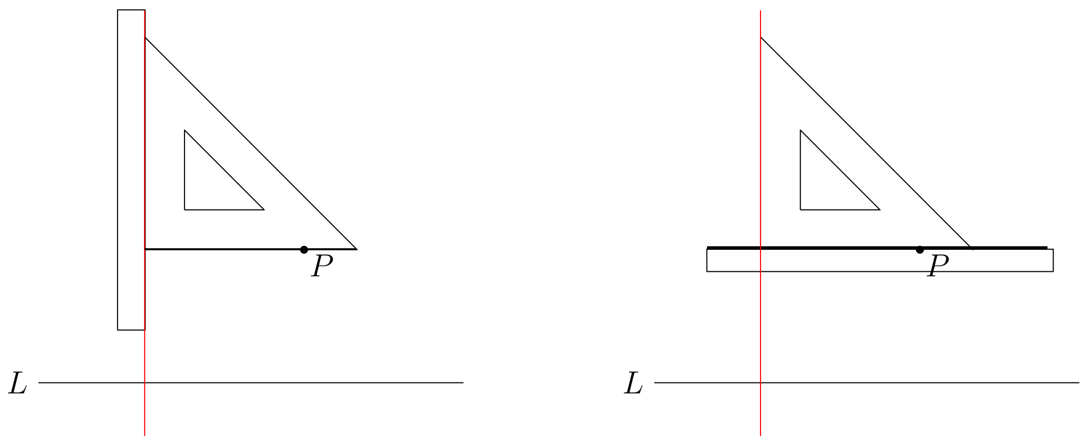
(3) Now hold the triangle firmly in place and put the ruler flush against the vertical side of the triangle, as shown in the right picture below.





(4) Hold the *ruler* firmly in place and slide the *triangle* along the ruler until the horizontal side passes though the point P, as shown in the picture on the left below.

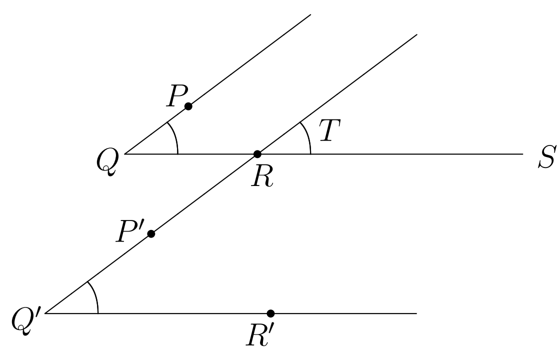
(5) Gently remove the ruler and draw the parallel line through P as in the following picture on the left, or place the ruler flush against the horizontal side of the triangle before drawing the line, as shown in the following picture on the right.



Now the explanation. We will use the red vertical line as the transversal of L and the line through P that was drawn in step (5). By the very nature of the drawing, any pair of corresponding angles of the red line with respect to these two lines will both be 90 degrees and therefore equal. By (b) of Theorem 1, the line drawn in step (5) is parallel to L.

We can now return to the original problem which inspired this detour into parallelism: How to prove that a dilation preserves degrees of angles (page 49). Let D be a dilation and let $\angle PQS$ be given. Let $D(QP) = Q'P'$ and if R is the intersection of L_{QS} and $L_{Q'P'}$, let $D(R) = R'$, so that $D(\angle PQS) = \angle P'Q'R'$. We have to prove that

$$|\angle PQS| = |\angle P'Q'R'|$$



Let the angle formed by $L_{Q'P'}$ and L_{QR} at R, as indicated in the picture, be denoted by $\angle T$. Since $D(QR) = Q'R'$, (iii) implies that $QR \parallel Q'R'$ (page 49), so that, by

Theorem 1(a),

$$|\angle P'Q'R'| \;=\; |\angle T$$

Since also $D(QP) = Q'P'$ by assumption, we have $QP \parallel Q'P$". So once more Theorem 1(a) implies that

$$|\angle T| \;=\; |\angle PQS|$$

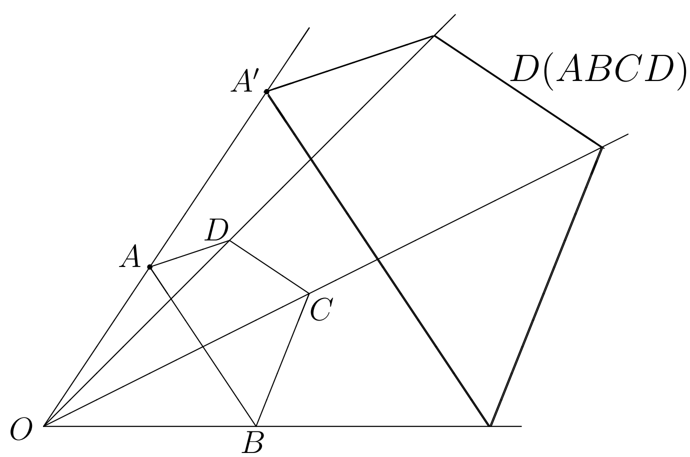
Hence $|\angle PQS| = |\angle P'Q'R'|$, as desired.

We have just given the essential idea of why a dilation preserves degrees of angles.

The dilated image of a figure

Property (ii) of a dilation makes it very easy to draw the dilated image of a **rectilinear** geometric figure, i.e., one that is the union of segments. Consider a segment PQ and a dilation D, then the image $D(PQ)$ by D is simply the segment $P'Q'$, where P', Q' are the images of P and Q by D, respectively. This is because (iii) says the image $D(PQ)$ is a segment joining $D(P) = P'$ and $D(Q) = Q'$; since $P'Q'$ is also a segment joining P' and Q', we must have $D(PQ) = P'Q'$ (there is only one line joining two distinct points).

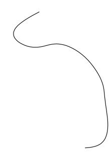
For example, if we have to get the dilated image of a given quadrilateral $ABCD$ with a scale factor of 2.1, we take a point O as the center of dilation, draw rays from O to the vertices. On each of these rays, say the ray from O to A, mark down A' so that $|OA'| = (2.1)|OA|$. We thus obtain a quadrilateral $A'B'C$" D'. By assertion (iii), $D(ABCD) = A'B'C'D'$.



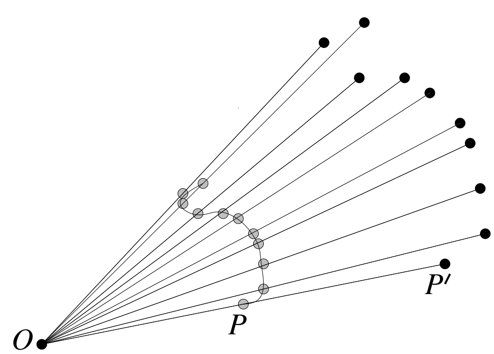
This $D(ABCD)$ is by definition the **magnification of $\boldsymbol{ABCD}$ to (2.1) times its size**. If the scale factor $r < 1$, then we'd speak of the **reduction to $\boldsymbol{r}$ times its size**. So for rectilinear figures, how to magnify or reduce them is straightforward.

Notice that, in an intuitive sense, $ABCD$ and $D(ABCD)$ do "look alike", i.e., they have "the same shape".

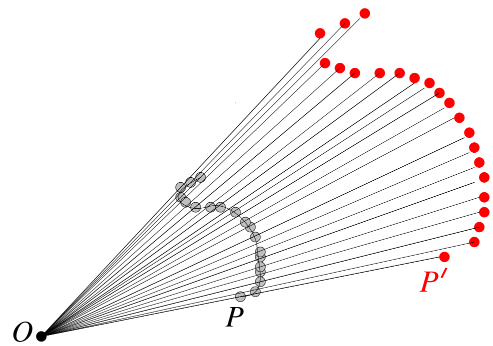
We can now return to the curve at the beginning of this section:



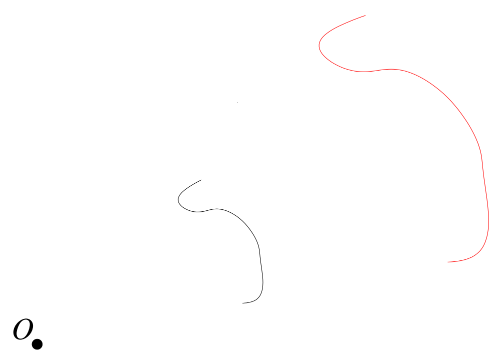
How to "double its size"? We choose an arbitrary point O outside the curve as center of a dilation and dilate the curve with a scale factor of 2. Now by definition, dilating the curve means dilating it point by point, and since the curve contains an infinite number of points, we must compromise by dilating only a *finite* number of points on the curve. We start simply: take a point P on the curve and on the ray OP, we mark off a point P' so that $|OP'| = 2|OP|$. Now repeat this for a small number of such P's and get something like the following. The contour of a curve that is bigger than, but "looks like" the original is unmistakable.



If we take more points on the original curve and dilate each of them, we get a better approximation to the curve.



Now if we choose, let us say 1500 points that are evenly spread out on the original curve,[6] and dilate them one-by-one, we get the usual curves that appear on the computer screen. (The visual perception of humans is so crude that it does not even "see" the missing points.) We have omitted the radial lines but retained the center of the dilation; on a normal computer screen, of course even the center is omitted.



By definition, the above dilated image of the curve with a scale factor of 2 is what is meant by **a curve double the size of the original**. Observe that it "has the same shape" as the original curve, but does look as if "it is twice as big". In general, the dilated image of a geometric figure by a dilation (with some chosen center) with scale factor $r > 0$ is called an ***r*-fold magnification** of the original figure if $r > 1$, and is called an ***r*-fold reduction** of the original figure if $r < 1$. What we have described here is the underlying principle of digital photography; this is how an image is magnified or reduced in the digital world.

In a classroom, getting students to do the magnification or reduction of a curvy figure by dilation (with a reasonable number of points chosen and strategically placed

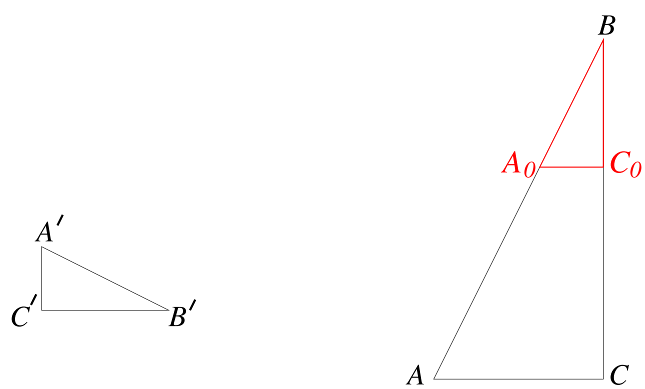
[6] This is an estimate of how many points the graphing software uses.

on the original figure) would be a very worthwhile learning experience. It reinforces their understanding of the definition of dilation, and it is also a "fun" activity because it is not at all obvious beforehand how a figure can be magnified or reduced.

We will not prove it here, but it is true that the dilated images of a given figure by dilations with a fixed scale factor r from different chosen centers are all congruent to each other, i.e., they are all, intuitively, of the "same size and same shape". This is why the preceding discussion made no reference to a particular choice of the center of dilation.

Similarity

Because we have just brought the concept of *congruence* into the discussion of dilation, this is the right place to take up the concept of *similarity*. What is a correct and useful definition of two figures being "similar"? We have seen that the dilated image of a figure "has the same shape" as the original, but can we say that two figures are "similar" *only* if one is the dilated image of the other? Consider, for example, the dilated image of a triangle ABC to a triangle $A_0B_0C_0$ by a dilation D centered at O as shown, with a scale factor $r < 1$. Of course these two triangles "have the same shape". Now let a congruence F move $\triangle A_0B_0C_0$ to $\triangle A'B'C'$, as shown (more precisely, F is the composition of a 90 degree clockwise rotation around B_0 followed by a translation).



Because $\triangle A_0B_0C_0$ and $\triangle A'B'C'$ "have the same size and the same shape", we have to agree that $\triangle ABC$ and $\triangle A'B'C'$ also have the same shape. Yet we can show that there is no dilation D' that maps $\triangle ABC$ to $\triangle A'B'C'$ because if there were, we'd have

$$D(A) = A' \quad \text{and} \quad D(B) = B'$$

so that by property (iii) of a dilation (page 49), we'd have $AB \parallel A'B'$, which is not the case. Therefore similarity between geometric figures cannot be limited to those so that one is obtained from the other by a dilation. At the same time, the preceding example also suggests how to define similarity correctly: we should include the composition with a congruence in the definition.

We define a figure $\mathcal{S}$ in the plane to be **similar** to another figure $\mathcal{S}'$ if there is a dilation D and a congruence F so that $(F \circ D)(\mathcal{S}) = \mathcal{S}'$. According to this definition, we now see that $\triangle ABC$ is similar to $\triangle A'B'C'$ because if D is the dilation that maps $\triangle ABC$ to $\triangle A_0B_0C_0$ and F is the congruence that maps $\triangle A_0B_0C_0$ to $\triangle A'B'C'$, then

$$(F \circ D)(\triangle ABC) = F(D(\triangle ABC)) = F(\triangle A_0B_0C_0) = \triangle A'B'C'$$

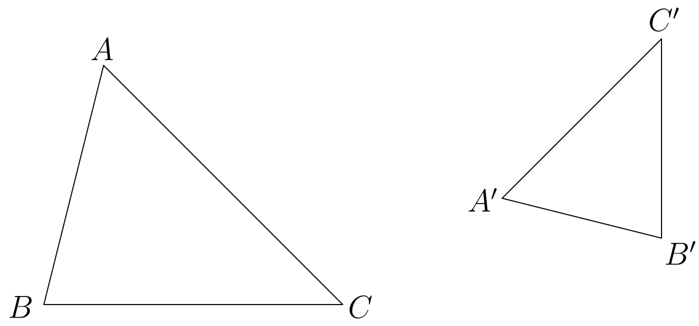
According to this definition, it is also the case that if a dilation D maps a figure $\mathcal{S}$ to another figure $\mathcal{S}'$, then $\mathcal{S}$ is similar to $\mathcal{S}'$ because we can let the congruence F be the identity basic rigid motion I (page 16) so that $(I \circ D)(\mathcal{S}) = \mathcal{S}'$.

The composition $F \circ D$ of a dilation followed by a congruence is called a **similarity**.

We remark that the definition of similarity could equally well be formulated as a congruence followed by a similarity, $D \circ F$. It can be shown that the two definitions are equivalent, in the sense that for any two figures $\mathcal{S}$ and $\mathcal{S}'$, there is a dilation D followed by a congruence F so that $(F \circ D)(\mathcal{S}) = \mathcal{S}'$, if and only if there is a congruence F_1 followed by a dilation D_1 so that $(D_1 \circ F_1)(\mathcal{S}) = \mathcal{S}'$. In addition, it can be proved that if a figure $\mathcal{S}$ is similar to another figure $\mathcal{S}'$, then $\mathcal{S}'$ is also similar to $\mathcal{S}$. To understand this statement, we have to unravel the definition of similarity, and what it says is this: if there is a dilation D and a congruence F so that $(F \circ D)(\mathcal{S}) = \mathcal{S}'$, then there is a dilation D' and a congruence F' so that $(F' \circ D')(\mathcal{S}') = \mathcal{S}$. These proofs are not suitable for eighth grade, but both statements are conceptually important, and they should be invoked when necessary.

3. The angle-angle criterion (AA) for similarity

Let $\triangle ABC$ be similar to $\triangle A'B'C'$.



Thus there is a dilation D and a congruence F so that $(F \circ D)(\triangle ABC) = \triangle A'B'C'$. It is convenient to denote the similarity $F \circ D$ by a single letter, say $\varphi = F \circ D$. Recalling the convention regarding the notation of congruent triangles, we explicitly point out that the notation $\varphi(\triangle ABC) = \triangle A'B'C'$ carries the convention that

$$\varphi(A) = A', \quad \varphi(B) = B', \quad \text{and} \quad \varphi(C) = C'$$

Let the scale factor of the dilation D be r. Then this r will also be called the **scale factor of the similarity $\boldsymbol{\varphi}$**. Let

$$D(A) = A^*, \quad D(B) = B^*, \quad \text{and} \quad D(C) = C^*$$

By properties (i) and (iv) of dilations (pages 47 and 49), we get

$$|\angle A| = |\angle A^*|, \quad |\angle B| = |\angle B^*|, \quad |\angle C| = |\angle C^*|$$

and

$$\frac{|AB|}{|A^*B^*|} = \frac{|BC|}{|B^*C^*|} = \frac{|AC|}{|A^*C^*|} \quad (= r)$$

Now F is a congruence which preserves lengths and degrees. Therefore all this information about $\triangle A^*B^*C^*$ will be transferred to $\triangle A'B'C'$. We summarize this discussion in the following theorem.

Theorem 2. *Let $\triangle ABC$ be similar to $\triangle A'B'C'$. Then*

$$|\angle A| = |\angle A'|, \quad |\angle B| = |\angle B'|, \quad |\angle C| = |\angle C'|$$

and

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|AC|}{|A'C'|}$$

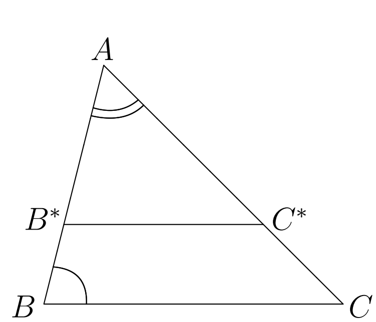
It is worth remarking that whereas the content of this theorem is usually taken to be the *definition* of similar triangles, for us this theorem is a logical consequence of the precise definition of similarity that confirms our intuition about what it means for two figures to have "the same shape".

The converse of Theorem 2 is also true. However, as in the case of congruence (page 39), much more is true. The following are the counterparts in similarity of the SAS, ASA and SSS criteria for congruence, respectively.

> **SAS criterion for similarity.** Given two triangles ABC and $A'B'C'$, suppose
> $$|\angle A| = |\angle A'| \quad \text{and} \quad \frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|}.$$
> Then they are similar.
>
> **AA criterion for similarity.** Given two triangles ABC and $A'B'C'$, suppose two pairs of corresponding angles are equal. Then they are similar.
>
> **SSS criterion for similarity.** Given two triangles ABC and $A'B'C'$, suppose the ratios of the (lengths) of three pairs of corresponding sides are equal. Then they are similar.

The proofs of these theorems are more suitable for a high school course than an eighth grade class, but we are going to give a proof of the AA criterion because it is so central to the discussion of the slope of a line. Fortunately, it is relatively short.

Proof of AA for similarity. So suppose we are given triangles ABC and $A'B'C'$ such that $|\angle A| = |\angle A'|$ and $|\angle B| = |\angle B'|$. We have to show that the triangles are similar.



If $|AB| = |A'B'|$, then triangles ABC and $A'B'C'$ are congruent because of the ASA criterion for congruence (pages 39 and 40); there would be nothing to prove. Thus we may assume that they are not equal, let us say, $A'B'$ is shorter than AB. On the segment AB, let B^* be the point so that AB^* is the same length as $A'B'$, i.e., $|AB^*| = |A'B'|$. Also let $r = |AB^*|/|AB|$. Denote the dilation with center A and scale factor r by D, and let C^* be the point in the segment AC so that $C^* = D(C)$. By FTS (page 46), $B^*C^* \parallel BC$ and therefore $|\angle AB^*C^*| = |\angle B|$, by Theorem 1 (page 50). But by hypothesis, $|\angle B| = |\angle B'|$, so

$$|\angle AB^*C^*| \;=\; |\angle B'|\,.$$

The triangles AB^*C^* and $A'B'C'$ now satisfy the conditions of ASA and are congruent. Hence there is a congruence F so that $F(\triangle AB^*C^*) = \triangle A'B'C'$. But by the definition of D, we already have $D(\triangle ABC) = \triangle AB^*C^*$. Thus $(F \circ D)(\triangle ABC) = \triangle A'B'C'$ because

$$\begin{aligned}(F \circ D)(\triangle ABC) &= F(D(\triangle ABC)) \qquad \text{(by the definition of } F \circ D\text{)}\\ &= F(\triangle AB^*C^*) \qquad \text{(because } D(\triangle ABC) = \triangle AB^*C^*\text{)}\\ &= \triangle A'B'C'. \qquad \text{(because } F(\triangle AB^*C^*) = \triangle A'B'C'\text{)}\end{aligned}$$

The two triangles ABC and $A'B'C'$ are therefore similar and the proof is complete.

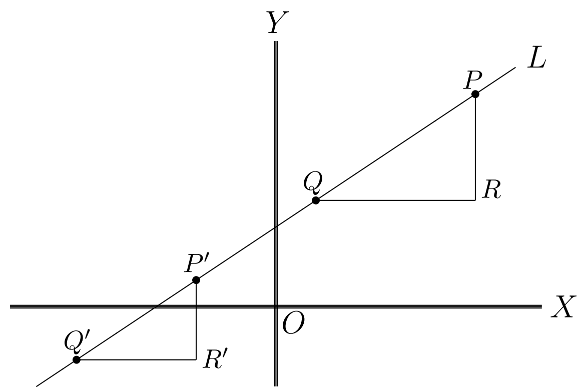
We remark that this proof, while correct as is, depends on the validity of the ASA criterion for congruence, FTS, and Theorem 1.

4. The Pythagorean Theorem

Definition of slope

For eighth grade, the significance of the above three criteria for similarity lies not so much in getting students to know how to prove them as in their ability to put them to use. In this section, we give two examples of such applications: the first is to correct a longstanding misconception of the slope of a line, and the second one is a proof of the Pythagorean theorem and its converse.

A typical example arising from algebra is the following. Given a line L in the coordinate plane, take two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ on L and let lines parallel to the coordinate axes be drawn so that they meet at R, as shown.



For this line, the **slope of L** is defined as the ratio

$$\frac{|PR|}{|QR|}$$

Now we come to a *serious issue* that is overlooked in almost all school textbooks: Is this definition of slope **well-defined**, in the sense that, if two other points P', Q' on L are chosen and we get a point of intersection R' in like manner (see picture), is it true that the corresponding ratio remains the same? Because this issue is important, we give it greater exposure: if by taking a different pair P', Q', we get a different number $|P'R'|/|Q'R'|$, then the ratio $|PR|/|QR|$ cannot be called *the slope of the line* L but rather

the slope of the two points P and Q *that happen to be chosen on* L*.*

Fortunately, we are going to prove that

$$\frac{|PR|}{|QR|} = \frac{|P'R'|}{|Q'R'|},$$

so this ratio is a property of the line L after all and not of the pair of points chosen.

Knowing the independence of the ratio from the choice of the pair of points gives a completely different perspective on the concept of slope; it suggests a different pedagogy on teaching the graph of linear equations of two variables. But that is a different story.

To prove the preceding equality, we are going to use the AA criterion for similarity (page 59) to prove that triangles PQR and $P'Q'R'$ are similar. Indeed, the lines QR and $Q'R'$, being both parallel to the x-axis. are parallel to each other. Theorem 1 (page 50) implies that $|\angle PQR| = |\angle P'Q'R'|$. Since $\angle PRQ$ and $\angle P'R'Q'$ are right angles, they are also equal. So the triangles PQR and $P'Q'R'$ have two pairs of equal angles and are therefore similar. By Theorem 2 (page 58), we get

$$\frac{|PR|}{|P'R'|} = \frac{|QR|}{|Q'R'|}$$

This implies

$$\frac{|PR|}{|QR|} = \frac{|P'R'|}{|Q'R'|}$$

by the cross-multiplication algorithm. We are done.

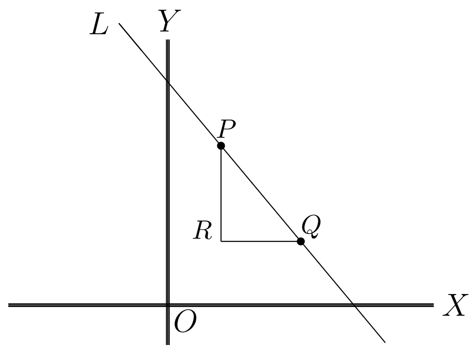
We should make contact with the usual definition of slope for $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ on a line L as the ratio

$$\frac{p_2 - q_2}{p_1 - q_1}$$

In case $(p_2 - q_2)(p_1 - q_1) > 0$, which is the case of the above graph, then either both $q_2 - p_2$ and $q_1 - p_1$ are positive or both are negative. So $(p_2 - q_2) = \pm|PR|$ and $(p_1 - q_1) = \pm|QR|$, so that

$$\frac{p_2 - q_2}{p_1 - q_1} = \frac{|PR|}{|QR|}.$$

This why the preceding argument is a valid one about the slope of a line. Of course, it could happen that $(p_2 - q_2)(p_1 - q_1) < 0$, which is the case for the following line L:



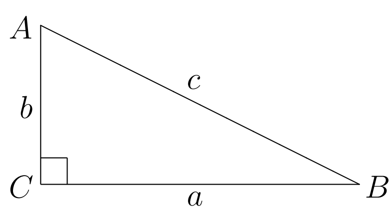
The same argument now shows that

$$\frac{p_2 - q_2}{p_1 - q_1} \;=\; -\,\frac{|PR|}{|QR|}\,.$$

Therefore except for the matter of strategically placing negative signs in the appropriate places, the same reasoning shows that the definition of slope *is* independent of the pair of points chosen.

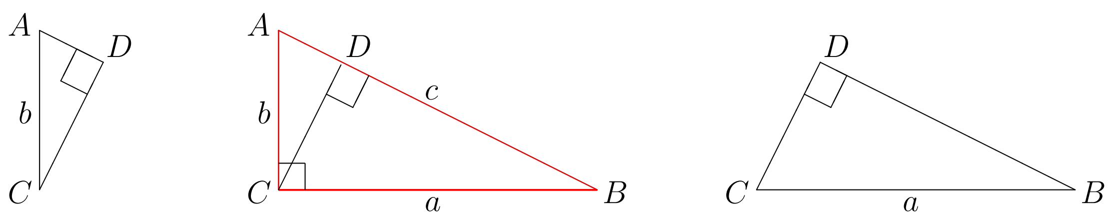
Proof of the Pythagorean Theorem

As our second application of the AA criterion for similarity, we prove the Pythagorean Theorem. This is one among many proofs of this theorem. Let us fix the terminology. Given a right triangle ABC with C being the vertex of the right angle. Then the sides AC and BC are called the **legs** of $\triangle ABC$, and AB is called the **hypotenuse** of $\triangle ABC$.



Theorem 3 (Pythagorean Theorem). *If the lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c, then $a^2 + b^2 = c^2$.*

The basic idea of the proof is very simple. Referring to the same picture, we draw a perpendicular CD from C to side AB, as shown:



We draw this perpendicular because it creates, from the point of view of the AA criterion for similarity, three similar triangles. For example, right triangles CBD and ABC are similar because they share $\angle B$ in addition to having equal right angles. Likewise, right triangles ACD and ABC are similar because they share $\angle A$. All this is clearly laid out in an animation of the proof created by Larry Francis:

Pythagorean Theorem proof from similar right triangles

Specifically, for the similar triangles $\triangle ABC$ and $\triangle ACD$, in order to set up the correct proportionality of sides, Theorem 2 (page 58) tells us that we need the correct correspondences of the vertices. The vertices of the two right angles obviously correspond, so C of $\triangle ABC$ corresponds to D of $\triangle CDB$. The two triangles share $\angle B$, so B of $\triangle ABC$ corresponds to B of $\triangle CDB$. Now there is no choice but that A of $\triangle ABC$ corresponds to C of $\triangle CDB$. Thus we have:

$$C \leftrightarrow D, \quad B \leftrightarrow B, \quad A \leftrightarrow C$$

Hence $\frac{|BA|}{|BC|} = \frac{|BC|}{|BD|}$, so that by the cross-multiplication algorithm,

$$|BC|^2 = |AB| \cdot |BD|$$

By considering the similar right triangles ABC and ACD, we conclude likewise that $\frac{|AC|}{|AB|} = \frac{|AD|}{|AC|}$ and

$$|AC|^2 = |AB| \cdot |AD|$$

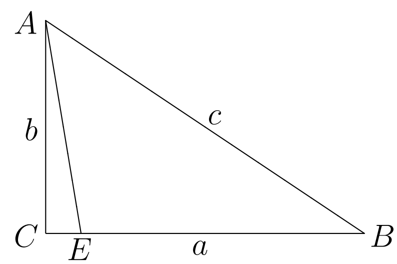
Adding, we obtain

$$|BC|^2 + |AC|^2 = |AB| \cdot |BD| + |AB| \cdot |AD| = |AB|\,(\,|BD| + |DA|\,) = |AB|^2$$

This is the same as $a^2 + b^2 = c^2$. The proof is complete.

We note that the **converse of the Pythagorean Theorem** is also correct, and its proof—surprisingly—depends on the Pythagorean Theorem itself. So suppose, in the notation and the picture above, $c^2 = a^2 + b^2$. We have to prove $AC \perp CB$.

Let E be the point on L_{BC} so that $AE \perp CB$. A priori, we have no idea if E is equal to C or not, and the goal is to show that it is. E could be on either side of C, and it makes no difference as far as the proof is concerned. So let us say E is between C and B.



We are given $c^2 = a^2 + b^2$, while the Pythagorean Theorem applied to $\triangle AEB$ gives $c^2 = |AE|^2 + |EB|^2$, which then becomes $c^2 = |AE|^2 + (a - |CE|)^2$. Applying the Pythagorean Theorem to $\triangle ACE$ (remember $AE \perp CB$) gives $|AE|^2 = b^2 - |CE|^2$. Therefore

$$c^2 = |AE|^2 + (a - |CE|)^2 = (b^2 - |CE|^2) + (a - |CE|)^2 = (a^2 + b^2) - 2a\,|CE|$$

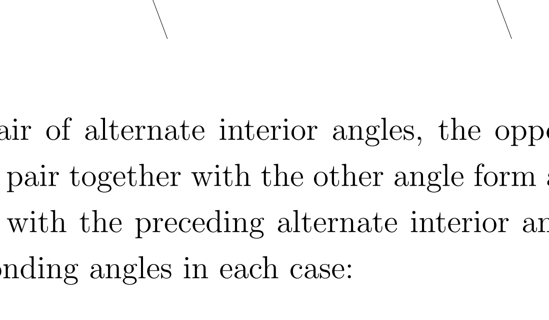
Comparing with $c^2 = a^2 + b^2$, we get $-2a\,|CE| = 0$. Since $a > 0$, necessarily $|CE| = 0$. Thus $C = E$, and the converse of the Pythagorean Theorem is proved.

One should give many exercises on the applications of the Pythagorean Theorem and its converse, including the distance formula in a coordinate system.

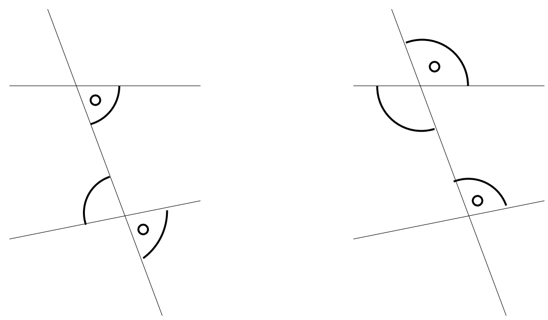
5. The angle sum of a triangle

We bring closure to the discussion of the AA criterion for similarity. If you look at all six theorems for congruence or similarity (page 39 and 59), you would notice that the hypothesis of each of them consists of three equalities except for the AA criterion, which has *two* equalities for angles. It is time to point out that the apparent difference is an illusion because we will prove the Angle Sum Theorem: The **angle sum** of a triangle (i.e., the sum of the degrees of the angles in a triangle) is always 180 degrees. Thus if two pairs of angles in the triangles are equal, then all *three* pairs of angles are equal.

To this end, recall that we made use of the concept of *corresponding angles* of a transversal relative to a pair of lines in Theorem 1 (page 50) . We now introduce a related concept of **alternate interior angles**. For eighth grade, it is best to dispense with the rather cumbersome precise definition and simply draw a picture and point to a pair of angles as examples of alternate interior angles, such as the angles that are marked down in each picture below:



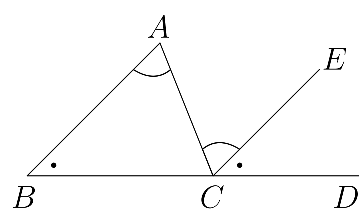
Note that, given a pair of alternate interior angles, the opposite angle (or *vertical angle*) of either of the pair together with the other angle form a pair of corresponding angles. We illustrate with the preceding alternate interior angles by marking down the resulting corresponding angles in each case:



In view of Theorem 1 (page 50) and the fact that opposite angles are equal, we have:

> (a) *Alternate interior angles of a transversal with respect to parallel lines are equal.* (b) *Conversely, if a pair of alternate interior angles of a transversal with respect to two lines are equal, then the two lines are parallel.*

We are now in a position to prove that the angle sum of a triangle is always equal to 180 degrees. Let triangle ABC be given. On the ray from B to C, take a point D so that the segment BD contains C. Through the point C, draw a line CE parallel to AB, as shown.



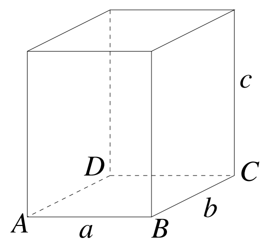
Now $|\angle A| = |\angle ACE|$ as they are alternate interior angles of AC relative to the parallel lines AB and CE. In addition, $|\angle B| = |\angle ECD|$ because they are corresponding angles of BD. Therefore the angle sum of triangle ABC is equal to the sum of the angles that make up the straight angle $\angle BCD$, and we are done.[7]

[7]Without going into details, there are subtle issues inherent in this proof that we have intentionally neglected.

6. Volume formulas

In grade 7, we explained why if a (right) rectangular prism has dimensions a, b, c, its volume is abc cubic units (i.e., if the linear unit is inches, the unit of the volume measure is inches^3, if the linear unit is cm., then the volume measure is in terms of cm.^{3}), etc. In grade 8, we expand the inventory of volume formulas to include those of a (generalized) right cylinder, a cone, and a sphere.

First we recall an interpretation of the volume formula for a rectangular prism. If we have such a prism, as shown,



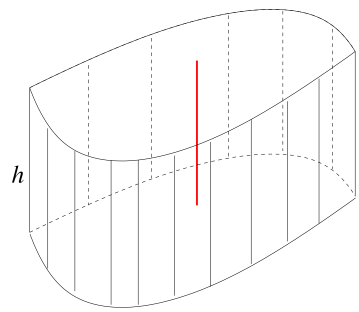
and if we call the rectangle $ABCD$ the **base** of the prism and c its height, then

(A) volume of rectangular prism = (area of base) $\times$ height

In this form, this formula can be generalized, as follows. Let $\mathcal{R}$ be a region in the plane, then the **right cylinder over $\boldsymbol{\mathcal{R}}$ of height $\boldsymbol{h}$** is the solid which is the union of all the line segments of length h lying above the plane, so that each segment is perpendicular to the plane and its lower endpoint lies in $\mathcal{R}$. When a right cylinder is understood, we usually say "cylinder" rather than "right cylinder". The region $\mathcal{R}$ is called the **base** of the cylinder. Notice that the **top** of a right cylinder (i.e., the points in the cylinder of maximum distance from the base) over $\mathcal{R}$ is also a planar region which is *congruent to* $\mathcal{R}$, but we will not spend time to explain what "congruent" means in three dimensions and will use the term in a naive sense.
Then we have:

(B) volume of right cylinder over $\mathcal{R}$ of height h = (area of $\mathcal{R}$) $\times$ h

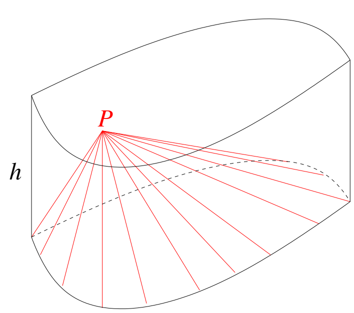
So if $\mathcal{R}$ is a rectangle, this yields volume formula (A) for a rectangular prism, but if $\mathcal{R}$ is a circle of radius r, then the right cylinder over a circle of radius r is called a **right circular cylinder**. The preceding formula then implies

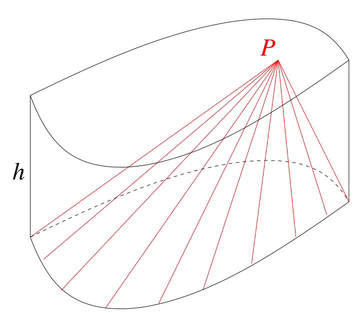


(C) volume of right circular cylinder of radius r and height $h = \pi\, r^2 h$

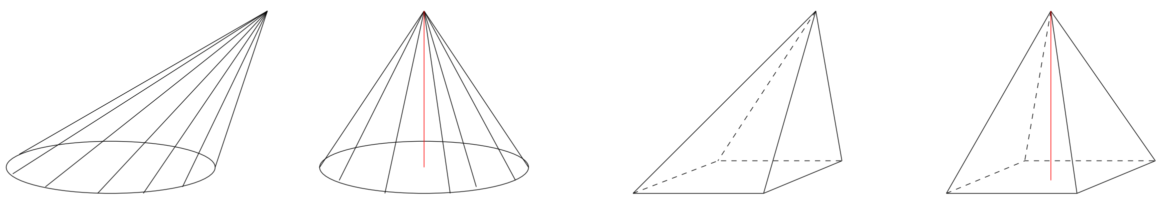
The case of a right circular cylinder is the most important example of a "cylinder" in school mathematics, but the reason we introduce the more general concept of a cylinder over an arbitrary planar region is that the explanations of the volume formulas (B) and (C) are the same. It is also important to recognize that there is only one general volume formula for cylinders, i.e., (B).

Let P be a point in the plane that contains the top of a cylinder of height h. Then the union of all the segments joining P to a point of the base $\mathcal{R}$ is a solid called a **cone with base $\boldsymbol{\mathcal{R}}$ and height $\boldsymbol{h}$**. The point P is the **top vertex** of the cone. Here are two examples of such cones.





One has to be careful with this use of the word "cone" here. If the base $\mathcal{R}$ is a circle, then this cone is called a **circular cone** (see left figure below). If the vertex of a circular cone happens to lie on the line perpendicular to the circular base at its center, then the cone is called a **right circular cone** (see the figure second from left below). In everyday life, a "cone" is implicitly a right circular cone, and in many textbook, this is how the word "cone" is used. If the base is a square, then the cone is called a **pyramid** (see middle figure below). If the vertex of a pyramid lies on the line perpendicular to the base at the **center** of the square (the intersection of the diagonals), the pyramid is called a **right pyramid** (see the figure second from right below). If the base is a triangle, the cone is called a **tetrahedron** (see right picture below).



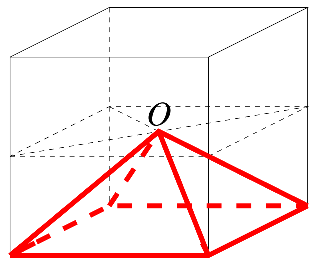
The fundamental formula here is

$$\text{(D)}\quad \text{volume of cone with base } \mathcal{R} \text{ and height } h = \frac{1}{3} \text{ volume of cylinder with same base and same height}$$

Of great interest here is the factor $\frac{1}{3}$, which is independent of the shape of the base. How this factor comes about is most easily seen through the actual computations using calculus. However, one can see the geometric reason for the $\frac{1}{3}$ in an elementary way, as follows. Consider the **unit cube**, i.e., the rectangular prism whose sides all have length 1. The unit cube has a **center** O, and the simplest definition of O may be through the use of the **mid-section**, which is the square that is halfway between the top and bottom faces (see the dashed square in the following picture), and let O be the intersection of the diagonals of the mid-section. It is easy to convince oneself that O is equidistant from all the vertices and also from all six faces.

Then the cone obtained by joining O to all the points of one face is congruent[8] to the cone obtained by joining O to all the points of any other face. There are six such cones.

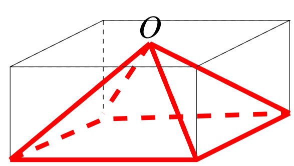
[8] Again we leave undefined the meaning of "congruent" in this context and allow it to be understood in a naive sense.



Let $\mathcal{C}$ be the cone joining O to the base of the unit cube; it is the red cone above. Of course congruent geometric figures have the same volume, and since six cones congruent to $\mathcal{C}$ make up the unit cube, and the unit cube has volume 1 by definition, we obtain:

$$\text{volume of } \mathcal{C} \;=\; \frac{1}{6}$$

The right way to interpret this formula is to consider the rectangular prism which is the lower half of the unit cube, i.e., the part of the unit cube that is below the mid-section:



This particular rectangular prism has volume $\frac{1}{2}$, and since $\frac{1}{6}$ is equal to $\frac{1}{3} \times \frac{1}{2}$, we have

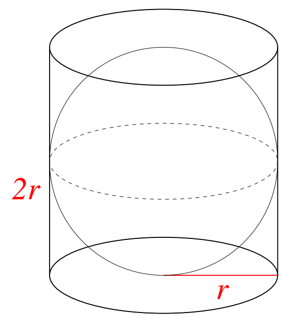
$$\text{volume of cone } \mathcal{C} = \frac{1}{3} \text{ (volume of cylinder with same base, same height)}$$

Here we see the emergence of the factor of $\frac{1}{3}$, and this is no accident because, using ideas from calculus, one can show that if the preceding formula is true for *one* cone $\mathcal{C}$, then it is true for *all* cones.

Finally, we come to the volume formula of a sphere of radius r:

(E) volume of sphere of radius $r \;=\; \frac{4}{3}\,\pi\, r^3$

The derivation of this formula is sophisticated and will have to be left to a high school course. The discovery of this formula was a major event in the mathematics of antiquity, and this honor goes to Archimedes (287–212 B.C.). However, this formula was also independently discovered in China by Zu Chongzhi (429–501 AD) and his son Zu Geng (circa 450–520 AD), by essentially the same method.[9] Archimedes' formulation of his result — one that he was proudest of — is the following:

> Let a sphere of radius r be given, then it has a **circumscribing right circular cylinder**, i.e., a right circular cylinder so that its radius is r and its height is $2r$. Then:
>
> 
>
>
> volume of sphere $= \frac{2}{3}$ (volume of circumscribing right cylinder)
>
> surface area of sphere $= \frac{2}{3}$ (surface area of circumscribing right cylinder)

It would make a good exercise for an eighth grader to verify the first assertion.

[9]This method has come to be known as *Cavalieri's Principle.* Bonaventura Cavalieri, 1598–1647, was eleven centuries behind the Zus.

HIGH SCHOOL

Congruence G-Co

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of

angles are congruent.

8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems

9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

10. Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

11. Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, right triangles, and trigonometry G-Srt

Understand similarity in terms of similarity transformations

1. Verify experimentally the properties of dilations given by a center and a scale factor:

a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar. Prove theorems involving similarity.

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

7. Explain and use the relationship between the sine and cosine of complementary angles.

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

Apply trigonometry to general triangles

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles G-C

Understand and apply theorems about circles

1. Prove that all circles are similar.

2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles

5. Derive using similarity the fact that the length of the arc intercepted by an angle

is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties with equations G-GPe

Translate between the geometric description and the equation for a conic section

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

2. Derive the equation of a parabola given a focus and directrix.

3. $(+)$ Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Use coordinates to prove simple geometric theorems algebraically

4. Use coordinates to prove simple geometric theorems algebraically. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point $(0, 2)$.

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

Geometric measurement and dimension G-Gmd

Explain volume formulas and use them to solve problems

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

Visualize relationships between two-dimensional and three-dimensional objects

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry G-mG

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

Goals of high school geometry

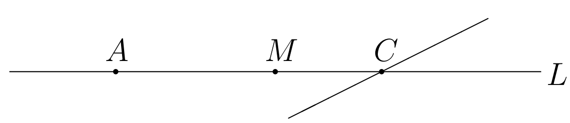
There are many standards here, and we will not be able to discuss all of them. Instead, we will concentrate only on the development of the theorems on congruence, similarity, and circles, i.e., the standards in G-Co, G-Srt, and G-C.

The initial part of the following discussion would appear to be just a review of topics that we have already gone over in grade 8, but there will be a different emphasis. Whereas concepts like rotation, reflection, and translation were treated in grade 8 mostly in the context of hands-on activities and with an emphasis on geometric intuition, the high school course will put equal weight on the precision of their definitions. In addition, it will be more explicit and more systematic in clarifying the starting point of the geometric discussion by clearly stating the assumptions. Nevertheless, it is not the goal of the Common Core Standards to treat plane geometry axiomatically. Standard G-Co 8 makes this perfectly clear:

> Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

In a sense, the real starting point of the geometric discussion is the collection of basic rigid motions: rotations, reflections, and translations. The purpose of such a starting point is to bring out the true mathematical significance of these concepts. They will be shown to lie at the foundation of the proofs of standard theorems in plane geometry, including the common length and area formulas. Such an approach has the added pedagogical advantage of avoiding the doldrums of axiomatic treatments that devote more than a hundred pages to proving theorems of the following type:

- Any two right angles are congruent.
- Every angle has exactly one bisector.
- If M is a point between points A and C on a line L, then M and A are on the same side of any other line that contains C.



There is something to be said about the value of being able to *prove* these geometrically obvious facts, but for most students, it is hardly inspiring to spend two months of the school year to learn about such proofs. The goal of the Common Core Standards is to steer clear of such an approach to geometry by putting geometry on an equal footing with any other part of school mathematics: there should be reasoning and and there should be proofs, but there should also be a minimum of formalism. Geometry can be learned the same way fractions or algebra is learned. It will be seen below that, by building on the geometric foundation in grade 8, we can achieve this goal with ease.

It is perhaps worth noting that *the ensuing discussion of high school geometry is entirely self-contained, in the sense that every step of the reasoning depends only on what comes before.* The sections are as follows.

1. Basic assumptions and definitions

We are going to start from the beginning and go through, one by one, all the geometric concepts we are going to use so that we can come to a common agreement.

By a **line**, we mean a *straight line*. A line will be assumed to be *infinite in both directions*. We will begin with a precise enunciation of ***what we assume to be known about the plane***. The eight assumptions are listed as (A1), . . . , (A8). The first six are to be found in section 1, (A7) is in section 2 (page 110), and (A8) is in section 3 (page 119). Every single one of them is intuitively obvious, and the only reason we enunciate them is to make sure that we all have a clearly defined common starting point.

(A1) *Through two distinct points passes a unique line.*

Two lines are said to be **distinct** if there is at least one point that belongs to one but not the other; otherwise we say the lines are **the same**. Lines that have no point in common are said to be **parallel**. In symbols, L_1 parallel to L_2 is denoted by $\boldsymbol{L_1 \parallel L_2}$. The following lemma is a simple consequence of (A1):

Lemma 1. *Given two distinct lines, either they are parallel, or they have exactly one point in common.*

An equivalent way of stating the lemma is that *two distinct lines either do not intersect, or intersect at exactly one point.* Naturally one needs to know when two lines intersect and when they don't. It turns out that this issue cannot be settled except by an explicit assumption.

(A2) (Parallel Postulate) *Given a line L and a point P* not *on L but lying in the same plane, there is exactly one line in the plane passing through P which is parallel to L.*

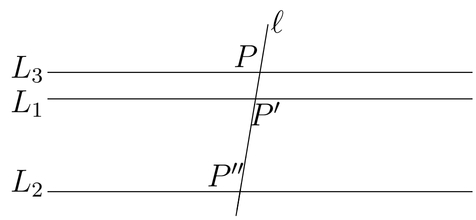
In other words, we assume as obvious that in the plane that we normally work with, for a point P not on a line L, every line that contains P intersects L except for one line. This postulate assumes explicitly that there *is* a line passing through P and parallel to L. However, we shall see in the Corollary to Theorem 1 on page 103 that the existence of such a parallel line can in fact be proved once we know there are enough rotations in the plane. So the main weight of the Parallel Postulate is the assertion that there is *no more than one such parallel line.*

We know intuitively that if three lines L_1, L_2, and L_3 are given so that $L_1 \parallel L_2$ and $L_2 \parallel L_3$, then $L_1 \parallel L_3$. It is less known that this intuitive fact has to be justified by the Parallel Postulate. More formally, we state:

Lemma 2. *If three lines L_1, L_2, and L_3 have the property that $L_1 \parallel L_2$ and $L_2 \parallel L_3$, then $L_1 \parallel L_3$.*

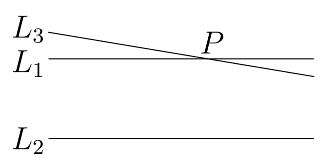
Proof. We will give two proofs. The first proof is a direct proof. Take a point P on L_3. Suppose we can prove that *any* line ℓ passing through P distinct from L_3 must intersect L_1. Then any line passing through P that is not L_3 is not parallel to L_1. But since the Parallel Postulate implies that there *is* a line passing through P that is parallel to L_1, L_3 must be that line. This then proves $L_3 \parallel L_1$.

So let ℓ be a line passing through P and distinct from L_3.



By the Parallel Postulate, through P passes only one line parallel to L_2. Since by hypothesis, L_3 is that line, ℓ is not parallel to L_2. Let ℓ intersect L_2 at some point P''. Now by the Parallel Postulate again, through P'' passes only one line parallel to L_1, and since by hypothesis L_2 is that line, ℓ is not parallel to L_1. Thus ℓ must intersect L_1 at some point P'. By the remark above, this proves the lemma.

This lemma can be more simply proved by a contradiction argument, as follows. If L_1 is not parallel to L_3, they intersect at a point P.

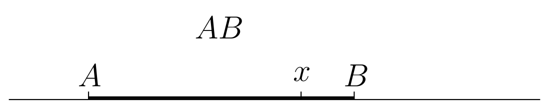


The point P does not lie on L_2 because P lies on L_3 and L_3 has no point in common with L_2 because $L_2 \parallel L_3$. Thus through P now pass two distinct lines L_3 and L_1, both parallel to L_2, a contradiction. Again the theorem is proved.

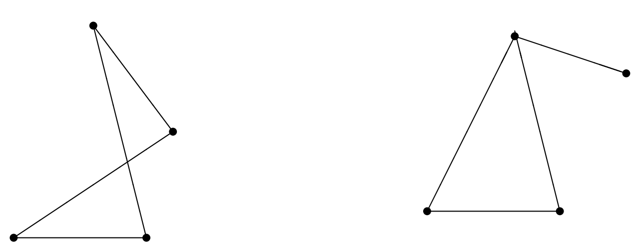
If A and B are two distinct points, then by (A1), there is a unique line containing A and B. We denote this line by $\boldsymbol{L_{AB}}$ and call it **the line joining $\boldsymbol{A}$ and $\boldsymbol{B}$**. On

$\boldsymbol{L_{AB}}$, denote by $\boldsymbol{AB}$ the collection of all the points *between* A and B together with the points A and B themselves. We call AB the **line segment**, or more simply the **segment joining** A **and** B, and the points A and B are called the **endpoints** of the segment AB. The term **segment** will be used in general to refer to the segment joining a pair of points.

Note that there is no universal agreement on the notation used to denote lines, segments, and later on, rays. For example, some books use AB to denote the line passing through A and B, $\overline{AB}$ to denote the segment between A and B, and $\overrightarrow{AB}$ to denote the ray from A to B. It can be confusing. Note also that it makes sense to talk about points on L_{AB} **between** A and B, because L_{AB} may be regarded as a number line and A and B then become numbers. For example, if $A = 0$ and $B > 0$, then the points between A and B would be all the numbers x so that $0 < x < B$.

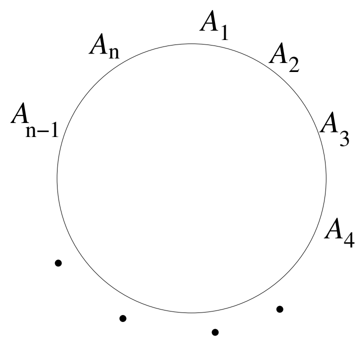


We are now in a position to define a polygon. (In the classroom, one would start with the definition of a triangle and a quadrilateral before tackling the general case, and care should be given to motivating the use of subscripts.) Let n be any positive integer ≥ 3. An **n-sided polygon** (or more simply an **n-gon**) is by definition a geometric figure consisting of n distinct points $A_1, A_2, \ldots, A_n$ *in the plane*, together with the n segments A_1A_2, A_2A_3, $\ldots$, $A_{n-1}A_n$, A_nA_1 so that *none of these segments intersects any other except at the endpoints as indicated*, i.e., A_1A_2 intersects A_2A_3 at A_2, A_2A_3 intersects A_3A_4 at A_3, etc. In symbols: the polygon will be denoted by $A_1A_2 \cdots A_n$. If $n = 3$, the polygon is called a **triangle**; $n = 4$, a **quadrilateral**; $n = 5$, a **pentagon**; and if $n = 6$, a **hexagon**. If this definition of a polygon seems too complicated, remember that we are trying to rule out the following as polygons and we have to do so precisely:



Given polygon $A_1 A_2 \cdots A_n$, the A_i's are called the **vertices** and the segments A_1A_2, A_2A_3, etc. the **edges** or sometimes the **sides**. For each A_i, both A_{i-1} and A_{i+1} are called its **adjacent vertices** (except that in the case of A_1, its adjacent vertices are A_n and A_2, and in the case of A_n, its adjacent vertices are A_1 and A_{n-1}). Thus the sides of a polygon are exactly the segments joining adjacent vertices. Any segment joining two nonadjacent vertices is called a **diagonal**.

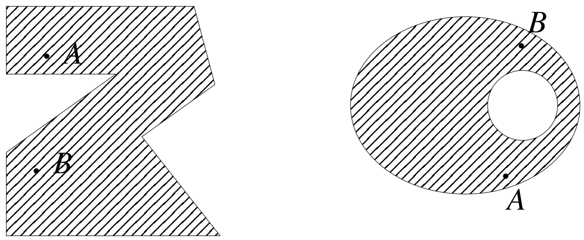
The best way to remember the notation associated with a polygon is to think of the points A_1, A_2, $\ldots$, A_n as being placed consecutively on a circle (note that we are only using the concept of a "circle" in an informal way here), for example, in clockwise (or counterclockwise) direction:



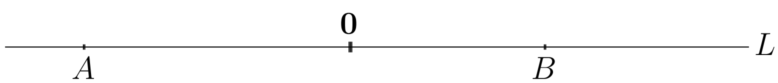
Then it is quite clear from this arrangement whether or not two vertices are adjacent.

In order to define angles, we need to know a little bit more about lines. We want to say that a point on a line separates the line into "two halves", just as 0 on the number line separate all numbers into positive and negative numbers. It would not be practical to do so by invoking the number line each time because we should allow geometry to speak for itself. Then the proper way to say this is by making an explicit assumption. To this end, we first introduce a definition.

A subset $\mathcal{R}$ in a plane is called **convex** if given any two points A, B in $\mathcal{R}$, the segment AB lies completely in $\mathcal{R}$. The definition has the obvious advantage of being simple to use, so the concern with this definition is whether or not it captures the intuitive feeling of "convexity". Through applications and lots of drawings, you will see that it does. For example, the shaded figures below are not convex.



Every line and the plane itself are of course convex. Many common figures, such as the interior of a triangle or a rectangle or a circle, once they have been properly defined, will also be seen to be convex. It is also a simple exercise to show that *the intersection of two convex sets is convex.* If we have a number line L, then "the positive half-line" L^+ consisting of all the positive numbers is convex: indeed if a and b are in L^+, then the segment joining a to b is the interval $[a, b]$ consisting of all the numbers x satisfying $a < x < b$. Since a is positive, x has to be positive and therefore every point in this segment also lies in L^+. For analogous reasons, "the negative half-line" L^- consisting of all the negative numbers is convex. Observe also that the number line L is now broken up into three parts: L^-, L^+, and the set $\{0\}$ consisting of the number 0 alone, so that every point of L is in one, and only one, of these parts. Furthermore, the line segment joining a positive number B to a negative number A must contain 0.



This example of L serves as a model of what we expect to be true in general of every line in the plane. Since we are starting from scratch, the only way we can guarantee this to be true is to make an assumption.

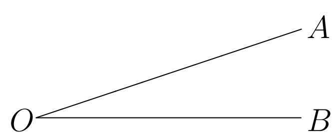
(A3) (Line separation) *A point P on a line L separates L into two* non-empty *convex subsets L^+ and L^-, called* **half-lines***, so that:*

> (i) *Every point of L is in one* and only one *of the sets L^+, L^-, and the set $\{P\}$ consisting of the point P alone.*
>
> (ii) *If two points A and B belong to different half-lines, then the line segment AB contains P.*

It follows from (i) that any two of the sets L^+, L^-, and $\{P\}$ are **disjoint**, i.e., do not share a point in common. It also follows from the convexity of L^+ and L^- that if two points A, B belong to the same half-line, then the line segment AB does not contain P:

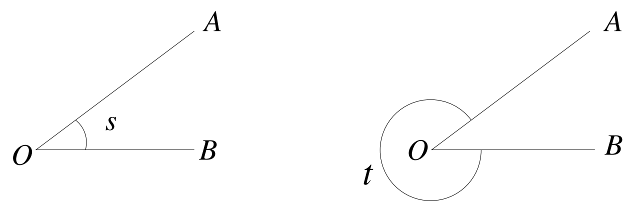
The set consisting of the point P and the points from a half-line, L^+ or L^-, is called a **ray**. We also say these are **rays issuing from $\boldsymbol{P}$**. If we want to specifically refer to the ray containing A, we use the symbol $\boldsymbol{R_{PA}}$. We will also refer to $\boldsymbol{R_{PA}}$ as the **ray from $\boldsymbol{P}$ to $\boldsymbol{A}$**. Similarly, the ray containing $\boldsymbol{B}$ issuing from $\boldsymbol{P}$ is denoted by $\boldsymbol{R_{PB}}$. The point P is the **vertex of $\boldsymbol{R_{PB}}$**. If P is between A and B, then the two rays R_{PA} and R_{PB} have only the vertex P in common, and each ray is, intuitively, infinite in only one direction.

Two rays are **distinct** if there is a point in one that does not lie in the other. An **angle** is the union of two distinct rays with a common vertex. The angle formed by the two rays R_{OA} and R_{OB} will be denoted by $\angle \boldsymbol{AOB}$.



If A, O, B are **collinear** (i.e., lie on a line so that O is between A and B), we say the angle is a **straight angle**. If R_{OA} and R_{OB} coincide, then we do not have an angle according to the definition above, but we make an exception and call it the **zero angle**. Now we have to face up to the fact that the intuitive concept of an angle is not just “two rays with a common vertex” but “the space between these two rays”. In other words, if $\angle AOB$ is neither the zero nor the straight angle, which of the following two subsets of the plane do we have in mind when we say $\angle AOB$, the space indicated by s or the one indicated by t?

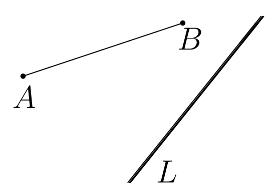
To resolve this difficulty, we need a precise way to differentiate between the two. We want to be able to say that a line separate the plane into “two halves”. If we already have coordinates axes in the plane, then this would be easily said. For example, the x-axis separates the plane into the upper half and lower half. But we are developing geometry from the beginning, so we won't get to set up coordinate axes until much



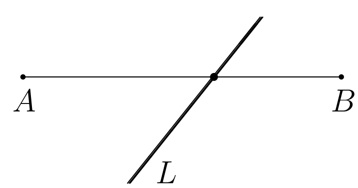
later. For now, here is our next assumption about the plane that makes possible the discussion of "half-planes".

(A4) (Plane Separation) *A line L separates the plane into two* non-empty *convex subsets, $\mathcal{L}$ and $\mathcal{R}$, called* **half-planes***, so that*:

(i) *Every point in the plane is in one* and only one *of the sets $\mathcal{L}$, $\mathcal{R}$, and L.*



(ii) *If two points A and B in the plane belong to different half-planes, then the line segment AB must intersect the line L.*



Two points that lie in the same half-plane of L are said to be **on the same side of $\boldsymbol{L}$**, and two points that lie in different half-planes are said to be **on opposite sides of $\boldsymbol{L}$**. The net effect of (i) and (ii) is to provide a recipe for testing whether two points, that are not on L, lie on the same side or opposite sides of the line L, in the following sense: if the segment AB does not intersect L, then by (ii), they are on the same side, but if AB intersects L, then A and B do not lie on the same side (a half-plane is convex and disjoint from L) and therefore must lie on opposite sides.

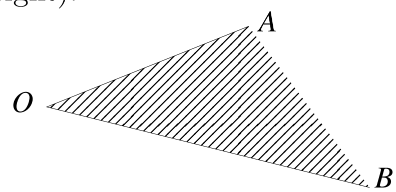
The union of either $\mathcal{L}$ or $\mathcal{R}$ with L is called a **closed half-plane**.

Now we return to an angle $\angle AOB$ which is neither the zero angle nor a straight angle. The rays R_{OA}, R_{OB} determine two subsets of the plane, one of them is the intersection of the following two closed half-planes:

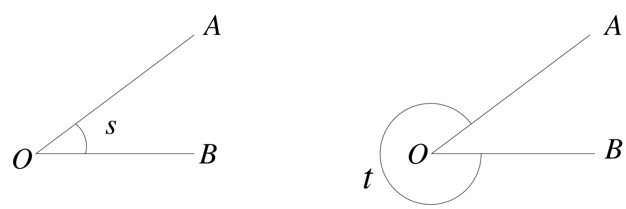
the closed half-plane of the line L_{OA} containing B, and

the closed half-plane of the line L_{OB} containing A.

By the observation above, this is a convex set, and is suggested by the shaded set in the following figure (note that the shading only covers a finite portion of a set extending infinitely to the right).



We will refer to this set as the **convex part** of $\angle AOB$, and this is the set that corresponds to our intuitive notion of what $\angle AOB$ is. When we refer to the convex part of an angle, we sometimes denote it be a single letter, e.g., $\angle \boldsymbol{O}$, if there is no danger of confusion. See the part indicated by s in the following:



On the other hand, there will be occasions to use the other subset of the plane determined by the rays R_{OA}, R_{OB}. This would be the **nonconvex part** of $\angle AOB$ as indicated by t above. Precisely, this is all the points that do not lie in the convex part of $\angle AOB$, together with all the point on both of the rays, R_{OA} and R_{OB}.

Unless stated otherwise, a (nonzero and non-straight) angle will refer to the convex part of the angle.

Our next goal is to introduce the measurement of angles in terms of *degree*. For a better understanding of degree, it is best to first introduce the concept of *distance* in the plane. Another reason we should be interested in the concept of *distance* is to make sense of the the concept of *the length of a segment in the plane.* Recall that for a segment on a given line, which is regarded as a number line, "length" on that line depends on the choice of a unit segment. Thus far, we have only measured the length of a segment on each line, one line at a time. Now there are many lines in the plane, and if the unit segment on each line is chosen at random—so that the length of a segment also varies randomly—there would be chaos. By introducing the concept of *distance*, we can define the length of a segment in the plane in a uniform way. This will be done after the next asssumption (A5).

(A5) *To each pair of points A and B of the plane, we can assign a number* $\mathbf{dist}(\boldsymbol{A},\boldsymbol{B}) \geq \mathbf{0}$ *so that*

(*i*) $\operatorname{dist}(A,B) = \operatorname{dist}(B,A)$.

(*ii*) $\operatorname{dist}(A,B) \geq 0$, *and* $\operatorname{dist}(A,B) = 0 \iff A$ and B *coincide.*

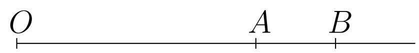
(*iii*) *If A, B, C are collinear points, and C is between A and B, then*

$$\operatorname{dist}(A,B) = \operatorname{dist}(A,C) + \operatorname{dist}(C,B)$$

Of course, condition (*iii*) is what prevents the assignment of a nonnegative number $\operatorname{dist}(A,B)$ to each pair of points A and B from being random or arbitrary. On each line of the plane, the **length of a segment** AB, denoted by the symbol $|\boldsymbol{AB}|$, will henceforth be defined to be $\operatorname{dist}(A,B)$. Thus "length of a segment" retains the intuitive meaning of "the distance between the endpoints". We say **two segments are equal** if they have the same length.

Observe that, in the presence of the distance function, to make a line into a number line once a point has been chosen to be 0, the choice of the unit 1 will now be limited to only one of two points. In other words, what we claim is that *there are exactly two points on the given line that are of distance 1 from O.* Intuitively, these are the points A and A' in the following picture so that $|AO| = |OA'| = 1$.

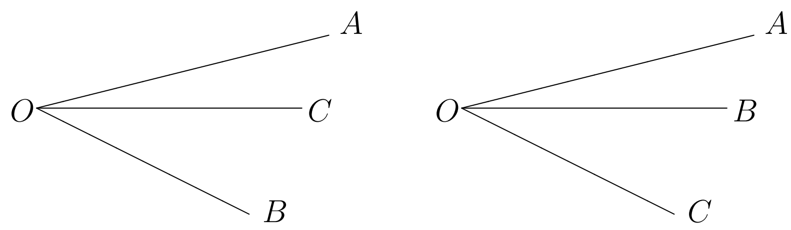
In the classroom, drawing the picture should be sufficient to convince students of this fact because its proof is hardly exciting. Nevertheless, here is the proof. Fix a ray with vertex O on the given line L (see (A3) on page 85), and suppose points A and B on this half-line are of distance 1 from O.



The segment $[A, B]$ does not contain O because A, B are in the same half-line (see (*ii*) of (L3)). Thus O is not between A and B, and the only possibility is that A is between O and B, or B is between O and A. Suppose the former holds. Then by (*iii*) of (A5), $|OA| + |AB| = |OB|$. But we are assuming $|OA| = |OB| = 1$, so we get $1 + |AB| = 1$, which implies $|AB| = 0$. Now by (*ii*) of (A5), $A = B$. Thus on each half-line there is only one point of distance 1 from O. Since there are exactly two half-lines issuing from O, the claim is proved.

Thus once a point has been chosen to be 0 on a line, there are only two ways to make the line into a number line corresponding to the two choices of the number 1 on this line.

Having introduced the concept of the length of a segment in the plane, we can now introduce the concept of the *degree* of an angle to measure its magnitude. Intuitively, every angle has a degree, a straight angle should be 180 degrees, and the "full" angle is 360 degrees. In order to say this precisely, some preparation is necessary. For entirely technical reasons, it is simpler if we only deal with the convex part of every angle under discussion, so intuitively, we avoid dealing with angles with degree > 180. *Let it be understood, therefore, that in the succeeding discussion, we only take the convex part of an angle* (page 88). We define two angles $\angle AOC$ and $\angle COB$ to be **adjacent** if they have a side in common and if C lies in the convex part of $\angle AOB$, e.g., $\angle AOC$ and $\angle COB$ in the left figure below are adjacent but the same angles in the right figure below are not adjacent.



Adjacent angles $\angle AOC$ and $\angle COB$ are the analogs, among angles, of segments AC, CB so that A, B, C are collinear and C is between A and B; they will allow us to formulate the analog of condition (iii) in assumption (A5) above. (This is one reason why we introduce *distance* before *degree*.) Our assumption about the degree of an angle now takes the following form (reminder: we only deal with the convex part of an angle in this discussion):

(A6) *To each angle $\angle AOB$, we can assign a number* $|\angle \boldsymbol{AOB}|$, *called its* **degree**, *so that*

(i) $0 < |\angle AOB| < 360°$, *where the small circle ° is the abbreviation of* degree. *Moreover, if one side of the line L_{OB} is given and a number x is given so that $0 < x < 360$ but $x \neq 180$, then there is a unique angle AOB so that $|\angle AOB| = x°$ and the ray R_{OA} lies on that side of L_{OB}.*

(ii) $|\angle AOB| = 0° \iff \angle AOB$ *is the zero angle, and* $|\angle AOB| = 180° \iff \angle AOB$ *is a straight angle.*

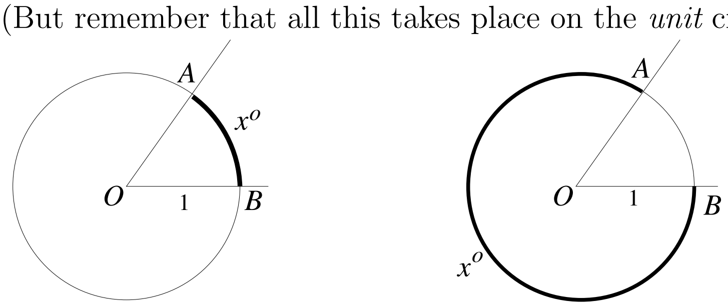
(iii) *If $\angle AOC$ and $\angle COB$ are adjacent angles then*

$$|\angle AOC| + |\angle COB| \ = \ |\angle AOB|$$

We note that by themselves, assumptions (A5) on distance and (A6) on degree do not have much substance. Their significance will be revealed only when we make the additional assumption that the basic rigid motions (in the next section) are distance-preserving and degree-preserving (see Lemma 8, page 110), and prove that there are "plenty of" basic rigid motions in the plane (see Lemmas 4, 5, and 7 on page 106, page 106, page 110, respectively).

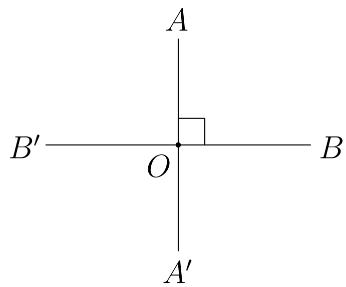
We now give a more intuitive discussion of the degree of an angle. The distance function allows us to introduce the concept of a *circle*. Fix a point O. Then the set of all the points A in the plane so that $\text{dist}(O, A)$ is a fixed positive constant r is called the **circle of radius r about O.** The point O is called the **center** of the circle. A line passing through the center O will intersect the circle at two points, say P and Q. The segment PQ is called a **diameter** of the circle, and the segment OP (or OQ) is also called a **radius** of the circle.

A circle whose radius is of length 1 is called a **unit circle**. Using a unit circle, we now describe how to assign to each angle a degree. Given $\angle AOB$, let $\mathcal{C}$ be the unit circle centered at O and we may as well assume that both A and B lie on $\mathcal{C}$. Let $\overset{\frown}{AB}$ denote the intersection of $\mathcal{C}$ with an angle, say, $\angle AOB$. *Here, we have to use both the convex part and nonconvex part of the angle, depending on the situation.* $\overset{\frown}{AB}$ is called an **arc** on $\mathcal{C}$; if it is the intersection of $\mathcal{C}$ with the convex part (respectively, the nonconvex part) of an angle, it is called a **minor arc** (respectively, **major arc**). It is possible, using the distance function in the plane, to define the *length* of any arc. An arc whose length is $\frac{1}{360}$ of the length of $\mathcal{C}$ is called **one degree**. Then we can subdivide a degree into n equal parts (where n is any whole number), thereby obtaining $\frac{1}{n}$ of a degree, etc. *It is exactly the same as the division of the chosen unit on a number line into unit fractions*, except that in this case, we have a "circular number line" so that, once a point has been chosen to be 0, the number 360 coincides with 0 again. If A is chosen as the 0 of this circular number line, the value of B on this circular number line is exactly the the degree of $\angle AOB$, which is of course the length of $\overset{\frown}{AB}$. Thus in the following picture, if the length of this arc $\overset{\frown}{AB}$ is x, then $|\angle AOB| = x^\circ$. (But remember that all this takes place on the *unit* circle around O.)



With the measurements of angles available, we can introduce some standard terminology for angles and polygons. Two angles are defined to be **equal** if they have the same degree. An angle of 90° is called a **right angle**. An angle is **acute** if it is less than 90°, and is **obtuse** if it is greater than 90°. There are analogs of these names for triangles, namely, a triangle is called a **right triangle** if one of its angles is a right angle, an **acute triangle** if all of its angles are acute, and an **obtuse triangle** if (at least) one of its angles is obtuse. (In view of the Angle Sum Theorem in grade 8, page 66, at most one angle of a triangle can be obtuse.)

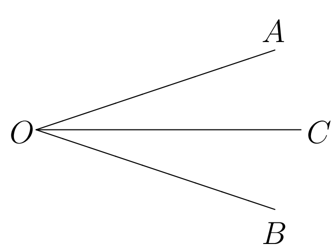
Let two lines meet at O, and suppose one of the four angles, say $\angle AOB$ as shown, is a right angle.



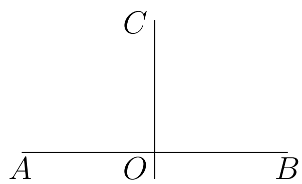
Then we claim that all the remaining angles are also right angles, i.e., $|\angle BOA'| = |\angle A'OB'| = |\angle B'OA| = 90°$. This is because by (*ii*) of (A6), $|\angle AOA'| = 180°$, so that by (*iii*) of (A6),

$$|\angle BOA'| = |\angle AOA'| - |\angle AOB| = 180° - 90° = 90°.$$

Similarly, the remaining two angles are also 90°. It follows that when two lines meet and if any one of the four angles so produced is a right angle, then all four angles at the point of intersection are right angles. It is therefore unambiguous to define the two lines to be **perpendicular** if an angle formed by the two lines at the point of intersection is a right angle. In symbols: $\boldsymbol{L_{AO} \perp L_{OB}}$ in the notation of the preceding figure, although it is equally common to write instead, $\boldsymbol{AO \perp OB}$. A ray R_{OC} in the convex part of an angle AOB is called an **angle bisector** of $\angle AOB$ if $|\angle AOC| = |\angle COB|$. Sometimes we also say less precisely that **the line $\boldsymbol{L_{OC}}$ (rather than the ray $\boldsymbol{R_{OC}}$) bisects the angle $\boldsymbol{AOB}$.**



It is clear that an angle has one and only one angle bisector (by (i) of assumption (A6)). Therefore if $CO \perp AB$ where O is a point of AB, as shown below, then CO is the unique angle bisector of the straight angle $\angle AOB$.



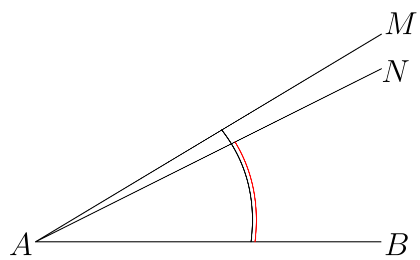
For a later reference, we make a separate statement of this observation:

> *Let L be a line and O a point on L. Then there is one and only one line passing through O and perpendicular to L.*

We can now complete the list of standard definitions about lines and segments. If AB is a segment, then the point C in AB so that $|AC| = |CB|$ is called the **midpoint** of AB. Analogous to the angle bisector, the **perpendicular bisector** of a segment AB is the line perpendicular to L_{AB} and passing through the midpoint of AB. It follows from the uniqueness of the line perpendicular to a line passing through a given point that *there is one and only one perpendicular bisector of a segment.*

We now introduce some common names for certain triangles and quadrilaterals. An **equilateral triangle** is a triangle with three sides of the same length, and an **isosceles triangle** is one with at least two sides of the same length. (Thus by our definition, an equilateral triangle is isosceles.) A quadrilateral all of whose angles are right angles is called a **rectangle**. A rectangle all of whose sides are of the same length is called a **square**. Be aware that at this point, we do not know whether there is a square or not, or worse, whether there is a rectangle or not. (If it is the case that the sum of (the degrees) of the four angles of quadrilateral is 361°, then clearly no rectangle can exist, much less a square.) A quadrilateral with at least one pair of opposite sides that are parallel is called a **trapezoid**. A trapezoid with two pairs of parallel opposite sides is called a **parallelogram**. A quadrilateral with four sides of equal length is called a **rhombus**. It can be proved (using the SSS criterion for triangle congruence on page 123) that rhombi are parallelograms.

We conclude by making a general observations about angles. Consider two angles $\angle MAB$ and $\angle NAB$ with a side, the ray R_{AB}, in common. Let us assume that M and N are on the *same side* of L_{AB} (see page 87).



Then it is believable that the rays R_{AM} and R_{AN} coincide if and only if the angles are equal. In a school classroom, the proof of something this boring should be skipped, but here is the proof. If M and N are on the *same side* of L_{AB}, then either M is in the convex part of $\angle NAB$ or N is in the convex part of $\angle MAB$ (this is a routine argument using (A4)). Let us say it is the latter, as shown. Then, by definition, R_{AM} and R_{AN} coincide if and only if $\angle MAN$ is the zero angle, and the latter happens if and only if $|\angle MAN| = 0$ by (*ii*) of (A6). Now using (*iii*) of (A6), we have,

$$|\angle MAN| \;=\; |\angle MAB| - |\angle NAB|$$

Thus $|\angle MAN| = 0$ if and only if $|\angle MAB| = |\angle NAB|$, i.e., the angles $\angle MAB$ and $\angle NAB$ are equal. We state this formally:

Lemma 3. *Given two angles $\angle MAB$ and $\angle NAB$, suppose they have one side R_{AB} in common and M and N are on the same side of the line L_{AB}. Then the other sides R_{AM} and R_{AN} coincide if and only if the angles are equal.*

2. Definitions of basic rigid motions and assumptions

The concept of transformation

In eighth grade, we introduced the **basic rigid motions** (i.e., rotations, reflections, and translations) mostly through the use of transparencies. Now we are going to define them precisely, and in so doing, we will be more careful with the order of the definitions because we have to make sure that each is **well-defined** in a sense that to be explained later. This particular presentation begins with rotations, then reflections, and then translations.[10] We first give the definitions, and then look back to decide what theorems need to be proved in order to ensure that each definition is logically sound.

We formally introduce the concept of a **transformation** F of the plane as a rule that assigns to each point P of the plane a point $F(P)$ of the plane. We note for a future reference that, according to this definition, to each point, a transformation can only assign *one* unambiguous point; thus, by definition, it cannot happen that for a given transformation F and a given point P, the assigned point $F(P)$ could be one of several possibilities. As in grade 8, $F(P)$ is called the **image of $\boldsymbol{P}$ by $\boldsymbol{F}$** and often we speak of **$\boldsymbol{F}$ mapping $\boldsymbol{P}$ to $\boldsymbol{F(P)}$.** If $\mathcal{S}$ is a **geometric figure** in the plane (i.e., a subset of the plane), then the collection of all the points $F(Q)$ where Q is a point of $\mathcal{S}$ is called the **image of $\boldsymbol{\mathcal{S}}$ by $\boldsymbol{F}$**, which is usually denoted by $\boldsymbol{F(\mathcal{S})}$. We likewise say **$\boldsymbol{F}$ maps $\boldsymbol{\mathcal{S}}$ to $\boldsymbol{F(\mathcal{S})}$.**

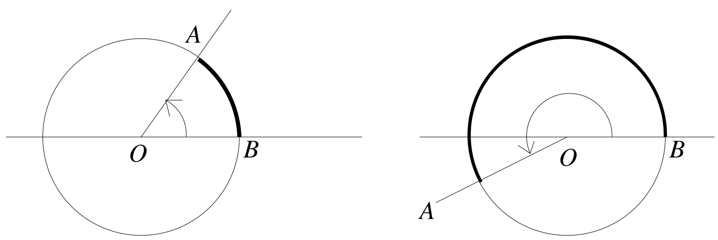
In a classroom, we suggest the use of a coordinate system to give students drills on the concept of a transformation. For example, the **identity transformation $\boldsymbol{I}$**, which is the transformation that maps each point to itself, can be described as $I(x, y) = (x, y)$ for any numbers x and y. To make students better appreciate the basic rigid motions, some standard distance-distorting transformations can be introduced, e.g., the transformation G so that $G(x, y) = (x + 3, y)$ or $G(x, y) = (x + y, y)$, and students can check that, in general, the distance between two points is not preserved and the degree of an angle is also not preserved. In particular, the image of a rectangle is in general not a rectangle. At the same time, it must be pointed out that, insofar as we are trying to build up geometry from the beginning, such drills should be used only for illustrations. Students should be aware that *these drills are not part*

[10] There are other ways to do this, such as starting with reflections.

of the logical development because, at this point, there is as yet no coordinate system in the plane.

Definitions of basic rigid motions

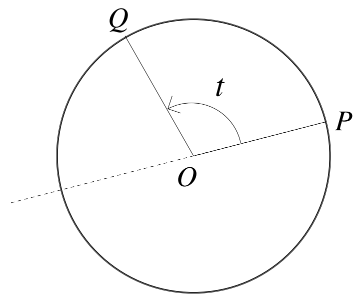
We now give in succession the definitions of the basic rigid motions: rotation, reflection, and translation. Before we give the definition of rotation, we mention explicitly that we will freely avail ourselves of the concepts of **clockwise** direction and **counterclockwise** direction on a circle. We will also take as self-evident that if a point B is fixed on a circle with center O, then all the points A so that A is in the counterclockwise (respectively, clockwise) direction of B and so that $0 < |AOB| < 180°$ will lie in a half-plane of the line L_{OB}. In the following picture, this would be the upper half-plane.



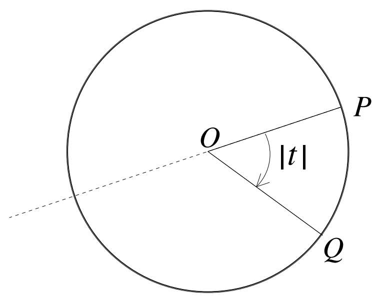
The whole discussion can be made precise by going through some elaborate definitions and proofs, but by common consent, it is better to skip them.

1. The **rotation *Ro* of *t* degrees** $(-180 \leq t \leq 180)$ around a given point O, called the **center of the rotation**, is a transformation of the plane defined as follows. Given a point P, we have to define what $Ro(P)$ is. The rotation is counterclockwise or clockwise depending on whether the degree is positive or negative, respectively. For definiteness, we first deal with the case where $0 \leq t \leq 180$. If $P = O$, then by definition, $Ro(O) = O$. If P is distinct from O, then by definition, $Ro(P)$ is the point Q on the circle with center O and radius $|OP|$ so that $|\angle QOP| = t°$ and so that Q is in the counterclockwise direction of the point P. We claim that this assignment is unambiguous, i.e., there cannot be more than one such Q. Indeed, if $t = 180°$, then Q is the point on the circle so that PQ is a diameter of the circle. If $t = 0$, then $Q = P$. Now if $0 < t < 180°$, then all the Q's with the stated properties (i.e.,

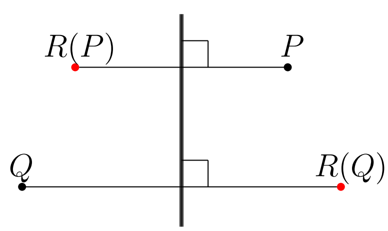
$0 < |\angle QOP| < 180°$ and Q is in the counterclockwise direction of the point P) lie in a fixed half-plane of the line L_{OP}. By Lemma 3 (page 95), there is only one such Q. Thus Ro is **well-defined**, in the sense that the rule of assignment is unambiguous. Notice that if $t = 0$, then Ro is the identity transformation of the plane I.



Now suppose $t < 0$. Then by definition, we rotate the given point P *clockwise* on the circle that is centered at O with radius $|OP|$. Everything remains the same except that the point Q is now the point on the circle so that $|\angle QOP| = |t|°$ and Q is in the clockwise direction of P (see picture below). We define $Ro(P) = Q$.

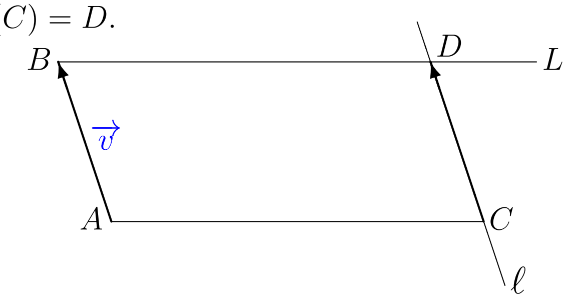


2. The **reflection R across a given line L**, where L is called **the line of reflection**, assigns to each point on L the point itself, and to any point P not on L, R assigns the point $R(P)$ which is **symmetric to it with respect to L**, in the sense that L is the perpendicular bisector (page 94) of the segment joining P to $R(P)$.

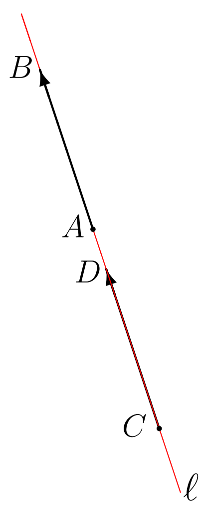


3. The **translation T along a given vector $\overrightarrow{v}$** assigns the point D to a given

point C in the following way. First, a vector is defined as in Grade 8, page 13. Let the starting point and endpoint of $\overrightarrow{v}$ be A and B, respectively. First assume C does not lie on line L_{AB}. Draw the line ℓ parallel to line L_{AB} passing through C; the Parallel Postulate guarantees that there is such an ℓ. The line L passing through B and parallel to the line L_{AC} then intersects line ℓ at a point D (L and ℓ must intersect because the Parallel Postulate says that there is only one line passing through C and parallel to L, which is L_{AC}, so ℓ is not parallel to L). By definition, T assigns the point D to C, i.e., $T(C) = D$.



Next, suppose C lies on the line L_{AB}, then the image D is by definition the point on the line L_{AB} so that the direction from C to D and the direction from A to B are the same, i.e., both of them point either to the positive direction or to the negative direction, and so that $|CD| = |AB|$. Or, if we regard L_{AB} as a number line so that all the points are now numbers, then we want D to be the number so that $D-C = B-A$.



Observe that if $\overrightarrow{0}$ is the zero vector, i.e., the vector with 0 length, then the translation along $\overrightarrow{0}$ is the identity transformation I.

A critical look at the definitions

We now take a critical look at the preceding definitions and expose the logical interconnections behind the formal statements by proving a few theorems. It will be noticed that the *fourth* theorem is already one of immense interest: the opposite sides of a parallelogram are equal (page 107).

The definitions of rotation and translation are straightforward, but the definition of a reflection raises a question. Let a line L be given and let P be a point not lying on L. Let the reflection across L be denoted by R. The definition of the point $R(P)$, to be denoted more simply by P', is that L is the perpendicular bisector of the segment PP'. Implicit in this definition is the fact that

(a) there is such a point P' so that L is the perpendicular bisector of the segment PP', and

(b) there is only one such point P'.

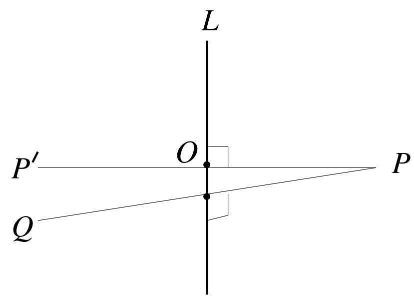
Neither is obvious at the moment. The need of (a) is obvious, but the need of (b) maybe less so. The fact is, if there is another point Q distinct from P' so that L is the perpendicular bisector of PQ, then the definition of a reflection implies that we can also define $R(P) = Q$. This raises the question: which point does R assign to P, P' or Q?

If we cannot verify that both (a) and (b) are valid, then the concept of a reflection is **not well-defined** on two levels. Given a line L and a point P in the plane, either the putative reflection R across L cannot assign a point to P (this would be the case if (a) fails), or there is more than one candidate of such a P' so that the assignment of R to P becomes ambiguous (this would be the case if (b) fails).

We will resolve this difficulty by proving the following theorem.

Theorem. *Given a line L and a point P, there is one and only one line passing through P and perpendicular to L.*

Assuming this theorem, (a) is easily seen to be true because if there is such a line, we simply let P' be the point on this line on the other side of L, so that $|PO| = |P'O|$, where O is the intersection of this line with L. Moreover, (b) is also true because, if



there is another point Q so that L is also the perpendicular bisector of PQ, then in particular $PQ \perp L$. But we know there is only one such line, so the two lines $L_{PP'}$ and L_{PQ} coincide and the point Q falls on $L_{PP'}$. It follows that Q and P' are two points on the same half-line of the line $L_{PP'}$ with respect to O and $|QO| = |P'O| (= |PO|)$. Hence $Q = P'$ and (b) is also true.

Thus, in order to show that the concept of reflection is well-defined, it remains to prove the Theorem. In addition, because we want to define reflection right after the definition of rotation, we have to prove the Theorem by making use of only properties of rotations. To this end, and for the development of plane geometry as a whole, we have to rely on some new assumptions about rotations that are, on the basis of the experience with basic rigid motions in grade 8, completely unexceptional. Precisely, we assume that:

Ro1. Rotations map lines to lines, rays to rays, and segments to segments.

Ro2. Rotations are **distance-preserving**, meaning that the distance between the images of two points is always equal to the distance between the original two points.

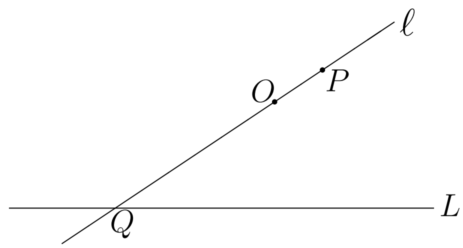
Ro3. Rotations are **degree-preserving**, meaning that the degree of the image of an angle is always equal to the degree of the original angle.

Note that, as in grade 8, assumption Ro1 guarantees that a rotation maps an angle to an angle (see page 23), so that assumption Ro3 makes sense.

We will leave assumptions Ro1 to Ro3 in this informal status for now but will summarize them in a comprehensive assumption **(A7)** below (page 110). Our immediate goal is to demonstrate how to make use of Ro1–Ro3 to prove the Theorem above. The next theorem is a critical first step toward this goal.

Theorem 1. *Let L be a line and O be a point not lying on L. Let $\mathcal{R}$ be the 180-degree rotation around O. Then $\mathcal{R}$ maps L to a line parallel to L itself.*

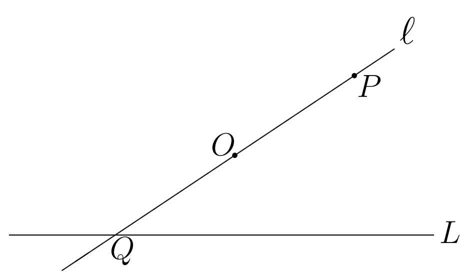
The truth of Theorem 1 depends on Lemma 1 on page 81, to the effect that *two distinct lines either do not intersect, or intersect at exactly one point.* Let us consider the situation of Theorem 1 where a line L and a point O are given and O does not lie on L. Let a line ℓ pass through O and intersect L at a point Q, as shown.



Now we make an observation: *if P is any point on the line ℓ not equal to Q, then P does not lie on L.* This is because L and ℓ, being distinct lines, already have one point Q in common and so Lemma 1 says no other point can be common to both lines. In particular, P does not lie on L, and the observation is proved.

With this observation in place, we can now prove Theorem 1.

Proof. First of all, we know that rotations map a line to another line (assumption Ro1 on page 101), so with assumptions and notation as in Theorem 1, $\mathcal{R}$ maps the line L to a line to be denoted by $\mathcal{R}(L)$. We have to show that $\mathcal{R}(L)$ and L have no point in common. Thus, if P is any point on $\mathcal{R}(L)$, we must show that P does not lie on L. By definition of $\mathcal{R}(L)$, there is a point Q of L so that P is the rotated image of Q by $\mathcal{R}$.



By the definition of Q, the segment OP is the 180-degree rotated image of the segment OQ; this means $|\angle POQ| = 180°$ and therefore P, O, and Q are collinear (see (*ii*) of Assumption (A6) on page 91). Let the line which contains P, O, Q be denote by ℓ. The preceding observation then tells us that P does not lie on L. This then proves Theorem 1.

We also give a proof by contradiction. Suppose $\mathcal{R}(L)$ and L have a point Q in common. Because Q is in $\mathcal{R}(L)$, there is a point P in L, so that $\mathcal{R}(P) = Q$. Because $\mathcal{R}$ is a 180-degree rotation around O, the three points P, Q, and O lie in a line ℓ. But Q is by assumption also a point in L, so ℓ and L have two distinct points in common: P and Q. But L and ℓ are distinct because O is in ℓ but not in L. This contradicts Lemma 1 (page 81) and Theorem 1 is proved.

Theorem 1 has an unexpected consequence. The Parallel Postulate assures us that if P is a point which does not lie on a given line L, then there is one and only one line passing through P and parallel to L. With Theorem 1 at our disposal, we now see that *there is in fact no need to assume the existence of such a line* because the said existence already follows from Theorem 1 (see the remark on page 81):

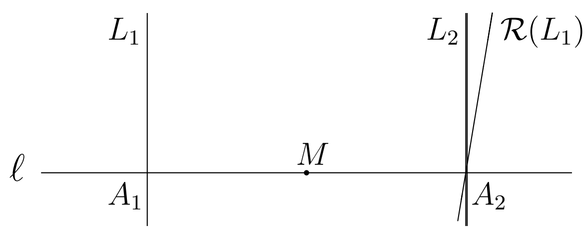
Corollary. *Given a line L and point P not on L, there is a line parallel to L and passing through P.*

Proof. Indeed, referring to the preceding picture, we take a point Q on L and let O be the midpoint of the segment PQ. If $\mathcal{R}$ is the 180-degree rotation around O, then Theorem 1 says the rotated image $\mathcal{R}(L)$ of L is parallel to L. But since a rotation preserves length (assumption Ro1, page 101)), $\mathcal{R}$ maps Q to P, so that $\mathcal{R}(L)$ in fact passes through P. The Corollary is proved.

Theorem 1 is deceptive because it is not obvious how it can be put to use. We will see that it is in fact a central theorem with numerous interesting consequences, including the very fact we are after, namely, that from a point outside a given line L, there cannot be two distinct lines passing through P and both perpendicular to L.

Theorem 2. *Two lines perpendicular to the same line are either identical or parallel* *to each other.*

Proof. Let L_1 and L_2 be two lines perpendicular to a line ℓ at A_1 and A_2, respectively. We have noted in an observation on page 94 that the the line passing through a given point of a line and perpendicular to that line is unique. Thus if $A_1 = A_2$, L_1 and L_2 are identical. So suppose $A_1 \neq A_2$. We need to prove that $L_1 \parallel L_2$. Let $\mathcal{R}$ be the rotation of 180 degrees around the midpoint M of A_1A_2. If we can show that the image of L_1 by $\mathcal{R}$ is L_2, then we know $L_2 \parallel L_1$ by virtue of Theorem 1.



To this end, note that $\mathcal{R}(L_1)$ contains A_2 because $\mathcal{R}(A_1) = A_2$. We are given that $L_1 \perp \ell$. Since $\mathcal{R}(A_1) = A_2$ and $\mathcal{R}(A_2) = A_1$, we see that $\mathcal{R}(\ell) = \ell$ (because of assumption (A1)). By assumption Ro3 on page 101, rotations map perpendicular lines to perpendicular lines. Thus we have $\mathcal{R}(L_1) \perp \ell$. It follows that each of $\mathcal{R}(L_1)$ and L_2 is a line that passes through A_2 and perpendicular to ℓ. By the preceding observation about the uniqueness of the line perpendicular to a line ℓ at a given point of ℓ, we see that, indeed, $\mathcal{R}(L_1) = L_2$ and therefore $L_1 \parallel L_2$. Theorem 2 is proved.

Corollary 1. *Through a point P not lying on a line ℓ passes at most one line L perpendicular to ℓ.*

Proof. Suppose in addition to L, there is another line L' passing through P and also perpendicular to ℓ. Since these lines are not parallel (they already have P in common), they have to be identical, by Theorem 2. Thus $L = L'$. Corollary 1 is proved.

We will make a digression. Recall that earlier we introduced the concept of a rectangle as a quadrilateral whose adjacent sides are all perpendicular to each other.

As a result of Theorem 2, we now have:

Corollary 2. *A rectangle is a parallelogram.*

Corollary 1 addresses one half of the concern about a reflection being well-defined by proving half of the Theorem on page 100. Now we prove the other half as well .

Theorem 3. *Given a point not lying on a line L, there is a line that passes through the point and perpendicular to L.*

Proof. Take any point $A \in \ell$ and let L' be the line passing through A and perpendicular to ℓ (see the observation on page 94). If L contains P, we are done, so we may assume that P does not lie on L'. By the Corollary to Theorem 1, there exists a line L passing through P and parallel to L'. Let L intersect ℓ at B.

The line passing through B and perpendicular to ℓ is parallel to L' by Theorem 2, and must therefore coincide with L, by the Parallel Postulate. Thus $L \perp \ell$. This proves Theorem 3.

Theorem 3 and Corollary 1 to Theorem 2 together prove completely the Theorem on page 100. As we pointed out above, this shows that the concept of reflection is well-defined.

We make some general remarks about rotations and reflections. First of all, we expect reflections to behave like rotations with respect to distance and degree. Precisely, we assume the following about reflections.

R1. Reflections map lines to lines, rays to rays, and segments to segments.

R2. Reflections are distance-preserving.

R3. Reflections are degree-preserving.

Now we point out a feature common to both rotations and reflections. We note that, as a result of assumption (A6), part (i), and the definition of a rotation, there are "plenty of" rotations, in the following sense:

Lemma 4. *Given a point and a number t so that $-180 \le t \le 180$, there is a rotation of degree t around the point.*

Analogously, the same can be said of reflections as a result of Theorem 3 and the definition of a reflection:

Lemma 5. *Given a line in the plane, there is a reflection across that line.*

Finally, we analyze the concept of a translation T along a given vector $\overrightarrow{v}$. Recall the definition: suppose the vector $\overrightarrow{v}$ has starting point A and endpoint B, then if C does not lie on L_{AB}, the image $D = T(C)$ is by definition the intersection of the line ℓ that is parallel to L_{AB} and the line passing through B parallel to L_{AC}.

An immediate question is, how do we know the line ℓ intersects the line passing through B and parallel to L_{AC}? Call the latter line ℓ'. Now according to the Parallel Postulate, there can be only one line passing through the point C that is parallel to ℓ'. By construction, L_{AC} is that line, so ℓ is not parallel to ℓ'. This is why we get the point of intersection D.

We have to know more about the vector $\overrightarrow{CD}$ (i.e., the vector with starting point C and endpoint $D = T(C)$). We claim that $|CD| = |AB|$. Granting this for the moment, we now have an intuitive understanding of the translation T_{AB} along the vector $\overrightarrow{AB}$: it moves every point "in the same direction" as $\overrightarrow{AB}$, and moves it the same distance as that from A to B.

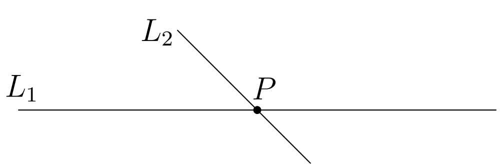
The fact that $|CD| = |AB|$ is a consequence of a theorem of great intuitive appeal:

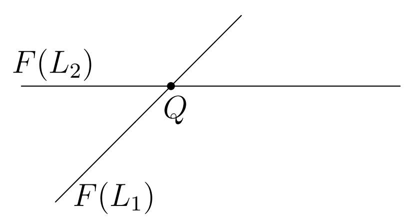
Theorem 4. *Opposite sides of a parallelogram are equal.*

Theorem 4 implies $|CD| = |AB|$ because the opposite sides of the above quadrilateral $BACD$ are parallel by construction: indeed $L_{BA} \parallel L_{CD}$ and $L_{BD} \parallel L_{AC}$ by construction. Therefore $BACD$ is a parallelogram and Theorem 4 is aplicable to give $|CD| = |AB|$. It remains therefore to prove Theorem 4.

The idea of the proof of Theorem 4 is to exploit Theorem 1, for the most practical of reasons: at this point, what else are we going to use? Of course, the presence of parallel lines in a parallelogram already suggests that something like Theorem 1 should be relevant. It will be obvious from the proof of Theorem 4 below why the following lemma is needed.

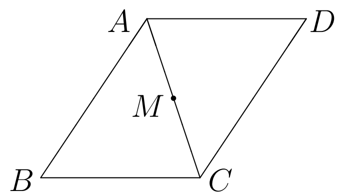
Lemma 6. *Let F be a transformation of the plane that maps lines to lines. Suppose two distinct lines L_1 and L_2 intersect at P and the image lines $F(L_1)$ and $F(L_2)$ intersect at a single point Q, then $F(P) = Q$.*





Proof of Lemma 6. Since P is a point in L_1, we see that $F(P)$ is a point on $F(L_1)$, by the definition of the image of L_1 by F. Similarly, $F(P)$ lies on the line $F(L_2)$. Therefore $F(P)$ lies in the intersection of $F(L_1)$ and $F(L_2)$. But by hypothesis, the latter intersection is exactly the point Q. So $F(P) = Q$.

Proof of Theorem 4. Given parallelogram $ABCD$, we have to prove that $|AD| = |BC|$ and $|AB| = |CD|$. It suffices to prove the former as the proof of the latter is similar. Let M be the midpoint of the diagonal AC and we will use Theorem 1 to explore the implications of the 180-degree rotation $\mathcal{R}$ around M.



Because $|MA| = |MC|$ and rotations preserve distance (assumption Ro1 on page 101), we have $\mathcal{R}(C) = A$ so that $\mathcal{R}(L_{BC})$ is a line passing through A and parallel to L_{BC} (by Theorem 1). Since the line L_{AD} has exactly the same two properties by assumption, the Parallel Postulate implies that $\mathcal{R}(L_{BC}) = L_{AD}$. Similarly, $\mathcal{R}(L_{AB}) = L_{CD}$. Thus, using the usual symbol $\cap$ to denote intersection, we have:

$$\mathcal{R}(L_{BC}) \cap \mathcal{R}(L_{AB}) \;=\; L_{AD} \cap L_{CD} = \{D\}$$

On the other hand, $L_{BC} \cap L_{AB} = \{B\}$. By Lemma 6, we have

$$\mathcal{R}(B) = D$$

Recall we also have $\mathcal{R}(C) = A$. Therefore $\mathcal{R}$ maps the segment BC to the segment joining D (which is the image of B) to A (which is the image of C), by the property that a rotation maps segments to segments (see assumption Ro1 on page 101). The latter segment has to be the segment DA, by (A1) (page 81). Thus $\mathcal{R}(BC) = DA$, so that by assumption Ro2 that rotations preserve distance (page 101), we have $|BC| = |AD|$, as desired. The proof of Theorem 4 is complete.

Corollary. *The angles of a parallelogram at opposite vertices are equal.*

The proof is implicit in the proof of Theorem 4: we already have $\mathcal{R}(\angle ABC) = \angle CDA$, so simply use assumption Ro3 on page 101 to conclude the proof. The proof of $\angle BAD = \angle DCB$ is similar.

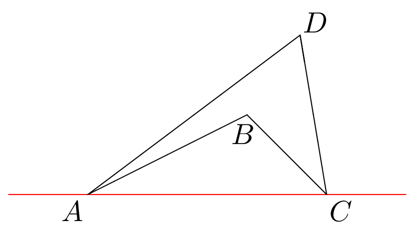
Theorem 4 together with the Corollary 2 to Theorem 2 (page 105) also imply that the *opposite sides of a rectangle are equal.* This reconciles the usual definition in school mathematics of a rectangle (a quadrilateral with four right angles and equal opposite sides) with our definition of a rectangle (a quadrilateral with four right angles).

Remark. We wish to make explicit something that is already contained in the above proof of Theorem 4:

> *If $ABCD$ is a parallelogram, then the vertices B, D lie on opposite sides of the diagonal line L_{AC}.*

This is because, if $\mathcal{R}$ is the 180-degree rotation around the midpoint M of the diagonal AC, we have just seen that $\mathcal{R}(B) = D$. Since it is easy to see that $\mathcal{R}$ maps each half-plane of L_{AC} to the other half-plane (see (A4) on page 87), we have the desired conclusion that B and D lie in opposite half-planes of L_{AC}.

The significance of such an observation lies in the fact that many quadrilaterals do not share this property, e.g.,



At this point, we pause to note that with Theorems 2 and 4 at our disposal, we at in a position to set up a coordinate system in the plane. Recall again: we are talking about a systematic and logical development of geometry *ab initio*, so the idea of setting up a coordinate system at this point simply means that only now do we have all the tools to define all the concepts correctly and have the necessary theorems (such as Theorems 2 and 4) to make sense of the coordinates (a, b) of a point P, e.g., a is the number on the x-axis which is the intersection of the x-axis and the line passing through P and parallel to the y-axis, but $|a|$ is also the distance of P from the y-axis. Because all this information is standard—except perhaps for the global perspective—we will not tarry on this topic here.

We note, as in the case of rotations and reflections, that:

T1. Translations map lines to lines, rays to rays, and segments to segments.

T2. Translations are distance-preserving.

T3. Translations are degree-preserving.

Because we know that given a line L and a point not on L, there is always a line passing through that point and parallel to L (Corollary to Theorem 1, page 103), the definition of the translation along any vector is well-defined. Therefore:

Lemma 7. *Given any vector, there is a translation along that vector.*

In summary, Lemmas 4, 5, and 7 (respectively, page 106, page 106, page 110) guarantee that there are "plenty of" basic rigid motion for any occasion. They will be the main tools for proving theorems, as one can see by the next section.

It is also time for us to summarize our assumptions about rotations Ro1–Ro3 (page 101), reflections R1–R3 (page 105), and translations T1–T3 (page 109) into one comprehensive assumption. Here it is:

(A7) *All basic rigid motions* (*rotations, reflections, and translations*)

(*i*) *map lines to lines, rays to rays, and segments to segments,*

(*ii*) *are distance-preserving,*

(*iii*) *are degree-preserving.*

3. Congruence criteria for triangles

[11] "HL" referes to hypotenuse-leg".

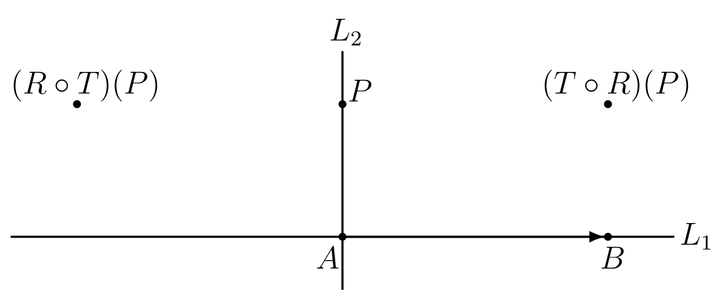
The concept of congruence

The main concern of this section is the proof of the three basic criteria for triangle congruence: SAS, ASA, and SSS. We begin by elucidating the concept of congruence.

We need the concept of *composing transformations*. Let F and G be transformations of the plane. We define a new transformation $F \circ G$, called the **composition of $\boldsymbol{G}$ and $\boldsymbol{F}$**, to be the rule which assigns to each point P of the plane the point $F(G(P))$. Schematically, we have;

$$P \longrightarrow G(P) \longrightarrow F(G(P))$$

i.e., we first let G send P to $G(P)$, and then let F send the point $G(P)$ to $F(G(P))$. Notice the peculiar feature of the notation: the symbol $F \circ G$ suggests that F comes before G, but in fact the definition itself, which assigns to P the point $F(G(P))$, requires that G act first. As far as the terminology is concerned, there is unfortunately no uniformity in how to indicate that G comes before F. So each time you come across the phrase "composing F and G", you have to find out precisely which is meant, is it $F \circ G$ or is it $G \circ F$? There is indeed a need to be careful, for the following reason. First, we say two **transformations $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ are equal**, in symbols $\boldsymbol{F_1 = F_2}$, if for every point P, it is true that $F_1(P) = F_2(P)$. Now define two transformations T and R as follows. Let L_1 and L_2 be perpendicular lines and let $\overrightarrow{AB}$ be a vector in L_1, as shown.



Define T to be the translation along the vector $\overrightarrow{AB}$, and R to be the reflection across L_2. It is now easily seen that for a point P on L_2, $(T \circ R)(P)$ and $(R \circ T)(P)$ are distinct points, as shown. Therefore $T \circ R \neq R \circ T$.

It would be instructive to give students many such examples to work on to let them experience firsthand the phenomenon of "noncommutativity".

The composition of more than two transformation is defined similarly. For example, if F, G, H, K are transformations, then **the composition $\boldsymbol{F \circ G \circ H \circ K}$** is defined to be the rule which assigns to each point P the point $F(G(H(K(P))))$.

Definition. *A* **congruence** *in the plane is a transformation of the plane which is equal to the composition of a finite number of basic rigid motions.*

The definition of congruence immediately implies that a composition of congruences is still a congruence. One can also verify, with a bit more effort, that if $\mathcal{S}$ is congruent to $\mathcal{S}'$, then also $\mathcal{S}'$ is congruent to $\mathcal{S}$.

Because each basic rigid motion is assumed to satisfy the three properties of (A7) (page 110), we would expect that so does a congruence. This is the content of the next lemma.

Lemma 8. *A congruence*

(*i*) *maps lines to lines, rays to rays, and segments to segments,*

(*ii*) *is distance-preserving,*

(*iii*) *is degree-preserving.*

In a classroom, a valid alternative to a proof of this lemma by an abstract argument given on the basis of (A7) and the definition of a composition of transformations would be a few concrete verifications of the behavior of the composition of two basic rigid motions on many points. If we want students to learn to prove geometric facts, it is probably more effective to make them prove interesting ones (such as Theorem 4 and its Corollary, or Theorems 5–7 below) rather than boring ones.

Lemma 8 explains why this *precise definition* of a congruence is the correct mathematical concept that captures our intuition of "same size, same shape". We will further discuss the pedagogical implications of this definition of congruence at the end of this section (page 125).

If $\mathcal{S}$ is congruent to $\mathcal{S}'$, we write $\boldsymbol{\mathcal{S} \cong \mathcal{S}'}$.

At this point, many simple facts about basic rigid motions can be given as exercises. For example,

- If R is the reflection across a line L, then $R \circ R = I$, where I is the identity transformation.
- If T_1 and T_2 are translations (along certain vectors), then there is a translation T_3 so that $T_1 \circ T_2 = T_3$.
- If T is the translation along a vector $\overrightarrow{AB}$ and if T' is the translation along the "opposite" vector $\overrightarrow{BA}$, then $T \circ T' = T' \circ T = I$.
- If $\mathcal{R}$ and $\mathcal{R}'$ are rotations around the same point O, of s and $-s$ degrees, respectively, then $\mathcal{R} \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{R} = I$.
- If $\mathcal{R}$ and $\mathcal{R}'$ are rotations around the same point O, of s and s' degrees, respectively, then $\mathcal{R} \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{R}$. Moreover, if $-180 \leq s + s' \leq 180$, then $\mathcal{R} \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{R} = \mathcal{R}_0$, where $\mathcal{R}_0$ is the rotation around O of $s + s'$ degrees.
- If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two rotations with different centers, is $\mathcal{R}_1 \circ \mathcal{R}_2$ always a rotation (around some point of a certain degree)?
- Can a translation be expressed as a composition of two reflections?
- [Hard] Can a rotation be expressed as a composition of two reflections?

SAS and ASA

To demonstrate the power of this definition of congruence, we now prove the following two theorems.

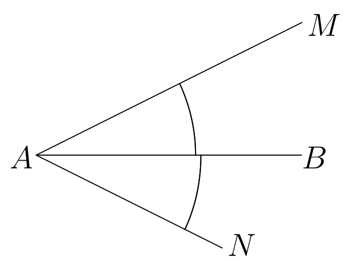
Theorem 5 (SAS). *Given two triangles ABC and $A_0B_0C_0$ so that $|\angle A| = |\angle A_0|$, $|AB| = |A_0B_0|$, and $|AC| = |A_0C_0|$. Then the triangles are congruent.*

Theorem 6 (ASA). *Given two triangles ABC and $A_0B_0C_0$ so that $|AB| = |A_0B_0|$, $|\angle A| = |\angle A_0|$, and $|\angle B| = |\angle B_0|$. Then the triangles are congruent.*

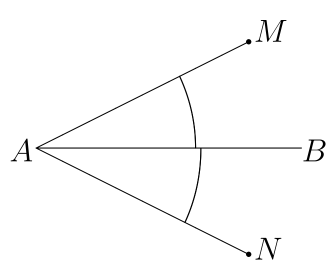
The proofs of these two theorems are very similar. Because we have already given an informal proof of ASA back in grade 8 (page 40), we will only give the proof of Theorem 5 (SAS) here.[12]

We begin with two simple observations on the behavior of angles under a reflection. They are nothing more than variations on the theme of Lemma 3 (page 95). As we shall we, they will be useful for other purposes as well.

Lemma 9. *Given two equal angles $\angle MAB$ and $\angle NAB$, suppose they have one side AB in common and M and N are on opposite sides of the line L_{AB}. Then the reflection across the line L_{AB} maps $\angle NAB$ to $\angle MAB$ (and also maps $\angle MAB$ to $\angle NAB$).*



Lemma 10. *Suppose two angles $\angle MAB$ and $\angle NAB$ are equal, and they have one side AB in common. Assume further that the* segments *AM and AN are equal. Then either $M = N$ (if M and N are on the same side of L_{AB}) or the reflection across L_{AB} maps N to M (if M and N are on opposite sides of L_{AB}).*



For the proof of Lemma 9, observe that the reflection R across L_{AB} maps $\angle NAB$ to $\angle N_0AB$, where $N_0 = R(N)$, so that $\angle N_0AB$ and $\angle MAB$ are now equal angles with one side R_{AB} in common. So $\angle N_0AB = \angle MAB$, by Lemma 3 (page 95). This

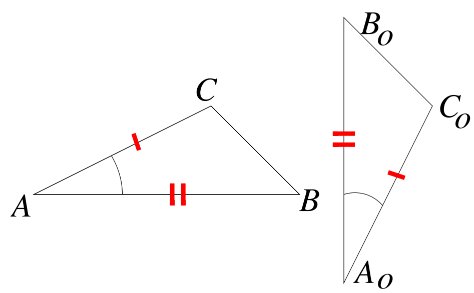
[12] It should be mentioned that using SAS, one can also prove ASA.

proves Lemma 9. As for Lemma 10, suppose M and N are on the same side of L_{AB}. By the same Lemma 3, we know that the *rays* AM and AN coincide. But since $|AM| = |AN|$, necessarily $M = N$. Now if M and N are on opposite sides of L_{AB}, then Lemma 9 shows that the reflection across L_{AB} maps the ray R_{AN} to the ray R_{AM}. Since a reflection preserves distance, the reflection maps the segment AN to a segment of length equal to $|AM|$, and therefore maps N to M by the preceding argument. This proves Lemma 10.

We are now in a position to begin the **proof of SAS.** An animation of this proof, due to Larry Francis, is given in

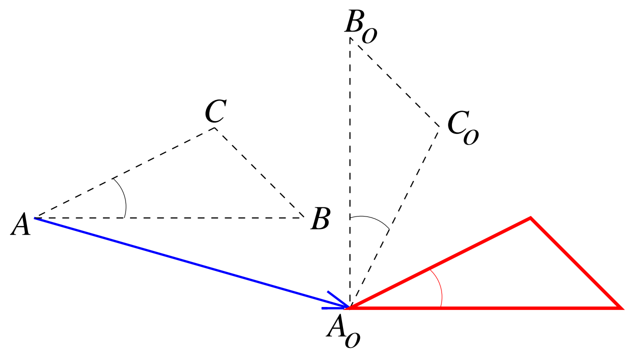
Side-Angle-Side Congruence by basic rigid motions

Now, suppose we are given triangles ABC and $A_0B_0C_0$ in the plane so that $\angle A$ and $\angle A_0$ are equal, and furthermore, $|AB| = |A_0B_0|$ and $|AC| = |A_0C_0|$ (see below). We have to explain why the triangles are congruent.

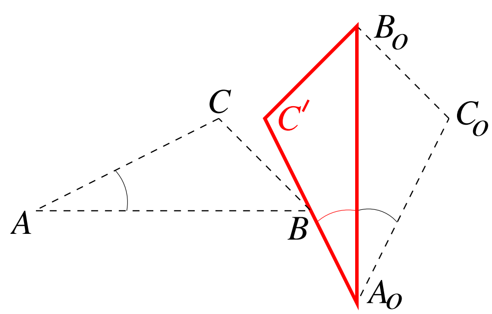


By our definition of congruence, this means we must exhibit a sequence of basic rigid motions so that their composition brings (let us say) $\triangle ABC$ to coincide exactly with $\triangle A_0B_0C_0$. For ease of comprehension, we will first prove the theorem for the pair of triangles in the above picture. At the end we will address other possible variations. We will first move vertex A to A_0 by a translation. Let T be the translation along the vector $\overrightarrow{AA_0}$ (from A to A_0, shown by the blue vector). We show the image of $\triangle ABC$ by T in red and use dashed lines to indicate the original positions of $\triangle ABC$ and $\triangle A_0B_0C_0$.

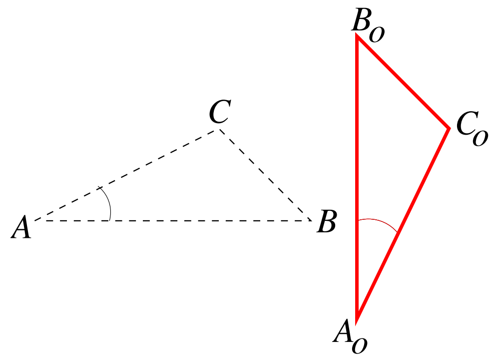
Next, we will use a rotation to bring the horizontal side of the red triangle (which is the translated image of AB by T) to A_0B_0. If the angle between the horizontal red side and A_0B_0 is t degrees (in the picture above, $t = 90$), then a rotation of t degrees



around A_0 will map the horizontal ray issuing from A_0 to the ray $R_{A_0B_0}$. Call this rotation $\mathcal{R}$. Now it is given that $|AB| = |A_0B_0|$, and we know a translation preserves lengths (Lemma 8, page 112). So the horizontal side of the red triangle has the same length as A_0B_0 and therefore $\mathcal{R}$ will map the horizontal side of the red triangle to the side A_0B_0 of $\triangle A_0B_0C_0$, as shown.



Two of the vertices of the red triangle already coincide with A_0 and B_0 of $\triangle A_0B_0C_0$. We claim that after a reflection across line $L_{A_0B_0}$ the third vertex of the red triangle will be equal to C_0. Indeed, the two marked angles with vertex A_0 are equal since basic rigid motions preserve degrees of angles (Lemma 8) and, by hypothesis, $\angle CAB$ and $\angle C_0A_0B_0$ are equal. Moreover, the left side of the red triangle with A_0 as endpoint has the same length as A_0C_0 because basic rigid motions preserve length (Lemma 8 again), and by hypothesis $|AC| = |A_0C_0|$. Therefore our claim follows from Lemma 10. Thus after a reflection across $L_{A_0B_0}$, the red triangle coincides with $\triangle A_0B_0C_0$, as shown:



Thus the desired congruence for the two triangles ABC and $A_0B_0C_0$ in this particular picture is the composition of a translation, a rotation, and a reflection.

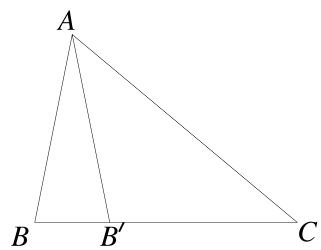
It remains to address the other possibilities and how they affect the above argument. If $A = A_0$ to begin with, then the initial translation would be unnecessary. It can also happen that after the translation T, the image $T(AB)$ (which corresponds to the horizontal side of the red triangle above) already coincides with A_0B_0. In that case, the rotation $\mathcal{R}$ would be unnecessary. Finally, if after the rotation the image of C is already on the same side of $L_{A_0B_0}$ as C_0, then Lemma 3 (page 95) implies that the image of C and C_0 already coincide and the reflection would not be needed. In any case, Theorem 5 is proved.

The perpendicular bisector and the HL criterion

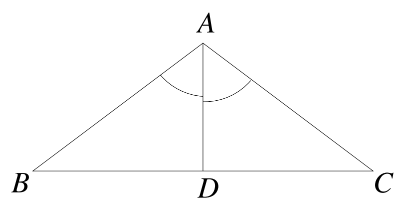
Before taking up the third major criterion for triangle congruence, SSS, we pause to observe a sometimes useful congruence criterion for right triangles. Note that the SAS criterion has a special requirement about the pair of equal angles: each of these two equal angles must be the angle "included" between the two sides in question. Otherwise the theorem fails as the following example shows. Let $\triangle ABB'$ be an isosceles triangle so that $|AB| = |AB'|$. On the line $L_{BB'}$, let C be a point outside the segment BB'. Now consider triangles ABC and $AB'C$.

These triangles are clearly not congruent, yet they have two pairs of equal sides ($|AB| = |AB'|$ and $|AC| = |AC|$) and one pair of equal angles ($|\angle ACB| = |\angle ACB'|$).

We are going to show that for *right* triangles, essentially the equality of two pairs of sides in addition to the given pairs of right angles are enough to guarantee congruence.



To this end we prove the following theorem, which is interesting in its own right. If $\triangle ABC$ is an isosceles triangle so that $|AB| = |AC|$, then it is common to refer to $\angle B$ and $\angle C$ as its **base angles**, $\angle A$ as its **top angle**, and BC as its **base**.



We will also refer to the line joining the midpoint of a side of a triangle to the opposite vertex as a **median** of the side, and the line passing through the opposite vertex and perpendicular to this side as **the altitude on this side.** Note that sometimes the segment from the vertex to the point of intersection of this line with the (line containing the) side is called the *median* and the *altitude*, respectively.

Theorem 7. (a) *An isosceles triangle has equal base angles.* (b) *In an isosceles triangle, the perpendicular bisector of the base, the angle bisector of the top angle, the median from the top vertex, and the altitude on the base all coincide.*

Proof. Referring to the preceding picture, let $|AB| = |AC|$ in $\triangle ABC$, and let the angle bisector of the top angle $\angle A$ intersect the base BC at D.[13] Let R be the reflection across the line L_{AD}. Since $|\angle BAD| = |\angle CAD|$, and since $|AB| = |AC|$, we have $R(B) = C$ by Lemma 10. Now it is also true that $R(D) = D$ and $R(A) = A$ because D and A lie on the line of reflection of R, so $R(BD) = CD$ and $R(BA) = CA$

[13]The fact that the angle bisector of $\angle A$ intersects BC is implied by assumption (A8) to be made explicit presently.

because a reflection maps a segment to a segment (by assumption (A7), page 110). Consequently, $R(\angle B) = \angle C$. Since a reflection preserves the degree of angles (again by assumption (A7), page 110), we have $|\angle B| = |\angle C|$. This proves part (a). For part (b), observe that since L_{AD} is the line of reflection and $R(B) = C$,

$$R(\angle ADB) = \angle ADC \quad \text{and} \quad R(BD) = (CD)$$

Therefore $|\angle ADB| = |\angle ADC| = 90°$, and $|BD| = |CD|$, so that L_{AD} is the perpendicular bisector of BC. Since L_{AD} is, by construction, also the angle bisector of $\angle A$, every statement in (b) follows. The proof is complete.

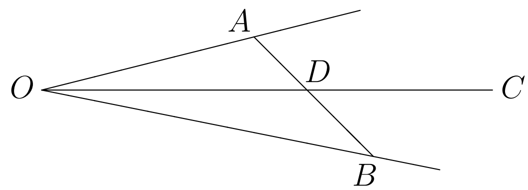
As an immediate corollary, we have the following useful characterization of the perpendicular bisector of a segment:

Corollary. *A point is on the perpendicular bisector of a segment if and only if it is equidistant from the endpoints of the segment.*

Proof. Let the segment be BC and let the point be A. If A is on the perpendicular bisector ℓ of BC, then by the definition of the reflection $\mathcal{R}$ across ℓ ((A7), page 98), $\mathcal{R}(B) = C$ and $\mathcal{R}(A) = A$. Thus $\mathcal{R}(AB) = AC$, and since reflection is distance preserving (page 110), $|AB| = |AC|$ and A is equidistant from the endpoints B and C. Conversely, suppose $|AB| = |AC|$. Thus triangle ABC is isosceles and the angle bisector of $\angle A$ is the perpendicular bisector of BC, by Theorem 7. But the angle bisector of $\angle A$ passes through A, so the perpendicular bisector of BC passes through A. The proof is complete.

We now backtrack a bit and deal with the assertion in the proof that the angle bisector of $\angle A$ must intersect side BC. While this fact is *intuitively* obvious, we also realize that there is no way we can explain why this must be true except to point to a picture. If we want to take this for granted, the way to do so is to add another assumption. This will be our last assumption.

(A8) (Crossbar axiom). *Given angle AOB, then for any point C in* (*the convex part of*) *$\angle AOB$, the ray R_{OC} intersects the segment AB* (at the point D in the following figure).



It is now clearly that (A8) implies that the angle bisector of an angle in a triangle must intersect the opposite side.

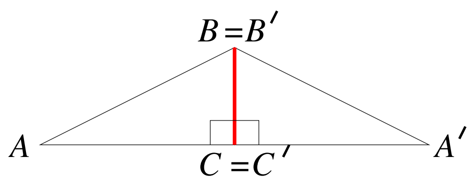
We are now ready for the congruence criterion for right triangles.

Theorem 8 (HL). *If two right triangles have equal hypotenuses and one pair of equal legs, then they are congruent.*

Proof. We give two proofs. The first uses basic rigid motions and it has the virtue of directly producing the congruence. The second one is the traditional argument using SAS.

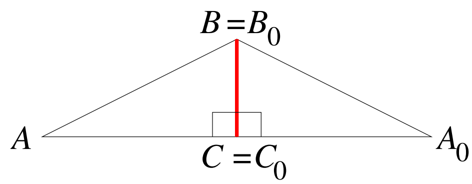
Suppose that the right triangles ABC and $A'B'C'$ satisfy $|\angle C| = |\angle C'| = 90°$ and $|B'C'| = |BC|$ in addition to $|AB| = |A'B'|$, and we will produce a sequence of basic rigid motions so that their composition maps $\triangle A'B'C'$ to $\triangle ABC$.

Case 1. We begin by proving the theorem under the special assumption that $B = B'$ and $C = C'$. Now A and A' are either on opposite sides of L_{BC} or on the same side. Let us begin by tackling the former case, A and A' being on opposite sides of L_{BC}, as shown.



Observe that A, C, and A' are collinear because $|\angle C| = |\angle C'| = 90°$, so that $|\angle ACA'| = |\angle C| + |\angle C'| = 90° + 90° = 180°$, by (iii) of assumption (A6) (page 91). Furthermore, because $BC \perp AA'$ and $|BA| = |B'A'| = |BA'|$, the Corollary to Theorem 7 implies that B lies on the perpendicular bisector of AA'. By the definition of a reflection (page 98), the reflection R across the line L_{BC} maps A' to A, B' to B and C' to C so that it maps $\triangle A'B'C'$ to $\triangle ABC$. The theorem is proved in this case.

Now suppose A, A' are on the same side of L_{BC}. Still with R as the reflection across L_{BC}, let $R(A') = A_0$, $R(B') = B_0$, and $R(C') = C_0$. Then $R(\triangle A'B'C') = \triangle A_0B_0C_0$. Now look at the two triangles ABC and $A_0B_0C_0$: the latter has the property that $B = B_0$, $C = C_0$, while A and A_0 are on opposite sides of L_{BC}.



Furthermore, $|AB| = |A_0B_0|$, $|\angle C| = |\angle C_0| = 90°$ (because a reflection preserves distance and degree, by assumption (A7) on page 110). The preceding argument then shows that

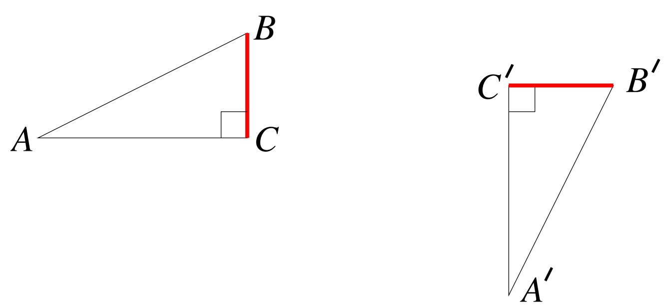
$$\triangle ABC \;=\; R(\triangle A_0B_0C_0)$$

In view of $\triangle A_0B_0C_0 = R(\triangle A'B'C')$ and the fact that $R \circ R$ is the identity transformation, we have

$$\triangle ABC \;=\; R(\triangle A_0B_0C_0) \;=\; R(R(\triangle A'B'C')) \;=\; \triangle A'B'C'$$

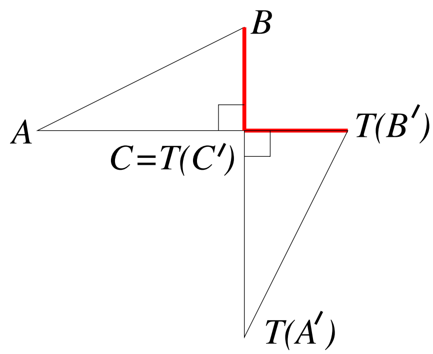
Thus $\triangle A'B'C'$ coincides with $\triangle ABC$ in the first place. Therefore the theorem is true if $B = B'$ and $C = C'$.

In general, we have a situation such as the following:

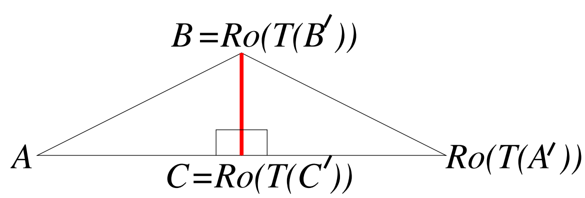


Let T be the translation so that $T(C') = C$ (note that T would be the identity transformation if C and C' already coincide). Then we have:

Because $|BC| = |B'C'|$ and because a translation preserves distance (assumption (A7), page 110), we see that BC and the segment $T(B'C')$ have the same length. Therefore a suitable rotation Ro will bring $T(B'C')$ to coincide with BC, as shown.



(Again, if BC already coincide with $T(B'C')$, then Ro would be the rotation of 0 degrees and would therefore be the identity transformation.)

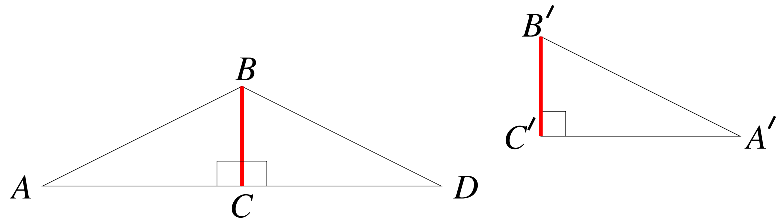


We now have two right triangles, $\triangle ABC$ and $Ro(T(\triangle A'B'C'))$ with the property that they share a leg—BC and $Ro(T(B'C'))$—and the hypotenuses AB and $Ro(T(A'B'))$ are equal. By Case 1, we see that there is a basic rigid motion F (which is either the identity transformation or the reflection across L_{AB}), so that F maps the triangle $Ro(T(A'B'C'))$ to $\triangle ABC$. Thus

$$(F \circ Ro \circ T)(\triangle A'B'C') \;=\; \triangle ABC$$

Since $F \circ Ro \circ T$ is a congruence, the proof of Theorem 8 is complete.

Next, we give the traditional proof of Theorem 8. Again, we have right triangles ABC and $A'B'C'$ so that $|\angle C| = |\angle C'| = 90°$ and $|B'C'| = |BC|$, $|AB| = |A'B'|$. On the line L_{AC}, take a point D so that $|CD| = |C'A'|$ and D and A are on the opposite half-planes of L_{BC}, as shown.



We claim that $\triangle BCD \cong \triangle B'C'A'$. This is because $BC \perp AD$ by hypothesis and therefore $\angle BCD$ and $\angle C'$ are equal as both are right angles. By hypothesis, $|BC| = |B'C'|$, and by construction, $|CD| = |C'A'|$. Thus SAS implies the desired congruence.

We next claim that $\triangle BCA \cong \triangle BCD$. We will use SAS again. The triangles have side BC in common. Moreover, we also have $|BA| = |BD|$; this is because the congruence $\triangle BCD \cong \triangle B'C'A'$ implies $|BD| = |B'A'|$ and $|B'A'| = |BA|$ by hypothesis. Finally we have to check that $|\angle DBC| = |\angle ABC|$, and this is so because $\triangle BAD$ being isosceles and BC being the altitude on AD, BC has to be also the angle bisector of $\angle ABD$ (Theorem 7(b)). All the conditions of SAS have been met and the sought-after congruence follows.

Putting the congruences $\triangle BCA \cong \triangle BCD$ and $\triangle BCD \cong \triangle B'C'A'$ together, we obtain $\triangle BCA \cong \triangle B'C'A'$ after all. The proof is complete.

The second (traditional) proof is so much shorter than the first proof using basic rigid motions that one may wonder why one should bother with the latter. We will discuss this issue at the end of the section.

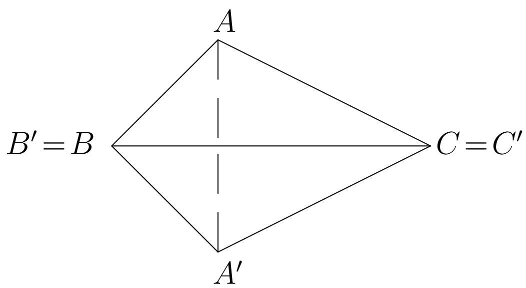
We now turn to the last major congruence criterion for triangles.

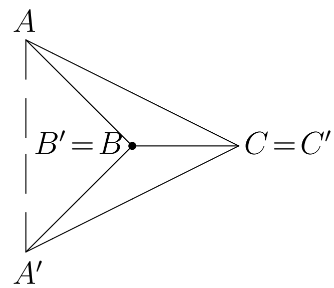
The SSS criterion

Theorem 9 (SSS). *Two triangles with three pairs of equal sides are congruent.*

Proof. In broad outline, this proof is very similar to the proof of Theorem 8. Suppose triangles ABC and $A'B'C'$ are given so that $|AB| = |A'B'|$, $|AC| = |A'C'|$, and $|BC| = |B'C'|$.

Case 1. We begin by assuming that the triangles satisfy an additional restrictive assumption: $B = B'$ and $C = C'$, and we will prove that there is a basic rigid motion that maps $\triangle A'B'C'$ to $\triangle ABC$. Either A and A' are on the same side of the line L_{BC} or on opposite sides; first assume they are on opposite sides. Here are two of the possibilities, but our proof will be valid in all cases.





By hypothesis, $|AB| = |A'B'|$, so B is equidistant from A and A'; by the Corollary to Theorem 7 (page 119), B lies on the perpendicular bisector of AA'. For the same reason, C lies on the perpendicular bisector of AA'. Because two points determine a line ((A1), page 81), L_{BC} is the perpendicular bisector of AA'. Thus the reflection R across L_{BC} maps A' to A, B to B and C to C (see the definition of reflection on page 98). Thus $R(\triangle A'B'C') = \triangle ABC$. This then proves the theorem under the stated restrictions that $B = B'$ and $C = C'$ and A, A' being on opposite sides of L_{BC}. Now suppose A, A' are on the same side of L_{BC}. Still with R as the reflection across L_{BC}, let $R(A') = A_0$, $R(B') = B_0$, and $R(C') = C_0$. Then $R(\triangle A'B'C') = \triangle A_0B_0C_0$, and the latter has the property that $B = B_0$, $C = C_0$, A, A_0 are on opposite sides of L_{BC}, and $|AB| = |A_0B|$, $|AC| = |A_0C|$ (because a reflection preserves distance, by assumption (A7) on page 110). The preceding argument then shows that

$$\triangle ABC \;=\; R(\triangle A_0B_0C_0)$$

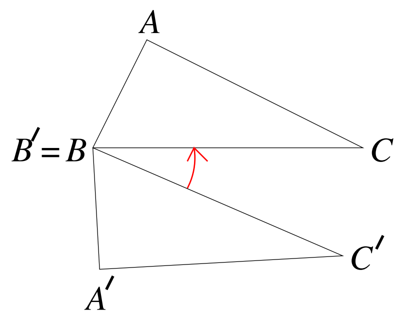
In view of $\triangle A_0B_0C_0 = R(\triangle A'B'C')$ and the fact that $R \circ R$ is the identity transformation, we have

$$\triangle ABC \;=\; R(\triangle A_0B_0C_0) \;=\; R(R(\triangle A'B'C')) \;=\; \triangle A'B'C'$$

Thus $\triangle A'B'C'$ coincides with $\triangle ABC$ in the first place. Therefore the theorem is true if, in addition to the equality of three pairs of sides, $B = B'$ and $C = C'$.

Case 2. Suppose we assume only that $B = B'$ but $C \neq C'$. Because $|BC| = |B'C'|$, a suitable rotation Ro around B will bring $B'C'$ to BC. Then the triangle

$Ro(\triangle A'B'C')$ and $\triangle ABC$ share a side BC, so that by Case 1, there is a basic rigid motion F (which is either a reflection or the identity) so that $F(Ro(\triangle A'B'C')) = \triangle ABC$.



Case 3. Finally, we treat the general case. In view of Case 2, we may assume that triangles ABC and $A'B'C'$ do not even share a vertex. Let T be the translation along the vector $\overrightarrow{B'B}$. Then $T(B') = B$, so that $T(\triangle A'B'C')$ and $\triangle ABC$ share a vertex B. Depending on whether $T(C')$ is equal to C or not, we are in either Case 1 or Case 2. Thus there is some basic rigid motion F and some rotation Ro (Ro would be the rotation of 0 degrees if $T(C') = C$), we have $F(Ro(T(\triangle A'B'C'))) = \triangle ABC$. This proves Theorem 9.

Pedagogical implications

We will briefly address the pedagogical implications of using basic rigid motions to define congruence and using them to prove the congruence of geometric figures. One of the problems encountered by beginners in geometry is the *formalism* inherent in the prevailing presentations of the subject. The two basic concepts of *congruence* and *similarity* come across as either formal and abstract, or pleasant but irrelevant. In the axiomatic presentations, congruence and similarity are defined only for polygons, and as such they are divorced from the way these terms are used in the intuitive context. In the other extreme, congruence is "same size and same shape", and similarity is "same shape but not necessarily the same size". What they have to do with *mathematics* is a question almost never addressed. These phrases are nothing but empty rhetoric, and students cannot relate them to the techniques of proving theorems using the *procedures* of SAS for both congruence and similarity, SSS for both congruence and similarity, etc. Ultimately, these concepts become synonymous with rote procedures.

The potential benefit of defining congruence using reflections, rotations, and translations is that they transform an abstract concept into one that is concrete and tactile. This is the whole point of the eighth grade geometry standards, which ask for the use of manipulatives, especially transparencies, to model reflections, rotations, and translations, i.e., to model congruence. It is for this reason that we used reflections, rotations, and translations to *prove* all three criteria of triangle congruence—SAS, ASA, and SSS—even when there was an option to use SAS to prove ASA and SSS. In the next section, we will give a few more examples of using reflections, rotations, and translations to prove theorems. In this way, theorem-proving in geometry will no longer be an exercise in formalism and abstraction. Congruence is something students can relate to in a tactile manner just by moving a transparency over a piece of paper. Later on, we will also ground the learning of similarity in similar tactile experiences.

The professional judgment of the practitioners in geometry is that geometric intuition is *built on* such tactile experiences rather than on abstract formalism. The goal of these standards is therefore to provide a sound foundation for the learning of geometry.

4. Some typical theorems

Overview

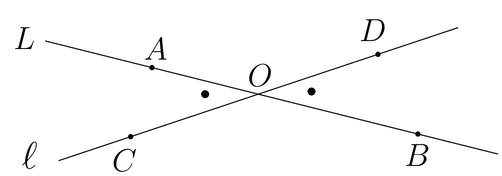
Having learned what it means for two geometric figures to be congruent, students now get to see some immediate applications. Since we are already in possession of all the general criteria for triangle congruence, we are free to develop the high school course on geometry at this point as in the classical treatment handed down to us by

Euclid if we so wish. However, in the spirit of the hands-on approach to geometry started in grade 8, we will continue to provide proofs, when it is appropriate, using basic rigid motions. Teachers can decide for themselves which kind of proofs are most appropriate for their students.

In addition to basic theorems about angles associated with a transversal with respect to a pair of parallel lines, section 4 proves the **concurrence** of angle bisectors, perpendicular bisectors, altitudes, and medians in a triangle (i.e., they all meet at a point). Along the way, various theorems of independent interest are proved, including some standard characterizations of a parallelogram and the fact that the angle sum of a triangle is 180 degrees. In the final subsection, we prove the triangle inequality.

Parallel lines and angles

First, we show that two intersecting lines L and ℓ give rise to some congruent angles. Suppose they meet at a point O. Let A, B be two points on L that lie on opposite sides of ℓ and, similarly, let C, D be two points on ℓ that lie on opposite sides of L.

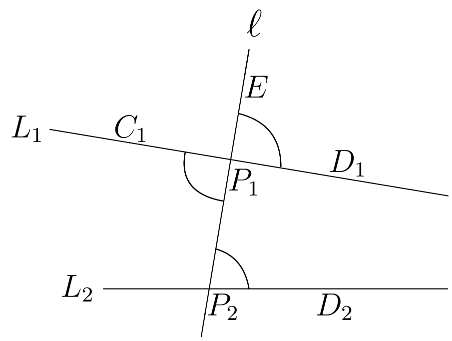


Then $\angle AOC$ and $\angle BOD$ are called **opposite angles** at O (sometimes called **vertical angles**). The following is standard.

Lemma 11. *Opposite angles are equal.*

Proof. The reason is that if $\mathcal{R}$ denotes the 180-degree rotation of the plane around O, then $\mathcal{R}$ moves the ray OA to the ray OB and the ray OC to the ray OD. Thus $\mathcal{R}$ moves the $\angle AOC$ to $\angle BOD$. Because basic rigid motions are assumed to preserve degree (page 110), the opposite angles $\angle AOC$ and $\angle BOD$ are equal. This proves Lemma 11.

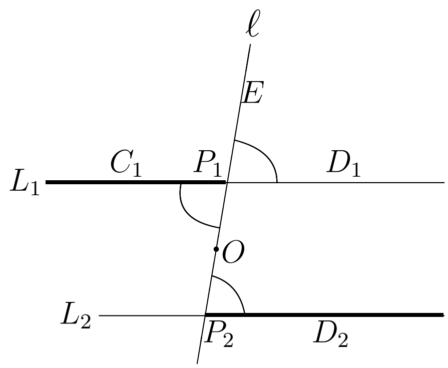
We next use this lemma to shed light on some basic properties of *parallel lines*. The whole discussion hinges on the deceptively simple Theorem 1 (page 102). First, we give more formal definitions of *corresponding angles* and *alternate interior angles* of a transversal with respect to a given pair of lines (they have been informally introduced earlier in grade 8 on page 49 and page 66). Let lines L_1 and L_2 be given and let ℓ be a line that intersects L_1 and L_2 at P_1 and P_2, respectively. The line ℓ is called a **transversal** of the lines L_1 and L_2. Let C_1, D_2 be points on L_1 and L_2, respectively, so that they lie on opposite sides of ℓ.



Then $\angle C_1P_1P_2$ and $\angle P_1P_2D_2$ are said to be **alternate interior angles** of the transversal ℓ with respect to L_1 and L_2. If E, D_1 are points on ℓ and L_1, respectively, so that $\angle EP_1D_1$ and $\angle C_1P_1P_2$ are opposite angles, then $\angle EP_1D_1$ and $\angle EP_2D_2$ are said to be **corresponding angles** of the transversal ℓ with respect to L_1 and L_2. We now come to one of the characteristic properties of the plane.

Theorem 10. *Alternate interior angles and corresponding angles of a transversal with respect to a pair of parallel lines are equal.*

Proof. Let the parallel lines be L_1 and L_2 and let a transversal ℓ intersect L_1 and L_2 at P_1 and P_2, respectively. Let the alternate interior angles be $\angle C_1P_1P_2$ and $\angle P_1P_2D_2$, as shown. It suffices to prove that these angles are equal because Lemma 11 then takes care of the statement about the corresponding angles $\angle EP_1D_1$ and $\angle EP_2D_2$.

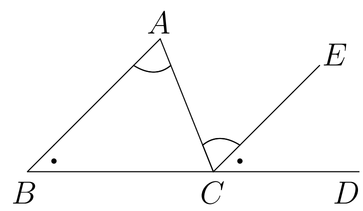


Let O be the midpoint of the segment P_1P_2 and let $\mathcal{R}$ be the 180-degree rotation around O. Furthermore, let $\mathcal{R}$ map the line L_2 to $\mathcal{R}(L_2)$. Now consider the two lines $\mathcal{R}(L_2)$ and L_1. We have $\mathcal{R}(L_2) \parallel L_2$ by Theorem 1, and also $L_1 \parallel L_2$ by hypothesis. Of course L_1 passes through the point P_1, but so does $\mathcal{R}(L_2)$ because a rotation preserves distance ((A7), page 110) and therefore $\mathcal{R}$ maps P_2 to P_1, so that $\mathcal{R}(L_2)$ contains P_1. By the Parallel Postulate, $\mathcal{R}(L_2) = L_1$. Because $\mathcal{R}$ is a 180-degree rotation, it maps the *ray* P_2D_2 to the *ray* P_1C_1. Therefore $\mathcal{R}$ maps $\angle P_1P_2D_2$ to $\angle P_2P_1C_1$. Since a rotation preserves degree ((A7) again), we see that $\angle P_1P_2D_2$ is equal to to $\angle P_2P_1C_1$. Theorem 10 is proved.

There is a noteworthy consequence of Theorem 10. By a common abuse of language, we abbreviate "the sum of the degrees of angles" to **the sum of angles**, and "the sum of the degrees of all three angles of a triangle" to **the angle sum of a triangle**.

Theorem 11 (Angle Sum Theorem). *The angle sum of a triangle is 180 degrees.*

Proof. This is the same proof as the one given in eighth grade, page 67, but rephrased in slightly more formal language. Let $\triangle ABC$ be given. On the ray BC, let a point D be chosen so that B and D lie on opposite sides of line L_{AC}. Then $\angle ACD$ is called an **exterior angle** of $\triangle ABC$. Let CE be the line parallel to line AB and passing through C (for the fact that there is such a line, see Corollary of Theorem 1, page 103).



By Theorem 10, $\angle A$ is equal to $\angle ACE$, and $\angle B$ is equal to $\angle ECD$. Hence the angle sum of $\triangle ABC$ is

$$|\angle A| + |\angle B| + |\angle C| \;=\; |\angle ACE| + |\angle ECD| + |\angle C| \;=\; 180^\circ.$$

Theorem 11 is proved.

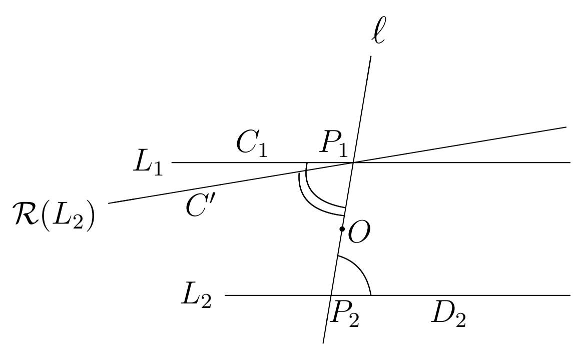
The reasoning in the preceding proof also proves the following Corollary. In the notation above, $\angle A$ and $\angle B$ are called the **remote interior angles** of the exterior angle $\angle ACD$. We have:

Corollary. *An exterior angle is equal to the sum of its remote interior angles.*

Theorem 10 has a converse, which is useful for deciding if two lines are parallel.

Thoerem 12. *If a pair of alternate interior angles or a pair of corresponding angles of a transversal with respect to two lines are equal, then the lines are parallel.*

Proof. Since equality of corresponding angles implies the equality of a pair of alternate interior angles by virtue of Lemma 11, it suffices to prove the theorem assuming the equality of a pair of alternate interior angles.

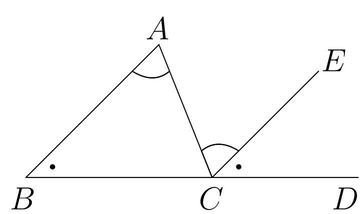


Thus let $\angle C_1P_1P_2$ and $\angle P_1P_2D_2$ be equal alternate interior angles of the transversal ℓ with respect to the lines L_1 and L_2, and we have to prove that $L_1 \parallel L_2$. As before, let O be the midpoint of the segment P_1P_2 and let $\mathcal{R}$ be the 180-degree rotation around O. If the rotated image of L_2 is denoted by $\mathcal{R}(L_2)$ and the rotated image of D_1 is denoted by C', then $\mathcal{R}(L_2)$ passes through P_1 and $\angle C'P_1P_2$ is equal to $\angle P_1P_2D_2$ (rotation preserves degree by (A7), page 110). By hypothesis, $\angle C_1P_1P_2$ is also equal to $\angle P_1P_2D_2$. Hence $\angle C'P_1P_2$ and $\angle C_1P_1P_2$ are equal angles with a common side P_1P_2. Moreover, since both C_1 and C' are on the opposite side of line $L_{P_1P_2} = \ell$ relative to D_2, we see that C_1 and C' are on the same side of ℓ. Therefore the rays P_1C_1 and P_1C' coincide (by Lemma 3, page 95), or what is the same thing, the lines L_1 and $\mathcal{R}(L_2)$ coincide. But $\mathcal{R}(L_2) \parallel L_2$, by Theorem 1, so $L_1 \parallel L_2$ after all. This proves Theorem 12.

Notice that Theorem 12 generalizes Theorem 2 on page 103.

Remark. At this point, we should make some comments that belong strictly to a Handbook for Teachers. For an introductory course in geometry, the proof of Theorem 11 on the angle sum of a triangle given above, or one similar to it, is an appropriate one. A teacher should be aware, however, that while this proof is not wrong, it is nevertheless incomplete in a subtle way, namely,

> how do we know that $|\angle ACD|$ is always bigger than $|\angle A|$ so that the ray R_{CE} lies in the convex part (page 88) of $\angle ACD$?



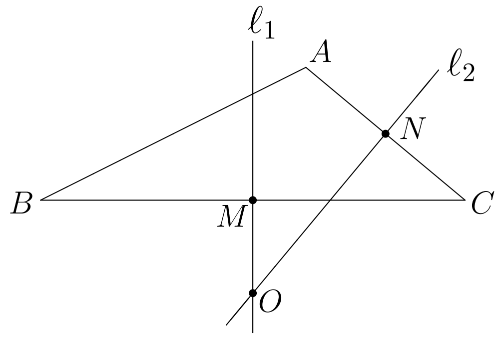
Because the picture is so seductive, questions about the validity of this fact is probably never going to be raised in a beginning class on geometry. One can give a rigorous proof, of course, but it turns out to be quite subtle, and the details would not be instructive for beginners. Moreover, in order to present such a proof, we would have to reformulate, with greater precision, assumptions (A1) to (A8), and we would also be forced to first establish some results of a purely technical nature that are devoid of geometric interest. From a mathematical standpoint, such foundational issues should not be taken lightly; setting a correct axiomatic system for the geometry that Euclid left with us took mankind all of 22 centuries (the *decisive*, finishing touch was supplied by David Hilbert in 1899). But what is good for mathematics may not be good for school education. A geometry course for *school students* should not worry about these technical details, anymore than students in elementary school should worry about the logical structure of whole numbers in the form of the *Peano axioms.* In a school classroom, it would be justified to mention in passing this subtle gap in the proof of Theorem 11, and let students know that an explanation can be found in upper division college mathematics courses.

Circumcenter, orthocenter, and incenter

We now turn to the standard concurrence theorems related to a triangle. We need a definition: three or more lines lines are **concurrent** if they meet at one point.

Theorem 13. (*i*) *The perpendicular bisectors of the three sides of a triangle meet at a point, called the* **circumcenter** *of the triangle.* (*ii*) *There is a unique circle that passes through the vertices of a triangle, and the center of this circle* (*the* **circumcircle** *of the triangle*) *is the circumcenter.*

Proof. Let the triangle be ABC, as shown, and let M, N be the midpoints of BC and AC, respectively. Also let the perpendicular bisectors of BC, AC be ℓ_1 and ℓ_2, respectively. Let ℓ_1 and ℓ_2 meet at O.



Since O lies on the perpendicular bisector of BC, $|OB| = |OC|$ (Corollary to Theorem 7 on page 119). Similarly, $|OC| = |OA|$. Together, we have $|OB| = |OA| = |OC|$, i.e., O is equidistant from A, B and C. In particular, O is equidistant from A and B. By the Corollary to Theorem 7 again, O lies on the perpendicular bisector of AB. As O already lies on the perpendicular bisectors of BC and AC, this proves the first part of the theorem.

To prove the second part, we have to first prove that there is a circle with center at O that passes through the vertices A, B, and C, and that any such circle must coincide with this circle. Consider the circle $\mathcal{K}$ with center O and radius $|OA|$. $\mathcal{K}$ must pass through B and C, on account of $|OB| = |OA| = |OC|$, so we have proved the existence of such a circle. Next,we have to show that any other circle $\mathcal{K}'$ passing through A, B and C must coincide with $\mathcal{K}$. Let the center of $\mathcal{K}'$ be O'. Since O' is by definition equidistant from B and C, O' lies on the perpendicular bisector of BC (Corollary to Theorem 7), and therefore lies on ℓ_1 . For exactly the same reason, O' must also lie on ℓ_2, the perpendicular bisector of AC. Therefore O' is the point of intersection of ℓ_1 and ℓ_2, which is O. This shows $O = O'$. Since $\mathcal{K}'$ passes through A, the radius of $\mathcal{K}'$ is also $|OA|$. Hence $\mathcal{K}' = \mathcal{K}$ because they have the same center and the same radius. The proof of the theorem is complete.

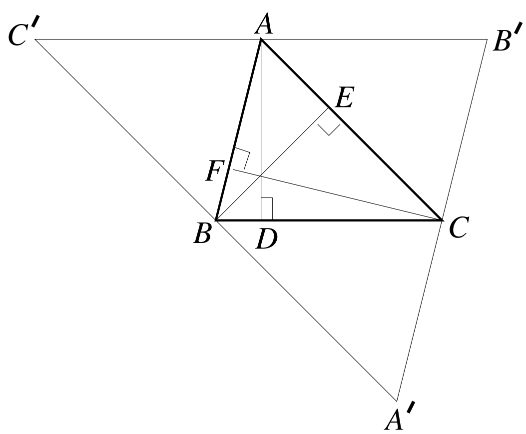
Remark. In a school classroom, the assertion in the preceding proof, to the effect that ℓ_1 and ℓ_2 must intersect, will likely be taken for granted and draw no attention whatsoever. A teacher may wish to use this as a teachable moment, however, and show students that the assertion *can* be proved. The reasoning is as follows. If ℓ_1 and ℓ_2 do not intersect, then $\ell_1 \parallel \ell_2$. By the Parallel Postulate, no line passing through M other than ℓ_1 can be parallel to ℓ_2. In particular, L_{BC} is not parallel to ℓ_2. Thus both ℓ_1 and ℓ_2 intersect L_{BC}, and Theorem 10 implies that $L_{BC} \perp \ell_2$. Since also $L_{AC} \perp \ell_2$, Theorem 12 implies that $L_{BC} \parallel L_{AC}$. But this is impossible because L_{BC} and L_{AC} are distinct lines with a point C in common. Therefore ℓ_1 and ℓ_2 must intersect.

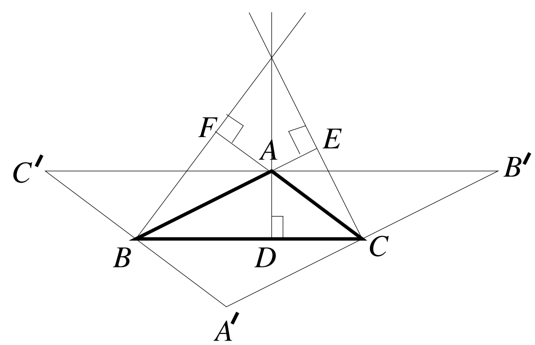
We will now use Theorem 13 to prove the concurrency of the altitudes of a triangle. Such a proof is not likely one that students can "discover" by themselves, but they can learn from it.

Theorem 14. *The three altitudes of a triangle meet at a point, called the* **orthocen-**

ter *of the triangle.*

Proof. Let the altitudes of $\triangle ABC$ be AD, BE and CF. Through each vertex, draw a line parallel to the opposite side, resulting in a triangle which we denote by $\triangle A'B'C'$. If the triangle is acute, it is illustrated on the left below. However if (let us say) $\angle A$ is obtuse, then we have a situation illustrated on the right below.





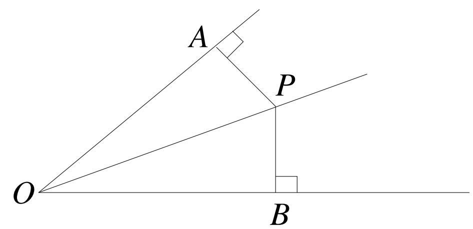
By construction, the quadrilaterals $AC'BC$, $ABCB'$ are parallelograms. Therefore, by Theorem 4 (page 107), $|C'A| = |BC| = |AB'|$. Thus A is the midpoint of $C'B'$. Moreover, since $AD \perp BC$ and $BC \parallel C'B'$, we also know that $AD \perp C'B'$ (Theorem 10 on page 128). It follows that L_{AD} is the perpendicular bisector of $C'B'$. Similarly, L_{FC} and L_{BE} are perpendicular bisectors of $A'B'$ and $C'A'$, respectively. By Theorem 13, L_{AD}, L_{FC} and L_{BE} meet at the circumcenter of $\triangle A'B'C'$. The proof is complete.

In approaching our next topic, the concurrence of the angle bisectors of a triangle, we should keep in mind the analogy between the angle bisector of an angle and the perpendicular bisector of a segment. To push the analogy further, we now prove the following lemma, which is the counterpart of the Corollary to Theorem 7 (page 119). We first need a definition. Let a line L and a point P not on L be given. From Theorem 3 (page 105) and Theorem 2 (page 103), we know there is a unique line passing through P and perpendicular to L; let this line intersect L at Q. The length $|PQ|$ is called the **distance of P from L** and the point Q is called the **foot of the**

perpendicular from P to L.

Lemma 12. *The angle bisector of (the convex part of) an angle is the collection of all the points equidistant from the two sides of the angle.*

Proof. First, we prove that the points on the angle bisector of the convex part of an angle (see page 88) are equidistant from its sides. So let the ray R_{OP} be the angle bisector of $\angle AOB$, and we may as well assume that the feet of the perpendicular from P to both sides of the angle are A and B, as shown. We have to prove that $|PA| = |PB|$.



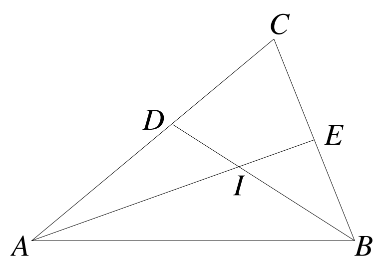
Let R be the reflection across L_{OP}. Since $|\angle POA| = |\angle POB|$, Lemma 9 on page 114 implies that R maps the ray OB to the ray OA. Using the degree-preserving property of R once more, we see that $R(PB)$ must be perpendicular to OA. Since there is only one perpendicular from P to the the line L_{OA} according to Theorem 2 (page 103), we conclude that $R(PB) = PA$. Because reflections preserve distance too, we get $|PA| = |PB|$, as desired.

Conversely, if a point P in the convex part of $\angle AOB$ is equidistant from the rays OA and OB, we must show OP bisects $\angle AOB$. Let $PA \perp OA$ and $PB \perp OB$ as before, then the hypothesis means $|PA| = |PB|$. The right triangles POA and POB, having the hypotenuse PO in common, are therefore congruent because of HL (Theorem 8, page 120). Consequently, $\angle POB$ and $\angle POA$ are equal. The proof of Lemma 12 is complete.

We now come to the concurrence theorem for angle bisectors. It has been noted that the term *median* or *altitude* could mean a ray, a line, or a segment (see page 118). The same is true for *angle bisector*. It was defined to be a ray (see page 93) or a line, but in the next theorem, the term *angle bisector* should be interpreted to mean the *segment between the vertex of the given angle and the point of intersection* *with the opposite side.* (The fact that the angle bisector of a triangle must intersect the opposite side is a consequence of the crossbar axiom; see page 119.) Or, referring to the picture below, the *angle bisector of* $\angle A$ in the statement of Theorem 15 will mean the *segment* AE.

Theorem 15. *The three angle bisectors of a triangle meet at a point, called the* **incenter** *of the triangle. The incenter is the unique point equidistant from the three sides.*

Proof. Let the angle bisectors AE and BD of $\angle A$ and $\angle B$ in $\triangle ABC$, respectively, intersect at I.



By Lemma 12, since I lies on the angle bisector of $\angle A$, it is equidistant from AC and AB. Because I also lies on the angle bisector of $\angle B$, it is equidistant from BA and BC. Together, these two facts imply that I is equidistant from CA and CB. By Lemma 12 again, I must also lie on the angle bisector of $\angle C$. So all three angle bisectors are concurrent. The fact that it is equidistant from all three sides is already contained in the preceding proof. Now suppose there is another point I' equidistant from all three sides. Because I' is equidistant from AB and AC, Lemma 12 implies that I' lies on the angle bisector of $\angle A$. The same reasoning then shows that I' lies on all the angle bisectors, i.e., $I' = I$. Theorem 15 is proved.

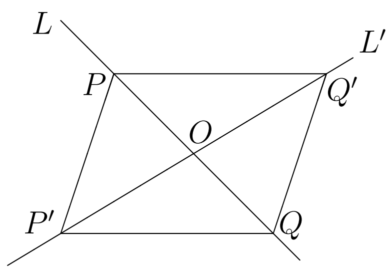
Remark. Theorem 15 asserts that not just the rays of the angle bisectors are concurrent, but that the *segments* themselves are already concurrent. This is significant because it implies that the incenter is always *inside* the triangle, as one would expect by looking at a drawing of angle bisectors of a triangle. Consider then the assertion at the beginning of the preceding proof, that the *segments* AE and BD meet at I. Pictorially, there is no room for doubt, but is there a *reason* behind it?

This is what assumption (A8) (page 119), the crossbar axiom, is for: By (A8), the angle bisector of $\angle A$ must intersect BC at a point E; then in $\angle B$, we see that the angle bisector from B must intersect segment AE, again because of (A8). This kind of information can be given out judiciously, but probably not as a point of emphasis, in school classroom instruction.

The centroid of a triangle

Finally we come to the concurrence of the medians in a triangle. For the proof, we need three theorems, all of which are of independent interest. The following two theorems are different characterizations of a parallelogram. The first one says that *parallelograms are the quadrilaterals whose diagonals bisect each other.*

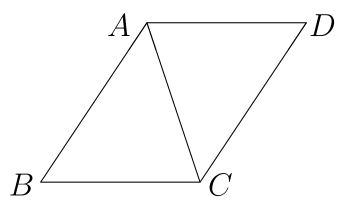
Theorem 16. *Let L and L' be two lines meeting at a point O. P, Q (resp., P', Q') are points lying on opposite half-lines of L (resp., L') determined by O. Then $|PO| = |OQ|$ and $|P'O| = |OQ'|$ $\iff$ $PP'QQ'$ is a parallelogram.*



Proof. We begin by proving that if $|PO| = |OQ|$ and $|P'O| = |OQ'|$, then $PP'QQ'$ is a parallelogram. Let $\mathcal{R}$ be the rotation of $180°$ around O. Then $\mathcal{R}$ clearly maps P to Q and P' to Q', and therefore $\mathcal{R}(PP') = QQ'$. By Theorem 1 (page 102), $PP' \parallel QQ'$. In the same way, we can prove $PQ' \parallel P'Q$. This proves that $PP'QQ'$ is a parallelogram. Conversely, suppose $PP'QQ'$ is a parallelogram. Then we have to prove that its diagonals bisect each other. This would follow if we can prove that $\triangle OPQ' \cong \triangle OQP'$. We appeal to ASA: By Theorem 4 (page 107), $|PQ'| = |QP'|$. By Theorem 10 (page 128), $|\angle PQ'O| = |\angle QP'O|$ and $|\angle Q'PO| = |\angle P'QO|$. The conditions of ASA are thus satisfied and we have the desired congruence. The theorem is proved.

Theorem 17. *A quadrilateral is a parallelogram $\Longleftrightarrow$ it has one pair of sides which are equal and parallel.*

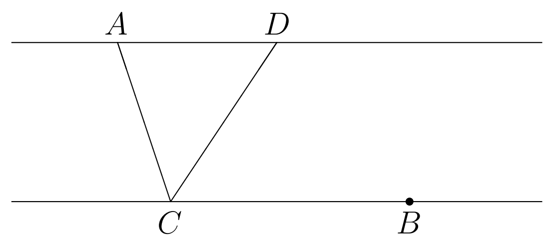
Proof. The fact that a parallelogram has a pair of sides which are equal and parallel is implied by Theorem 4 (page 107). We prove the converse. Let $ABCD$ be a quadrilateral so that $|AD| = |BC|$ and $AD \parallel BC$. We have to prove that $ABCD$ is a parallelogram. It suffices to prove that $AB \parallel CD$.



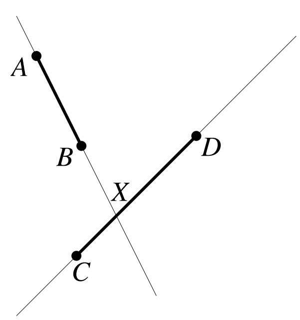
Observe that $\triangle ACD \cong \triangle CAB$ by virtue of SAS. Indeed, the triangles share a side AC, $|AD| = |BC|$ by hypothesis, and finally $|\angle CAD| = |\angle ACB|$ because of Theorem 10 (page 128) and the hypothesis that $AD \parallel BC$. This proves the congruence. Consequently, $|\angle BAC| = |\angle DCA|$. By Theorem 12 (page 130), $AB \parallel CD$. The proof is complete.

Remark. We have by now gotten used to the fact that a seemingly simple geometric proof can hide some unpleasant subtleties. In the case of the preceding proof, the fact that B and D lie in opposite half-planes of the diagonal line L_{AC} plays a critical role. Without that, the angles $\angle DAC$ and $\angle ACB$ would not be *alternate interior angles* of the transversal L_{AC} with respect to the parallel lines AD and BC and Theorem 10 would not be applicable to guarantee their equality. In fact, it is for occasions like this that we took the trouble to define *alternate interior angles* so carefully on page 128. Again, the usual pictures such as the preceding one make us believe that B and D would automatically on opposite sides of L_{AC}, but we know from the Remark after the proof of Theorem 4 (page 109) that, while this is true for all parallelograms, it is not true for general quadrilaterals. Thus, here is an "obvious" fact that calls for a proof.

The following proof that B and D lie on opposite sides of L_{AC} under the hypothesis of Theorem 17 is not recommended for general use in the school classroom but, as usual, it is being offered for teachers' information. We are going to argue by contradiction. Suppose B and D lie on the same side of L_{AC}. Then we have the following picture, where $L_{AD} \parallel L_{CB}$.



We claim that B lies in the convex part (page 88) of the angle $\angle CAD$. Thus we need to prove that (1) B lies in the closed half-plane of L_{AD} containing C, and (2) B lies in the closed half-plane of L_{AC} containing D. The reason for (1) is that the segment CB does not intersect L_{AD}, because even the whole *line* L_{CB} does not intersect L_{AD} ($L_{CB} \parallel L_{AD}$ by hypothesis). So C and B belong to the same half-plane of L_{AD} (see the definition of half-planes in (A4), page 87). The reason for (2) is our assumption that B and D lie on the same side of L_{AC}. Therefore we know B lies in the convex part of $\angle CAD$. By the crossbar axiom ((A8) on page 119), the *ray* R_{AB} intersects the segment CD at a point X. We now show that, in fact, X lies on the *segment* AB so that the *segments* AB and CD intersect. To show this, we use another contradiction argument. If AB does not intersection the segment CD, then we have a situation shown by the following picture:

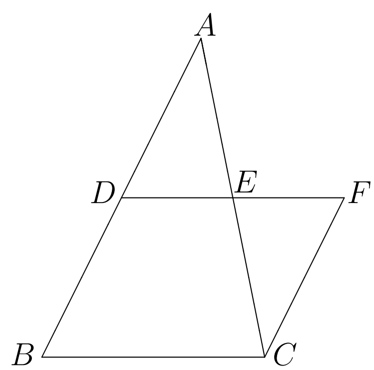


Now the line L_{AB}, having intersected line L_{CD} at X, cannot intersect L_{CD} elsewhere. Therefore the segment AB does not contain any point of the line L_{CD}. This means B lies in the half-plane of L_{CD} containing A. But we also have B lying in the half-pane of L_{AC} containing D because our hypothesis at the moment is that B and D lie on the same side of L_{AC}. Thus B lies in the convex part of the angle $\angle ACD$. The crossbar axiom (page 119) implies that the ray R_{CB} intersects AD. This contradicts the hypothesis of Theorem 17 that $L_{AD} \parallel L_{CB}$. Therefore it must be the case that the *segments* AB and CD intersect at a point X. However, the definition of a polygon does not allow the sides AB and CD of the quadrilateral $ABCD$ to intersect (see page 83). This contradiction shows that B and D must lie in opposite half-planes of L_{AC} after all.

Our proof of Theorem 17 is now complete in every respect.

We are now in a position to prove one of the central theorems of triangle geometry.

Theorem 18. *Let $\triangle ABC$ be given, and let D and E be midpoints of AB and AC, respectively. Then $DE \parallel BC$ and $|BC| = 2|DE|$.*



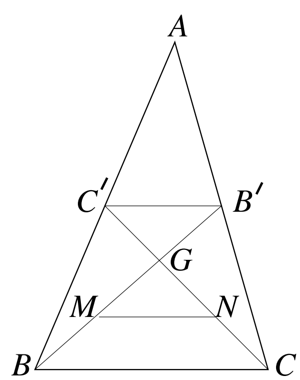
Remark. This theorem calls for a proof of $|BC| = 2|DE|$, i.e., that the length of one segment is twice that of another. We have no tools to prove something like this, so we must change this equality to something we can handle. For example, construct a segment twice as long as DE and then try to prove that this segment and BC have the same length. This explains the proof to follow.

Proof. On the ray R_{DE}, we take a point F so that $|DF| = 2|DE|$. Now E is the midpoint of DF (by construction), and also the midpoint of AC (by hypothesis), so Theorem 16 implies that $ADCF$ is a parallelogram. By Theorem 4 (page

107, $|CF| = |AD|$. Since $|AD| = |DB|$ by hypothesis, we have $|CF| = |BD|$. On the other hand, $CF \parallel AD$ because $ADCF$ is a parallelogram; this is of course the same as $CF \parallel BD$. The quadrilateral $DBCF$ therefore has a pair of sides which are equal and parallel. By Theorem 17, $DBCF$ is a parallelogram. Thus $DF \parallel BC$, which is the same as $DE \parallel BC$. Furthermore, $|DF| = |BC|$ (Theorem 4), and since $|DE| = |EF|$, we have $|BC| = 2|DE|$. The proof is complete.

Theorem 18 has a surprising consequence: if $ABCD$ is any quadrilateral, then the quadrilateral obtained by joining midpoints of the adjacent sides of $ABCD$ is always a parallelogram. It should be pointed out that Theorem 18 is important not only for proving that the medians are concurrent, but it is also central to the understanding of similarity. See Section 6 below (page 149).

Theorem 19. *The three medians of a triangle meet at a point G, called the* **centroid** *of the triangle. On each median, the distance of G to the vertex is twice the distance of G to the midpoint of the opposite side.*



Proof. We focus attention on the median issuing from B, to be called BB'. We claim that either of the two medians issuing from A and C will meet BB' at a point G so that $|BG| = 2|GB'|$. Once this is done, then we know that all three medians meet at the point G as described and the theorem will be proved.

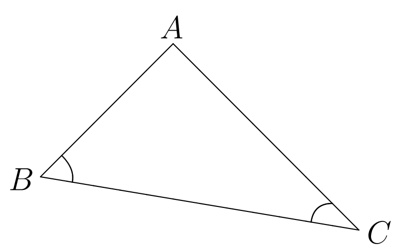
Let us consider the case of the median CC' issuing from C. We will prove something that is equivalent to the preceding assertion, namely, we let G be the point of intersection of CC' and BB' and then prove that $|BG| = 2|GB'|$. By Theorem 18, $C'B' \parallel BC$ and $|C'B'| = \frac{1}{2}|BC|$. Now let M, N be midpoints of BG and CG, respectively. Then by Theorem 18 again, $MN \parallel BC$ and $|MN| = \frac{1}{2}|BC|$. Therefore, $C'B' \parallel MN$ and $|C'B'| = |MN|$. By Theorem 17, $MNB'C'$ is a parallelogram and therefore the diagonals MB' and NC' bisect each other (Theorem 16). Thus, $|GB'| = |MG|$, but since M is the midpoint of BG, we have $|BM| = |MG| = |GB'|$, which is equivalent to $|BG| = 2|GB'|$. The proof is complete.

The triangle inequality

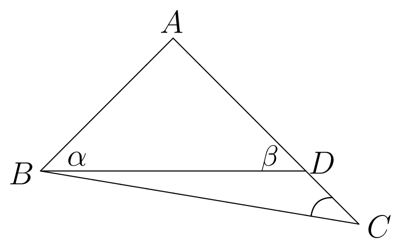
The goal of this subsection is to prove that the sum of (the lengths of) two sides of a triangle exceeds (the length of) the third, the so-called **triangle inequality**. This basic fact in Euclidean geometry is what gives rise to the common perception that "the shortest distance between two points is a straight line".

The proof of the triangle inequality requires a preliminary result. Given $\triangle ABC$, we say BC is the side **facing** $\angle \boldsymbol{A}$, and AB is the side **facing** $\angle \boldsymbol{C}$. The result in question is that, of the two sides facing two given angles in a triangle, the side facing the larger angle is longer. One can prove this by a contradiction argument, but it is far simpler to first prove its converse.

Theorem 20. *In a triangle, the angle facing the longer side is larger. More precisely, if in triangle ABC, $|AC| > |AB|$, then $|\angle B| > |\angle C|$.*



Proof. We are given that $\triangle ABC$ satisfies $|AC| > |AB|$, and we have to prove $|\angle ABC| > |\angle C|$. Since $|AC| > |AB|$, there is a point D between A and C so that $|AB| = |AD|$. Clearly, D is in the convex part of $\angle ABC$. Let $\angle ABD$ be denoted by $\angle\alpha$ and $\angle ADB$ be denoted by $\angle\beta$, as shown below.



Since $\triangle ABD$ is isosceles, by Theorem 7, $|\angle\alpha| = |\angle\beta|$. By the Corollary to Theorem 11 (page 130), $|\angle\beta| > |\angle C|$. Thus,

$$|\angle ABC| \; > \; |\angle\alpha| \; = \; |\angle\beta| \; > \; |\angle C|$$

Therefore $|\angle ABC| > |\angle C|$, as desired.

Corollary 1. *In a triangle, the side facing the larger angle is longer. That is, if in $\triangle ABC$, $|\angle B| > |\angle C|$, then $|AC| > |AB|$.*

Proof of Corollary 1. Between $|AC|$ and $|AB|$, there are three possibilities: $|AC| > |AB|$, $|AC| = |AB|$ and $|AC| < |AB|$. We will eliminate the second and the third. If $|AC| = |AB|$, then $|\angle B| = |\angle C|$ by Theorem 7 and this contradicts the hypothesis that $|\angle B| > |\angle C|$. If on the other hand $|AC| < |AB|$, then Theorem 20 implies $|\angle B| < |\angle C|$ and this too contradicts the hypothesis that $|\angle B| > |\angle C|$. So the only possible choice is $|AC| > |AB|$. The Corollary is proved.

The next corollary of Theorem 20 is the converse of Theorem 7(a).

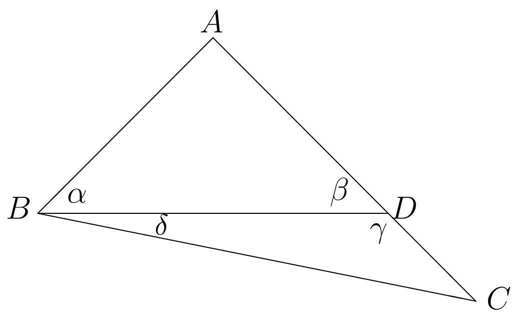
Corollary 2. *In a triangle, equal angles face equal sides.*

Proof of Corollary 2. Suppose in $\triangle ABC$, $|\angle B| = |\angle C|$. Then we have to prove $|AB| = |AC|$. If not, then either $|AB| < |AC|$ or $|AB| > |AC|$. But according to Theorem 20, we would have $|\angle C| < |\angle B|$ or $|\angle B| < |\angle C|$, respectively, and both contradict the hypothesis that $|\angle B| = |\angle C|$. Therefore $|AB| = |AC|$ and the corollary is proved.

Here then is the theorem we are after.

Theorem 21 (Triangle inequality). *The sum of the lengths of two sides of a triangle exceeds the length of the third.*

Proof. Let us prove that in $\triangle ABC$, $|AB|+|BC| > |AC|$. Now if $|AC| \le |AB|$, there would be nothing to prove. Therefore, let $|AB| < |AC|$. Then there is a point D between A and C so that $|AD| = |AB|$. We observe that D lies in the convex part of $\angle ABC$. Let us denote $\angle ABD$ by $\angle\alpha$, $\angle ADB$ by $\angle\beta$, $\angle BDC$ by $\angle\gamma$, and $\angle DBC$ by $\angle\delta$, as shown.



Now $|AB| + |BC| > |AC|$ is equivalent to $|AD| + |BC| > |AD| + |DC|$. Therefore, it suffices to prove $|BC| > |CD|$. By the preceding Corollary, it suffices in turn to prove $|\angle\gamma| > |\angle\delta|$. Note that because $|AB| = |AD|$, Theorem 7 implies that $|\angle\alpha| = |\angle\beta|$. Using this fact, and by repeated use of the the Corollary to Theorem 11 (page 130), we have:

$$|\angle\gamma| \;>\; |\angle\alpha| \;=\; |\angle\beta| \;>\; |\angle\delta|$$

That is, $|\angle\gamma| > |\angle\delta|$, as claimed.

A slightly different proof is the following. As before, we have $|\angle\alpha| = |\angle\beta|$. Looking at the angle sum of $\triangle ABD$, we see that $\angle\beta$ is an acute angle, because $|\angle\alpha| + |\angle\beta| + |\angle A| = 180°$ implies $2|\angle\beta| + |\angle A| = 180°$, which in turn implies $2|\angle\beta| < 180°$. So $|\angle\beta| < 90°$. Therefore $\angle\gamma$ is an obtuse angle. But in any triangle, there can only be one obtuse angle. Therefore, looking at $\triangle BCD$, we see that $\angle\delta$ is acute, and $|\angle\gamma| > |\angle\delta|$.

5. Constructions with ruler and compass

The game of seeing what geometric figures can be constructed using only a compass (for drawing circles of a given radius) and a ruler (for drawing lines without making use of the marked lengths on the ruler) was started by the Greeks perhaps before 500 B.C. It would have been forgotten long ago except for the fact that it led to some unsolved problems that spurred significant mathematical breakthroughs. For students, these construction problems are important for at least two reasons: they promote the learning of geometry through tactile experiences, and they provide a splendid opportunity for making proofs relevant by demanding verification that the constructions are correct. Both facts are very relevant in the teaching of constructions in the school classroom, especially in view of the fact that too often the verification of the correctness of the constructions is ignored.

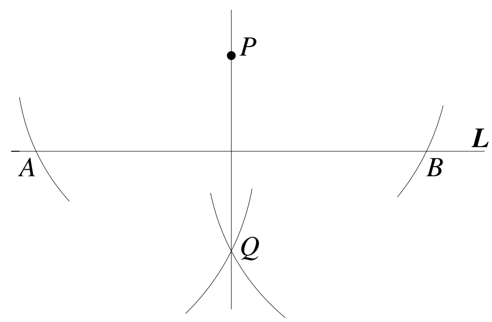
The following are some basic constructions that all students of geometry should know, but one should be aware that there is an endless list of intricate construction problems in the literature.

1. Reproduce a line segment on a ray with a specified endpoint.
2. Construct an equilateral triangle on a given side.
3. Reproduce an angle with one side specified.
4. Construct a line perpendicular to a given line L from a given point.
5. Construct the perpendicular bisector of a line segment.
6. Construct the angle bisector of an angle.
7. Construct a line parallel to a given line through a given point.
8. Divide a given line segment into any number of equal segments.
9. Construct a regular hexagon inscribed in a circle
10. Draw tangents to a circle from a point outside the circle.
11. Construct the sum, difference, product and quotient of two given positive numbers.
12. Construct the square root of a positive number.

Items 9–12 are on topics we will take up in the next two sections (see page 165 and page 199 ff.). In this section, we give the constructions for items 4 and 8. First item 4.

Construct a line perpendicular to a given line L from a given point.

Let the given point be P. There are two cases to consider: P lies on L, and P does not lie on L. It will be seen that the following construction takes care of both cases at the same time.



The construction:

(a). Draw a circle with P as center so that it intersects L at two points A and B.

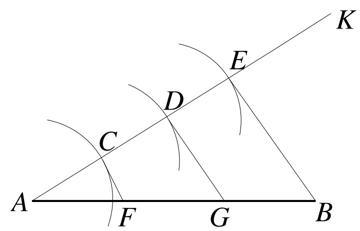
(b). Draw two circles with the same (sufficiently large) radius, and with centers at A and B, so that they intersect; let one of the points of intersection be Q, and make sure that $Q \neq P$.

(c). The line L_{PQ} is the line we seek.

Proof that $L_{PQ} \perp L$. By step (a), $|PA| = |PB|$, and by step (b), $|QA| = |QB|$. Therefore the Corollary to Theorem 7 (page 119) implies that L_{PQ} is the perpendicular bisector of AB. In particular, $L_{PQ} \perp L$.

Divide a given line segment into any number of equal segments.

Let segment AB be given. We show how to trisect AB. The construction can obviously be generalized to equal division into any number of parts.



The construction:

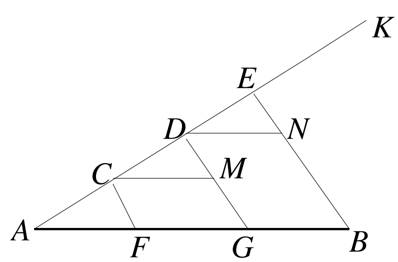
(a). Let R_{AK} be any ray issuing from A which is different from R_{AB}. Let AC be any segment on R_{AK}.

(b). Reproduce AC successively on AK so that AC is equal to CD and is equal to DE (see Construction 1 above).

(c). Join EB. From D and C, construct lines parallel to L_{EB} (see Construction 7 above, and also the discussion on page 50 ff.), and let these lines intersect AB at G and F, respectively.

(d). AF, FG, and GB have the same length.

Proof that $|AF| = |FG| = |GB|$. Let the lines passing through C and D and parallel to L_{AB} intersect DG and EB at M and N, respectively. Then $L_{AB} \parallel L_{CM} \parallel L_{DN}$ (Lemma 2, page 82). For the same reason, we also have $L_{CF} \parallel L_{DG} \parallel L_{EB}$ (by step (c) above).



Therefore $CFGM$ and $DGBN$ are parallelograms, and we see that $|CM| = |FG|$ and $|DN| = |GB|$ because opposite sides of a parallelogram are equal (Theorem 4 on page 107). It now suffices to prove

$$|AF| \;=\; |CM| \;=\; |DN|.$$

We will do so by using ASA to prove that

$$\triangle ACF \cong \triangle CDM \cong \triangle DEN.$$

By step (b), $|AC| = |CD| = |DE|$. Because $L_{FC} \parallel L_{GD} \parallel L_{BE}$ by step (c), $|\angle ACF| = |\angle CDM| = |\angle DEN|$ (Theorem 10 on page 128). For exactly the same reason, we have $L_{AB} \parallel L_{CM} \parallel L_{DN}$ by construction, so $|\angle CAF| = |\angle DCM| = |\angle EDN|$. Therefore $\triangle ACF \cong \triangle CDM \cong \triangle DEN$ because of ASA. It follows that $|AF| = |CM| = |DN|$. This completes the proof.

6. Definitions of dilations and similarity

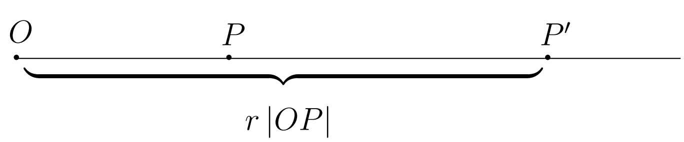
Dilations and FTS

In the study of geometry, there is also a need for transformations that are less rigid than congruences. These are the *dilations.* With the preparation on dilations in eighth grade in place, we will come straight to the definition of these transformations.

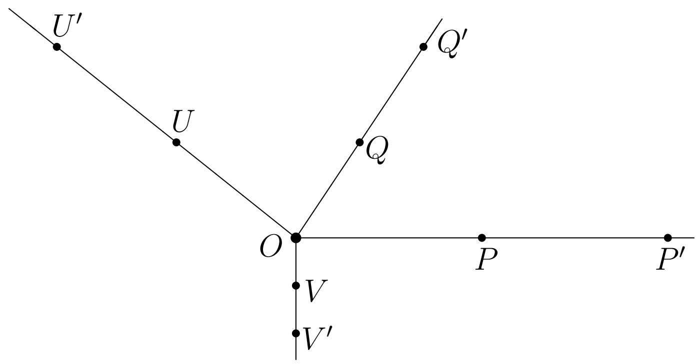
Definition. *A* **dilation $\boldsymbol{D}$ with center $\boldsymbol{O}$ and scale factor $\boldsymbol{r}$ ($\boldsymbol{r > 0}$)** is a transformation of the plane that assigns to each point P a point $D(P)$ so that

(1) *$D(O) = O$.*

(2) *If $P \neq O$, the point $D(P)$, to be denoted more simply by P', is the point on the ray R_{OP} so that $|OP'| = r|OP|$.*



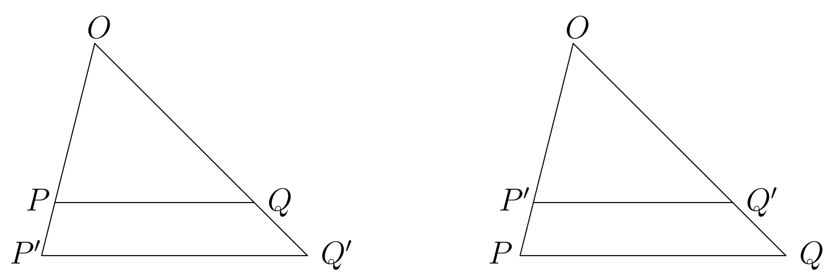
A dilation is intuitively some kind of "projection from the point O". Each ray issuing from O is mapped to the same ray (*caution:* all this says is that the *ray* is mapped to itself, but each point on the ray will in general be mapped to another point on the same ray). Here is an example of how a dilation with $r = 2$ maps four different points (for any point P, we let the corresponding letter with a prime, P', denote the image $D(P)$ of P):



Anything substantial we have to say about dilations will have to come from the following fundamental theorem (**FTS**).

Theorem 22. (Fundamental Theorem of Similarity (FTS)) *Let D be a dilation with center O and scale factor $r > 0$. Let P and Q be two points so that L_{PQ} does not contain O. If $D(P) = P'$ and $D(Q) = Q'$, then*

$$P'Q' \parallel PQ \quad \text{and} \quad |P'Q'| \;=\; r\,|PQ|$$



The case $r > 1$ The case $r < 1$

FTS sheds new light on Theorem 18: we now understand that Theorem 18 is the special case of FTS when $r = 2$. The crux of the proof of FTS is in fact the proof of the special case of FTS when the scale factor r is a *positive integer*. Let us isolate this special case.

Lemma 13. *FTS is valid when the scale factor r is a positive integer.*

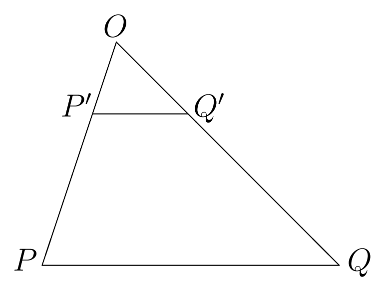
We will not be able to give a complete proof of Lemma 13, but we already have a proof when $r = 2$ and we will presently give a proof when $r = 3$. After that, we will make some comments about the proof of the general case when r is any positive integer.

Our immediate goal is to show that, if we assume the truth of Lemma 13, then we can give a proof of FTS for all *fractions* r. This is as far as we can go in school mathematics. The complete proof of FTS in general will depend on the so-called **Fundamental Assumption of School Mathematics (FASM)**, which guarantees that knowing the validity of FTS for all fractions r is enough to ensure the validity of FTS for all *positive real numbers* r. The proof of FASM is however beyond the scope of school mathematics. Let us therefore concentrate on proving FTS for all fractions r. To this end, we prove (always assuming Lemma 13):

Lemma 14. *FTS is valid for all* unit fractions $\frac{1}{n}$, *where n is a positive integer.*

Proof. Let D is a dilation with center O and scale factor $\frac{1}{n}$, and let P, Q be two points not collinear with O. Let $D(P) = P'$ and $D(Q) = Q'$, so that $|OP'| = \frac{1}{n}\,|OP|$ and $|OQ'| = \frac{1}{n}|OQ|$. Then we have to prove that

$$P'Q' \parallel PQ \quad \text{and} \quad |P'Q'| \;=\; \frac{1}{n}|PQ|.$$



Observe that if we let D_0 be the dilation with center O and scale factor n, then we have

$$D_0(P') = P \quad \text{and} \quad D_0(Q') = Q.$$

By Lemma 13, we have

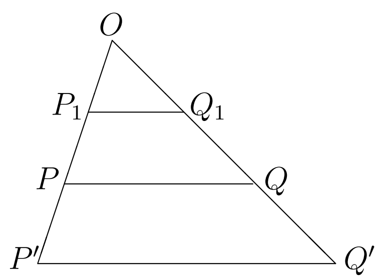
$$PQ \parallel P'Q' \quad \text{and} \quad |PQ| \;=\; n|P'Q'|.$$

But this is the same as the desired result that $P'Q' \parallel PQ$ and $|P'Q'| = \frac{1}{n}|PQ|$. The proof of Lemma 14 is complete.

Assuming Lemma 13, we are now in a position to **prove FTS for all fractions** $\boldsymbol{r = \frac{m}{n}}$, where m and n are positive integers. Thus let D be a dilation with center O and scale factor $\frac{m}{n}$. Let P and Q be two points so that L_{PQ} does not contain O. If $D(P) = P'$ and $D(Q) = Q'$, then we have to prove that

$$P'Q' \parallel PQ \quad \text{and} \quad |P'Q'| \ = \ \frac{m}{n}\,|PQ|$$

On the ray R_{OP}, let P_1 be the point so that $|OP_1| = \frac{1}{n}|OP|$ and, similarly, let Q_1 be the point on the ray R_{OQ} so that $|OQ_1| = \frac{1}{n}|OQ|$.



Because it is given that $|OP'| = \frac{m}{n}|OP|$, we see that $|OP'| = m|OP_1|$. Similarly, $|OQ'| = m|OQ_1|$. Now let D_1 be the dilation with center O and scale factor m. Then we have $D_1(P_1) = P'$ and $D_1(Q_1) = Q'$. By Lemma 13, we get

$$P'Q' \parallel P_1Q_1 \quad \text{and} \quad |P'Q'| \ = \ m\,|P_1Q_1|.$$

Now let D_2 be the dilation with center O and scale factor $\frac{1}{n}$. Then because $|OP_1| = \frac{1}{n}|OP|$ and $|OQ_1| = \frac{1}{n}|OQ|$, we get $D_2(P) = P_1$ and $D_2(Q) = Q_1$. By Lemma 14, we get

$$P_1Q_1 \parallel PQ \quad \text{and} \quad |P_1Q_1| \ = \ \frac{1}{n}\,|PQ|.$$

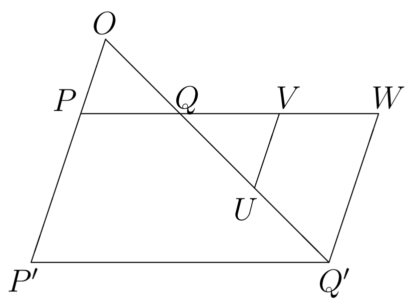
Putting these two sets of conclusions together, we therefore have

$$P'Q' \parallel PQ \quad \text{and} \quad |P'Q'| \ = \ \frac{m}{n}\,|PQ|.$$

This then completes the proof of FTS for scale factors that are fractions.

Now we return to the **proof of Lemma 13 for the case of $r = 3$**. Thus we assume that $|OP'| = 3|OP|$ and $|OQ'| = 3|OQ|$ and want to prove

$$P'Q' \parallel PQ \quad \text{and} \quad |P'Q'| \ = \ 3|PQ|.$$



The main idea is that we already have Theorem 18 (page 140) at our disposal and we should use it. Moreover, we should imitate the proof of Theorem 18 if possible. To this end, we do as we did in the proof of that theorem, namely, extend PQ along the ray R_{PQ} until $|PW| = 3|PQ|$. Join $Q'W$. Now if we can prove $PP'Q'W$ is a parallelogram, then we certainly get $P'Q' \parallel PQ$ and $|P'Q'| = |PW|$. The latter then implies $|P'Q'| = 3|PQ|$ in view of $|PW| = 3|PQ|$.

To prove $DBCF$ is a parallelogram, if we take the midpoints U and V of QQ' and QW, respectively, then Theorem 18 implies $UV \parallel Q'W$ and $|Q'W| = 2|UV|$. Since $|OQ'| = 3|OQ|$ implies $|QQ'| = 2|OQ|$, and $|PW| = 3|PQ|$ implies $|QW| = 2|PQ|$, we see that $|PQ| = |QV|$ and $|OQ| = |QU|$. Thus by Theorem 16 (page 137), $OPUV$ is a parallelogram and $UV \parallel OP$. Opposite sides of a parallelogram being equal (Theorem 4 on page 107), we have $|UV| = |OP|$. But we already know $UV \parallel Q'W$, we have $OP \parallel Q'W$. In other words, $PP' \parallel Q'W$. From $|OP'| = 3|OP|$, we get $|PP'| = 2|OP|$. But we also have $|Q'W| = 2|UV| = 2|OP|$, thus $|PP'| = |Q'W|$. Therefore $PP'Q'W$ is a quadrilateral with a pair of sides (PP' and $Q'W$) which are parallel and equal. By Theorem 17 on page 138, $PP'Q'W$ is a parallelogram. By a remark above, the proof of Lemma 13 for $r = 3$ is complete.

This proof strongly suggests that the same idea could be pushed one step further to prove Lemma 13 for $r = 4$, that is, if $|OP'| = 4|OP|$, $|OQ'| = 4|OQ|$, then $P'Q' \parallel PQ$, and $|P'Q'| = 4|PQ|$. The next step is $r = 5$, and so on. If students are comfortable with mathematical induction, it would make a wonderful extra assignment to ask them to finish the proof of the general case of Lemma 13, when r is any positive integer n, by mathematical induction on n.

In any case, we will freely make use of FTS from now on.

We now make more explicit a remarkable feature of FTS.

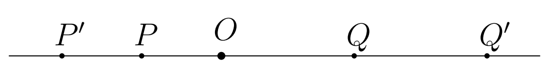
Lemma 15. *Let D be a dilation with scale factor $r > 0$. Then* ***D* changes distance by a factor of *r*** *in the sense that, for any two points P and Q in the plane, if we denote $D(P)$ by P' and $D(Q)$ by Q', then $|P'Q'| = r|PQ|$.*

Proof. First assume P and Q are collinear with O. If P and Q lie on the same ray issuing from O, let $|OQ| > |OP|$. Then let $P' = D(P)$ and $Q' = D(Q)$ as in the theorem.

We claim: $|P'Q'| = r\,|PQ|$ in this case because

$$|P'Q'| = |OQ'| - |OP'| = r\,|OQ| - r\,|OP| = r\,(|OQ| - |OP|) = r\,|PQ|$$

This proves the claim. Now suppose P and Q do not both lie in a ray issuing from O. Then the segment PQ contains O.



From the above argument, we have $|OP'| = r\,|OP|$ and $|OQ'| = r\,|OQ|$, so that

$$|P'Q'| \;=\; |P'O| + |OQ'| \;=\; r\,(|PO| + |OQ|) \;=\; r\,|PQ|,$$

as desired.

Finally, if L_{PQ} does not contain O, then the result is contained in FTS. The proof of Lemma 15 is complete.

The reason Lemma 15 is remarkable is that, by definition, we know that a dilation with scale factor $r > 0$ expands or contracts by a factor of r *along each ray issuing* *from O, and only along such rays.* Yet Lemma 15 tells us that the same phenomenon persists along *any* direction. It is this fact that makes dilation a "shape preserving" transformation. In a school classroom, students should be given the opportunity to do many drawings by hand (without a computer) magnifying or shrinking a given figure, as suggested back on page 55.

The following theorem summarizes the basic properties of dilations; it is the counterpart of Lemma 8 for congruences (page 112).

Theorem 23. *A dilation has the following properties:*

(i) *it maps lines to lines, rays to rays, and segments to segments,*

(ii) *it changes distance by a factor of r, where r is the scale factor of the dilation,*

(iii) *it maps every line passing through the center of dilation to itself, and maps every line not passing through the center of the dilation to a parallel line,*

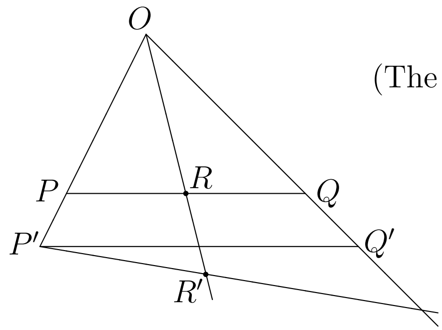
(iv) *it is degree-preserving.*

Proof. Let D be a dilation with center O and scale factor r.

The main point of (i) is that a dilation maps lines to lines; the proof of the rest of (i) is routine. The following proof that a dilation maps a line to a line is essentially the same as the one given in eighth grade on page 47. Given a line L_{PQ}, we have to show that $D(L_{PQ})$ is a line. If L_{PQ} contains the center of dilation O, it follows from the definition of a dilation that $D(L_{PQ}) = L_{PQ}$. We will therefore assume that L_{PQ} does not contain O so that FTS becomes applicable. We claim:

$$D(L_{PQ}) = L_{P'Q'}, \text{ where } D(P) = P' \text{ and } D(Q) = Q'.$$

In greater detail, this means (1) if R is a point on the line L_{PQ}, then the point $R' = D(R)$ lies on $L_{P'Q'}$, and (2) conversely, every point R' on line $L_{P'Q'}$ is the image of some point R on L_{PQ}, i.e., $D(R) = R'$. We begin with the proof of (1). To show that R' lies on $L_{P'Q'}$, it suffices to show that the line $L_{P'R'}$ and the line $L_{P'Q'}$ coincide.



(The case $r > 1$.)

Now,

$D(P) = P'$ and $D(Q) = Q'$ imply $P'Q' \parallel PQ$, by FTS.

$D(P) = P'$ and $D(R) = R'$ imply $P'R' \parallel PR$, by FTS.

Thus we have two lines, $P'Q'$ and $P'R'$, both parallel to PQ and both passing through P'. According to the Parallel Postulate, they must be one and the same line. This is exactly what we want to prove.

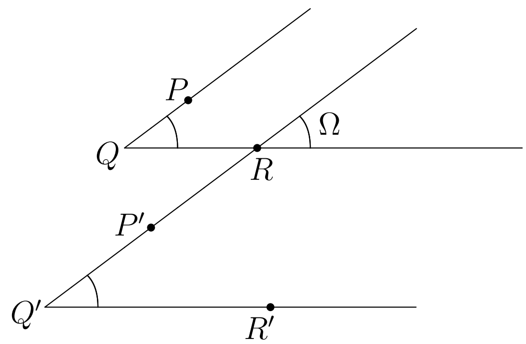
The reasoning for the converse (2) (i.e., every point R' on line $L_{P'Q'}$ is equal to $D(R)$ for a point R on L_{PQ}), is entirely similar if we look at the dilation D_1 with center O but with scale factor $\frac{1}{r}$ and observe that $P = D_1(P')$ and $Q = D_1(Q')$. So the same argument shows that $D_1(R')$ is a point of L_{PQ}. If we denote $D_1(R')$ by R, then this implies $D(R) = R'$. Thus $D(L_{PQ}) = L_{P'Q'}$, as desired.

Part (ii) is exactly the content of Lemma 15. The assertion about lines passing through the center of dilation in part (iii) follows from the definition, and the assertion about lines not passing through the center of dilation is implied by FTS.

For part (iv), we note first of all that it makes sense because, by (i) above, a dilation maps rays to rays and therefore angles to angles. The question is therefore whether the degrees of angles are preserved. Let a nonzero angle $\angle PQR$ be given. Let $D(P) = P'$, $D(Q) = Q'$, and $D(R) = R'$. We will prove that

$$|\angle PQR| = |\angle P'Q'R'|$$

The case of O being collinear with Q and P or with Q and R is simpler, so we will henceforth assume that O does not lie in L_{QP} or L_{QR}. By part (iii), $L_{Q'P'} \parallel L_{QP}$. The Parallel Postulate therefore implies that L_{QR} is not parallel to $L_{Q'P'}$ (because L_{QR} is the only line passing through Q parallel to $L_{Q'P'}$). Without loss of generality, we may assume R is the intersection of L_{QR} and $L_{Q'P'}$, as shown.



Let the angle formed by the ray $R_{Q'P'}$ and the ray R_{QR} at R be denoted by $\angle\Omega$, as indicated in the picture. Since $D(QR) = Q'R'$, (iii) implies that $QR \parallel Q'R'$, so that by Theorem 10 (page 128),

$$|\angle P'Q'R'| \;=\; |\angle\Omega|$$

Since also $D(QP) = Q'P'$, the same reasoning implies that

$$|\angle\Omega| \;=\; |\angle PQR|$$

Hence $|\angle PQR| = |\angle P'Q'R'|$, as desired. The proof of Theorem 23 is complete.

As a consequence of part (i) of Theorem 23, we see that the dilated image of a polygon is completely determined by the dilated images of the vertices of the polygon. In other words, if (for example) $PQRS$ is a quadrilateral and if D is a dilation so that $D(P) = P'$, $D(Q) = Q'$, $D(R) = R'$, and $D(S) = S'$, then we claim that

$$D(PQRS) \;=\; P'Q'R'S'$$

Let us understand what this means: $D(PQRS)$ is the image by D of the quadrilateral $PQRS$, while $P'Q'R'S'$ is the quadrilateral whose vertices are the images by D of P, Q, R, and S. Thus *a priori*, $D(PQRS)$ is related to $P'Q'R'S'$ only by their vertices but the sides of $P'Q'R'S'$ may have nothing to do with $D(PQRS)$. What we are asserting is, however, that each side of $P'Q'R'S'$ is exactly the image of the corresponding side of $PQRS$. The reason is of course the fact proved in part (i) of Theorem 23, to the effect that

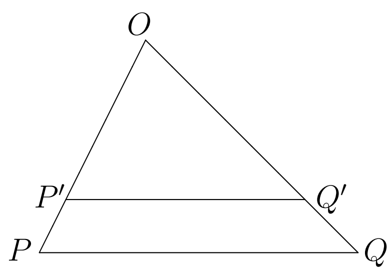
$$D(L_{PQ}) \;=\; L_{P'Q'}, \quad D(L_{QR}) \;=\; L_{Q'R'}, \quad \text{etc.}$$

FTS, another view

In applications, the following reformulation of FTS is often more useful.

Theorem 24. *Let $\triangle OPQ$ be given, and let P' be a point on the ray R_{OP} not equal to O. Suppose a line parallel to PQ and passing through P' intersects OQ at Q'. Then*

$$\frac{|OP'|}{|OP|} = \frac{|OQ'|}{|OQ|} = \frac{|P'Q'|}{|PQ|}$$



Proof. The point P' could be on the segment OP or could be outside OP; the proof is the same in either case. We have drawn the picture with P' in OP and will prove the theorem accordingly. Let $r = \frac{|OP'|}{|OP|}$. We are assuming that $r < 1$. Therefore there is a point Q_0 on the segment OQ so that $|OQ_0| = r|OQ|$. Since also $|OP'| = r|OP|$, it follows that if we let D be the dilation with center O and scale factor r, then $D(Q) = Q_0$ and $D(P) = P'$. By FTS,

$$L_{P'Q_0} \parallel L_{PQ} \quad \text{and} \quad |P'Q_0| = r|PQ|.$$

Thus $L_{P'Q_0}$ is a line passing through P' and parallel to L_{PQ}. By hypothesis, $L_{P'Q'}$ is also a line passing through P' and parallel to L_{PQ}. The Parallel Postulate therefore implies that the two lines $L_{P'Q_0}$ and $L_{P'Q'}$ coincide; in particular, $Q_0 = Q'$. Then the equalities

$$|OQ_0| = r|OQ|, \quad |OP'| = r|OP|, \quad |P'Q_0| = r|PQ|$$

now become

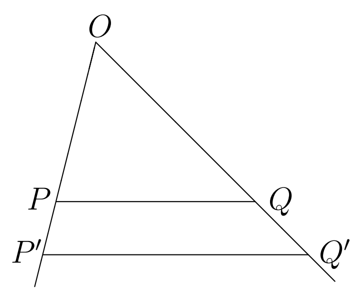
$$|OQ'| = r|OQ|, \quad |OP'| = r|OP|, \quad |P'Q'| = r|PQ|.$$

This can be expressed equivalently as

$$\frac{|OQ'|}{|OQ|} \;=\; \frac{|OP'|}{|OP|} \;=\; \frac{|P'Q'|}{|PQ|} \;=\; r.$$

Theorem 24 is proved.

Theorem 24 provides another way to think about a dilation $\mathcal{D}$ with a given center O and a given scale factor r. Take any ray issuing from O and fix a point P on this ray. Let P' be the point on this ray so that $|OP'| = r|OP|$. Then by definition, $\mathcal{D}(P) = P'$. We now show, once P and P' are known, how to determine the image $\mathcal{D}(Q)$ of any point Q so long as P, O, Q are not collinear. So given such a point Q, we claim that $\mathcal{D}(Q)$ is the point Q' that is the intersection of the ray R_{OQ} and the line passing through P' and parallel to L_{PQ}.



This is because by Theorem 24, we have

$$\frac{|OP'|}{|OP|} \;=\; \frac{|OQ'|}{|OQ|}$$

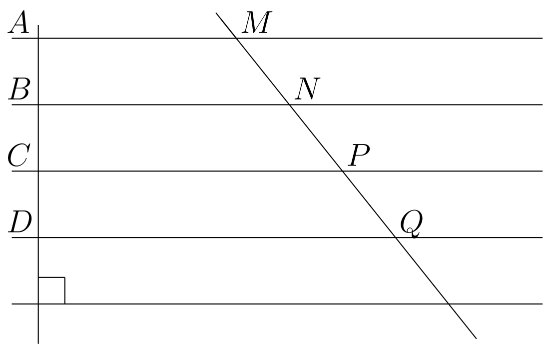
Therefore,

$$|OQ'| \;=\; \frac{|OP'|}{|OP|}\,|OQ| \;=\; r\,|OQ|,$$

and hence $Q' = \mathcal{D}(Q)$.

Theorem 24 also gives rise to a meaningful hands-on activity with notebook papers. We begin with a general observation about the ruling on these papers. The lines are supposed to be mutually parallel (see Lemma 2 on page 82) and **equidistant**, i.e., if you draw a line L_{AB} perpendicular to one line, then it is perpendicular to every line

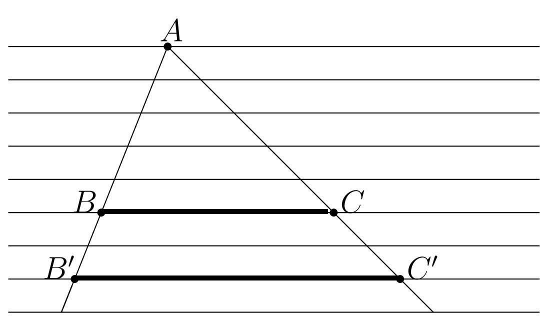
(Theorem 10 on page 128) and the segments intercepted by the lines on L_{AD} are theoretically all of the same length. Thus $|AB| = |BC| = |CD| = \ldots$ in the picture below. Now let L_{MQ} be a transversal of all the parallel lines. Then observe that the latter **intercept equal segments on $\boldsymbol{L_{MQ}}$** in the sense that $|MN| = |NP| = |PQ| = \ldots$; this is proved by the reasoning on page 147 which proves the validity of the equidivision of a segment .



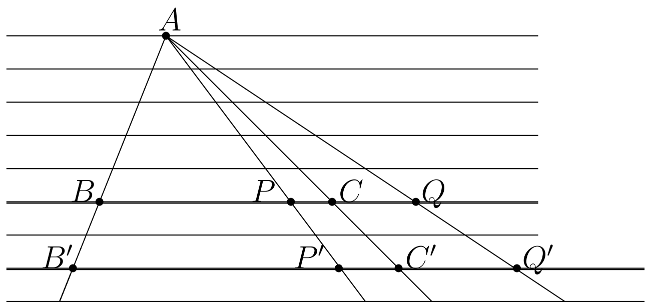
Now, the activity. Referring to the picture below, pick a point A on one of the lines and let two transversals through A intersect the fifth line below A at B and C, respectively. Let the intersections of the rays R_{AB} and R_{AC} with the seventh line below A be B' and C', respectively (see picture below). Then because the parallel lines intercept equal segments on the rays R_{AB} and R_{AC}, we have

$$\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|} = \frac{7}{5},$$

according to Theorem 24. Thus if D^* is the dilation with center A and scale factor $\frac{7}{5}$, then $D^*(B) = B'$ and $D^*(C) = C'$. Now check by direct measurements that $|B'C'| = \frac{7}{5}|BC|$. Repeat this activity by varying the numbers 5 and 7.



Here is a slight variation on this activity. Pick any point P on L_{BC}, and let the line joining A and P intersect $L_{P'Q'}$ at P'. Now measure $|AP|$ and $|AP'|$; is it true that $|AP'| = \frac{7}{5}|AP|$? Pick another point Q on L_{PQ} and get Q' on $L_{P'Q'}$ as shown. Again, is it true that $|AQ'| = \frac{7}{5}|AQ|$? Try other choices P and Q.



Similarity and basic criteria for similarity

Let $\mathcal{S}$ and $\mathcal{S}'$ be two figures (i.e., sets) in the plane. The following is the precise way to say "they have the same shape."

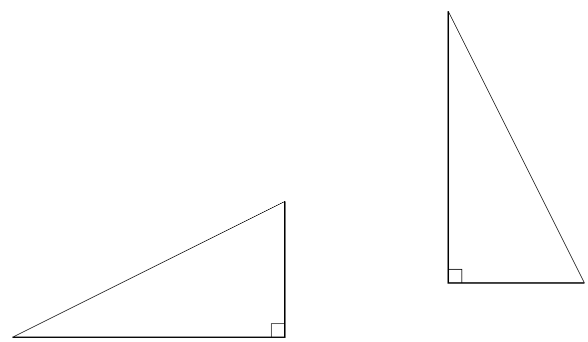
Definition. *We say $\mathcal{S}$ is* **similar** *to $\mathcal{S}'$, in symbols, $\boldsymbol{\mathcal{S} \sim \mathcal{S}'}$, if there is a dilation D so that*

$$D(\mathcal{S}) \cong \mathcal{S}'$$

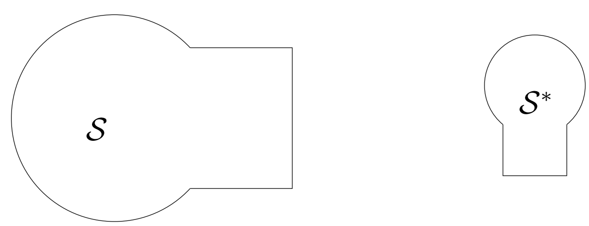
In greater detail, $\mathcal{S} \sim \mathcal{S}'$ means there is a congruence F and a dilation D so that $F \circ D$ maps $\mathcal{S}$ to $\mathcal{S}'$, i.e., $F(D(\mathcal{S})) = \mathcal{S}'$. A composition $F \circ D$ of a congruence F and a dilation D is called a **similarity**. The **scale factor of the similarity** $\varphi \circ D$ is by definition the scale factor of the dilation D.

> *We could have just as well defined a similarity by composing D and F in the reverse order, i.e., $D \circ F$. But of course, once so defined, one must be consistent throughout. The two definitions are equivalent, in the sense that for any two sets $\mathcal{S}$ and $\mathcal{S}'$, $F(D(\mathcal{S})) = \mathcal{S}'$ for a congruence F and a dilation D if and only if there is a congruence F' so that $D(F'(\mathcal{S})) = \mathcal{S}'$.*
>
> *This situation is similar to defining 3×5 as either $5 + 5 + 5$, or $3 + 3 + 3 + 3 + 3$, but once a choice has been made, we have to be consistent about it.*

Why must congruence enter into the discussion of similarity? First of all, two congruent figures clearly "have the same shape", but if we define "have the same shape" to mean "related by a dilation", then congruent figures such as these,



would not "have the same shape" and that would be absurd (neither right triangle can be the dilation of the other because if it were, the hypotenuses would have to be parallel, by FTS. An even better example is the following two sets $\mathcal{S}$ and $\mathcal{S}^*$:



Now $\mathcal{S}^*$ clearly has "the same shape" as $\mathcal{S}$, but it is not a dilation of $\mathcal{S}$, because if it were, then the horizontal segment of $\mathcal{S}^*$ would have to be parallel to the vertical segment of $\mathcal{S}$ according to FTS. In this case, a dilation of $\mathcal{S}$ by a scale factor of $\frac{1}{2}$, followed by a clockwise rotation of 90-degree around some point in $\mathcal{S}$ and a suitable translation would map $\mathcal{S}$ to $\mathcal{S}^*$.

What these simple examples show is that it is too restrictive to define *"similarity"* in terms of dilations alone. One must allow for a composition with a congruence as well. The above definition of similarity now appears to be more reasonable.

Because of Lemma 8 (page 112) and Theorem 23 (page 155), the following theorem holds for similar triangles.

Theorem 25. *Suppose* $\triangle ABC \sim \triangle A'B'C'$. *Then:*

$$|\angle A| = |\angle A'|, \quad |\angle B| = |\angle B'|, \quad |\angle C| = |\angle C'|$$

and

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|}$$

It is the various converses of Theorem 25 that are more interesting. The following are the similarity analogs of the SAS, ASA, and SSS criteria for congruence.

Theorem 26. **(SAS for similarity)** *Given two triangles* ABC *and* $A'B'C'$, *if* $|\angle A| = |\angle A'|$, *and*

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|}$$

then $\triangle ABC \sim \triangle A'B'C'$.

Theorem 27. **(AA for similarity)** *Two triangles with two pairs of equal angles are similar.*

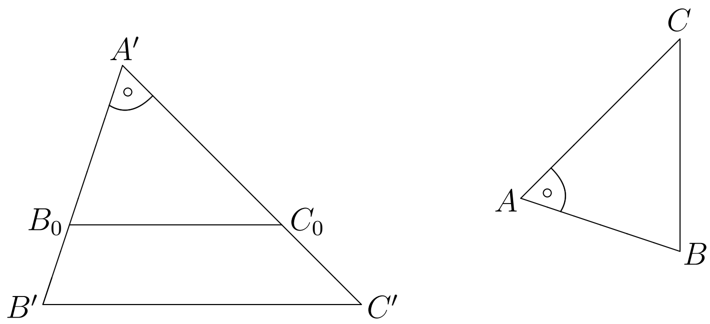
Theorem 28. **(SSS for similarity)** *If the corresponding sides of two triangles are proportional, the triangles are similar. More precisely, if in* $\triangle ABC$ *and* $\triangle A'B'C'$,

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|}$$

then the triangles are similar.

We have already given a proof of *AA for similarity* in grade 8 (page 59). The proofs of the other two criteria are similar in spirit, so we will try to be brief.

Proof of Theorem 26. If $|AB| = |A'B'|$, then the hypothesis would imply $|AC| = |A'C'|$ and we are reduced to the SAS criterion for congruence. Thus we may assume that $|AB|$ and $|A'B'|$ are not equal. Suppose $|AB| < |A'B'|$. Then the hypothesis that $|AB|/|A'B'| = |AC|/|A'C'|$ implies $|AC| < |A'C'|$. On $A'B'$, let B_0 be the point so that $|A'B_0| = |AB|$. Similarly, on $A'C'$, let C_0 be the point satisfying $|A'C_0| = |AC|$.



Because $|\angle A| = |\angle A'|$ by hypothesis, the SAS criterion for congruence (page 113) implies that $\triangle A'B_0C_0 \cong \triangle ABC$. Let F be the congruence that maps $\triangle A'B_0C_0$ to $\triangle ABC$. Moreover, if r denotes the common value of $|AB|/|A'B'|$ and $|AC|/|A'C'|$, then we have $|A'B_0| = |AB| = r\,|A'B'|$. Similarly, $|A'C_0| = r\,|A'C'|$. This means that the dilation $\mathcal{D}$ with center A' and scale factor r maps A' to A', and also B' to B_0 and C' to C_0, by the definition of dilation. Thus $\mathcal{D}$ maps $\triangle A'B'C'$ to $\triangle A'B_0C_0$, so that

$$(F \circ \mathcal{D})(\triangle A'B'C') \;=\; F(\mathcal{D}(\triangle A'B'C')) \;=\; F(\triangle A'B_0C_0) \;=\; \triangle ABC$$

This shows that $\triangle A'B'C' \sim \triangle ABC$ and Theorem 26 is proved.

Proof of Theorem 28. Let $|AB|/|A'B'|$ be denoted by r. Then by hypothesis,

$$\frac{|AB|}{|A'B'|} \;=\; \frac{|AC|}{|A'C'|} \;=\; \frac{|BC|}{|B'C'|} \;=\; r \tag{1}$$

Referring to the preceding picture, we may assume that $|AB| < |A'B'|$, so that $r < 1$. Thus also $|AC| < |A'C'|$. On the segments $A'B'$ and $A'C'$, we may therefore choose B_0 and C_0 so that $|A'B_0| = |AB|$ and $|A'C_0| = |AC|$. We want to prove that $\triangle A'B_0C_0 \cong \triangle ABC$ by using the SSS criterion, and for this, we have to prove $|B_0C_0| = |BC|$. To this end, let $\mathcal{D}$ be the dilation with center A' and scale factor r. Then because $|A'B_0| = |AB|$ and $|A'C_0| = |AC|$, equation (1) implies

$$\frac{|A'B_0|}{|A'B'|} \;=\; \frac{|A'C_0|}{|A'C'|} \;=\; r,$$

so that $|A'B_0| = r|A'B'|$ and $|A'C_0| = r|A'C'|$. It follows that $\mathcal{D}(B') = B_0$ and $\mathcal{D}(C') = C_0$. By FTS, we have $|B_0C_0| = r|B'C'|$, so that using equation (1) again, we have

$$
\begin{aligned}
|B_0C_0| &= r\,|B'C'| \\
&= r\left(\frac{1}{r}|BC|\right) \qquad \text{(equation (1))} \\
&= |BC|.
\end{aligned}
$$

Thus SSS implies that $\triangle A'B_0C_0 \cong \triangle ABC$, and there is a congruence F so that $F(\triangle A'B_0C_0) = \triangle ABC$. So finally,

$$
(F \circ \mathcal{D})(\triangle A'B'C') \;=\; F(\mathcal{D}(\triangle A'B'C')) \;=\; F(\triangle A'B_0C_0) \;=\; \triangle ABC
$$

This shows that $\triangle A'B'C' \sim \triangle ABC$. The proof of Theorem 28 is complete.

Applications of similarity

The first thing we want to do is to give a solution to the construction problem 11 in section 5 (page 145), namely,

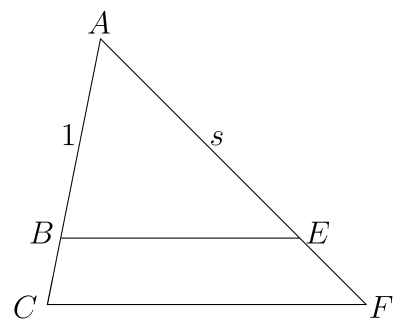
> **Construct the sum, difference, product and quotient of two given positive numbers.**

First, we make the problem more precise. *It is always understood that a segment of length* 1 *is given.* Now let two segments of length r and s be also given, both r and s assumed to be positive. The problem asks for the construction of a segment with length equal to $r+s$, $r-s$ (assuming $r>s$), rs, and $\frac{r}{s}$, respectively.

The constructions of segments of lengths $r \pm s$ are routine (though keep in mind FASM). We will concentrate on constructing segments of length rs and $\frac{r}{s}$. First rs.

The construction:

1. On a ray R_{AC}, let B and C be chosen so that $|AB| = 1$ and $|AC| = r$. (See Construction 1 on page 145).



2. On another ray R_{AE}, let $|AE| = s$ and let the line passing through C and parallel to L_{BE} intersect R_{AE} at F (see Construction 7 on page 145).

3. Then $|AF| = rs$.

Proof that $|AF| = rs$. By Theorem 24 (page 158),

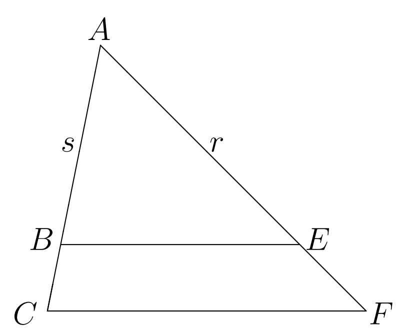
$$\frac{|AC|}{|AB|} = \frac{|AF|}{|AE|},$$

which then becomes $r/1 = |AF|/s$ (Steps 1 and 2), so that $|AF| = rs$, as desired.

We next construct a segment of length $\frac{r}{s}$ using the same idea.

The construction:

1. On a ray R_{AC}, let B and C be chosen so that $|AB| = s$ and $|AC| = 1$. (See Construction 1 on page 145).



2. On another ray R_{AE}, let $|AE| = r$ and let the line passing through C and parallel to L_{BE} intersect R_{AE} at F (see Construction 7 on page 145).

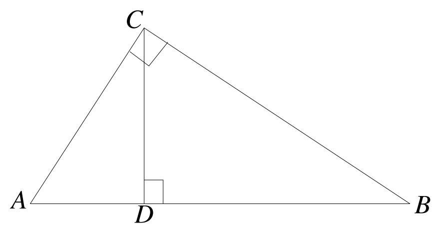
3. Then $|AF| = \frac{r}{s}$.

Proof that $|AF| = \frac{r}{s}$. By Theorem 24 (page 158),

$$\frac{|AC|}{|AB|} = \frac{|AF|}{|AE|},$$

which then becomes $1/s = |AF|/r$ (Steps 1 and 2), so that $|AF| = \frac{r}{s}$, as desired.

The next application is to a seemingly restrictive situation, the case of a right triangle ABC with an altitude CD on its hypotenuse AB:



However, one cannot fail to notice the abundance of triangles that look similar, ABC, ACD, and CBD, and one would expect good things to come out of it. Indeed this is the configuration that leads to a proof of the Pythagorean Theorem (see page 64). Let us first prove something basic and elementary.

Theorem 29. *Let $\triangle ABC$ be a triangle with a right angle at C, and let CD be the altitude on the hypotenuse AB. Then $|CD|^2 = |AD| \cdot |DB|$.*

Proof. We claim that $\triangle ACD \sim CBD$ by the AA criterion. This is so because the Angle Sum Theorem implies that in the right triangle ACD, we have $|\angle CAD| + |\angle ACD| = 90°$. But $\angle ACD$ and $\angle DCB$ are adjacent angles (see page 88), so also $|\angle ACD| + |\angle DCB| = 90°$ (assumption (A6)(iii), page **??**). Therefore $|\angle CAD| = |\angle DCB|$. For the same reason, we also get $|\angle CBD| = |\angle ACD|$. So we have the desired similarity. By Theorem 25, we obtain from these similar triangles the proportion

$$\frac{|CD|}{|DB|} = \frac{|AD|}{|CD|}$$

The theorem is now a consequence of the cross-multiplication algorithm.

As mentioned above, by pushing this reasoning with the same picture a step further, we would arrive at a proof of the Pythagorean Theorem. Because the proof given in grade 8 (page 64) is valid verbatim, we can simply restate the theorem here.

Theorem 30 (Pythagorean Theorem). *If the lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c, then* $a^2 + b^2 = c^2$.

The **converse of the Pythagorean Theorem** can also be proved in exactly the same way as in grade 8 (see page 65), but in the more formal setting of a high school course, it would be a good thing to point out to students that the deduction of $C = E$ from $|CE| = 0$ in that proof depends explicitly on assumption (A5)(ii) (page 89). This may increase their understanding of the role of assumptions (A1)–(A8), which is to make more explicit each step of the reasoning process.

We want to prove a generalization of the Pythagorean Theorem, which will also include the converse of the Pythagorean Theorem as a special case. To this end, we now use similar triangles to give a preliminary definition of the trigonometric functions.

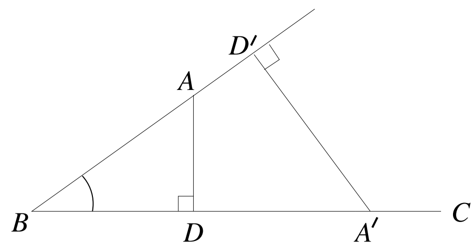
Given a nonzero *acute* angle $\angle ABC$, we will assign to it a number, to be denoted by $\mathbf{sin}\,\angle\boldsymbol{ABC}$, called the **sine of** $\angle\boldsymbol{ABC}$. Sometimes $\sin\angle ABC$ is also denoted by $\mathbf{sin}\,\boldsymbol{ABC}$ or even $\mathbf{sin}\,\boldsymbol{B}$. Here is the definition of $\sin\angle ABC$: take a point on one side of $\angle ABC$—which may as well be A—then

$$\sin\angle ABC \;=\; \frac{\text{the distance of } A \text{ from the other side of } \angle ABC}{|AB|}$$

(Recall that the *distance of a point from a line* is defined on page 134.) Thus in the following picture, if $AD \perp BC$, then $\sin\angle ABC = |AD|/|AB|$.

Of course, we have to make sure that the definition is well-defined, in the sense that, if we take another point A' on a side of the angle (which need not be the same side as the one containing the point A above) and $A'D' \perp$ the other side, the following proportion holds:

$$\frac{|A'D'|}{|A'B|} \;=\; \frac{|AD|}{|AB|}.$$



This is so because the right triangles $\triangle ABC$ and $\triangle A'BC'$ have $\angle B$ in common so that the AA criterion for similarity implies $\triangle ABD \sim \triangle A'BD'$, and the above proportion follows from Theorem 25 (page 163).

We remark that in a school classroom, care should be taken to check that the definition of sine (and that of *cosine* below) is well-defined. We have to promote the good habit among students of always making sure that what they do makes sense. The fundamental fact is that similarity lies at the heart of the definition of sine and cosine; there may be clever ways to camouflage this fact, but ultimately all students should be made aware of this simple truth.

In a similar vein, we define another number $\mathbf{cos} \angle \boldsymbol{ABC}$, called the **cosine of** $\angle \boldsymbol{ABC}$. Let A be a point on a side of $\angle ABC$, then if the foot of the perpendicular from A to the other side of $\angle ABC$ is D, then the definition of cosine is

$$\cos \angle ABC \;=\; \frac{|DB|}{|AB|}$$

Again, if A' is another point on a side of $\angle ABC$ and if the foot of the perpendicular from A' to the other side of $\angle ABC$ is D', then we must prove

$$\frac{|D'B|}{|A'B|} \;=\; \frac{|DB|}{|AB|}.$$

This also follows from the fact that $\triangle ABD \sim \triangle A'BD'$ as above.

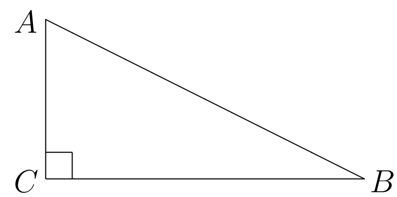
As in the case of sine, the other symbols for $\cos \angle ABC$ are $\mathbf{cos}\, \boldsymbol{ABC}$ and $\mathbf{cos}\, \boldsymbol{B}$. It is also common to assign a third number to $\angle ABC$, called the **tangent** of the angle. By definition: $\tan \angle ABC = \sin \angle ABC / \cos \angle ABC$. Referring to the above picture, we have

$$\tan \angle ABC \;=\; \frac{|AD|}{|BD|} \;=\; \frac{|A'D'|}{|BD'|}$$

We also write $\mathbf{tan}\, \boldsymbol{ABC}$ and $\mathbf{tan}\, \boldsymbol{B}$.

Thus in a right triangle ABC, we would have

$$\sin B = \frac{|AC|}{|AB|}, \quad \cos B = \frac{|CB|}{|AB|}, \quad \tan B = \frac{|AC|}{|CB|}$$



For the zero angle, we agree to define

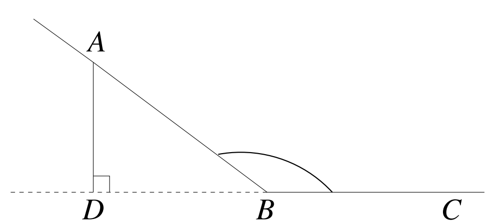
$$\mathbf{\sin 0 = 0, \quad \cos 0 = 1, \quad \tan 0 = 0}.$$

These definitions are of course influenced by the behavior of $|AC|$ and $|BC|$ when $\angle ABC$ is close to the zero angle: in an intuitive sense, $|AC| \to 0$ and $|BC| \to |AB|$ as $|\angle ABC| \to 0°$. If $\angle ABC$ is a right angle, which we will denote by "90" on this occasion for the sake of clarity, we will define

$$\mathbf{\sin 90 = 1, \quad \cos 90 = 0.}$$

Again, these definitions are influenced by the behavior of $|AC|$ and $|BC|$ when $\angle ABC$ is acute but almost 90 degrees: intuitively, $|BC| \to 0$ and $|AC| \to |AB|$ as $\angle ABC \to 90°$.

Now suppose $\angle ABC$ is obtuse and is not a straight angle (but we are still using the convex part of the angle; see page 88). We will define $\sin B$ and $\cos B$ in the same *formal* way, but with a mild twist. We drop a perpendicular from a point A on one side of $\angle ABC$ to *the line containing the other side*. (Recall: a side of an angle is a ray.) Let the foot of this perpendicular be D as before.



Then if we strictly follow the preceding discussion, we would define $\sin B$ as $|AD|/|AB|$ and $\cos B$ as $|DB|/|AB|$. However, if we think of B as the origin of a coordinate system in the plane, then the difference between the point D in the case $\angle ABC$ is acute and the point D in the case $\angle ABC$ is obtuse stands out: D lies on the positive x-axis in the former case, and lies in the negative x-axis in the latter case. It is then easy to understand why we now define *sine and cosine of an obtuse angle* as follows:

$$\sin ABC \;=\; \frac{|AD|}{|AB|}, \quad \cos ABC \;=\; -\,\frac{|DB|}{|AB|}.$$

In short, if $\angle ABC$ is obtuse, then referring to the preceding picture, the sine and cosine of $\angle ABC$ is related to the sine and cosine of the *acute* angle $\angle ABD$ by

$$\sin ABC \;=\; \sin ABD, \quad \cos ABC \;=\; -\,\cos ABD.$$

Consequently, for an obtuse $\angle ABC$,

$$\tan ABC \;=\; -\,\frac{|AD|}{|DB|} \;=\; -\,\tan ABD.$$

It remains to point out that $\tan 90$ has no definition, or in the terminology of functions, "the right angle is not in the domain of definition of the tangent function". But to complete our definitions of these three so-called **trigonometric functions**, we define for a straight angle, to be denoted by "180" on this occasion,

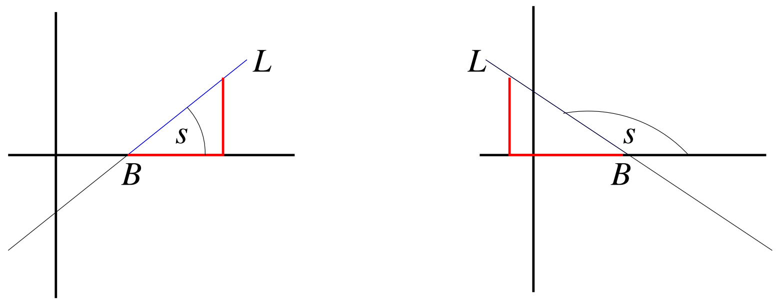
$$\mathbf{\sin 180 \;=\; 0, \quad \cos 180 \;=\; -1, \quad \tan 180 = 0.}$$

There is a noteworthy interpretation of tangent. Let L be a nonvertical line in a coordinate system. If L is also not horizontal, let L intersect the x-axis at B. Let s be the angle made by L and the x-axis in the way shown by the following pictures: the cases of L slanting to the right like this / is on the left, and the case of L slanting to the left like this \ is on the right:

By the definition of the tangent of angle s, we see that

$$\tan s \;=\; \text{slope of } \; L$$

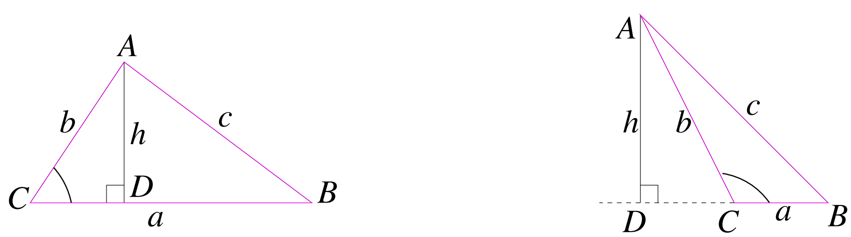
We can now prove the generalization of the Pythagorean Theorem.



Theorem 31 (Law of Cosines). *Given a triangle ABC, let the length of the side opposite vertex A be denoted by a, that opposite vertex B be denoted by b, and that opposite vertex C be denoted by c. Then*

$$c^2 \;=\; a^2 + b^2 - 2ab\,\cos C$$

where $\cos C$ *refers to the cosine of* $\angle ACB$.



Observe that if $\angle C$ is a right angle, then $\cos C = 0$ and Theorem 31 becomes the Pythagorean Theorem. Conversely, if $c^2 \;=\; a^2 + b^2$, then by Theorem 31, $\cos C = 0$. It follows from the definition of cosine that this is possible only if $|\angle C| = 90°$. We have therefore proved that in a $\triangle ABC$, if $c^2 \;=\; a^2 + b^2$, then $\angle C$ is a right angle. Thus Theorem 31 implies the converse of the Pythagorean Theorem.

Proof of Theorem 31. When $\angle C$ is acute (see the above picture on the left), the proof is entirely standard and we will be brief. Let AD be the altitude on base BC, so that $AD \perp BC$. Let the length of AD be h, Then the Pythagorean Theorem implies that

$$c^2 \;=\; h^2 + |DB|^2 \tag{2}$$

$$b^2 \;=\; h^2 + |CD|^2 \tag{3}$$

Since $\angle C$ is acute, $|DB| = (a - |CD|)$. A simple computation using this fact and equations (2) and (3) yields the desired conclusion. What is usually neglected, however, is the fact that the case of an obtuse $\angle C$ must also be considered. In that case, $|DB| = (a + |CD|)$ (see the above picture on the right), so that

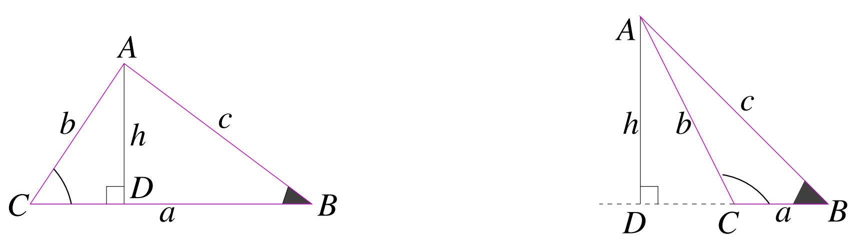
$$\begin{aligned} c^2 &= h^2 + |DB|^2 \qquad \text{(equation (2))} \\ &= h^2 + (a + |CD|)^2 \\ &= a^2 + (h^2 + |CD|^2) + 2a\,|CD| \\ &= a^2 + b^2 + 2a\,|CD| \qquad \text{(equation (3))} \end{aligned}$$

Therefore $c^2 = a^2 + b^2 + 2a\,|CD|$. But by definition, $\cos C = -\,|CD|/b$ when $\angle C$ is obtuse, so $|CD| = -\,b\,\cos C$. Substituting this into the preceding equation immediately gives the conclusion of Theorem 31.

Theorem 31 has a related theorem. We will explain presently how they are related.

Theorem 32. **(Law of Sines)** *Let $\triangle ABC$ be given. Notation as in Theorem 31, we have*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Proof. It suffices to prove

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

There are two cases: both $\angle B$ and $\angle C$ are acute (see the above picture on the left), and one of them is obtuse (the above picture on the right). As in the case of Theorem 31, it is necessary to prove both cases of the theorem. Fortunately, the following proof is valid for either situation. Let $AD \perp BC$ as before and let $|AD| = h$. We have:

$$\begin{aligned}
\frac{\sin B}{b} &= \frac{h}{c} \cdot \frac{1}{b} \qquad \text{(definition of sine)} \\
&= \frac{h}{b} \cdot \frac{1}{c} \\
&= (\sin C) \cdot \frac{1}{c} \qquad \text{(definition of sine for acute or obtuse angle)} \\
&= \frac{\sin C}{c}
\end{aligned}$$

The proof is complete.

The subject of trigonometry was designed to **solve triangles**, in the sense of computing the lengths of all the sides and the degrees of all the angles of a given triangle if only partial information is given about the sides and the angles. The need for solving triangles arose in Greek astronomy; ancient astronomers compiled elaborate tables of the trigonometric functions so that knowing the degree of an angle would be (essentially) equivalent to knowing the sine, or cosine, or tangent of this angle. Theorems 31 and 32 are two of the main tools for solving a triangle, and they complement each other. For example, suppose $|\angle A|$, $|\angle B|$, and c are given; this is the situation of SAS and we know that $\triangle ABC$ is completely determined (Theorem 5, page 113) so that all that remains is to get the explicit values of the other lengths and degrees. Theorem 31 is not immediately applicable for this purpose as a little reflection would reveal, but Theorem 32 is because we also know $|\angle C|$ on account of the Angle Sum Theorem (page 129). Therefore we know $(\sin C)/c$. Since we also know $\sin B$, from

$$\frac{\sin B}{b} = \frac{\sin C}{c},$$

we can compute b. Similarly, from

$$\frac{\sin A}{a} = \frac{\sin C}{c},$$

we can compute a. Thus $\triangle ABC$ is solved.

For another example, suppose a, b, and $|\angle C|$ are given instead (this is therefore the SAS situation). Theorem 32 will be of no help, but from Theorem 31, we can compute c right away. Then knowing a, b, and c, Theorem 31 allows us to compute $\cos A$, and therefore $|\angle A|$. We now get $|\angle B|$ from the Angle Sum Theorem.

Many other applications can be given as exercises on the basis of Theorems 31 and 32.

7. Some theorems on circles

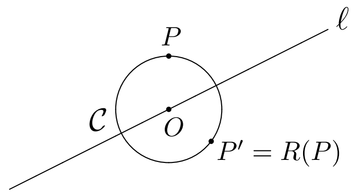
Basic properties of the circle

Recall from page 92 that the *circle* $\mathcal{C}$ *of radius* $\boldsymbol{r}$ *and center* $\boldsymbol{O}$ is the set of all the points P in the plane so that $|OP| = r$ (it will always be understood that $r \geq 0$). Still with $\mathcal{C}$, we also defined the **closed disk** $\overline{\mathcal{C}}$ of a given circle $\mathcal{C}$ to be all the points Q so that $|CQ| \leq r$. This definition has to be understood in the context of school mathematics, in which the word "circle" is used for both a closed disk and its circular boundary. There will come a time, e.g., in the discussion of area in the next section, when we won't be able to afford the presence of this confusion.

In the context of rotation (page 97), a circle has **maximum rotational symmetry**, in the sense that if Ro is *any* rotation around the center of a given circle $\mathcal{C}$, then Ro maps $\mathcal{C}$ onto itself, i.e., $Ro(\mathcal{C}) = \mathcal{C}$. This is clear. The next property about the intrinsic symmetry of a circle may be just as intuitive, but we had better prove it because we will have to use it later (see the proofs of Theorem 36 and 38 below).

Theorem 33. *A circle is symmetric with respect to any line passing through its center, i.e., the reflection* R *across any line* ℓ *passing through the center of a circle* $\mathcal{C}$ *maps* $\mathcal{C}$ *onto itself:* $R(\mathcal{C}) = \mathcal{C}$.

Proof Proving the equality of two sets, $R(\mathcal{C})$ and $\mathcal{C}$, means we have to prove two things: (i) $R(\mathcal{C}) \subset \mathcal{C}$ and (ii) $\mathcal{C} \subset R(\mathcal{C})$. Let O be the center of $\mathcal{C}$ and ℓ be a line containing O, as shown.



To prove (i), we have to show that if P is a point on $\mathcal{C}$, then $R(P)$ is also a point on $\mathcal{C}$. Let us denote $R(P)$ by P'. Since R preserves distance and $R(O) = O$, $|OP| = |R(OP)| = |OP'|$. Therefore P and P' lie on the same circle centered at O, i.e., $R(P)(= P')$ lies on $\mathcal{C}$. Next, we prove (ii). If P is on $\mathcal{C}$, we must prove $P = R(Q)$ for some Q on $\mathcal{C}$. With $P' = R(P)$ as above, the fact that $R \circ R$ is the identity transformation implies that $P = R(P')$. We have just seen that both P and P' lie on $\mathcal{C}$, so letting $Q = P'$ gets the job done. The proof is complete.

Next, we will verify that all circles "look alike". The precise theorem below illustrates the virtue of having a precise concept of "looking alike" in the form of a general definition of similarity.

Theorem 34. *Any two circles are similar.*

Proof. We break up the proof into three steps.

Step 1. Given two circles with the same center, there is a dilation mapping one to the other.

Step 2. The image of a circle by a translation is a circle with the same radius.

Step 3. The theorem is true in general.

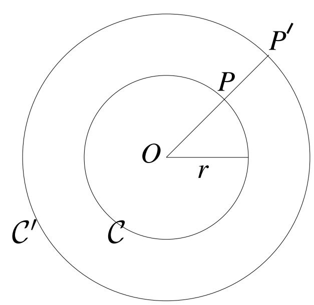
Both Step 1 and Step 2 are very intuitive. In an average classroom, it would be defensible to accept both on faith and concentrate on proving Step 3 on the basis of the first two steps. This goes as follows. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two circles with centers O_1 and O_2, respectively. Let T be the translation along the vector $\overrightarrow{O_1O_2}$. Then $T(O_1) = O_2$, and Step 2 implies $T(\mathcal{C}_1)$ and $\mathcal{C}_2$ are circles with the same center O_2. By Step 1, there is a dilation D so that $D(\mathcal{C}_2) = T(\mathcal{C}_1)$. We pause to observe that if T' is the translation along the "opposite" vector $\overrightarrow{O_2O_1}$, then $T' \circ T = I$, where I is the identity transformation (see page 113). In particular, $T'(T(\mathcal{C}_1)) = \mathcal{C}_1$. Consequently,

$$(T' \circ D)(\mathcal{C}_2) \;=\; T'(D(\mathcal{C}_2)) \;=\; T'(T(\mathcal{C}_1) \;=\; \mathcal{C}_1$$

Therefore the dilation D followed by the translation T' map $\mathcal{C}_2$ to $\mathcal{C}_1$, i.e., the two circles are similar (see definition on page 161). The proof of Theorem 34 is complete.

If students are truly curious, the following proofs of Steps 1 and 2 may be given. They are a bit tedious, but there is also value in teaching beginners to argue carefully and methodically as in the following proofs.

Let us first prove Step 1. Let $\mathcal{C}$ and $\mathcal{C}'$ be two circles with the same center O and with radii r and r', respectively. If $r = r'$, then $\mathcal{C} = \mathcal{C}'$ and the identity transformation (which is a dilation with scale factor 1) maps one to the other. We may therefore assume that $\mathcal{C}$ and $\mathcal{C}'$ have unequal radii and are therefore distinct circles. Let us prove that there is a dilation that maps $\mathcal{C}$ to $\mathcal{C}'$. Let $s = r'/r$, and let D be the dilation with center at O and scale factor s. We claim $D(\mathcal{C}) = \mathcal{C}'$. As usual, we must prove $D(\mathcal{C}) \subset \mathcal{C}'$ and $\mathcal{C}' \subset D(\mathcal{C})$. To show the former, let P be a point of $\mathcal{C}$ and we have to prove that $D(P)$ belongs to $\mathcal{C}'$.



Let $P' = D(P)$, then we have to show $|OP'| = r'$. By the definition of D, P' is the point on the ray R_{OP} so that $|OP'| = s\,|OP|$. Therefore,

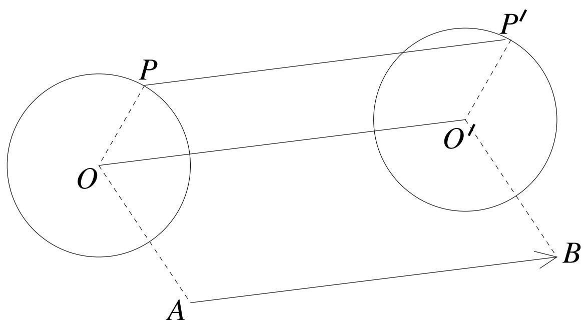
$$|OP'| \;=\; s\,|OP| \;=\; \left(\frac{r'}{r}\right)\, r \;=\; r',$$

as desired.

We prove next that $\mathcal{C}' \subset D(\mathcal{C})$, let P' be a point of $\mathcal{C}'$. We have to show that for some P on $\mathcal{C}$, $P' = D(P)$. On the ray $R_{OP'}$, let P be the point so that $|OP| = r$.

Then by the definition of $\mathcal{C}$, P is on $\mathcal{C}$. Moreover, by the definition of D, $D(P)$ is the point P_0 on the ray R_{OP} so that $|OP_0| = s\,|OP|$. But $s\,|OP| = (r'/r)r = r'$, so $|OP_0| = r'$. Since also $|OP'| = r'$ (because P' is a point of $\mathcal{C}'$), and since P_0 and P' are both points on the same ray R_{OP}, we conclude that $P' = P_0$ and therefore $D(P) = P_0 = P'$. Step 1 is proved.

Next, Step 2. This appears to be obvious but is actually subtle. So let $\mathcal{C}$ be a circle with center O and radius r. Let T be the translation along the vector AB. We claim that $T(\mathcal{C})$ is a circle of radius r and center O', where $O' = T(O)$.



Let $\mathcal{C}'$ be the circle of radius r around O', and we must prove $T(\mathcal{C}) = \mathcal{C}'$. This means we have to prove $T(\mathcal{C}) \subset \mathcal{C}'$ and $\mathcal{C}' \subset T(\mathcal{C})$. To prove the former, take a point P on $\mathcal{C}$ and let $P' = T(P)$. Because a translation preserves segments and distance, we have $|O'P'| = |T(OP)| = |OP| = r$. Thus by the definition of $\mathcal{C}'$, P' is a point on $\mathcal{C}'$. This proves $T(\mathcal{C}) \subset \mathcal{C}'$. Conversely, let P' be a point on $\mathcal{C}'$, and we must show that for some P on $\mathcal{C}$, we have $T(P) = P'$. Let the line passing through P' and parallel to L_{AB} meet the line passing through O and parallel to $L_{O'P'}$ at a point P. We claim that $T(P) = P'$. Now observe that because $O' = T(O)$, $L_{OO'}$ is parallel to L_{AB} by the definition of a translation. Thus both $L_{PP'}$ and $L_{OO'}$ are parallel to L_{AB}. By Lemma 2 (page 82), $L_{PP'} \parallel L_{OO'}$. It follows that $OO'P'O$ is a parallelogram so that by Theorem 4 (page 107), $|OP| = |O'P'| = r$. Therefore P is a point on the circle $\mathcal{C}$. In addition, the definition of $T(P)$ is that it is the intersection of

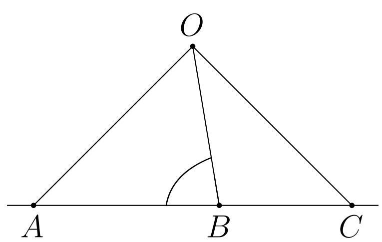
> the line L_1 passing through P and parallel to L_{AB}, and
>
> the line L_2 passing through O' and parallel to L_{OP}.

Since we already have $L_{PP'} \parallel L_{AB}$ and $L_{O'P'} \parallel L_{OP}$, the Parallel Postulate implies that $L_1 = L_{PP'}$ and $L_2 = L_{O'P'}$. Hence $T(P)$ is the intersection of $L_{PP'}$ and $L_{O'P'}$, which is of course P'. The proof of Step 2, and therewith the proof of Theorem 34 is complete.

Next, we prove another rather obvious property of the circle. This proof is very instructive because it makes use of substantive theorems.

Theorem 35. *A circle and a line meet at no more than 2 points.*

Proof. Let the given circle be $\mathcal{C}$ with center O, and the given line be L. Suppose there are at least three points A, B, C in the intersection of $\mathcal{C}$ with L. Since these three points lie on the line L, we may assume without loss of generality that B is between A and C.



Since all three points are also on $\mathcal{C}$, we have $|OA| = |OB| = |OC|$. However, we know that an exterior angle is greater than either remote interior angle (see Corollary on page 130), we have $|\angle OBA| > |\angle OCA|$. Since $\triangle OAC$ is isosceles, $|\angle OCA| = |\angle OAC|$ (Theorem 7(a), page 118). Thus in $\triangle OAB$, we have $|\angle OBA| > |\angle OAB|$, and therefore $|OA| > |OB|$. Contradiction. Theorem 35 is proved.

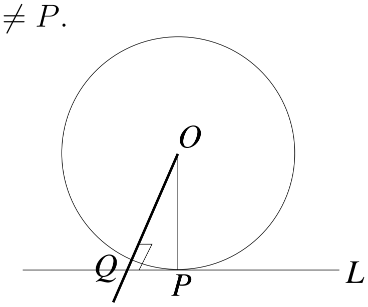
Tangents

Notice that Theorem 35 does not guarantee that there is a line that meets a given circle at 0, 1, or 2 points. All it says is that these are the only possibilities. Now if a circle is given, it is clear that there is a line meeting it at 0 points, i.e., *not* meeting it at all (proving this will be a good exercise). It is also easy to see that there is a line meeting it at exactly 2 points: take two points on the given circle and the line joining them will have the requisite property because Theorem 35 says the line and the circle cannot meet at another point. Showing that there is a line meeting the circle at exactly 1 point, however, takes a bit of work. This will be the content of the next theorem. A line that meets a given circle $\mathcal{C}$ at exactly one point is called a **tangent line of** $\mathcal{C}$ at that point. We are going to construct a tangent line to a given circle at a pre-assigned point of the circle, and the next theorem tells us how. Recall the standard terminology: the segment joining the center of a circle to a point P on the circle is called the **radius of the circle at $\boldsymbol{P}$**.

Theorem 36. *Let P be a point on a circle $\mathcal{C}$. A line containing P is a tangent line to $\mathcal{C}$ at P $\iff$ it is perpendicular to the radius of $\mathcal{C}$ at P.*

If we assume this theorem for a minute, then because through a point on a line there is only *one* line perpendicular to the given line, we see that *the tangent line to a circle at a given point is unique.* This allows us to speak of *the* tangent line at a point of a circle. The proof of Theorem 36 can be given as another straightforward application of the Corollary to Theorem 20 (page 142), but for a change of pace, we will invoke the Pythagorean Theorem instead.

Proof. First assume a line L is tangent to the circle $\mathcal{C}$ at a point P. If the center of $\mathcal{C}$ is O, we have to prove that $L \perp OP$. Suppose not, let the perpendicular from O to L meet L at Q, where $Q \neq P$.



Let R be the reflection across L_{OQ}. Because $L \perp L_{OQ}$, $R(L) = L$ (see definition of reflection on page 98). Since L_{OQ} passes through the center O of $\mathcal{C}$, also $R(\mathcal{C}) = \mathcal{C}$ (Theorem 33, page 175). Therefore, since P is the intersection of $\mathcal{C}$ and L, the point $P' = R(P)$, being the intersection of $R(\mathcal{C})$ and $R(L)$, is also the intersection of $\mathcal{C}$ and L. Moreover, since P lies in a half-plane of L_{OQ}, P' lies in the other half-plane of L_{OQ} and in particular, $P' \neq P$. Thus L intersects $\mathcal{C}$ at two distinct points. Contradiction.

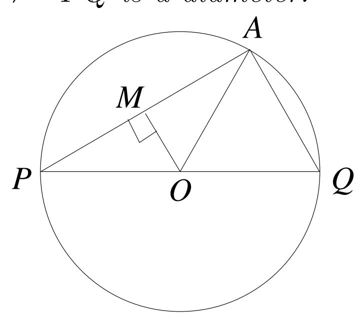
Conversely, if a radius of $\mathcal{C}$ is perpendicular to a line L at a point P on $\mathcal{C}$. Take any point Q on the line L so that $Q \neq P$. By the Pythagorean Theorem, the hypotenuse $|OQ|$ is greater than the radius $|OP|$ of $\mathcal{C}$ and therefore Q does not lie on the circle $\mathcal{C}$. This means that $\mathcal{C}$ and L intersect only at P. Hence L is tangent to $\mathcal{C}$, and Theorem 36 is proved.

In Theorem 15, we proved that the incenter of a triangle is equidistant from the three sides. It then follows from Theorem 36 that we have the following corollary.

Corollary. *There is a unique circle tangent to all three sides of a given triangle.*

Given a point outside a circle, a natural question is whether there is a line passing through P that is also tangent to the circle. We will answer this affirmatively, but quite surprisingly, we have to first find out a property of the diameters of a circle.

Theorem 37. *Let PQ be a chord on a circle $\mathcal{C}$ and let $A \in \mathcal{C}$ be distinct from P and Q. Then $|\angle PAQ| = 90° \iff PQ$ is a diameter.*



Proof. We first prove that if PQ is a diameter of $\mathcal{C}$, then $\angle PAQ$ is a right angle. As usual, let O be the center of $\mathcal{C}$, then because OA, OP, and OQ are all radii, $|OA| = |OP| = |OQ|$. From the isosceles triangles OPA and OQA, we get (Theorem 7, page 118)

$$|\angle OPA| = |\angle OAP|, \qquad |\angle OAQ| = |OQA| .$$

On the other hand, the angle sum of $\triangle PAQ$ is 180° (Theorem 11, 129), so

$$|\angle APO| + (|\angle PAO| + |\angle OAQ|) + |\angle OQA| = 180°.$$

Thus,

$$2\,(|\angle PAO| + |\angle OAQ|) \;=\; 180^\circ\,,$$

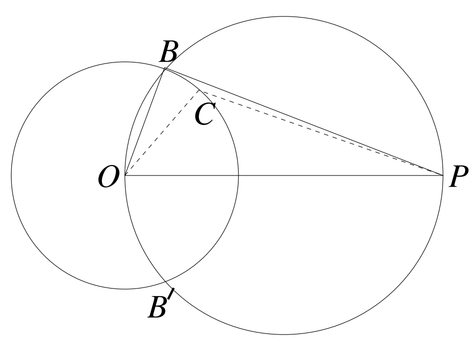
which is the same as saying $|\angle PAQ| = 90^\circ$..

Conversely, suppose PQ is a chord on $\mathcal{C}$ so that $\angle PAQ$ is a right angle. We must prove that PQ is a diameter of $\mathcal{C}$. Let O be the midpoint of PQ and let M be the midpoint of AP. By Theorem 18, $OM \parallel AQ$, and (as $QA \perp AP$) consequently also $OM \perp AP$. Thus OM is the perpendicular bisector of AP. In like manner, O also lies on the perpendicular bisector of AQ. Thus the midpoint O of PQ is in fact the circumcenter of $\triangle APQ$ (Theorem 13, page 132). In particular PQ is a diameter of $\mathcal{C}$. This completes the proof.

As an application of Theorem 37, we prove the following definitive result about tangent lines to a circle. For its statement, we follow the time-honored tradition on geometry of using ambiguous terminology in order to achieve brevity: if a tangent line from a point P outside a circle $\mathcal{C}$ intersects $\mathcal{C}$ at the point B, then **the tangent from P to $\mathcal{C}$** will refer to the *segment* PB.

Theorem 38. *From a point P outside a given circle $\mathcal{C}$, there are exactly two lines tangent to $\mathcal{C}$. Moreover, these two tangents from P to $\mathcal{C}$ have the same length.*

Proof. Let the center of $\mathcal{C}$ be O. We first prove that *there is at least one tangent line to $\mathcal{C}$ from P*. Let the circle having OP as diameter intersects $\mathcal{C}$ at some point B. By Theorem 37, $OB \perp PB$. By Theorem 36, L_{PB} is tangent to $\mathcal{C}$ at B.



Next, we prove that *there are at least two tangent lines from P to $\mathcal{C}$.* We know that $\mathcal{C}$ is symmetric with respect to L_{OP} (Theorem 33, page 175). So the reflection of B across L_{OP} is a point B' lying on $\mathcal{C}$, and also $OB' \perp PB'$ because $OB \perp PB$ and reflection is a congruence. Thus $L_{PB'}$ is a second tangent from P to $\mathcal{C}$ (Theorem 36 again). We also observe that because reflection is a congruence, $|PB| = |PB'|$.

Finally we prove that *there are no more than two tangent lines from P to $\mathcal{C}$.* Suppose C is another point on $\mathcal{C}$ so that L_{PC} is a third tangent line from P to $\mathcal{C}$. Then by Theorem 36 once more, $OC \perp PC$. There is no loss of generality in assuming that C is in the same half-plane of L_{OP} as B. Then we have $|OB| = |OC|$ and of course $|OP| = |OP|$. Hence the right triangles OPB and OPC are congruent (Theorem 8 (HL), page 120), and this implies $|\angle POB| = |\angle POC|$ and $|OB| = |OC|$. Consequently, $B = C$ on account of Lemma 10 (page 114). This proves that there are no more than two tangents from P to $\mathcal{C}$. The proof is complete.

Remark. In the preceding proof, we made use of the intuitively obvious fact that *because the circle with OP as diameter contains a point O inside circle $\mathcal{C}$ and a point P outside $\mathcal{C}$, the circle must intersect $\mathcal{C}$.* While this *is* intuitively obvious, its validity can only be affirmed by invoking a theorem in advanced mathematics. Because this is a high school course, the only way out is to make an explicit assumption to this effect. However, to do so would create a digression with little geometric content. For pedagogical reasons, it will therefore be more prudent to let such things slide and concentrate instead on the geometry.

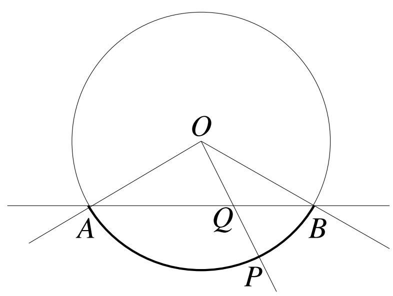
Angles subtended by chords and arcs

We now take a closer look at circles by investigating the *chords* and *arcs* on a circle and their associated angles. By a **chord** on a circle $\mathcal{C}$, we mean a segment joining two points on $\mathcal{C}$.

Theorem 37 makes the study of chords very easy in case the chord is a diameter. It must also be said that diameters are usually the annoying exceptions to general theorems about chords, and we are happy to dispose of them once and for all. *From now on, we shall ignore diameters in the consideration of chords.* Note that each chord AB of a circle $\mathcal{C}$ with center O gives rise to the **central angle $\angle AOB$ subtended by AB**, and the intersections of $\mathcal{C}$ with the convex (respectively, nonconvex) part of $\angle AOB$ is called a *minor arc* (respectively, *major arc*) **subtending $\angle AOB$** on $\mathcal{C}$. (See page 92 for the definition of *major* and *minor arcs.*). The following lemma gives a different characterization of minor and major arcs; *in a school classroom, one might consider skipping the proof of part (ii) of the lemma because the reasoning is technical and intricate.*

Lemma 16. (i) *Let $\widehat{AB}$ be a minor arc on a circle $\mathcal{C}$. Then $\widehat{AB}$ and the center O of $\mathcal{C}$ lie in opposite closed half-planes of the line L_{AB}. (ii) Let $\widehat{AB}$ be a major arc on a circle $\mathcal{C}$. Then $\widehat{AB}$ and the center O of $\mathcal{C}$ lie in the same closed half-planes of the line L_{AB}.*

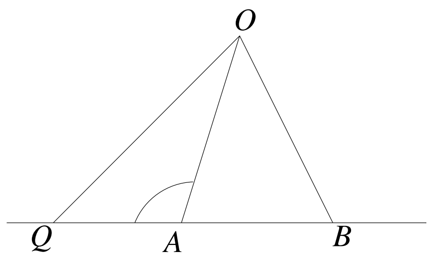
Proof. We first prove (i). Let $\widehat{AB}$ be a minor arc on $\mathcal{C}$. If P is a point on $\widehat{AB}$ not equal to A or B, then P lies in the convex part of $\angle AOB$. By the crossbar axiom (page 119), the ray R_{OP} intersects the segment AB at a point Q, $Q \neq A, B$.



Now $\triangle OAB$ is isosceles and Q is on the segment AB not equal to the endpoints. The reasoning in the proof of Theorem 35 (page 179) shows that $|OQ| < |OA|$, which implies $|OQ| < |OP|$. Therefore the segment OP intersects the line L_{AB} at Q, and by assumption (A4) (page 87), O and P lie in opposite half-planes of L_{AB}. This being true of every point P on $\widehat{AB}$ not equal to A and B), (i) follows.

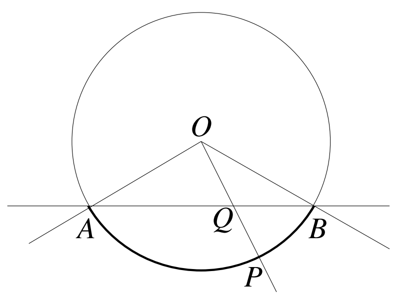
We will prove (ii) by a contradiction argument. Suppose P is now a point on the major arc $\widehat{AB}$ but lies on the opposite side of O relative to the line L_{AB}. Then the segment OP contains a point Q of the line L_{AB}. We claim that Q is in fact a point of the *segment* AB. If not, Q lies outside AB, let us say, A is between Q and B. We claim that $|OQ| > |OA|$.

By the Corollary to Theorem 20 (page 142), it suffices to prove that $|\angle OAQ| > |\angle OQA|$. According to the Angle Sum Theorem (page 129), no triangle can have



two angles $\geq 90°$. Since the base angles $\angle OAB$ and $\angle OBA$ of the isosceles triangle $\triangle OAB$ are equal (Theorem 7, page 118), we therefore see that both $\angle OAB$ and $\angle OBA$ are acute. It follows that $\angle OAQ$ is obtuse. In the triangle OQA, the Angle Sum Theorem again implies that $\angle OQA$ is acute. Thus $|\angle OAQ| > |\angle OQA|$, and we have $|OQ| > |OA|$ as claimed.

Now Q is between O and P and therefore $|OP| > |OQ| > |OA| = |OP|$, a contradiction. It follows that Q lies in the segment AB.



We are going to show that this implies P is in the convex part of $\angle AOB$, which then contradicts the assumption that P is in the major arc $\widehat{AB}$ and finishes the proof of the lemma. To this end, we recall the definition of the convex part of $\angle AOB$ (see page 88). To show that P belongs to the convex part of $\angle AOB$, we must show that

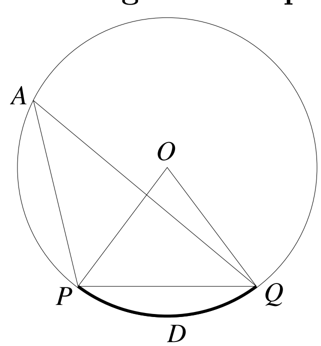
> P and A belong to the same side of L_{OB}, and
>
> P and B belong to the same side of L_{OA}.

The proof of both are similar, so we will only prove the first. Observe that the segment AQ cannot contain any point on the line L_{OB}: suppose it contains a point X on L_{OB}, then X would have to be different from B because B is a point on L_{AQ} outside AQ. Then the two lines L_{AQ} and L_{OB} would contain two distinct points B and X and are therefore identical, by Lemma 1 (page 81). A contradiction. It follows that

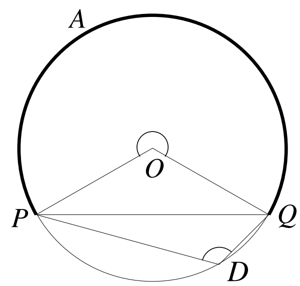
AQ does not intersect L_{OB}; this means A and Q lie on the same side of L_{OB} (see assumption (A4), page 87). But we can reason in exactly the same way to conclude that the segment PQ does not contain any point of L_{OB} and therefore P and Q lie on the same side of L_{OB}. Putting these two conclusions together, we see that P and A lie on the same side of L_{OB}. As mentioned above, P and B belong to the same side of L_{OA} for similar reasons. Thus P belongs to the convex part of $\angle AOB$, contrary to the hypothesis. Lemma 16 is proved.

It follows from Lemma 16 that given a chord AB on a circle $\mathcal{C}$ with center O, the minor arc determined by A and B can be characterized as the intersection of $\mathcal{C}$ with the half-plane of L_{AB} which does not contain the center O. Similarly the major arc determined by A and B can be characterized as the intersection of $\mathcal{C}$ with the half-plane of L_{AB} which contains the center O.

Given a chord PQ on a circle $\mathcal{C}$ with center O, we assume as always that PQ is not a diameter. Then PQ determines a minor arc and a major arc. We will refer to these arcs as **opposite arcs**. We can remove the ambiguity by adding a letter to each arc, as in the picture below, so that, e.g., $\widehat{PDQ}$ denote the minor arc. Let A be a point on the opposite (major) arc which is distinct from either P or Q. Then $\angle PAQ$ is said to be an **angle subtended by arc $\widehat{PDQ}$** on the circle $\mathcal{C}$. The angle $\angle PAQ$ is also called the **inscribed angle intercepting the arc $\widehat{PDQ}$**.



We note that the angle subtended by arc $\widehat{PDQ}$ could be equivalently defined as **the angle subtended by the chord PQ** provided the arc $\widehat{PDQ}$ rather than its opposite is understood. In the same spirit, $\angle PAQ$ could be equivalently defined as the **inscribed angle intercepting the the chord PQ**. We also note that in the case of a major arc $\widehat{PAQ}$, then the central angle subtended by this arc as well as an inscribed angle $\angle PDQ$ subtended by it would look like this:



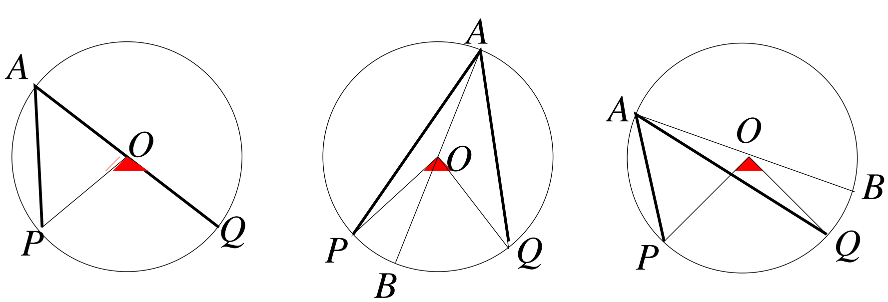
In this case, the central angle subtended by $\widehat{PQ}$ (indicated by the small arc around O) is greater than 180°.

The following theorem is among the theorems in geometry that are at once surprising and elementary.

Theorem 39. *Fix an arc on a circle $\mathcal{C}$. Then all angles subtended by this arc are equal to half of the central angle subtended by the arc.*

This proof is well known, so only a few brief comments are needed. Fix an arc $\widehat{PQ}$ on a circle $\mathcal{C}$ with center O, and let $\angle PAQ$ be an angle subtended by $\widehat{PQ}$. Then we want to prove that $|\angle PAQ| = \frac{1}{2}\,|\angle POQ|$, for *any* A in the opposite arc of $\widehat{PQ}$, $A \neq P, Q$. Let us take up the case where $\widehat{PAQ}$ is a major arc; the case of $\widehat{PAQ}$ being a minor arc is entirely similar.

If $\widehat{PAQ}$ is the major arc, then by definition, A and the center O are in the same half-plane of the line L_{PQ}. There are three possibilities: (a) the ray R_{AO} coincides with one side of $\angle PAQ$ (see figure below on the left), (b) the ray R_{AO} lies in the convex part of $\angle PAQ$ (see figure below in the middle), and (c) the ray R_{AO} lies in the nonconvex part of $\angle PAQ$ (see figure below on the right).

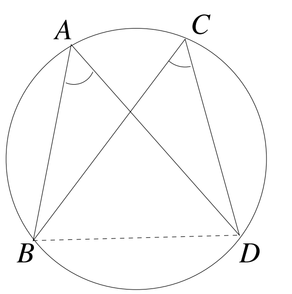


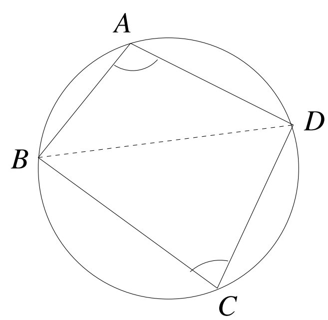
The proof then proceeds by making repeated use of the Corollary of Theorem 11 on exterior angles (page 130) and Theorem 7 on the base angles of isosceles triangles (page 118).

Concyclic points

In applications, we often have to decide whether a collection of points is **concyclic**, i.e., whether they lie on the same circle. Recall that any three noncollinear points lie on a unique circle (Theorem 13 on page 132). We should therefore begin with four points and ask if the circle passing through three of these points also passes through the fourth one. The following is the basic theorem in this direction.

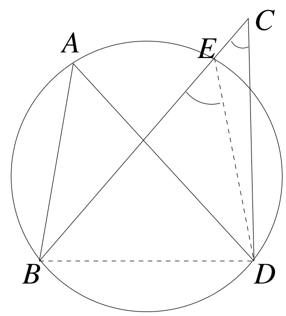
Theorem 40. *Let four points A, B, C, D be given.* (i) *If A and C lie on the same side of the line L_{BD}, then the four points are concyclic* $\iff$ $|\angle BAD| = |\angle BCD|$ (*see left picture below*). (ii) *If A and C lie on opposite sides of the line L_{BD}, then the four points are concyclic* $\iff$ $|\angle BAD| + |\angle BCD| = 180°$ (*see right picture below*).

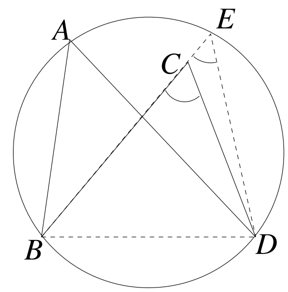




Proof. (i) If A, B, C, D are concyclic, then the hypothesis on A and C implies that they lie on the same (major or minor) arc determined by B and D. Theorem 39 then shows that $|\angle BAD| = |\angle BCD|$. Conversely, suppose A and C lie on the same side of the line L_{BD} and $|\angle BAD| = |\angle BCD|$. Then we have to prove that A, B, C, D are concyclic. Suppose not, and we shall deduce a contradiction. Let $\mathcal{C}$ be the circle passing through A, B, and D (Theorem 13, page 132), and suppose C does not lie on C. There are two possibilities, as depicted by the pictures below: either C lies outside $\mathcal{C}$ or C lies inside $\mathcal{C}$.[14]

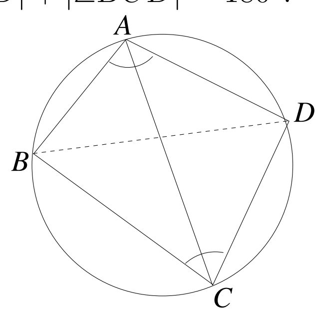
[14]A point lies **outside** (resp., **inside**) a circle with center O and radius r if its distance from O exceeds (resp., is less than) r.





The proofs for both situations are essentially the same, so we will take the case on the left, i.e., C is outside $\mathcal{C}$. Then BC intersects $\mathcal{C}$ at a point E. Clearly A and E also lie on the same side of L_{BD} and Theorem 39 implies that $|\angle BAD| = |\angle BED|$. By definition, E is between B and C so that $\angle BED$ is an exterior angle of $\triangle DEC$ (see page 129). But then $|\angle BED|$ is bigger than its remote interior angle $\angle BCD$ (Corollary on page 130). Together, we have $|\angle BAD| > |\angle BCD|$, and this contradicts the hypothesis that they are equal. Thus part (i) is proved.

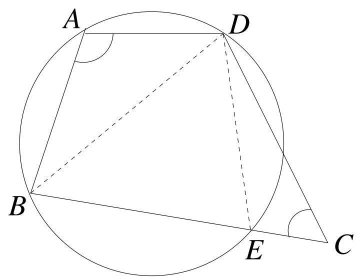
(ii) First suppose A, B, C, D are concyclic and A and C lie on opposite sides of L_{BD}, and we will show $|\angle BAD| + |\angle BCD| = 180°$.

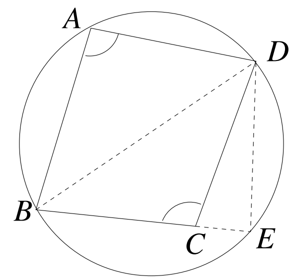


We have $|\angle BAC| = |\angle BAC| + |\angle CAD|$. Therefore,

$$
\begin{aligned}
|\angle BAD| + |\angle BCD| &= (|\angle BAC| + |\angle CAD|) + |\angle BCD| \\
&= |\angle BDC| + |\angle CBD| + |\angle BCD| \qquad \text{(Theorem 39)} \\
&= 180°, \qquad \text{(angle sum of } \triangle BCD\text{)}
\end{aligned}
$$

as desired. Next we prove the converse. Suppose A and C are on opposite sides of L_{BD} and $|\angle BAD| + |\angle BCD| = 180°$. We will show that C lies on the circle $\mathcal{C}$ passing through A, B, D. Suppose not, then there are two cases: C lies outside $\mathcal{C}$ and C lies inside $\mathcal{C}$.





Again, the proofs of both cases are entirely similar, and we therefore only prove one of the two cases, let us say, the latter. So let C be inside $\mathcal{C}$. Let the ray R_{BC} meet $\mathcal{C}$ at E. Clearly A and E lie on opposite sides of L_{BD} and A, B, E, D are concyclic. Therefore we have

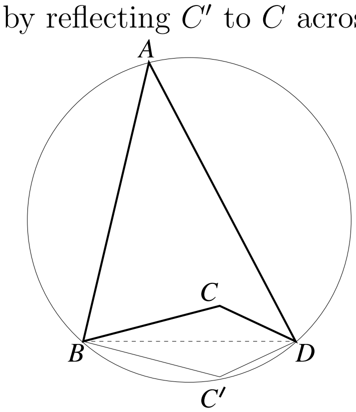
$$|\angle BAD| + |\angle BED| \;=\; 180^\circ$$

Now C is by assumption between B and E so that $\angle BCD$ is an exterior angle of the remote interior angle $\angle CED$ in $\triangle CED$. Therefore $|\angle BCD| > |\angle BED|$ (Corollary on page 130). Thus

$$180^\circ \;=\; |\angle BAD| + |\angle BED| \;<\; |\angle BAD| + |\angle BCD| \;=\; 180^\circ,$$

where the last step is from the hypothesis. This contradiction then completes the proof of Theorem 40.

Remarks. (1) Call a quadrilateral **cyclic** if its vertices are concyclic. Then part (ii) of Theorem 40 is usually presented to students as: "A quadrilateral is cyclic if and only if the sum of opposite angles is 180 degrees". The problem with this statement can be illustrated by the following quadrilateral $ABCD$ which is obtained from the cyclic quadrilateral $ABC'D$ by reflecting C' to C across the line L_{BD}.



Consider then the statement,

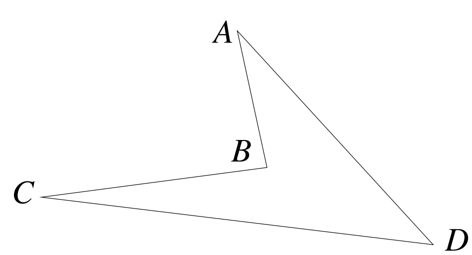
$$(\natural) \quad |\angle BAD| + |\angle BCD| = 180^\circ \text{ implies } ABCD \text{ is a cyclic quadrilateral.}$$

Because we automatically take the convex part of an angle, the hypothesis in ($\natural$), namely, $|\angle BAD| + |\angle BCD| = 180^\circ$, is still correct because $\angle BCD$ is equal to $\angle BC'D$ so long as we are taking only the convex part of an angle into account. Such being the case, ($\natural$) is a false statement and therewith the theorem that the sum of opposite angles of a quadrilateral being 180° guarantees concyclicity is also false. However, the convention is that, in the statement ($\natural$), the "interior angle $\angle BCD$"—the angle facing the "interior" of the quadrilateral $ABCD$, and therefore the *nonconvex part* of $\angle BCD$ in this case—must be used in the equation "$|\angle BAD| + |\angle BCD| = 180^\circ$". Under this convention, $|\angle BCD|$ refers to the nonconvex part of the angle and is thus bigger than 180°; this quadrilateral $ABCD$ no longer satisfies the hypothesis of ($\natural$) and is therefore no longer a counterexample to ($\natural$).

In a school course in geometry, taking every angle of a quadrilateral to mean the "interior angle" is the better choice pedagogically, although mathematically it is somewhat difficult to precisely define "interior".

(2) A minor problem arose twice in the proof of Theorem 40: for instance, how do we know that in proving the converse of part (i), the ray R_{BC} will always intersect $\mathcal{C}$ at a point E? This is a problem we have run into already on page 183; we know it can be fixed but we will ignore it now.

(3) Another problem in the preceding proof is more serious. In the proof of part (ii), we claimed on page 189 that $|\angle BAD| = |\angle BAC| + |\angle CAD|$. This tacitly assumes that C lies in the convex part of $\angle BAD$, and this is of course false in general:

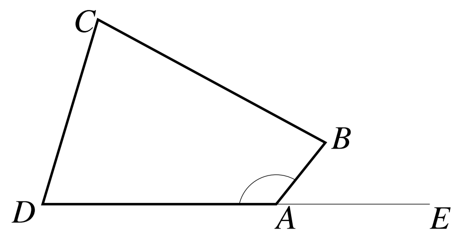


Fortunately, such counterexamples do not arise in the case of cyclic quadrilaterals, which is of course our present situation. The cruz of the matter is this:

If a quadrilateral is cyclic, then each vertex lies in the convex part of the *opposite angle.*

In a school classroom, it would be defensible to take this on faith or assign it as an exercise rather than spend instructional time on it.

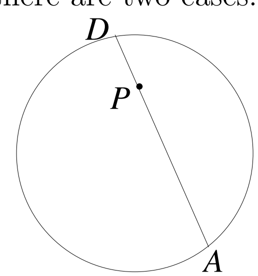
There is an equivalent formulation of part (ii) of Theorem 40 that is often used. Given a quadrilateral $ABCD$, we will henceforth agree that an angle of the quadrilateral will mean the **interior angle** of $ABCD$; see (1) of the preceding Remarks. If the interior angle at A is $< 180°$, then its **exterior angle** is by definition the angle $\angle BAE$ where E is a point on L_{DA} so that A is between D and E. See the picture below, *where the interior angle at A is the angle indicated by an arc.* The same discussion of course applies to every vertex of the quadrilateral.

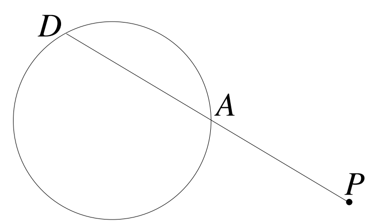


Corollary. *Let the interior angle at A of a quadrilateral $ABCD$ be $< 180°$. Then the exterior angle at A is equal to the opposite interior angle at C $\iff$ $ABCD$ is cyclic.*

We now explore a little bit the vast ramifications of Theorem 39.

Take a point P and let a circle $\mathcal{C}$ be given in the plane. Let a line passing through P intersect $\mathcal{C}$ at two points A and C (compare Theorem 35 on page 179). Without looking at any pictures, we may ask if the product $|PA| \cdot |PC|$ is always the same independent of the line (naturally, you are not likely to ask this question until after much experimentation that *suggests* that the product seems not to depend on the line used). Now there are two cases: P is inside $\mathcal{C}$ and P is outside $\mathcal{C}$, as shown.

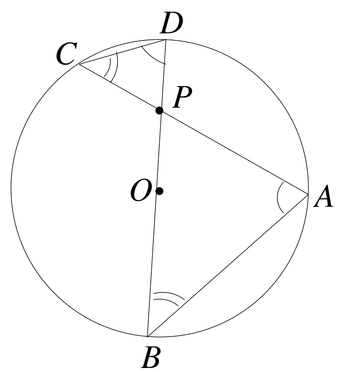




It turns out that the answer is always yes, and we can even give the exact value of the product in both cases.

Theorem 41. (i) *Let P be a point inside a circle $\mathcal{C}$ with center O and radius r, and let a line passing through P intersect $\mathcal{C}$ at A and C. Then $|PA| \cdot |PC| = r^2 - |OP|^2$.*
(ii) *Let P be a point outside a circle $\mathcal{C}$ with center O and radius r, and let a line passing through P intersect $\mathcal{C}$ at A and C. Then $|PA| \cdot |PC| = |OP|^2 - r^2$.*

Proof. (i) Let P be inside $\mathcal{C}$, and let the diameter passing through P intersect the circle at B and D.



By Theorem 39, $|\angle D| = |\angle A|$ and $|\angle C| = |\angle B|$. Therefore $\triangle PCD \sim \triangle PBA$ by the AA criterion for similarity. Therefore, by Theorem 25 (page 163),

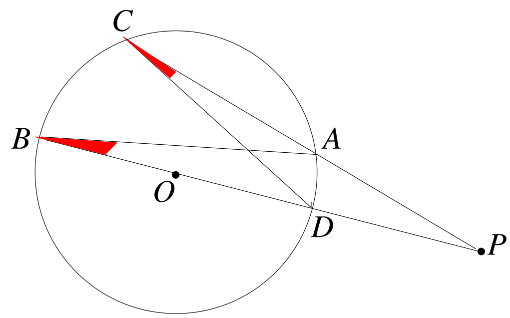
$$\frac{|PA|}{|PD|} = \frac{|PB|}{|PC|}.$$

By the cross-multiplication algorithm, we have $|PA| \cdot |PC| = |PB| \cdot |PD|$. Now because DB is a diameter of $\mathcal{C}$,

$$\begin{aligned} |PB| &= |BO| + |OP| = r + |OP| \\ |PD| &= |OD| - |OP| = r - |OP|. \end{aligned}$$

Thus,

$$|PA| \cdot |PC| = |PB| \cdot |PD| = (r + |OP|)(r - |OP|) = r^2 - |OP|^2$$



and (i) is proved.

(ii) Now suppose P is outside a circle $\mathcal{C}$ with center O and radius r. and let a line passing through P meet $\mathcal{C}$ at A and C.

We have $|\angle C| = |\angle D|$ because of Theorem 39, so that $\triangle PCD \sim \triangle PBA$ because of the AA criterion for similarity (the triangles share $\angle P$.) By Theorem 25 again,

$$\frac{|PA|}{|PD|} = \frac{|PB|}{|PC|}.$$

By the cross-multiplication algorithm, we have $|PA| \cdot |PC| = |PB| \cdot |PD|$. Now because BD is a diameter of $\mathcal{C}$,

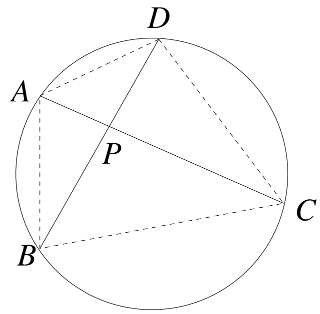
$$\begin{aligned} |PB| &= |OP| + |OB| = |OP| + r \\ |PD| &= |OP| - |OD| = |OP| - r. \end{aligned}$$

Thus,

$$|PA| \cdot |PC| = |PB| \cdot |PD| = (|OP| + r)(|OP| - r) = |OP|^2 - r^2$$

and (ii) is proved, and therewith Theorem 41.

Each of the two parts in Theorem 41 has a converse. Let us take up part (i) first. Let P be a point inside a circle with center O and radius r. If two chords BPD and APC pass through P, then we know that $|PA| \cdot |PC| = |PB| \cdot |PD|$ because they are both equal to $r^2 - |OP|^2$.



We now prove the converse.

Theorem 42. *Let two lines intersect at P and let A, C be points on one line separated by P and let B, D be points on another line also separated by P. If $|PA| \cdot |PC| = |PB| \cdot |PD|$, then the four points A, B, C, D are concyclic.*

Proof. Join B and C. Then the hypothesis easily implies that A and D lie on the same side of line L_{BC}. Now $\angle APB$ and $\angle DPC$ are equal because they are opposite angles. Moreover, we have

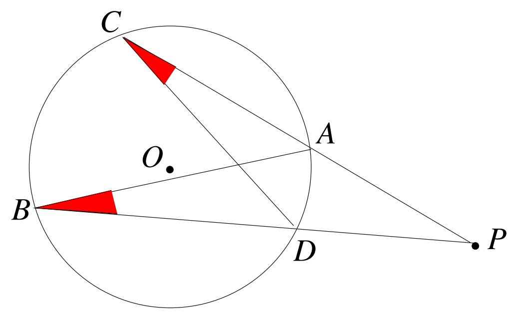
$$\frac{|PA|}{|PD|} = \frac{|PB|}{|PC|},$$

which is an immediate consequence of $|PA| \cdot |PC| = |PB| \cdot |PD|$ by the cross-multiplication algorithm. Therefore $\triangle PAB \sim \triangle PDC$ on account of SAS for similarity (Theorem 26 on page 163). By Theorem 25 (page 163), $|\angle PAB| = |\angle PDC|$. It follows from part (i) of Theorem 40 that A, B, C, D are concyclic, and the theorem is proved.

Sometimes Theorem 42 is stated in terms of a quadrilateral. Given a quadrilateral $ABCD$. *Suppose we know that the diagonals AC, CD,* as segments*, intersect at a point P.* Then Theorem 42 and part (i) of Theorem 41 are seen to be equivalent to the following statement:

> *Let $ABCD$ be a quadrilateral whose diagonals AC, BD intersect at a point P. Then $ABCD$ is a cyclic quadrilateral if and only if $|PA| \cdot |PC| = |PB| \cdot |PD|$.*

Next, let P be a point outside of a circle $\mathcal{C}$ with center O and radius r. Let two lines through P intersect $\mathcal{C}$ at A, C and B, D, respectively. Without loss of generality, we may assume that A is between C and P, and that D is between B and P, as shown:



Then $|PA| \cdot |PC| = |PB| \cdot |PD|$ because by Theorem 41(ii), both are equal to $|OP|^2 - r^2$. The converse then states:

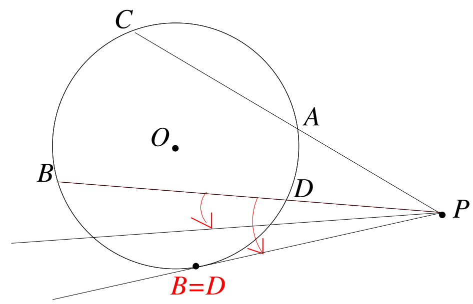
Theorem 43. *Suppose two lines intersect at a point P. Let A be between C and P on one line, and let D be between B and P on another. If $|PA| \cdot |PC| = |PB| \cdot |PD|$, then the four points A, B, C, D are concyclic.*

Proof. From $|PA| \cdot |PC| = |PB| \cdot |PD|$ and the cross-multiplication algorithm, we get

$$\frac{|PA|}{|PD|} = \frac{|PB|}{|PC|}.$$

The triangles PAB and PDC also share an angle, namely, $\angle P$. Therefore triangles PAB and PDC are similar because of SAS for similarity (Theorem 26 on page 163), and Theorem 25 (page 163) implies $|\angle C| = |\angle B|$. From the hypothesis that A is between C and P and D is between B and P, it is easy to see that B and C lie on the same side of line L_{AD}. Therefore Theorem 39 implies that A, B, C, D are concyclic. The proof is complete.

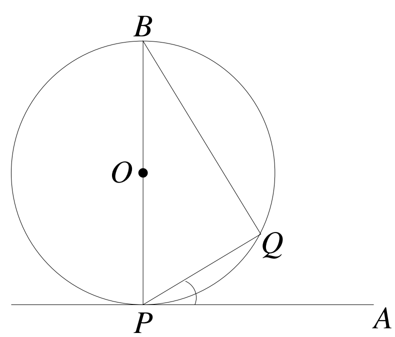
One can pursue the discussion of Theorem 43 in the following way. Hold the line PAC fixed, but now allow the line PDB to turn counterclockwise around P until it becomes tangent to the circle $\mathcal{C}$ and $B = D$, as suggested below.



Intuitively, the conclusion of Theorem 43, that $|PA| \cdot |PC| = |PB| \cdot |PD|$, "should" become $|PA| \cdot |PC| = |PB|^2$. Our next goal is to prove this assertion precisely. We first need a preliminary result which is interesting in its own right.

Theorem 44. *Let ℓ be a line tangent to a circle $\mathcal{C}$ at P. Let PQ be a chord of $\mathcal{C}$ and, furthermore, let A be a point on ℓ, B be a point on $\mathcal{C}$ so that A and B lie on opposite half-planes of the line L_{PQ}. Then*

$$|\angle APQ| = |\angle PBQ|$$



Proof. We first tackle the case that B lies on the major arc of PQ. By Theorem 39, $|\angle PBQ|$ doesn't depend on the location of B on the circle so long as it lies in the half-plane of L_{PQ} opposite to A. We may therefore assume that PB is a diameter of $\mathcal{C}$.

By Theorem 37 (page 181), $\angle Q$ is a right angle, so that

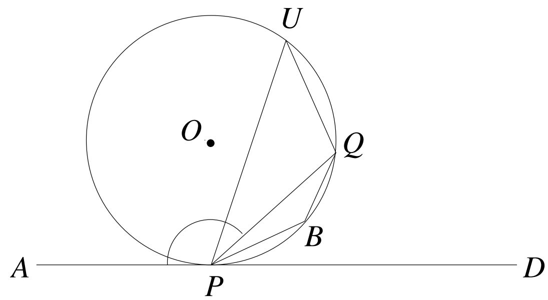
$$|\angle PBQ| + |\angle BPQ| \;=\; 90^\circ$$

But by Theorem 36 (page 180), $\angle BPA$ is also a right angle so that

$$|\angle APQ| + |\angle BPQ| \;=\; 90^\circ$$

Hence $|\angle APQ| = |\angle PBQ|$.

In case B lies on the minor arc of PQ, let U be any point on the major arc of PQ. Also let D be any point on ℓ on the opposite ray of R_{PA}.



Then the preceding argument shows that

$$|\angle DPQ| = |\angle PUQ|$$

By Theorem 40(ii), $|\angle PBQ| = 180° - |\angle PUQ|$, and obviously, $|\angle APQ| = 180° - |\angle DPQ|$. Hence also $|\angle APQ| = |\angle PBQ|$. The proof is complete.

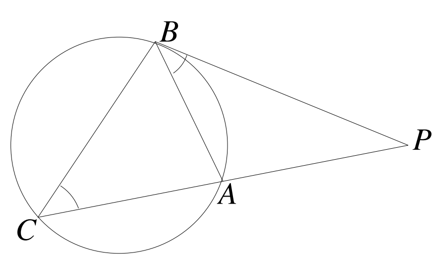
The following theorem is the main goal we are after.

Theorem 45. *Let P be a point outside a circle $\mathcal{C}$ and let line PB be tangent to $\mathcal{C}$ at B. If another line through P intersects $\mathcal{C}$ at A and C, then*

$$|PB|^2 = |PA| \cdot |PC|$$

Conversely, if a line through a point P outside a circle $\mathcal{C}$ meets $\mathcal{C}$ at two points A and C, and if a point $B \in \mathcal{C}$ satisfies $|PB|^2 = |PA| \cdot |PC|$, then line PB is tangent to the circle $\mathcal{C}$ at B.

Proof. Let PB be tangent to $\mathcal{C}$ at B and line L_{PA} intersects $\mathcal{C}$ at C, as shown.

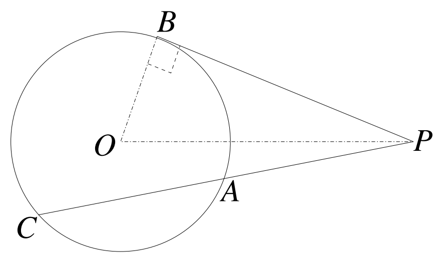


Observe that $\triangle PBC \sim \triangle PAB$ because $|\angle PBA| = |\angle PCB|$ (Theorem 44) and $|\angle P| = |\angle P|$. Thus

$$\frac{|PB|}{|PA|} = \frac{|PC|}{|PB|}$$

which is equivalent to $|PB|^2 = |PA| \cdot |PC|$.

Conversely, suppose $|PB|^2 = |PA| \cdot |PC|$ where L_{PB} and L_{PA} are lines which intersects the circle $\mathcal{C}$ at B and at A, C, respectively. We have to prove that L_{PB} is tangent to $\mathcal{C}$ at B.



By part (ii) of Theorem 41 (page 193), $|PA| \cdot |PC| = |OP|^2 - r^2$, and $|OB| = r$. Therefore, by the hypothesis that $|PB|^2 = |PA| \cdot |PC|$, we get

$$|OP|^2 = |PB|^2 + |OB|^2.$$

The converse of the Pythagorean Theorem now implies that $OB \perp PB$. By Theorem 36, PB is tangent to $\mathcal{C}$. The proof of Theorem 45 is complete.

Construction problems

We are now in a position to return to construction problems 9, 10, and 12 on page 145. First:

Construct a regular hexagon inscribed in a circle.

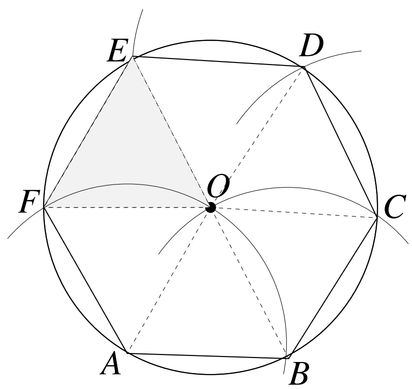
We begin by defining a *regular polygon*. A polygon (see page 83) is said to be **regular** if its vertices lie on a circle and all its sides are equal;[15] we then say that the regular polygon is **inscribed** in that circle.

[15]This is not the standard definition of a regular polygon, but is equivalent to it.

Since every triangle is inscribed in its circumcircle, a regular 3-gon is therefore an equilateral triangle. A regular 4-gon is a square; this is a straightforward consequence of Theorem 39 on page 187. The same theorem also leads to the fact that the degree of an angle in a regular n-gon is $\left(\frac{n-2}{n}\right)180°$. If $n = 6$, we therefore see that the angle of a regular hexagon is 120 degrees.

Now we can state the construction problem more precisely.

Given a circle $\mathcal{C}$ with center O, we have to locate six points on circle O so that they form the vertices of a regular hexagon.



The construction:

1. Take a point A on $\mathcal{C}$ whose radius will be denoted by r. With A as center and with r as radius, draw a circle which intersects $\mathcal{C}$ at B and F.

2. With B as center and r as radius, draw a circle which intersects circle O at an additional point C.

3. Repeat the drawing of circles with center C and then D, as shown, so that we obtain two more points D and E.

4. Connect the successive points A, B, …, F and A to get the desired hexagon.

Proof that $ABCDEF$ is a regular hexagon. By construction, every triangle in the picture—except $\triangle OEF$— is an equilateral triangle whose sides are all equal to r (see steps 1–3). We claim that $\triangle OEF$ is also equilateral. To this end, notice that, because every triangle (except $\triangle OEF$) is equilateral, each angle of these triangles is 60 degrees. Thus $|\angle FOA| = |\angle AOB| = |\angle BOC| = |\angle COD| = |\angle DOE| = 60°$. Since there are 360 degrees around O, we conclude that $|\angle EOF| = 60°$. But $\triangle EOF$ is isosceles ($|OE| = |OF|$), so by Theorem 7(a) (page 118),

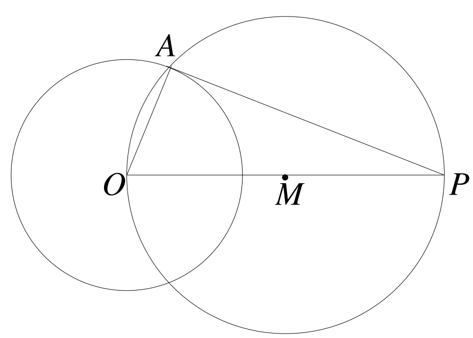
$$|\angle OEF| \;=\; |\angle OFE| \;=\; \frac{1}{2}(180° - 60°) \;=\; 60°$$

The angles of $\triangle OEF$ are therefore all equal to $60°$ and therefore $\triangle OEF$ is equilateral as well (Corollary 2 of Theorem 20, page 143). Thus all the angles and all the sides of the hexagon $ABCDEF$ are equal. This then proves that $ABCDEF$ is regular.

It remains to discuss the construction problems 10 and 12 from page 145.

Draw tangents to a circle from a point outside the circle.

More precisely, let P be a point outside a circle $\mathcal{C}$. The problem is to construct a line passing through P and tangent to circle $\mathcal{C}$.



The construction:

1. Let O be the center of $\mathcal{C}$. Join P to O to obtain segment OP.

2. Locate the midpoint M of OP (see Construction 4 on page 145).

3. With M as center and MP as radius, draw a circle that intersects circle O at two points.

4. If A is a point of intersection in Step 3, then the line L_{PA} is tangent to circle $\mathcal{C}$.

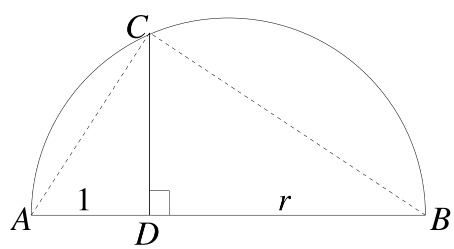
Proof that L_{PA} is tangent to $\mathcal{C}$. By Step 3 of the construction, OP is a diameter of the constructed circle and therefore $\angle PAO$ is a right angle (Theorem 37 on page 181). By Theorem 36 (page 180), L_{PA} is tangent to circle $\mathcal{C}$.

Construct the square root of a positive number.

Precisely, let a segment of length 1 be given. Also let a segment of length r be given. Then we have to construct a segment of length $\sqrt{r}$.

The construction:

1. Construct a segment AB of length $1 + r$.



2. Draw a circle with AB as diameter (see Construction 4 on page 145 for locating the center of this circle).

3. If D is the point in AB so that $|AD| = 1$ and $|DB| = r$, let D be one of the points of intersection of the line perpendicular to AB and the circle C (see Construction 3 on page 145).

4. Then $|CD| = \sqrt{r}$.

Proof that $|CD| = \sqrt{r}$. By Step 2 of the construction, AB is the diameter of the circle and therefore $\angle ACB$ is a right angle (Theorem 37 on page 181). By Theorem 29 on page 167, we therefore have $|CD|^2 = 1 \cdot r = r$. Hence $|CD| = \sqrt{r}$.