George M. Bergman
155 Dwinelle Hall

Fall 2008, Math 1B
Solutions to the **Second Midterm**

24 Oct., 2008
3:10-4:00 PM

1. (48 points: 12 points each.) Compute the following. If an expression is undefined, say so.

(a) $\int_{-1}^{1} x^{-2}\,dx$ *Answer: Undefined*

(b) $\lim_{n\to\infty} (2^n+3^n)/4^n$ *Answer:* 0

(c) $\Sigma_{n=0}^{\infty} (2^n+3^n)/4^n$ *Answer:* 6

(d) The first four terms (i.e., the constant term through the x^3 term) of the Maclaurin series for $(1+x+x^2)^{-1}$ *Answer:* $1-x+x^3$. (*You can also show a term* "$+0x^2$", *but you don't need to.*)

2. (36 points, 12 points each) For each of the items listed below, either *give an example* with the property stated, or give a brief reason why *no such example exists.* (If you give an example, you are *not* asked to show that it has the asserted property.)

(a) A power series whose interval of convergence is the open interval $(0, 100)$.

Answer: $\Sigma_{n=0}^{\infty} (x-50)^n/50^n$. (*For this and the other parts of this question, there are other examples you could have given than the ones shown here.*)

(b) A bounded sequence of real numbers $a_0, a_1, a_2, \ldots,$ which does not converge.

Answer: $a_n = (-1)^n$

(c) A power series which converges at $x=-1$ but at no other point. *Answer:* $\Sigma_{n=0}^{\infty} n!\,(x+1)^n$.

3. (16 points) Suppose $a_0, a_1, a_2, \ldots, a_n, \ldots$ are real numbers, and that r and s are nonzero real numbers with $|r| < |s|$. Prove that if the sequence $|a_0|, |a_1 s|, |a_2 s^2|, \ldots, |a_n s^n|, \ldots$ is bounded, then the series $\Sigma_{n=0}^{\infty} a_n r^n$ converges absolutely.

This was part of a lemma that I proved in class, in proving the theorem on radii of convergence of power series; so of course you cannot quote that lemma, or the theorem on radii of convergence, in proving this. But you can use any of the earlier results we had concerning sums of series.

Though remembering the lecture might help you do this problem, it is not necessary. The proof is a straightforward application of earlier results in the text.

Answer: The boundedness of the sequence of terms $|a_n s^n|$ *means that there exists a constant* $C \geq 0$ *such that* $|a_n s^n| < C$ *for all* n. *Now* $|a_n r^n| = |a_n s^n|\,|r/s|^n \leq C\,|r/s|^n$, *so the series* $\Sigma_{n=0}^{\infty} |a_n r^n|$ *has each term* $\leq$ *the corresponding term of the series* $\Sigma_{n=0}^{\infty} C\,|r/s|^n$. *The latter is a geometric series with ratio* $|r/s| < 1$, *and so converges.*

So $\Sigma_{n=0}^{\infty} |a_n r^n|$ *converges by the comparison test, which says that* $\Sigma_{n=0}^{\infty} a_n r^n$ *converges absolutely.*

George M. Bergman
961 Evans Hall

Fall 1999, Math 1B

Second Midterm

26 October, 1999
8:10-9:30 AM

1. (36 points, 6 points apiece) Find the following. If an expression is undefined, say so.

(a) $\Sigma_{n=2}^{\infty} 5^{-n}$.

(b) $\Sigma_{n=1}^{\infty} (2^n + 2^{-n})$.

(c) The set of all real numbers p such that $\Sigma_{n=2}^{\infty} n^{-1}(\ln n)^p$ converges.

(d) The Maclaurin series for 2^x.

(e) The Taylor series for $1/x^2$ centered at $x = 1$.

(f) The solution to the differential equation $xy' = (x+1)\,y$ satisfying the initial condition $y(1) = 1$.

2. (16 points) Let a and b be real numbers. Prove that $\Sigma_{n=1}^{\infty} (\frac{a}{n} + \frac{b}{n+1})$ converges if and only if $a+b=0$.

3. (30 points, 6 points apiece) For each of the items listed below, give either *an example*, or a brief reason why *no example exists*. (If you give an example, you are *not* asked to show that it has the asserted property.)

(a) A power series $\Sigma_{n=0}^{\infty} a_n (x-1)^n$ with radius of convergence 3.

(b) A power series $\Sigma_{n=0}^{\infty} a_n x^n$ which converges for all $x \geq -1$ and no other x.

(c) A power series $\Sigma_{n=0}^{\infty} a_n (x-2)^n$ which converges for all real numbers x.

(d) A series $\Sigma_{n=1}^{\infty} a_n$ which is convergent but not absolutely convergent.

(e) Two series $\Sigma_{n=1}^{\infty} a_n$ and $\Sigma_{n=1}^{\infty} b_n$ such that $a_n \geq b_n$ for all n, and $\Sigma_{n=1}^{\infty} a_n$ converges, but $\Sigma_{n=1}^{\infty} b_n$ diverges.

4. (18 points) (a) (7 points) Find the first three terms (i.e., the constant, linear, and square terms) of the Taylor series for $\ln x$ centered at $x=2$.

(b) (11 points) Prove using the formula for the remainder (''Taylor's Formula'') that for all x in the interval $[1.5, 2.5]$, the sum of the above three terms approximates $\ln x$ to within $1/81$.

Second Midterm Exam—60 points

1 *(6 points each)*. Find the sums of the following series:

a. $\displaystyle\sum_{n=1}^{\infty}\left[\ln\left(\frac{n+1}{n}\right)-\ln\left(\frac{n+2}{n+1}\right)\right].$

b. $\displaystyle\sum_{n=3}^{\infty}\left(3\left(\frac{3}{4}\right)^{n}-4\left(-\frac{1}{2}\right)^{n+1}\right).$

2 *(7 points each)*. Determine whether each series is divergent, conditionally convergent, or absolutely convergent:

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n+1)!-n!}{(n+2)!-(n+1)!}$

b. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}\tan\left(\frac{\pi}{4}+\frac{1}{n}\right).$

3 *(5 points each)*. Decide whether the following infinite series converge or diverge:

a. $\displaystyle\sum_{n=1}^{\infty}\left(1-\cos\frac{1}{n}\right).$

b. $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^{n^2}.$

4a *(6 points)*. Consider the curve with polar equation $r=1+2\cos(\theta)$. Express in terms of definite integrals the area inside the small loop. [The problem was accompanied by a small graph of the curve, produced by Mathematica.]

4b *(6 points)*. Find the slope of the line tangent to the polar curve $r=1+2\cos(\theta)$ at the point with $\theta=\pi/2$.

5a *(5 points)*. Express in terms of definite integrals the length of the entire curve $r=1+2\cos(\theta)$ (both loops).

5b *(7 points)*. Suppose that $f(x)=\displaystyle\sum_{n=0}^{\infty}a_nx^n$ for all x such that $|x|\leq\frac{1}{2}$. What function has Taylor series $\displaystyle\sum_{n=2}^{\infty}n^2a_nx^n$?

Department of Mathematics, University of California, Berkeley

Math 1B

Alan Weinstein, Spring 2002

Second Midterm Exam, Thursday, April 11, 2002

Instructions. Be sure to write on the front cover of your blue book: (1) your name, (2) your Student ID Number, (3) your GSI's name (Tathagata Basak, Tameka Carter, Alex Diesl, Clifton Ealy, Peter Gerdes, John Goodrick, Matt Harvey, George Kirkup, Andreas Liu, Rob Myers, or Kei Nakamura).

Read the problems very carefully to be sure that you understand the statements. Show all your work as clearly as possible, and circle each final answer to each problem. **When doing a computation, don't put an "=" sign between things which are not equal. When giving explanations, write complete sentences. Remember: if we can't read it, we can't grade it.**

1. [8 points] Find the general solution of the differential equation $dy/dx = x^2y^2$, and find the particular solution for which $y = 1$ when $x = 1$.

2. [5 points] For which values of p is the series $\sum_{n=1}^{\infty} \sin(1/n)n^p$ convergent?.

3. [5 points] Show that absolute value $|\sin x - x - x^3/6|$ is less than $1/100$ for all x in the interval $(-1, 1)$.

4. [9 points] For each of the following statements, tell whether it is true or false, and give a justification for your answer. In particular, the falsity of an "if/then" statement should be justified by a counterexample. (A correct T or F gets no credit without a correct justification.)

(A) If the sequence $\{a_n\}$ converges to L, then a_{100} is closer to L than a_{99} is.

(B) If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then the sequence $\{a_n\}$ is also convergent.

(C) If $a_n \to 0$ as $n \to \infty$, then the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent.

5. [11 points] The velocity $v = \frac{dx}{dt}$ of a vehicle on a experimental highway is controlled by signals transmitted from the roadway so that, when the vehicle is at the position x miles from the beginning of the highway, its velocity is $20x + 10$ miles per hour.

(A) Find the position of the vehicle as a function of t if it starts at $x = 0$ when $t = 0$.

(B) How long does it take the vehicle to travel from $x = 0$ to $x = 3$ if it starts at $x = 0$ when $t = 0$?

(C) How long does it take the vehicle to travel from $x = 0$ to $x = 3$ if it starts at $x = 0$ when $t = 1$?

6. [7 points] Find an explicit formula for the function of x represented by the power series $\sum_{n=0}^{\infty}(3^n + 1)x^n$. What is the radius of convergence of the series?

MATH 1B, Lecture 3
Saranson

March 14, 1996

MIDTERM EXAMINATION

Name (Printed)____________________

Signature____________________

TA____________________

Section time____________________

1	
2	
3	
4	
5	
6	
Total	
Grade points	

Closed book. No calculators.

SHOW YOUR WORK. Cross out anything you have written that you do not want the grader to consider.

The points for each problem are in parentheses. Perfect score = 65.

1. (5) Evaluate the limit: $\lim\limits_{n\to\infty} n^2 \ln(1+\frac{1}{n^2})$

Name______________________________

2. (15) Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series whose n-th partial sum is given by $s_n = 1 + \frac{(-1)^{n+1}}{n+1}$.

(a) What is a_n? In particular, what are a_1, a_2, a_3?

(b) Does the series converge? If so, what is its sum? Explain.

(c) Does the series converge absolutely? Explain.

Name______________________________

3. (15) Determine whether the following series converge or diverge. Explain your reasoning. Be sure to make clear which convergence test or tests you are using.

(a) $\sum_{n=1}^{\infty} e^n e^{(-e^n)}$

(b) $\sum_{n=1}^{\infty} \ln(1 + \frac{1}{n^2})$

Name______________________________

4. (15) Find the radii of convergence of the following power series. (Do not check endpoints.) Justify your answers.

(a) $\sum_{n=0}^{\infty} 2^{4n}(x-3)^{3n}$

(b) $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^{3n}$

Name______________________________

5. (10) Write down the first three nonvanishing terms in the Maclaurin series of the function $f(x) = x^2(1-x^2)^{-1/3}$. Do the same for the functions f' and f''.

6. (5) Let $\{c_n\}_{n=0}^{\infty}$ be a convergent sequence with a nonzero limit L. Find the radius of convergence of the power series $\Sigma_{n=0}^{\infty} c_n x^n$. Explain your reasoning.

Name ____________________

TA ____________________

Math 1B — Second Midterm
V.Jones, Spring 1998

150 points total. The first 5 questions are Multiple Choice.
For each question mark an $\times$ in the most correct place
in the grid below. No partial credit for 1–5.
Questions 6, 7 and 8 are not multiple choice.

	a	b	c	d	e
1					
2					
3					
4					
5					

MC ________

6 ________

7 ________

8 ________

Math 1B Midterm

Multiple Choice Questions. Each multiple choice question worth 15 points.

1. Consider the series $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n}$. Which of the following is correct?

   a) The series converges absolutely.

   b) The series converges but not absolutely.

   c) The series does not converge because the n^{th} term does not tend to zero.

   d) The series does not converge, by the ratio test.

   e) The series does not converge, by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$.

2. Suppose numbers a_n, for $n \geq 0$, are given such that $\sum_{n=0}^{\infty} 2^n a_n$ converges. Which of the following is a consequence?

a) $\lim_{n\to\infty} a_n = 0$

b) $\lim_{n\to\infty} a_n = 1$

c) $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$ converges

d) $\sum_{n=0}^{\infty} a_n$ diverges

e) $\lim_{k\to\infty} \left(\sum_{n=0}^{k} 2^n a_n \right) = 0$

3. The MacLaurin series for $\sin^2 x$ is

a) $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$

b) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

c) $x + \frac{x^5}{5!} + \frac{x^9}{9!} + \frac{x^{13}}{13!} + \dots$

d) $1 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots$

e) $\frac{1}{2}\left\{ \frac{4x^2}{2!} - \frac{16x^4}{4!} + \frac{64x^6}{6!} - \frac{256x^8}{8!} + \dots \right\}$

4. Which of the following is an example of a sequence with $\lim_{n\to\infty} a_n = -\infty$?

a) $a_n = (-1)^n n$

b) $a_n = (-1)^{n^2} n^2$

c) $a_n = \frac{1}{n}(e^n - e^{n^2})$

d) $a_n = \dfrac{n \ln(n+5)}{(n+1)(n+5)}$

e) $a_n = \cos n$.

5. Let $J_0(x)$ be the Bessel function $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$. Which of the following is correct?

a) $x^2 \dfrac{d^2 J_0}{dx^2} + x \dfrac{dJ_0}{dx} + x^2 J_0 = 0$

b) $J_0(x) = e^x \cos(\ln x)$.

c) $\lim_{x\to 0} J_0(x) = 2$

d) $\int_0^t J_0(x)dx = J_0(t)$

e) $\dfrac{dJ_0}{dx} = J_0$

Longer Questions

6. *(15 pts)* (a) Find the MacLaurin series for $\dfrac{x^2}{(1-x)^2}$.

(15 pts) (b) Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} n\left(\frac{1}{2}\right)^n$.

7. *(15 pts)* Consider the series $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots$. What is the last term of the series that would have to be included to obtain $\pi/4$ to within 0.01. Justify your answer.

8. For each of the following series, say whether it converges absolutely, conditionally, or diverges. Give reasons.

(15 pts) (i) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{n}{n+50}$

(15 pts) (ii) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$