Theorem 3 page 449

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) the series converges only when $x=a$. $(R=0)$

(ii) the series converges for all x. $(R=\infty)$

(iii) there is a positive number R such that

(a) the series converges for $|x-a|<R$

(b) " " diverges for $|x-a|>R$

proof: Let R be the least upper bound of the set of all $r \geq 0$ such that the sequence $|c_n|r^n \to 0$ as $n \to \infty$.

This set is non-empty since $r=0$ satisfies $|c_n|r^n \to 0$. If the set is unbounded, i.e. if there are arbitrarily large values of r such that $|c_n|r^n \to 0$ as $n \to \infty$, then we define $R=\infty$. Otherwise, the completeness axiom (see page 418) guarantees that R exists and is finite.

cases (ii) and (iii a): suppose $|x-a|<R$ (guaranteed in case ii where $R=\infty$)

- must show $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges
- since $|x-a|$ is not an upper bound for the set (R is the least upper bound) there is an $r>|x-a|$ such that $|c_n|r^n \to 0$ as $n \to \infty$.

- convergent sequences are bounded, so there is a constant C such that $|c_n| r^n \leq C$ for all n.

- $$\sum_{n=0}^{\infty} |c_n(x-a)^n| = \sum_{n=0}^{\infty} |c_n| r^n \frac{|x-a|^n}{r^n} \leq C \sum_{n=0}^{\infty} \left(\frac{|x-a|}{r}\right)^n$$

comparison test

convergent geometric series

- series is absolutely convergent, hence convergent,

cases (i) and (iiib): suppose $|x-a| > R$

- must show $\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges

- $|x-a|$ does not belong to the set as R is an upper bound for the elements in the set

- so $|c_n||x-a|^n$ does not converge to zero

- neither does $c_n(x-a)^n$ (if it did, so would $|c_n||x-a|^n$)

- $\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges