George M. Bergman
155 Dwinelle Hall

Fall 1999, Math 1B
Final Examination

16 December, 1999
8:00-11:00 AM

1. (40 points, 4 points apiece) Find the following. If an expression is undefined, say so.

(a) $\int \frac{1}{x^4+x^3}\,dx$

(b) $\int_2^\infty x^{-3}e^{x^{-1}}\,dx$

(c) $\int_{-2}^{2} x^{-3}e^{x^{-1}}\,dx$

(d) $\lim_{n\to\infty} (0.999)^n$.

(e) The set of all real numbers p such that $\Sigma_{n=1}^{\infty} (-1)^n/(1+n^p)$ converges.

(f) The Maclaurin series (power series centered at 0) for $x\cos x$.

(g) The Maclaurin series for $\int_0^x e^{t^2}\,dt$ (as a function of x).

(h) $(1+i)^{-6}$

(i) The general solution to the differential equation $y''+4y'+4y = x^2$.

(j) The solution to the differential equation $y'' + y' + y = 1$ satisfying the initial conditions $y(0) = y'(0) = 1$.

2. (24 points, 8 points apiece) Find the following.

(a) The set of real numbers x such that the power series $\Sigma_{n=0}^{\infty} (n^5/8^n)(x-5)^n$ converges.

(b) The general solution to the differential equation $y' - y\tan x = e^{5x}$.

(c) The Maclaurin series for the solution to the differential equation $y'' - xy' - 2y = 0$ satisfying $y(0) = 0$, $y'(0) = 1$.

3. (12 points) For all $a\in(-\infty,\infty)$ and $x\in[0,\pi/2)$, find $\int_0^x (\cos t)^a(\sin t)^3\,dt$. You will find one formula that is valid for all x as above and *almost* all values of the exponent a, and other formulas for one or two special values of a. For full credit, you should obtain formulas covering all values of a, and specify the values of a for which each is valid, showing your work.

4. (12 points) Suppose y_1 is a solution to a second-order linear differential equation $y'' + P(x)y' + Q(x)y = G(x)$, and y_2 is a solution to an equation $y'' + P(x)y' + Q(x)y = H(x)$ with the same left-hand side, but in general a different right-hand side. Prove that y_1+y_2 is a solution to the equation $y'' + P(x)y' + Q(x)y = G(x) + H(x)$.

5. (12 points) Find the general solution of the differential equation $y'' + y = 1/x$ $(x>0)$. The solution will involve integrals that cannot be evaluated in terms of elementary functions, so in your answer, simply show those integrals, but do not attempt to evaluate them.

P. Vojta

Math 1BM Final Exam

Sat 20 May 2000

Some Formulas

1. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
2. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
3. $\sin A \cos B = \frac{1}{2}\left[\sin(A - B) + \sin(A + B)\right]$
4. $\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$
5. $\cos A \cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$
6. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$
7. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$
8. $\displaystyle\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$
9. $\displaystyle\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$
10. Weierstrass substitution: $t = \tan\left(\frac{x}{2}\right)$; $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$.
11. Binomial series:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \qquad \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad (n \ge 1); \qquad \binom{k}{0} = 1$$

12. $\ln(1 - x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \cdots.$
13. $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots.$

1. (35 points) Find:
   (a). $\int_0^2 \sqrt{x^2+4}\,dx$
   (b). $\int_1^e (\ln x)^2 dx$
   (c). $\int_0^{1/2} \frac{\arctan x}{x}\,dx$

2. (12 points) If the curve $y = \sqrt{2x - x^2}$, $0 \le x \le 1$, is rotated about the x-axis, find the area of the resulting surface.

3. (14 points) Describe how one can compute $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ to within 0.01.
   (You do not need to actually carry out the computation, but if your answer involves, say, the n^{th} partial sum, then you should say what n is.)

4. (25 points) Determine whether the following series converge absolutely, converge conditionally, or diverge:
   (a). $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n+1}}$
   (b). $\frac{2}{1} - \frac{1}{2} - \frac{1}{3} + \frac{2}{4} - \frac{1}{5} - \frac{1}{6} + \frac{2}{7} - \dots$

5. (25 points) Determine whether the following series converge absolutely, converge conditionally, or diverge:
   (a). $\sum_{n=1}^{\infty} \left(\arctan\left(7 + \frac{1}{n}\right) - \arctan 7\right)$
   (b). $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

6. (14 points) Find the Maclauren series for $\frac{1}{\sqrt{1-x^2}}$.

7. (20 points) Solve the differential equation
$$y' = \frac{\ln x}{xy + xy^3}\ .$$

8. (20 points) Solve the boundary-value problem $y'' + 2y' + 2y = 0$, $y(0) = 0$, $y(\pi/2) = 1$.

9. (24 points) Find the general solution of the differential equation $y'' + 2y' + y = \frac{e^{-x}}{x}$.

10. (24 points) Find the (series) solution of the initial-value problem
$$y'' - xy' - y = 0 \qquad y(0) = 0\ , \qquad y'(0) = 1\ .$$

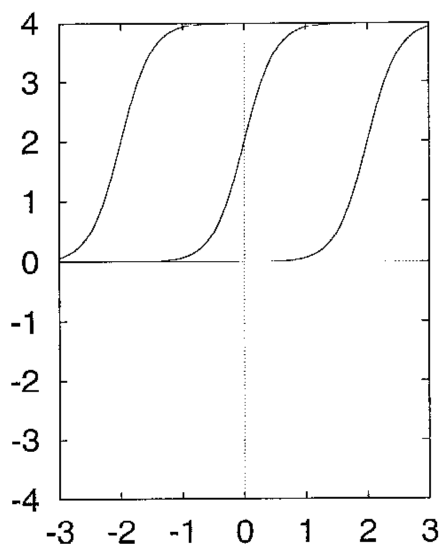
11. (12 points) Match the equation to the graph. You may assume that each graph belongs to one equation.

____ $y' = x^2 + y$ ____ $y' = x^3 + xy$

____ $y' = y(4 - y)$ ____ $y' = x^2 - y$

Suggestion: Do not solve the equations.

A.



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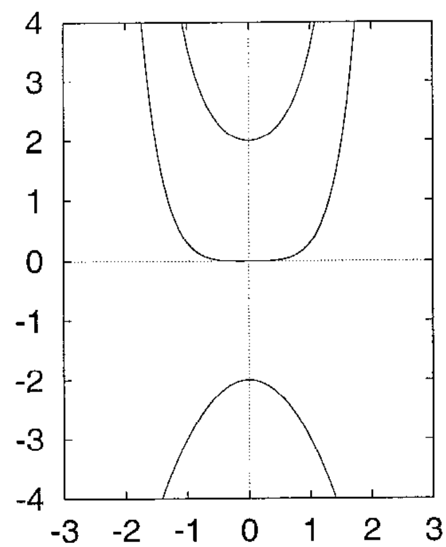

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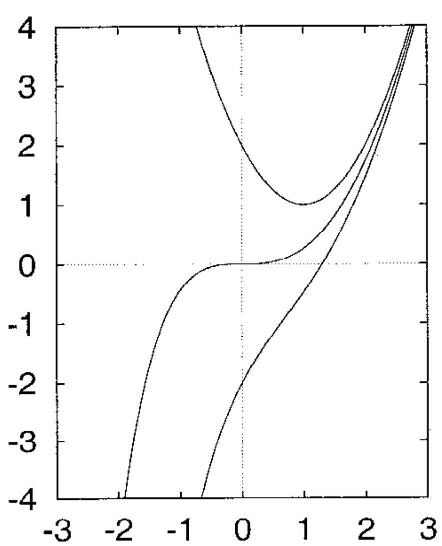

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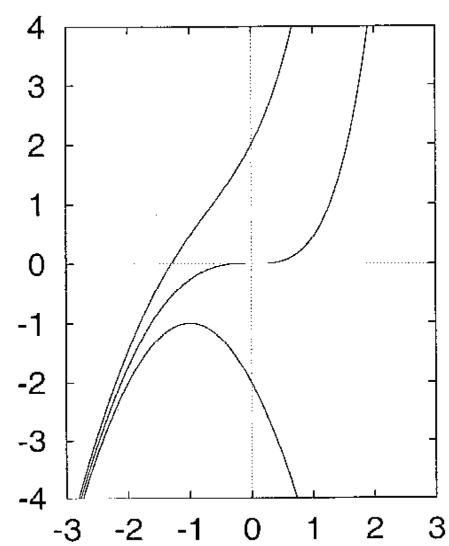

NAME ____________________

STUDENT ID NUMBER ____________________

TA's name or section number ____________________

MATH 1B Final Exam Fall 2001

V.F.R. Jones

There are 450 points altogether.

The first 15 questions are multiple choice, each worth 15 points. Choose the most correct answer to each question and mark the corresponding box in the grid ON THE BACK OF THIS PAGE. Mark only one box per question. No partial credit.

TA use only:

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2	
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4	
5	
6	
Total	

Question	a	b	c	d	e
1					
2					
3					
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12					
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14					
15					

Multiple choice questions:

1) Which of the following is correct for any convergent series $\sum_{n=1}^{\infty} a_n$ with positive terms?

(a) $\displaystyle\sum_{n=1}^{\infty}(a_n+1)$ converges.

(b) $\displaystyle\sum_{n=1}^{\infty} a_n^2$ converges.

(c) $\displaystyle\sum_{n=1}^{\infty} 1/a_n$ converges.

(d) $\displaystyle\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

(e) $\displaystyle\sum_{n=1}^{\infty} \sqrt{a_n}$ diverges.

2) The recurring decimal 0.12121212.... is the rational number

(a)3/11

(b)3/22

(c)1/12

(d)3/25

(e)4/33

3) Which of the following strategies is most likely to succeed to find a particular solution of the differential equation $y'' - y = \dfrac{1}{x}$, for $x \neq 0$?

(a) Try a solution of the form $\dfrac{A}{x} + B$ for some constants A and B.

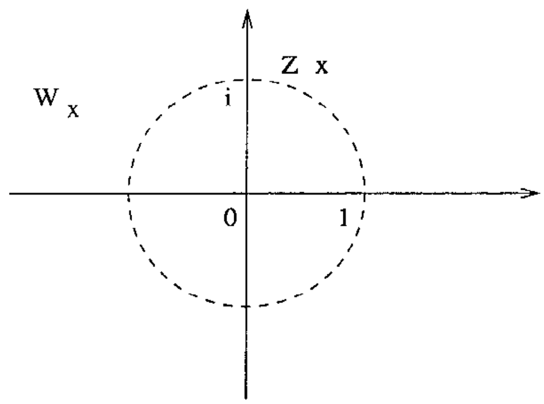
(b) Try a power series solution $\sum_{n=0}^{\infty} a_n x^n$.

(c) Try a solution of the form $v_1(x)e^x + v_2(x)e^{-x}$ for some functions v_1 and v_2.

(d) Try a solution of the form $\frac{A}{x}e^x + \frac{B}{x^2}e^x$ for some constants A and B.

(e) Try a solution of the form $\frac{A}{x}e^x + \frac{B}{x}e^{-x}$ for some constants A and B.

(4) Which of the following is most correct for the complex numbers Z and W marked with x's in the picture of the complex numbers below? (The dashed circle represents the unit circle - that is to say all complex numbers of modulus 1.)
(a)$Z = W + i$
(b)$Z = W^2$
(c)$W = Z^2$
(d)$Z = W - 1$
(e)$Z = 2W$



5)Which of the following is true for any sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = 4$?

(a) There is an $N > 0$ for which $a_n < 2$ for all $n \leq N$.

(b) There is an N for which $|a_n - 4| < 1$ for all $n \geq N$.

(c) $\lim_{n\to\infty}(a_n + a_{n+1}) = \infty$.

(d) For no value of n is a_n bigger than 300.

(e)For any $\epsilon > 0$ there is an N with $|a_n - 4| > \epsilon$ for all $n \geq N$.

6)The integral $\int_2^\infty \frac{1}{x^2 - 6x + 9}$ is
(a)divergent
(b)1/2
(c)1
(d)2
(e)$ln(3)$

7)The general solution to the differential equation $y'' + 2y' + 5y = 0$ is

(a) $c_1e^{3x} + c_2e^{-x}$

(b) $c_1e^{-3x} + C_2e^{x}$

(c) $e^{\sqrt{5}x}(c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}})$

(d) $Ae^{-x}\cos(2x + \phi)$

(e) $c_1xe^{3x} + c_2e^{3x}$

8) Which of the following integrals gives the area of the surface obtained by rotating the curve $y = e^x$ for y between $1/e$ and e about the line $x = 2$?

(a)$2\pi \int_{-1}^{1} (2 - e^x)\sqrt{1 + e^{2x}}\, dx$

(b)$2\pi \int_{1/e}^{e} (e^x - 2)\sqrt{1 + e^{2x}}\, dx$

(c)$2\pi \int_{1/e}^{1} (2 - ln(x))\sqrt{1 + e^{2y}}\, dy$

(d)$2\pi \int_{1/e}^{e} (2 - ln(y))\sqrt{1 + \frac{1}{y^2}}\, dy$

(e)$2\pi \int_{-1}^{1} (x - 2)\sqrt{1 + e^{2x}}\, dx$

9) To integrate the function $\frac{x^3}{x^3 - 1}$ by partial fractions one should try to express it in the form

(a) $1 + \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$

(b)$\frac{A}{x^3} - \frac{B}{x^2} + \frac{C}{x}$

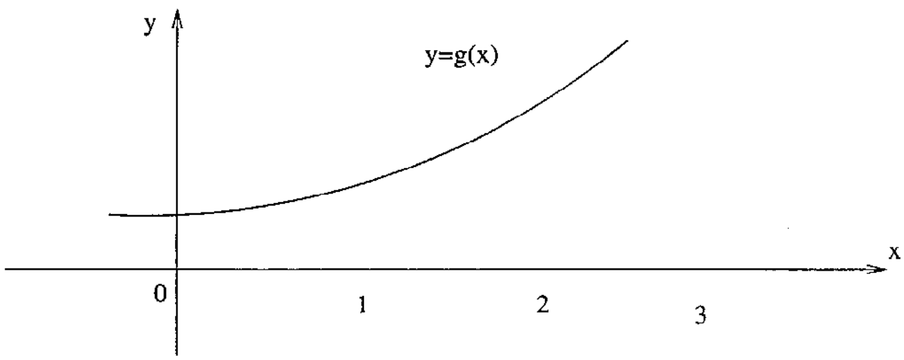
(c)$\frac{A}{x^3} + \frac{B}{x^2} - \frac{C}{x}$

(d)$\frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$

(e)$\frac{A}{x - 1} - \frac{Bx + C}{x^2 + x + 1}$

10) A solution $y = g(x)$ to the equation $dy/dx = f(x, y)$ is given by the curve sketched in the figure below. Which of the following is most likely to be correct concerning the approximation y_1 to $g(1)$ that would be obtained by Euler's method starting at $x = 0$ with step size 0.1?

(a) $y_1 = g(1)$
(b) $y_1 > g(1)$
(c) $y_1 < g(1)$
(d) $y_1 > g(1) + 0.1$
(d) $y_1 > g(1) + 1$



11) The complex number e^{2+i} is equal to

(a) $e^{\sqrt{5}}(\cos(\tan^{-1}(1/2)) + i\sin(\tan^{-1}(1/2)))$
(b) $e^{\sqrt{5}}(\cos(\tan^{-1}(2)) + i\sin(\tan^{-1}(2)))$
(c) $-e^2$
(d) $e^2(\cos 1 + i \sin 1)$
(c) a shoe

12) Consider the differential equation $\frac{dP}{dt} = 5P(1-P)$. Which of the following is correct?
(a) If $P(0) = 0.5$, then $P(t) < 1$ for all positive t.
(b) If $P(0) = 0.5$ then $P(1) = 1$.
(c) If $P(2) = 6$ then $P(t)$ is increasing when $t = 7$.
(d) If $P(0) = 0.5$ then $P(t) = 0.5$ for all t.
(e) Squash is the most popular sport in the United States.

13) Which of the following is correct concerning a linear homogeneous second order differential equation with constant coefficents?
(a) The boundary value problem always has a unique solution.
(b) The initial value problem always has a unique solution.
(c) The initial value problem may not have a solution but if one exists it is unique.
(d) The boundary value problem may not have a solution but if one exists it is unique.
(e) The boundary value problem always has a solution but there may be infinitely many different ones.

14) Suppose the sequence (a_n) is such that $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ converges. It then follows that:
(a)$lim_{n\to\infty} a_n = 0$

(b)$\sum_{n=1}^{\infty} a_n$ converges absolutely.

(c)$\sum_{n=1}^{\infty} a_n$ converges.

(d)$\sum_{n=1}^{\infty} \frac{a_n}{4^n}$ converges absolutely.

(e)$\sum_{n=1}^{\infty} a_n$ diverges.

15) $lim_{x\to 0} \quad \dfrac{e^x - \cos^2 x}{\sin^2 x - ln(1-x)}$ equals
(a) ∞
(b) 0
(c) 1
(d) -1
(e) 2

The next six questions are *not* multiple choice. Show your reasoning and give your answers in the space provided.

1.(50 points)

Find two linearly independent solutions to the differential equation

$$y'' + x^2 y = 0$$

2.(40 points)
A vat has a volume of 100 liters. It initially contains 50 liters of pure water. Brine with a concentration of 1 gram per liter begins to flow into the vat at a rate of 2 liters per minute. The mixed solution escapes through a leak at a rate of 1 liter per minute. How much salt is there in the vat when it begins to overflow?

3.(30 points) Solve the initial value problem

$$y'' + y' + y = 0, \quad y(0) = 0, y'(0) = 1$$

4. (a)(20 points) Is the series $\sum_{n=3}^{\infty} \frac{1}{n^2 - n}$ convergent? If so find its sum.

(b)(10 points) Suppose f is a continuous positive strictly decreasing function for $x \geq 1$ and $a_n = f(n)$.

By drawing a picture, rank the following in increasing order:

$$\int_1^6 f(x)\, dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i.$$

5)(30 points) Find a particular solution of the differential equation

$$y'' - y' - y = e^{2x} + 1$$

6)(45 points-15 each) Evaluate the following integrals:

(i) $\int_0^1 x\sqrt{1-x^4}dx$

(ii) $\int_1^\infty \frac{1}{x^{\sqrt{2}}}dx$

(iii) $\int x^3 e^{x^2}\,dx$

Mathematics 1BM
Final Exam

Professor K. A. Ribet
May 17, 1997

10 Evans
8–11 AM

Your Name: ______________________________ TA: __________________________

This booklet comprises a cover sheet, eight pages of questions, and a formula sheet at the back. Please check that your booklet is complete and write your name on this cover sheet and the question sheets. Some of the questions have low point values; leave them until the end if you can't do them right away. As usual:

- You need not simplify your answers unless you are specifically asked to do so.
- It is essential to write legibly and *show your work*.
- If your work is absent or illegible, and your answer is not perfectly correct, then no partial credit can be awarded.
- Completely correct answers which are given without justification may receive little or no credit.
- During this exam, you are not allowed to use calculators or consult your notes or books.

Problem	Maximum	Your Score
1	8	
2	8	
3	9	
4	7	
5	19	
6	8	
7	9	
Total	**68**	

At the conclusion of the exam, hand in this exam paper to your TA.

Your Name: ______________________________

1a *(3 points)*. Express as a definite integral the length of the curve $y = \sin x$, $0 \le x \le \pi/2$.

1b *(5 points)*. Decide whether $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdots (3n+2)}$ converges absolutely, converges conditionally, or diverges.

Your Name: ______________________________

2a *(4 points)*. Find $f^{(666)}(0)$ if $f(x) = \sec\left(x^{333}\right)$.

2b *(4 points)*. A tank contains 500 L of pure water. Brine that contains 0.05 kg of salt per liter flows into the tank at the rate of 8 L/min. The solution is kept thoroughly mixed and drains from the tank at the rate of 8 L/min. Let $y(t)$ be the amount of salt in the solution at time t, measured in kg; time is measured in minutes. What differential equation is satisfied by y? Without solving this equation, guess the value of $\lim_{t\to\infty} y(t)$.

Your Name: ______________________________

3a *(4 points)*. Find one solution to $y'' + 5y' + 6y = \cos x$.

3b *(5 points)*. Use power series methods to solve the initial-value problem:
$y'' + xy' - 2y = 0$, $y(0) = 1$, $y'(0) = 0$.

Your Name: ______________________________

4a *(3 points)***.** Let $y = f(x)$ be the solution to the initial-value problem $y' = y^2 + 2y$, $y(2) = 1$. Without finding a formula for $f(x)$, compute $f'(2)$ and $f''(2)$.

4b *(4 points)***.** Find the equation of the curve which passes through $(1, 1)$ and which is orthogonal to the family of curves $2x^2 + y^2 = k$.

Your Name: ______________________________

Evaluate each of the following integrals whenever possible (beware of divergent improper integrals):

5a *(5 points)***.** $\displaystyle\int_0^2 \sqrt{2x - x^2}\, dx.$

5b *(5 points)***.** $\displaystyle\int_{-5}^{5} \frac{4\, dt}{t^2 - 2t - 3}.$

Your Name: ______________________________

5c *(5 points)***.** $\int \cos\sqrt{x}\,dx$.

5d *(4 points)***.** $\int_e^\infty \frac{dx}{x(\ln x)^2}$.

Your Name: ______________________________

6a *(4 points)*. Test for convergence: $\displaystyle\sum_{n=0}^{\infty} \sqrt{\frac{2^n + 3^n}{6^n}}$.

6b *(4 points)*. How many terms of the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ are required to approximate the actual sum with an error of less than 5×10^{-4}?

Your Name: ______________________________

7a *(4 points)***.** Using a series representation for $(1+x)^{1/2}$, evaluate
$\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{\big(1\cdot 3\cdot 5\cdots(2n-3)\big)\,9^n}{n!(32)^n}$.

7b *(5 points)***.** Solve the differential equation $xy' = 2\sqrt{xy} - y$.

MATH 1B, Lecture 3
Sarason

May 10, 1996

FINAL EXAMINATION

SIGNATURE____________________

PRINTED NAME____________________

SID Number____________________

TA____________________

Section time____________________

Closed book except for crib sheet.

No calculators.

SHOW YOUR WORK. Cross out anything you have written that you do not want the grader to consider.

The points for each problem are in parentheses. Perfect score = 145.

1	
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11	
Total	
Grade points	

1. (10) Determine whether the improper integral

$$\int_1^\infty \frac{d}{dx}(e^{-x} \ln x)\, dx$$

converges, and if it does, find its value.

Name________________________________

2. (10) Perform the integration: $\int \frac{x^2+2x-1}{x(x^2+1)}\, dx$

Name______________________________

3. (10) Perform the integration: $\int \cos^3 x \sin^3 x \, dx$

Name______________________________

4. (15) Find the area of the surface of revolution one obtains by revolving about the x-axis the portion of the curve $y^2 = e^x + 1$ that lies between the lines $x = 0$ and $x = 1$.

Name________________________________

5. (15) Do the following infinite series converge? Explain your answers. In particular, make clear which convergence tests you are using.

(a) $\sum_{n=1}^{\infty} (n^{-2} + 2^{-n})$

(b) $\sum_{n=1}^{\infty} e^{-(n^{1/2})}$

Name________________________________

6. (15) Let k be a real number. (a) Find the n-th Taylor coefficient of the function $f(x) = (1+x)^k$ about the point $x = -\frac{1}{2}$ $(n = 0, 1, 2, \cdots)$. (b) Determine the radius of convergence of the Taylor series of f about the point $x = -\frac{1}{2}$ (same f as in part (a)).

Name________________________________

7\. (10) Find the general solution of the differential equation

$$(x^2+4x+5)^{1/2}y' = y^2.$$

Name________________________________

8. (15) Let a vibrating system consist of a weight attached to a spring. Assume that, in suitable units, the mass of the weight is 1, the damping constant is 2, and the spring constant is 2. Assume the system is subjected to the driving force $F(t) = F_0 \sin t$ (F_0 = constant), and that it starts from rest in its equilibrium position at time $t = 0$. Find the formula giving the displacement as a function of time.

Name________________________________

9. (10) What is the 16th derivative of the function $f(x) = e^{(-x^2)}$ at the origin?

Name________________________________

10. (15) Find a numerical approximation to

$$\int_0^{1/10} \cos(x^2)\, dx$$

accurate to within 10^{-10}. Explain your method. (NO CALCULATORS)

Name___________________________

11. (20) (a) Use the method of power series to solve the initial value problem

$$(1-x^2)y'' - 6xy' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

(b) What is the radius of convergence of the series you obtain?

(c) Can you sum the series?