Reading #29, to be read for November 12

1. A USEFUL PRINCIPLE IN SOLVING DIFFERENTIAL EQUATIONS.

Consider a first-order differential equation

(1) $$y' = F(x,y).$$

As we have seen, this is in general satisfied, not just by one function $y = f(x)$, but by a *family* of functions. A case of this phenomenon that we are long familiar with is when (1) has the form

(2) $$y' = F(x).$$

Then we know that the family of solutions has the form

(3) $$y = G(x) + C,$$

where $G(x)$ is any antiderivative of $F(x)$, and C ranges over all real numbers. Since we have learned many techniques of integration (finding antiderivatives), it is helpful when we can reduce the solution of other sorts of differential equations to that process. One situation where we have seen we can do so is when the given equation is separable. Another very wide class of cases is based on paying attention to how the solutions of our given equation relate to one another. Here is the general principle.

(4) *Given a differential equation* (1)*, examine the way different solutions must be related, and look for a* ***change of variables*** *that will turn the solutions of* (1) *into a family of functions related to one another simply by addition of constants. After this change of variables, the differential equation* (1) *will reduce to one of the form* (2)*, and hence can be solved by integration.*

In the next two sections, I will give two examples of this principle, which together lead to a standard technique for solving first-order *linear* differential equations.

2. FIRST ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.

A first-order differential equation (1) is said to be **linear** if $F(x,y)$ is linear as a function of y; in other words, if the equation has the form

(5) $$y' = P(x)\,y + Q(x).$$

Note that in this equation, the terms y' and $P(x)\,y$ are of degree 1 in y and its derivative, while $Q(x)$ is of degree 0 in those variables. An equation is called **homogeneous** in a set of variables if all the terms have the same degree in those variables. Thus, (5) is homogeneous in y and its derivative only if $Q(x) = 0$; i.e., when the equation has the form

(6) $$y' = P(x)\,y.$$

In this section we will see how to solve equations (6). In the next section we will climb on the shoulders of that result, and solve general equations (5).

I claim that the solutions to (6) are

(7) $$y = c\,e^{\int P(x)\,dx},$$

for real numbers c; that is, if A is any antiderivative of $P(x)$, then the solutions to (6) are the functions

(8) $$y = c\,e^{A(x)}.$$

It is easy to check that these functions satisfy (6); but how would we discover this solution? Here are three independent approaches, each of which helps us in a different way to think about differential equations generally.

Motivation #1. Our general principle (4) says that we should look at how solutions to (6) are related. Clearly, if $y = f(x)$ is a solution to (6), then so is $y = c f(x)$ for every real constant c. Hence (4) says that we should look for a change of variables that will turn solutions related in this way into solutions related by an additive constant. Multiplying a function by a positive constant corresponds to adding a constant to its logarithm; so let us make the substitution

(9) $$u = \ln y; \quad \text{equivalently,} \quad y = e^u.$$

When we put this into (6), it becomes $u' e^u = P(x) e^u$, which simplifies to $u' = P(x)$. This is, as we hoped, a differential equation of the form (2), hence we can solve it by integration, getting $u = \int P(x)\, dx + C$, i.e., $A(x) + C$ for $A(x)$ an antiderivative of $P(x)$. Substituting this into the right-hand equation of (9), we get (8), where $c = e^C$.

Actually, this only gives the case of (8) where c is positive, since our substitution $u = \ln y$ only makes sense for positive-valued functions y. However, once we have found this family of solutions, it is easy to see that zero and negative values of c also give solutions to (6). So if we simply think of our general principle (4) as a guide, it has indeed guided us to the solution (8).

Assuming the domain of definition of our functions is an interval (possibly infinite), (8) in fact gives all solutions to (6). To see this, let $z = e^{A(x)}$ and let y be any other solution to (6). We want to show that y/z is constant. If we differentiate y/z, and use the fact that both y and z are solutions to (6), we get

$$\begin{aligned}(y/z)' &= (y'z - yz')/z^2 \\ &= ((P(x)\,y)\,z - y(P(x)\,z))/z^2 \\ &= 0/z^2 = 0.\end{aligned}$$

So y/z is indeed a constant c, so $y = cz = c\,e^{A(x)}$.

Let us now look at a different way we could have discovered the solution (8) of (6).

Motivation #2. Notice that the equation (6) is *separable*, as defined on p.398 of our text. Hence following the method shown there, we divide by the function of y appearing on the right-hand side of (6), in this case y itself, getting

(10) $$y'/y = P(x),$$

and integrate both sides of this equation, getting

(11) $$\ln y = \int P(x)\, dx + C.$$

To get y we exponentiate, which again gives (7). (The same comments about negative c and uniqueness given in Motivation #1 apply here.)

Here is a brief sketch of a third avenue leading to (8):

Motivation #3. Assume we want to solve (6) on some interval $[a, b]$. We may subdivide $[a, b]$ using points $a = x_1 < x_2 < \ldots < x_{n+1} = b$, and think of $P(x)$ as approximately constant on each interval $[x_i, x_{i+1}]$, with some constant value $P(x_i^*)$ $(x_i \le x_i^* \le x_{i+1})$. Then on that interval, (6) can be approximated by the equation $y' = P(x_i^*)\,y$, for which we know the general solution is any constant times the exponential function $e^{P(x_i^*)\,x}$ (§3.4 of our text). Hence, as x moves from x_i to x_{i+1}, the value of y will be multiplied by approximately $e^{P(x_i^*)\,\Delta x_i}$, where $\Delta x_i = x_{i+1} - x_i$. So as x goes all the way from a to b, y is multiplied by approximately

$$e^{P(x_1^*)\,\Delta x_1} \cdot e^{P(x_2^*)\,\Delta x_2} \cdot \ldots \cdot e^{P(x_n^*)\,\Delta x_n}$$

$$= \ e^{\Sigma_i P(x_i^*)\,\Delta x_i}.$$

As our subdivision of $[a, b]$ becomes finer and finer, this should approach $e^{\int_a^b P(t)\,dt}$. Replacing the upper index of integration by the variable x to get a function, we see that it will have the form (7).

The above discussion is too sketchy to be a proof; but again, it guides us to the solution (7), which we can then verify by substituting it into the given equation.

Here are some examples of this method.

Example 1. Solve $y' = y \sin x$. Find the particular solution for which $y = 1$ when $x = 0$.

Solution. An antiderivative of $\sin x$ is $-\cos x$, so (7) gives the general solution $y = c\,e^{-\cos x}$.

To find the particular solution with $y = 1$ when $x = 0$, we substitute $x = 0$, $y = 1$, getting $1 = c\,e^{-\cos 0} = c\,e^{-1}$. So $c = e$, so the particular solution is $y = e \cdot e^{-\cos x} = e^{1-\cos x}$.

Example 2. Solve $y' = -y\,x$.

Solution. An antiderivative of $-x$ is $-x^2/2$, so the expression (7) gives $y = c\,e^{-x^2/2}$.

Example 3. Solve $y' = r\,y/x$, where r is any real number.

Solution. An antiderivative of r/x is $r \ln |x|$, so (7) gives $y = c\ e^{r \ln |x|} = c\,|\,x^r\,|$.

3. THE NONHOMOGENEOUS CASE.

We are now ready to tackle the more general equation (5),

$$y' \ = \ P(x)\,y \ + \ Q(x).$$

Our general principle (4) says that we should look at the relationship among solutions to this equation. I claim that if $f(x)$ is one such solution, and $g(x)$ is any nonzero solution to the corresponding *homogeneous* equation (6),

$$y' \ = \ P(x)\,y,$$

(which we learned how to solve in the preceding section), then for every real constant c, the function $f(x) + c\,g(x)$ is again a solution to the nonhomogeneous equation (5).

To see this, let $y_0 = f(x)$ and $y_1 = g(x)$, let c be any constant, and let us write $y_2 = y_0 + c\,y_1$, the function we want to show is also a solution to (5). We calculate

$$\begin{aligned} y_2' \ &= \ y_0' + c\,y_1' \\ &= \ P(x)\,y_0 \ + \ Q(x) \ + \ c\,P(x)\,y_1 \\ &= \ P(x)\,(y_0 + c\,y_1) \ + \ Q(x) \\ &= \ P(x)\,y_2 \ + \ Q(x), \end{aligned}$$

as required. It is also not hard to check, essentially by turning the above calculation around, that if y_0 and y_2 are any two solutions to our nonhomogeneous equation, then they differ by a solution to the homogeneous equation, i.e., by a multiple of y_1.

So the solutions to (5) differ by constant multiples of y_1. The principle (4) tells us to make a change of variables that will transform these into functions that differ by constants. To do this we should clearly divide by y_1; i.e., let

$$u \ = \ y/y_1, \quad \text{equivalently,} \quad y \ = \ u\,y_1. \tag{12}$$

Let us substitute the latter equation into (5) and simplify.

$$(u\,y_1)' = P(x)\,(u\,y_1) \ + \ Q(x)$$

$$u'y_1 + u(y_1 P(x)) = P(x)(uy_1) + Q(x)$$
$$u'y_1 = Q(x)$$
$$u' = Q(x)/y_1$$

We can now find u by integrating $Q(x)/y_1$, and get y by multiplying u by y_1 (see (12)). In summary,

(13) *To solve a nonhomogeneous linear differential equation* (6), *first find a nonzero solution* y_1 *to the corresponding homogeneous linear differential equation* (7). *Then make the substitution* $y = uy_1$ *in* (6), *getting an equation that can be solved by integration, and substitute back to obtain* y.

The resulting family of functions is described by

$$y = y_1 \left(\int (Q(x)/y_1)\, dx + C\right).$$

(You might prefer to learn the substitution used in the above procedure, or memorize the final formula, or both. If you memorize the formula, you should be careful to remember that y_1 denotes a solution to the corresponding homogeneous equation.)

Example 4. Solve $y' = xy + x^3$.

Solution. We must first find a nonzero solution y_1 to the corresponding homogeneous equation, i.e., a function satisfying $y_1' = xy_1$. The method of §2 tells us that such a solution is $e^{\int x\,dx} = e^{x^2/2}$. (Since we only need one such solution, we have left out the constant ''c'' of the formula in that section.) The formula at the end of (13) now gives

(14) $$y = e^{x^2/2} \left(\int x^3/e^{x^2/2}\, dx + C\right).$$

Writing the integral as $\int x^3 e^{-x^2/2}\, dx$, we make the substitution $u = -x^2/2$, getting $\int 2ue^u\, du$. Integration by parts gives $2ue^u - 2e^u = 2(u-1)e^u$. Expressing this in terms of x, we find that our integral equals $2(-x^2/2-1)e^{-x^2/2} = -(x^2+2)e^{-x^2/2}$. Thus, (14) gives

$$\begin{aligned} y &= e^{x^2/2}\left(-(x^2+2)e^{-x^2/2} + C\right) \\ &= -x^2 - 2 + Ce^{x^2/2}. \end{aligned}$$

After all this computation, it is worth checking that these functions do indeed satisfy the given differential equation. You will not find it hard do so.

Remark: In many texts, first-order linear differential equations are written in the forms

(15) $$y' + P(x)y = Q(x),$$

(16) $$y' + P(x)y = 0$$

with the $P(x)y$ on the left, instead of on the right as in (5) and (6). This is because they will eventually be looked at in the context of higher order equations,

(17) $$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_0(x)y = Q(x).$$

I used the forms (5) and (6) because I wanted to discuss these equations as instances of (1). There is no essential difference in the method of solution (and in real life, such equations are at least as likely to appear in this form as the other). But note that because $P(x)$ effectively has opposite signs in the two formulations, it appears preceded by a minus sign in the expression corresponding to (7) in such texts.

4. A FEW MORE WORDS ON THE GENERAL PRINCIPLE (4).

In the two cases where we used the general principle (4) above, the change of variables that we made replaced the dependent variable y by a new dependent variable u, but kept the independent variable x unchanged. The method is not limited to such cases, so I will give below one class of examples of a different sort.

This section is not required reading for Math 1B – all you are required to learn from these pages are the techniques for solving homogeneous and nonhomogeneous first-order linear differential equations, developed in §§2-3 above. But (4) constitutes an important insight into the theory of differential equations, which you may find worth knowing.

Suppose a differential equation has the form

$$y' = F(y/x). \tag{18}$$

Notice that this means that the slope of the direction field is constant along all lines through the origin: on the line $y = cx$, the slope shown by the direction field is $F(c)$.

It is not hard to see from this that the operation of enlarging the picture of any solution-curve by a fixed nonzero factor r (relative to the origin) will give the picture of another solution-curve. That is, if the curve $y = f(x)$ is one solution to the equation, then for any nonzero r, another solution will be the curve $y/r = f(x/r)$; in other words $y = r\,f(x/r)$.

How do we make a change of variables that will turn this system of curves into curves that differ by additive constants?

The first step is to note that the process of ‘‘enlarging’’ a solution as described above takes a curve containing a point (x, y) to a curve contain the point (rx, ry). We would like corresponding points on different curves to have the same value of the independent variable. We can achieve this by letting our new independent variable be

$$u = y/x. \tag{19}$$

We can then take x as our new dependent variable, and eliminate y, using (19) in the form $y = ux$. To express (18) in terms of our new independent variable, it is easiest to pass to differential notation:

$$\begin{aligned} dy/dx &= F(y/x) \\ dy &= F(u)\,dx \\ d(ux) &= F(u)\,dx \\ u\,dx + x\,du &= F(u)\,dx \\ x\,du &= (F(u) - u)\,dx \\ dx/du &= x/(F(u) - u). \end{aligned} \tag{20}$$

The original ‘‘enlarging’’ operation, which leaves our new independent variable u unchanged, still multiplies our new dependent variable x by the constant r; so (as in §2) we can convert it into addition of a constant by taking logarithms. Thus, we pass to the new dependent variable $v = \ln x$, i.e., we let $x = e^v$. The last line of (20) then becomes $(dv/du)\,e^v = e^v/(F(u) - u)$, i.e., $dv/du = 1/(F(u) - u)$. This we can at last solve by integration, getting $v = \int du/(F(u) - u) + C$. We then substitute back, to get an equation relating x and y.

The above discussion points us to a somewhat more detailed description of the procedure (4) for cases where our system of transformations do not, initially, preserve the independent variable:

(21) *Given a differential equation* (1), *find a family of transformations of the plane which carries solution-curves of* (1) *to other solutions-curves of* (1). *Then make a change of variables so that this family of transformations preserves the new independent variable. Finally, choose a new dependent variable so that the transformations are given by addition of a constant to that variable. The differential equation* (1) *will then reduce to one that can be solved by integration.*

The transformations referred to above, that carry solution-curves to solution-curves, are called by specialists in the theory of differential equations ‘‘symmetries’’ of the given equation. (A Google Book search for ‘‘symmetries’’ together with ‘‘differential equations’’ gives, at the moment, 3,012 results, for your reading pleasure.) The above ideas are also used in connection with higher-order differential equations, where each family of symmetries allows an integration that decreases the order of the equation by 1.

EXERCISES

These are based on the techniques of §§2-3 above. Answers at bottom of page.

In Exercises 1-4, find the general solution to the differential equation.

1. $y' = (x+1)\,y$.

2. $y' = y/(x^2 - 9)$.

3. $y' = y \tan x$.

4. $y' = y \sec x$.

In Exercises 5-6, find the general solution to the differential equation, and then the particular solution having $y = 0$ when $x = 0$.

5. $y' = x + y$.

6. $y' = y \sec x + 1 + \sin x$.

7. Express the general solution to the differential equation $y' = xy + 1$ in terms of an integral. (The function given by that integral is not an elementary function, so you cannot carry the solution further.)

Answers. **1.** $y = c\, e^{(x+1)^2/2}$. **2.** $y = c\,((x-3)/(x+3))^{1/6}$. **3.** $y = c \sec x$. **4.** $y = c\,(\sec x + \tan x)$. **5.** General solution: $y = -1 - x + c\, e^x$. Particular solution: $y = -1 - x + e^x$. **6.** General solution: $y = (\sec x + \tan x)(\sin x + C)$. Particular solution: $y = (\sec x + \tan x) \sin x$. **7.** $y = e^{x^2/2}\,(\int e^{-x^2/2}\, dx + C)$.