Building better mathematics teachers*

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Abstract. We will explain why the emphasis of Elizabeth Green's article, [Green1], on *teaching* as a panacea for curing the ills of school mathematics education is misleading. In fact, math teachers have to contend with far more fundamental issues: the *seriously flawed mathematics* in standard school textbooks and the education establishment's failure to provide them with the requisite content knowledge for teaching. A related article by Mark Thames and Deborah Ball, [Thames-Ball], on *skilled teaching* will also be discussed in the last section.

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Section headings:

The July 24, 2014 edition of the New York Times Magazine contains an article by Elizabeth Green ([Green1]) that makes the claim that Americans "stink at math" because "the traditional way of teaching math simply doesn't work". The only way out, according to her, is to radically change the way teachers teach math by following a prescription made by NCTM (National Council of Teachers of Mathematics) in the 1980s. Because she was writing for the popular press, Green may have intentionally oversimplified the narrative for her audience. That said, one cannot help but be a bit alarmed by the ill-effects of potential misinterpretations of such an oversimplification.

"I, We, You" versus "You, Y'all, We"

To Green, the traditional way of teaching mathematics is encoded in the pattern

"I, We, You", i.e., *I* (the teacher) show *you* (the students) how to perform a skill, then together, *we* follow my instructions in a sample problem or two, and finally *you* the students work through similar problems on your own. It is taken as a given that "I, We, You" is synonymous with the mind-numbing memorization of incomprehensible rules. The proposed radical change, which Green attributes mainly to Magdalene Lampert (cf. [Lampert]), is to abandon "I, We, You" in favor of "You, Y'all, We". Roughly, this means each lesson begins with a problem that *you* the students work on individually, then in peer groups (*y'all*), and finally as a whole class (*we*). "You, Y'all, We" is supposed to bring with it "passionate discussions" about mathematics in order "to uncover math's procedures, properties and proofs". Green would have us believe that Americans stink at math because most school math classrooms have been in the "I, We, You" mode whereas mathematics teaching should follow the "You, Y'all, We" paradigm.

Is that it? Even allowing for unavoidable oversimplifications, most people would not consider this to be a satisfactory answer to the question "Why do Americans stink at math?".

Green actually knows better. She describes how, in the 1980s, California's schools adopted a similar approach to teaching math as "You, Y'all, We". Unfortunately, most classrooms never experienced any change in the mathematical substance of the classroom discussions but got stuck in the *formal* changes from "I, We, You" to "You, Y'all, We" (e.g., students were seated in small groups rather than in rows facing the teacher). She makes the passing comment that "Without the right training, most teachers do not understand math well enough to teach it the way Lampert does." She does not elaborate on what the "right training" is. Many people have been kept awake at night over exactly this question: What actually constitutes this *right training?* But Green leaves it unanswered, and the lasting impression a reader is likely to get from the article is that all classrooms should *look like* "You, Y'all, We". Teachers' *understanding of math?* It hardly seems something worth worrying about.

The case of division-with-remainder

But the "right training" does matter, at least to the National Mathematics Advisory Panel (see the Recommendation on page 37 of [NMP]). We can see why by considering a problem used in Green's article to illustrate the "I, We, You" pattern:

"Let's try out the steps for $242 \div 16$". (We).

Because 16 is not known to divide 242 evenly in advance, this is what is called a **division-with-remainder**[1] Few, if any, school textbooks bother to explain clearly that this is asking for the biggest whole number q (called the **quotient**) so that $q \times 16 \leq 242$, and that the difference $242 - (q \times 16)$ is called the **remainder**. Instead,

[1] The school mathematics terminology for "**division algorithm** in an integral domain" in abstract algebra.

these books simply home in on the procedure of long division—"Draw a division house, put '242' on the inside and '16' on the outside, etc."—obtain the number 15 on top of the house and the number 2 in the basement, declare by *fiat* that this means the quotient is 15 and the remainder is 2, and write the answer as $242 \div 16 = 15R2$. Green bemoans the teaching-by-rote of division-with-remainder in general as "an arbitrary process wholly divorced from the real world of numbers". She is right, but her insinuation that the root cause of this kind of teaching is the adoption of the "I, We, You" pedagogy is shockingly wide of the mark. She has confused cause with effect.

School textbooks have traditionally taught the long division algorithm as nothing more than this procedure of "drawing a division house"; more recent ones may add a few plausibility arguments, but none addresses the algorithm as *mathematics* in order to give students a sense of what they are doing with the "division house". Teachers are not to blame for this, because they were taught the same way when they were students in K–12, and they were never taught anything different in college about long division. So they simply follow the same script and teach long division as nothing more than a rote procedure. (After all, pianists are expected to perform a piece of music according to the notes of a score, good or bad.) It therefore comes to pass that elementary teachers may heed the clarion call of "You, Y'all, We" and make their classrooms more lively, but they cannot improve student learning of long division by explaining to students, clearly, how to go logically from the "division house" to $242 \div 16 = 15R2$.

One reason for teachers' failure is simple: the equality $242 \div 16 = 15R2$ inherently makes no sense because the symbol $\div$, *within the context of whole numbers,* cannot be used for division-with-remainder. We can write $6 \div 3 = 2$ and $32 \div 8 = 4$, because 3 divides 6 and 8 divides 32, and the quotients 2 and 4 can be effortlessly explained by the fact that $6 = 2 \times 3$ and $32 = 4 \times 8$, respectively. In general, given whole numbers m and d, the symbol "$\div$" can be used for $m \div d$ *only* when we know ahead of time that the divisor d divides the dividend m, so that the equation $m \div d = q$ for some whole number q has an unambiguous meaning: $m = q \times d$. But to write $242 \div 16$? Not so good, because 16 does not divide 242.[2] Therefore to say $242 \div 16 = 15R2$ is to imply that $242 = (15R2) \times 16$. However, this equality doesn't make any sense because $15R2$ is not a number (i.e., not a point on the number line) but *a pair of numbers*—15 and 2—linked together in some mysterious way by R. Furthermore, ordinary arithmetic does not show students how to multiply a pair of numbers—$15R2$—by a single number (i.e., 16) to get the number 242. We can appreciate the nonsensical nature of "$242 \div 16 = 15R2$" yet another way: if this equality makes

[2] Suppose there is no prior assurance that 242 is divisible by 16, but we want students to perform the long division of 242 by 16. We can avoid the abuse of the "$\div$" symbol by simply asking them to "find the quotient and the remainder of 242 divided by 16". If it turns out that the remainder of a long division is 0 (i.e., if the dividend is divisible by the divisor), e.g. the long division of 272 by 16, then one has a choice of writing either "$272 = (17 \times 16) + 0$" or "$272 \div 16 = 17$". Either is correct (and they are also equivalent to each other according to the definition of "$272 \div 16 = 17$").

sense, then so does $47 \div 3 = 15R2$, which therefore leads to the inevitable—and absurd—conclusion that $242 \div 16 = 47 \div 3$ (since both sides are equal to $15R2$). [3]

Teachers (and, ultimately, their students) need to be taught that "division-with-remainder" is not a "division" (in the sense of $6 \div 3$), in the same way that a *sea lion* is not a *lion.* Therefore the symbol $\div$ does not transfer to "division-with-remainder" as is. The proper way to express "the division-with-remainder of 242 by 16 has quotient 15 and remainder 2" is to write the following equation of whole numbers:

$$242 = (15 \times 16) + 2 \qquad (1)$$

where the **remainder** 2 satisfies $2 < 16$, 16 being the divisor. Teachers should also get to see how to break down the "division house" into a sequence of *simpler* divisions-with-remainder (see Section 7.3 of [Wu2011a]). Then the long division of 242 by 16 can be shown to lead *directly and logically* to equation (1) (see Sections 7.4–7.5 of [Wu2011a]).

Without a knowledge of the proper way to express division-with-remainder in the form of equation (1), and without any understanding that the "division house" is actually the first in a chain of logical steps leading inexorably to equation (1), teachers are handicapped. They may know heuristic arguments and attractive metaphors and analogies to make the "division house" appear palatable to students. They may also

[3] In the context of *fractions,* both $242 \div 16$ and $47 \div 3$ are well-defined (see, e.g., the definition on page 239 of [Wu2011a]) and are proved to be equal to $\frac{242}{16}$ and $\frac{47}{3}$, respectively. Of course these fractions are not equal.

possess great pedagogical skills to get students actively involved. But when all is said and done, we owe it to students to give them a straightforward explanation—without metaphors or analogies—of why the "division house" is a correct procedure that produces the quotient and remainder of a division-with-remainder.

If elementary teachers are given the chance to learn the mathematics in this and related topics, they may not become super-teachers overnight, but the amount of rote memorization in their classrooms can be expected to decrease substantially. On the other hand, not knowing that the equality $242 \div 16 = 15R2$ is mathematically nonsensical, and especially not knowing the reasoning that connects the long division algorithm to equation (1), will certainly doom any effort to lead students to a real conceptual understanding of this algorithm, regardless of whether it is "You, Y'all, We" or "I, We, You".[4]

Now we know why teaching long division in the elementary classroom is "an arbitrary process wholly divorced from the real world of numbers": it is because the education establishment has consistently ignored the basic need of all teachers for correct *mathematical* information. Without this support, teachers have no choice but to fall back on the "I, We, You" routine to better hide this lack of knowledge. *Please don't blame the teachers, because it is our collective fault.*

[4] In the Common Core Standards ([CCSSM]), understanding the long division algorithm is a topic in the fifth grade; at that point of students' learning trajectory, it would be entirely appropriate to expose them to the reasoning, at least in the special case of a one-digit divisor and a two-digit dividend, that leads from the "division house" to the analog of equation (1).

It is unrealistic to expect teachers to wake up to the defect of "$242 \div 16 = 15R2$" all by themselves when school textbooks and college textbooks for professional development reassure them that it is correct. It doesn't help that such misuse of the division symbol "$\div$" even finds its way into leading education documents (e.g., p. 37 of [NCTM1989] and p. 153 of [NCTM2000]). Furthermore, the reasoning that leads from the long division algorithm to the equation $242 = (15 \times 16) + 2$ is sophisticated and it would not be fair to expect teachers to recreate it for themselves. This fact also points to the inherent limitation of lesson study that Green describes at some length in the article. She hints that through lesson study, teachers can also learn correct mathematics; the unspoken expectation is that, for example, teachers will somehow learn to replace "$242 \div 16 = 15R2$" by "$242 = (15 \times 16) + 2$" and *prove* the latter. In her words, "By the end, the teachers had learned not just how to teach the material from that day but also about math and the shape of students' thoughts and how to mold them." This assertion could ultimately be proved or disproved by research data, but until that day comes, I will think of it as unfettered optimism running amok, nothing more.

Some may put down such a concern about the nonsensical nature of "$242 \div 16 = 15R2$" as nothing more than a mathematician's nitpicking, but it is not. To understand the seriousness of this concern, we have to look further down students' learning path. The equality $242 \div 16 = 15R2$ is an egregious abuse of the *equal sign*: there is no way that an undefined quantity "$242 \div 16$" could be "equal" to "$15R2$", which is a pair of numbers, 15 and 2, locked together with "R" in some kind of clandestine relationship (what is it?). By writing "$242 \div 16 = 15R2$", school textbooks send an unmistakable signal to elementary students that they may abuse the equal sign any which way they wish. By the time students come to algebra, their misconception of what the equal sign means will interfere, inevitably, with their ability to learn how to solve *equations* in algebra (see, e.g., [Knuth-Stephans-McNeil-Alibali]). It is better to nip this bad practice in the bud early on.

The concept of Textbook School Mathematics (TSM)

The fact that teachers have been denied access to a correct mathematical exposition of the long division algorithm is by no means an isolated example. The same can be said about almost every major topic of K–12 mathematics. If we agree to use the term **content knowledge** to refer to the body of knowledge about the *mathematics of the K–12 curriculum*—and keep in mind that *mathematics* is not just a collection of facts but includes the reasoning for every assertion and the attendant skills and applications, all of which are packaged together in a coherent way—then teachers' content knowledge must be the crux of the matter in discussing math teaching as of 2014. It is somewhat shocking but nevertheless true that the education establishment has failed to provide teachers with the content knowledge they need for teaching mathematics correctly. (This echoes the sentiment expressed in the last line of [Thompson-Thompson].) What is at issue here is not a few instances of minor errors or misconceptions in the generic school textbooks and college textbooks for professional development,[5] but a wide-ranging promulgation of fundamentally incorrect school mathematics. We call the mathematics contained in these books **Textbook School Mathematics**, or more simply, **TSM** ([Wu2011b] and [Wu2014c]). TSM distinguishes itself from (correct) school mathematics by *not offering definitions* to most basic mathematical concepts such as fraction, percent, or constant rate; by the *absence of correct reasoning* for most basic facts such as why the product of two negative numbers is positive or why the graph of a linear equation is a line; by *blurring the line between heuristics and proof* in too many instances, including $a^0 = 1$ for a positive number a, $\frac{a}{b} = a \div b$ for a fraction $\frac{a}{b}$, and that two lines are perpendicular if and only if the product of their slopes is -1; and by generally *presenting mathematics as a collection of tricks* with no internal coherence so that, instead of presenting fractions as a direct continuation of whole numbers, fractions are singled out for being *different kinds of numbers* (cf. [Wu2014a] and [Wu2014b]).

In this light, the use of the division symbol "$\div$" in $m \div n$ between two whole numbers m and n ($n \neq 0$)—with a total disregard for whether or not m is a multiple

[5]There are exceptions, including [Jensen], [Parker-Baldridge], [Wu2002], and [Wu2011a].

of n—is vintage TSM. The writing of $242 \div 16 = 15R2$ is also vintage TSM, and so is the absence of any explanation for why, by constructing the "division house", one can get the quotient (i.e., 15) and the remainder (i.e., 2).

The case of teaching fractions

Perhaps the easiest way to see TSM in action is to consider the teaching of fractions in the upper elementary grades. In Green's article, there is a striking anecdote about how the A&W restaurant chain lost the "hamburger war" to McDonald's in the 1980s because the American public failed to recognize that $\frac{1}{3}$ is more than $\frac{1}{4}$. As usual, Green attributes this innumeracy simply to the "traditional way of teaching mathematics", but such unfocused diagnosis does not address the real issue. Instead, let us take a long, hard look at how fractions are taught in elementary school.

A fraction is usually presented in TSM as a piece of pizza (or a pie). For sure, students can see at the beginning, by drawing pictures, that $\frac{1}{3}$ is more than $\frac{1}{4}$ because $\frac{1}{3}$ of a pizza is bigger than $\frac{1}{4}$ of a pizza. But trouble begins almost immediately as soon as they encounter the basic fact about equivalent fractions, to the effect that $\frac{a}{b} = \frac{na}{nb}$ for all fractions $\frac{a}{b}$ and nonzero whole numbers n. The usual explanation from TSM is that

$$\frac{a}{b} \stackrel{(1)}{=} 1 \times \frac{a}{b} \stackrel{(2)}{=} \frac{n}{n} \times \frac{a}{b} \stackrel{(3)}{=} \frac{na}{nb} \tag{2}$$

Now equality (1) is halfway acceptable, and the fact that $1 = \frac{n}{n}$ in equality (2) is solid. But equality (3) is a problem: First of all, in TSM, multiplying two fractions $\frac{n}{n}$ and $\frac{a}{b}$ means multiplying two pieces of pizza,[6] and that is hard to imagine. So seeds of doubt are beginning to be planted in students' minds. Furthermore, it is too good to be true: just multiply the numerators and the denominators separately. Is that all there is to fractions? Next comes the addition of fractions: putting two pieces of pizzas together. Simple, just like adding two whole numbers, right? *But*, the rule says

$$\frac{5}{12} + \frac{7}{18} = \frac{(3 \times 5) + (2 \times 7)}{36} = \frac{29}{36}$$

because 36 is the *least common denominator* of 12 and 18. What has happened to just putting two pieces of pizza together? What does the *least* common denominator have to do with pizzas? And why not just do it like multiplication: add the numerators and the denominators separately?

$$\frac{5}{12} + \frac{7}{18} = \frac{5+7}{12+18}\,?$$

Elementary school students naturally wonder why something this good works only for multiplication—why can't it also work for addition? The same thought occurs to

[6] This is the delightful observation in [Hart].

teachers, too, of course, but nowhere in their K–12 education or teacher training are these types of questions ever addressed or answered. If they consult the literature, they would find on page 96 of [NCTM1989] the following passage:

> This proficiency in the addition, subtraction, and multiplication of fractions and mixed numbers should be limited to those with simple denominators that can be visualized concretely or pictorially and are apt to occur in real-world settings . . . This is not to suggest, however, that valuable instruction time should be devoted to exercises like $17/24 + 5/18$ or $5\frac{3}{4} \times 4\frac{1}{4}$, which are much harder to visualize and unlikely to occur in real-life situations.

So TSM is as clear about fractions as the day is long that there is no way to make sense of the basic facts about fractions. Teachers have two options: only do simple additions such as $\frac{1}{2} + \frac{2}{3}$ by drawing lots of pictures, or resort to the "I, We, You" routine to forestall questions from students and make the lessons on fractions safe and simple.[7] Students get the idea too: don't ask questions, just do it. This is why, by the time they come to the division of fractions, they are more than willing to go down the path of least resistance: "Ours is not to reason why, just invert and multiply." In this state of mind, how would students do a word problem like the following?

[7] The addition of fractions *is* as simple as the addition of whole numbers, provided it is done the right way. See, for example, page 5 of [Wu2011b] or, in greater detail, Section 14.1 of [Wu2011a].

> *Two faucets, A and B, together fill a tub in* $5\frac{1}{3}$ *minutes. Faucet A alone can fill the tub in* $8\frac{3}{4}$ *minutes. How long will it take faucet B to fill it alone?*

They have $5\frac{1}{3}$ and $8\frac{3}{4}$ pizzas in front of them, and somehow they have to figure out how these pizzas show up in faucets. They already know going in that it won't make any sense, and it is too late in the day to try anything different. Why not just forget the pizzas and simply follow the teacher's directions? *Get it done and hand it in.* By the way, is $\frac{1}{3}$ bigger or smaller than $\frac{1}{4}$? At this point, does it matter anymore? Who knows anyway? Oh, well, since $3 < 4$, probably $\frac{1}{3} < \frac{1}{4}$.

How could the "You, Y'all, We" paradigm have changed the preceding outcome, given that teachers' minds were filled with TSM and students got nothing but TSM from their textbooks? We are here talking about how to restore students' faith in reasoning and make them believe that school mathematics makes sense and is learnable. For that, teachers have to be capable of rectifying the errors in TSM. Consider, again, equation (2) on page 13,

$$\frac{a}{b} = 1 \times \frac{a}{b} = \frac{n}{n} \times \frac{a}{b} = \frac{na}{nb}$$

It is most unlikely that passionate discussions among students in an average fifth grade classroom will be enough to uncover the serious mathematical error in this all-too-facile piece of pseudo-reasoning; less likely still would such discussions lead to a simple and correct explanation of equivalent fractions (for the latter, see pp. 209–211 of [Wu2011a]). The resolution of this impasse would require a teacher with strong content knowledge who can help shape and give guidance to students' discussions. Such a teacher must know *more than TSM*. Of course the same remark applies equally well to the earlier example, namely, the teaching of the division-with-remainder of 242 by 16.

The A&W chain lost the "hamburger war" because TSM ruled the classrooms of the nation and the education establishment did not see fit to provide our teachers with correct, non-TSM content knowledge. Green seems not to realize that the pervasiveness of the "I, We, You" routine in the American classrooms is a *consequence* of these two facts; if she wants to root out "I, We, You", then she will have to strike at its root cause. When all is said and done, so long as teachers only know TSM but not correct mathematics, "I, We, You" and rote memorization will follow as surely as night follows day. A prescription of the "You, Y'all, We" pedagogy as a cure for our educational malaise while leaving TSM intact is a prescription for disaster, because great pedagogy and poor content knowledge make a lethal combination: they leave students powerfully convinced that the incorrect information they are fed is actually correct. Let us not replay the scenarios from the nineties when pedagogy often overwhelmed content knowledge.

Green is likely aware that, behind "You, Y'all, We", there has to be a teacher with strong content knowledge in order to make it work; her comment (quoted above) that, "Without the right training, most teachers do not understand math well enough to teach it the way Lampert does", betrays this awareness. Yet, by downplaying (or is it *failing to bring out*?) the important role of content knowledge, Green will mislead—and in some cases, further encourage—teachers and district administrators to believe that the silver bullet for our math education crisis is *great teaching*, and therefore, it must be *pedagogy* that matters the most. This would be a profound disservice to the critical issue of the day (and the title of her book [Green2]): *Building a Better Teacher.*

I have illustrated the need of robust and correct content knowledge for the making of better math teachers by using two examples from elementary school mathematics, because elementary school mathematics is the concern of [Green1]. However, it would be wrong to infer from these examples that the corrosive effect of TSM is felt only in the early grades. On the contrary, TSM has warped the school curriculum from K to 12, with no exceptions ([Wu2014a] and [Wu2014c]). Students' fear of *variables* that begins in the middle grades is a case in point, and their well-documented learning difficulties with the *slope of a line* in Algebra I is another. Of course, there are fundamental issues with how TSM treats the laws of (rational) exponents, the relationship between exponential and logarithmic functions, the introduction of axiomatic systems in high school geometry, etc. (see [Wu2014c]). The list goes on and on. If we are serious about building better mathematics teachers, then the first step has to be the eradication of TSM from every phase of mathematics education.

The importance of content knowledge

The emphasis on teachers' content knowledge usually elicits a standard outcry: "knowing math is not all there is to teaching". Because this outcry has gained some currency in the recent past, I would like to take this opportunity to put such reaction in the proper perspective.

First and foremost, it is indeed correct that "knowing math is not all there is to teaching". But the role of content knowledge in the teaching of mathematics must be highlighted, because not only is a firm command of mathematics the *sine qua non* of effective mathematics teaching, but also because the importance of content knowledge seems not to be fully recognized in mathematics education as of 2014. Let us address the former first.

What is being claimed is that sound content knowledge is *necessary* for good mathematics teaching. Part of this claim is the rather innocuous statement that a *math* teacher with inferior content knowledge is *necessarily* an ineffective teacher. In this form, the claim is probably not controversial. Unfortunately, the standard outcry mistakes this claim for one that says a robust content knowledge is *sufficient* for good teaching. The distinction between what is *necessary* and what is *sufficient* is absolutely fundamental in mathematics, and one of the main justifications for requiring all school students to learn mathematics is to help them—through the use of reasoning in mathematics—learn how to reason in the real world. The fact that so many have failed to make this distinction in the critical issue of teacher education is an eloquent testimony to our collective failure in school mathematics education. There is probably no better illustration of the pernicious impact of TSM than this spectacular failure.

It goes without saying that sound content knowledge *alone* is not sufficient for good mathematics teaching because good pedagogy still has a critical role to play. For example, a sensible fifth grade teacher will not begin her explanation of the "division house" using the division-with-remainder of 242 by 16; something like 47 by 3 would be far more appropriate. Above and beyond deep content knowledge, such decisions involve considerations of human communication and the learning trajectory of an average fifth grader.

What is important for the future of mathematics education is to recognize that a firm command of mathematics is *necessary* for good mathematics teaching, in the same way that a firm command of whole numbers is *necessary* for learning fractions. Let us not try to contradict the latter assertion by saying that *knowing whole numbers is not all there is to the learning of fractions.*

The education establishment has had a difficult relationship with the issue of content knowledge in every facet of mathematics education, including teaching, assessment, and evaluation of textbooks. One reason that the issue of teachers' content knowledge is being more vigorously discussed today than (let us say) twenty-five years ago is that there has been a greater participation of working mathematicians in the education debate. Because of their professional training, these mathematicians cannot help but immediately notice many blatant mathematical flaws in school math education and, concomitantly, the inaction of the education community in response to these flaws. Even when "content" has become an acceptable word in *recent* education discussions, there remains the persistent perception that the education community is inured to teachers' content knowledge deficit and, in any case, seems to feel no urgency in replacing teachers' flawed knowledge base (i.e., TSM) with correct mathematical knowledge. For example, there is so far no visible movement from Schools of Education across the land to begin the educational reconstruction of the preservice professional development of mathematics teachers: Is a long-term plan being devised to give these teachers the needed content knowledge? Is any form of collaborative effort with the mathematics departments being actively pursued?[8] If I may put it plainly, the fact that a firm command of *correct* mathematics is *necessary* for good mathematics teaching does not seem to resonate with educators.

Green's article is a case in point. When such a widely read article about mathematics teaching treats the issue of content knowledge in such a cavalier manner, it cries out for damage control. The picture she draws is a hazy one: replace "I, We,

[8]In [Wu2011c] and [Wu2014b], one can find a fairly detailed indication of some of the difficulties that would lie ahead in such an undertaking.

You" with "You, Y'all, We", replace "answer-getting" with "sense-making", and get students to "realize, on their own, why their answers were wrong". It is all about *teaching*. We can imagine the education administrators who succumb to Green's siren song going all out to train their teachers on "teaching". Green's offhand caveat that, "without the right training, most teachers do not understand math well enough to teach it the way Lampert does" will be the first casualty in administrators' mad rush to "teach teaching", especially because Green does not spell out what this "training" might be. Teachers may become more accomplished pedagogically, but with their TSM intact, they will continue to tell their students that $284 \div 16 = 12R2$ and that $3 \div 5$ is part of the definition of the fraction $\frac{3}{5}$. Under the circumstance, how would "You, Y'all, We" make these mathematical errors go away? How would vigorous classroom discussions help students make sense of these errors and realize on their own that their teacher's teaching is wrong? As long as teachers' mathematical knowledge is restricted to TSM, there is nothing we can do that is more fundamental than to provide them with the needed content knowledge.

One of the biggest obstacles that mathematics education faces at present is, indeed, how to produce better teachers, and within this obstacle course, the highest hurdle is the seemingly intractable problem of replacing prospective teachers' TSM knowledge with correct mathematical knowledge. The latter claim is an anecdotal one, based on my fourteen-year experience of doing professional development with

K–12 teachers, but without a doubt, education research should be able to confirm or refute whether teachers' lack of the requisite content knowledge for teaching K–12 is the most dominant factor—as of 2014—in students' nonlearning of mathematics. I believe it is.

The standard reaction from the education establishment to the insistent clamoring for better content knowledge for teachers is that there are many things to attend to in the making of a good teacher: pedagogical strategies, students' mathematical thinking, communication among students, use of technology, equity concerns, content knowledge, etc. Therefore mathematics teachers should learn about all of them *together*, at each step, and not single out any one item. So the thinking goes. However, just like the building of a good football team, one does not simply go out to get some thirty good players and call it a team; rather, one does so with the *top priority* of getting a franchise quarterback so that a team can be built around him. Now, it is elementary that one good quarterback does not a team make. A good quarterback with a porous offensive line would spell disaster, and to win games, excellent defense and special teams are also indispensable. It is undeniable that *every one of the players is important*. Nevertheless, the prevailing conviction among NFL coaches is to get a good quarterback first. We believe that content knowledge is the quarterback on the team of "prerequisites for good *mathematics* teaching."

Green clearly disagrees. It would be wrong to imagine that Green's position on this issue is an outlier; like any good reporter, she faithfully reports what she observes in the world of education. Educators are generally not aware of the pervasive damage perpetrated by TSM on the most fundamental parts of mathematics learning, such as the need for precise definitions in order to carry out any kind of legitimate reasoning, the sharp distinction between a theorem and a definition, etc. An example of the former is the fact that TSM tries to talk about division-with-remainder without first saying precisely what division-with-remainder is; see equation (1) on page 7. As to the latter, TSM makes believe that part of the "meaning" (definition) of a fraction $\frac{m}{n}$ is that it is a "division", $m \div n$. The education literature seems oblivious to the fact that $\frac{m}{n} = m \div n$ is a theorem that must be *explained* (proved) to students in order to clarify their concept of a fraction. Of equal importance is the fact that such an explanation cannot be given until we first give a *definition* of $m \div n$ for *all* whole numbers m and n ($n \neq 0$), because a mathematical explanation only uses concepts whose meaning is already known. (Compare the footnote on page 7.) This kind of transparency is what is needed to help erase students' fear of fractions.

As of 2014, there are a few local and sporadic attempts among universities at teaching prospective teachers correct school mathematics. However, there is as yet no established and widely adopted program in institutions of higher learning for pre-service professional development that aims at systematically correcting prospective teachers' misconceptions accumulated over thirteen years of immersion in TSM. Setting up such a program is without question a tall order. Extra funding for special mathematics classes for this purpose will be necessary, and so will be the willingness of the schools of education and departments of mathematics to collaborate on such a long-term project. But the biggest challenge in this endeavor may be getting enough *competent* mathematicians to take an interest in teaching future teachers. (See pp. 63–73 of [Wu2014b] for a more elaborate discussion of the inherent difficulties.) Sadly, as yet we have no solution to this challenge.

Skilled teaching

A recent article by Mark Thames and Deborah Ball ([Thames-Ball]) looks at the same issue of building better mathematics teachers in a way that complements ours. We differ from [Thames-Ball] in the level of specificity, but there is also some common ground between the two approaches that will be discussed in due course. Thames and Ball identify four elements of high-quality mathematics instruction that they believe will help solve what they call "the mathematics education problem", and the one among the four that they discuss at length is (unsurprisingly) *skilled teaching*. They draw a sharp distinction between *teacher quality* and *teaching quality*:

> Another impediment to progress is the inclination to persist with outdated and refuted ideas about teacher quality, especially with respect to content

> knowledge. . . . The focus tends to be on *teacher* quality, particularly when it comes to teachers' inadequate content knowledge. However, the issue is not teacher quality, but *teaching* quality . . . . If teachers could be selected in ways that were predictive of how well those teachers would teach, then teacher quality could be taken as indicative of the quality of teaching, but because there are currently no effective ways of identifying the characteristics of teachers that will predict whether their teaching will be good, the focus should be on assessing the quality of teaching. ([Thames-Ball], page 34)

We will comment on this passage presently. But to continue with the discussion of [Thames-Ball], the article lists five features that characterize skilled teaching. We paraphrase them as follows:

(A) It focuses on core concepts.

(B) It is culturally and linguistically sensitive.

(C) It gets students to be active and engaged.

(D) It attends to mathematical language and reasoning.

(E) It requires the diagnostic skill to understand whether students are "getting it".

Since the nurturing of skilled teaching requires a strong system of teacher education, the authors recommend five "strategic elements" for designing such a system ([Thames-Ball], page 35):

(1) Build a common mathematics curriculum. [9]

(2) Develop valid and reliable assessments coordinated to the curriculum.

(3) Build a system of supplying skilled teachers to every school to teach that curriculum.

(4) Center teacher licensure and training on practice.

(5) Organize schools to support beginning teachers.

Although [Thames-Ball] uses a vignette to illustrate the fine points of skilled teaching, it says very little overall about how to make the strategic elements (1)–(5) a reality, or what steps it would take to ensure that teachers become *skilled* in the sense of (A)–(E). In particular, it never gets down to the level of discussing the role of content knowledge in skilled teaching. Apparently, Thames and Ball do not regard the flawed body of mathematical knowledge (i.e., TSM) that pervades the school system

[9] While our nation's need of a common mathematics curriculum is beyond dispute, the message on page 36 in [Thames-Ball] should not be misinterpreted to mean that the lack of such a common mathematics curriculum can completely account for teachers' content knowledge deficit at present. There is no doubt that, with such a curriculum in place, prospective teachers' content instruction "would be able to get much closer to assuring that teachers know mathematics and pedagogical practice well enough to deliver mathematics to their students" ([Thames-Ball], page 36). However, the basic mathematics underlying arithmetic, geometry, and algebra is not curriculum-dependent, nor is—for example—the ability to distinguish between a definition or a theorem. I believe teachers' content deficit at this basic level can be traced to the misinformation they get from the TSM imbedded in the K–12 textbooks and in the leading math textbooks for teachers that are used in colleges nationwide.

to be a major issue. Their starting point seems to be that teachers come equipped with a mathematical knowledge base that they can work with, so their focus is on molding this knowledge into an effective educational tool through skilled teaching.

However, we look at the same issue from the trenches and find teachers' knowledge of—and exclusive reliance on—TSM to be so alarming that we are obligated to call for its replacement with correct mathematical knowledge. For example, [Thames-Ball] wants a common mathematics curriculum that is coherent (see strategic element (1) above), but if teachers, educators, and administrators who design this common curriculum only know TSM—as they generally do at the moment—an *incoherent* common curriculum will be the inevitable result. Far from an idle speculation, all the incoherent state mathematics standards [10] in the years 1990–2010 bear witness to the validity of this assertion. Likewise, even if one were to conscientiously design a system to produce skilled teachers (see strategic element (3) above), would teachers skilled in teaching TSM to students be part of the intended design outcome?

Let us go a step further. In their list of five characteristic features of skilled teaching, (A)–(E), we recognize that a robust content knowledge is absolutely essential for (A), (D), and (E). Thames and Ball recognize that too. Regarding (D), for example, they point out that "reasoning is important for student learning and is central to skillful teaching". If the [Thames-Ball] article were to get down to our level

[10] With maybe two or three exceptions.

of specificity, it would have to face—without a doubt—the same dilemma of finding ways to instill in prospective teachers the ability to reason mathematically when they have been essentially denied the opportunity to reason for the first thirteen years of their schooling. Take another example: (A) calls for a skilled teacher to focus on core concepts. But what is a core concept in TSM may not be a core concept in correct school mathematics. TSM considers *order of operations* a core concept, and most teachers believe it too. [11] But do we really want skilled teachers to make a song and dance of *order of operations* and mesmerize their students in the process? With this in mind, the details of the design that promises to deliver skilled teachers now begin to occupy center stage. We await a sequel to [Thames-Ball] that will wade into these details. Given the authors' formidable scholarship, and especially their willingness to recommend specific actions to realize their vision (as in [Thames-Ball]), this sequel cannot come a day too soon.

One cannot fail to notice at this point that the present article is far less ambitious than [Thames-Ball]. We merely try to point out a roadblock (TSM) on the path toward producing *competent* teachers, whereas [Thames-Ball] aspires to build a system of *skilled* teachers. This difference will explain our reaction to the passage from [Thames-Ball] cited on page 25 about *teaching quality*. We believe that so long as TSM rules all the school classrooms and mathematics education research, it will

[11] I am writing from firsthand experience: [Wu2004] was written in frustration after one such encounter in a Content Review meeting of a state assessment panel.

be premature to discuss *teaching quality*. For example, how to judge the *teaching* quality of a lesson that gives very convincing arguments for why equation (2) on page 13 is a correct proof of equivalent fractions? After all, this argument has been in textbooks for several decades; educators have allowed it to stay in the classroom, and generations of students (and teachers) have come to believe its validity. Therefore we are not yet at the stage where we can afford to worry about getting skilled teaching, only *competent teachers* who are in possession of correct content knowledge and will not corrupt students with TSM-type pseudo-reasoning such as equation (2). In other words, we want to ensure that, before considering the fine points of teaching, all *teachers* have a TSM-free version of (correct) mathematical knowledge. After that, the ambitious (and much needed) program of [Thames-Ball] to bring about skilled teaching can take over.

For the purpose of measuring teachers' mathematical understanding (regardless of the minor variations in the curricula), we need a competent assessment system for teacher credentialing. In an age filled with cynicism about assessment, it is a pleasure to report that [Thames-Ball] shares our belief that assessment still has the capacity to do good (see pp. 16–17 of [Wu2012]). In addition, [Thames-Ball] also shares our dim view of the sorry state of professional development for math teachers. Specifically, [Thames-Ball] points to the fact that our math teachers "haven't been prepared to be successful" ([Thames-Ball], page 39; also see the conclusion of

[Thompson-Thompson]). There is perhaps no need to elaborate on this obvious fact except to repeat our earlier statement: "It is somewhat shocking but nevertheless true that the education establishment has failed to provide teachers with the content knowledge they need for teaching mathematics correctly."

Some thirty years ago, Lee Shulman made an insightful observation in his influential address on *pedagogical content knowledge* ([Shulman]) that content was still the "missing paradigm" in research on teaching (ibid., p. 7). We can only hope that, thirty years from now, both mathematicians and educators will have made enough progress along these lines that all will consider such a discussion about mathematical content knowledge to be passé.

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